

## DONALDSON–THOMAS INVARIANTS OF CALABI–YAU ORBIFOLDS UNDER FLOPS

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ABSTRACT. We study the Donaldson–Thomas type invariants for the Calabi–Yau threefold Deligne–Mumford stacks under flops. A crepant birational morphism between two smooth Calabi–Yau threefold Deligne–Mumford stacks is called an orbifold flop if the flopping locus is the quotient of weighted projective lines by a cyclic group action. We prove that the Donaldson–Thomas invariants are preserved under orbifold flops.

### 1. Introduction

The goal of this paper is to prove a natural property that the Donaldson–Thomas (DT) type invariants of Calabi–Yau threefold Deligne–Mumford (DM) stacks are preserved under orbifold flops. The techniques we use are Bridgeland’s Hall algebra identities inside the motivic Hall algebra of some abelian categories, Joyce–Song’s integration map from the motivic Hall algebra to the ring of functions on the quantum torus, and Calabrese’s method of Hall algebra identities under threefold flops.

**1.1. Motivation and the DT-invariants.** Let  $X$  be a proper smooth Calabi–Yau threefold. Fixing the topological data  $(\beta, n)$  for  $\beta \in H_2(X, \mathbb{Z})$ , and  $n \in \mathbb{Z}$ , the DT invariant  $\text{DT}_{n, \beta}$  is defined by the virtual count of the Hilbert scheme of curves on  $X$  with topological data  $(\beta, n)$ :

$$\text{DT}_{n, \beta} = \int_{[I_n(X, \beta)]^{\text{virt}}} 1,$$

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where  $I_n(X, \beta)$  is the Hilbert scheme of curves  $C$  on  $X$  (the Donaldson–Thomas moduli space) such that

$$[C] = \beta, \quad \chi(\mathcal{O}_C) = n.$$

Here  $[I_n(X, \beta)]^{\text{virt}}$  is the zero dimensional virtual fundamental class of  $I_n(X, \beta)$ , constructed by R. Thomas in [41] since the scheme  $I_n(X, \beta)$  admits a perfect obstruction theory in the sense of Li–Tian in [33] and Behrend–Fantechi in [5].

In [4], Behrend provides another way to the DT-invariants of Calabi–Yau threefolds, which are not necessarily proper. Behrend proves that the scheme  $I_n(X, \beta)$  admits a symmetric obstruction theory and if it is proper, the virtual count is given by the weighted Euler characteristic:

$$\text{DT}_{n, \beta} = \int_{[I_n(X, \beta)]^{\text{virt}}} 1 = \chi(I_n(X, \beta), \nu_I),$$

where  $\nu_I : I_n(X, \beta) \rightarrow \mathbb{Z}$  is an integer valued constructible function which we call the *Behrend function* of  $I_n(X, \beta)$ . If  $I_n(X, \beta)$  is not proper, the weighted Euler characteristic  $\chi(I_n(X, \beta), \nu_I)$  is defined as the DT-invariant for  $X$ . Behrend’s theory works for any moduli schemes of objects on the derived category of coherent sheaves  $D(X)$  admitting a symmetric obstruction theory. This makes the DT-invariants into motivic invariants.

A very important variation of DT-invariant is the Pandharipande–Thomas (PT) stable pair invariant.

**DEFINITION 1.1** ([39]). A stable pair  $[\mathcal{O}_X \xrightarrow{s} F]$  is a two-term complex in  $D^b(X)$  satisfying:

- (1)  $\dim(F) \leq 1$  and  $F$  is pure;
- (2)  $s$  has zero-dimensional cokernel.

The moduli scheme  $\text{PT}_n(X, \beta)$  of stable pairs with fixed topological data  $[F] = \beta \in H_2(X, \mathbb{Z})$ ,  $\chi(F) = n$  is a scheme and the PT-invariant is defined by

$$\text{PT}_{n, \beta}(X) = \chi(\text{PT}_n(X, \beta), \nu_{\text{PT}}),$$

where  $\nu_{\text{PT}}$  is the Behrend function on  $\text{PT}_n(X, \beta)$ . Both DT-invariants and PT-invariants are curve counting invariants of  $X$ . The famous DT/PT-correspondence conjecture in [39] equates these two invariants in terms of partition functions.

The conjecture was proved by Bridgeland [8], and Toda [42] using the wall crossing idea, under which the DT-moduli space and the PT moduli space correspond to different (limit) stability conditions in Bridgeland’s space of stability conditions.

We pay more attention to Bridgeland’s method for the proof. In [8] Bridgeland uses identities in the motivic Hall algebra  $H(\mathcal{A})$  of the abelian category of coherent sheaves  $\mathcal{A} = \text{Coh}(X)$  such that the DT-moduli space and the

PT-moduli space are both elements in the Hall algebra. Then applying the integration map as in [28], [9] Bridgeland gets the DT/PT-correspondence.

The same idea works for threefold flops. Let

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X' \\ & \searrow \psi & \swarrow \psi' \\ & Y & \end{array}$$

be a flopping contraction, such that  $\psi, \psi'$  all contract rational curves  $\mathbb{P}^1$  to singular points, with normal bundle  $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ . The local model is the Atiyah flop. In [15], Calabrese studies and proves the flop formula of the DT-type invariants using the method of the Hall algebra identities and the integration map, generalizing the idea in [8].

More precisely, for the flop  $\phi : X \dashrightarrow X'$ , Bridgeland [7] proves that their derived categories are equivalent:

$$\Phi : D(X) \rightarrow D(X'),$$

where  $\Phi$  is given by the Fourier–Mukai type transformation. Furthermore, the equivalence  $\Phi$  sends the category of perverse sheaves to perverse sheaves, that is,

$$\Phi(q \operatorname{Per}(X)) = {}^p \operatorname{Per}(X'),$$

where  $q = -(p+1)$  is the perversity. Usually we take  $p = -1, 0$ . In [15], Calabrese proves some identities in the Hall algebra  $H({}^p \mathcal{A})$  for  ${}^p \mathcal{A} := {}^p \operatorname{Per}(X)$ . Since  $\Phi$  preserves perverse sheaves, applying the integration map he gets the flop formula for the DT-invariants. A proof of the flop formula for DT-type invariants using Joyce’s wall crossing formula is given by Toda in [43]; and the study of DT-invariants under blow-ups and flops using J. Li’s degeneration formula was given by Hu–Li in [22].

**1.2. Flops of Calabi–Yau threefold stacks.** In this paper, we consider the orbifold flop of Calabi–Yau threefold DM stacks. The reason to consider Calabi–Yau threefold stacks (or orbifolds), on one hand, is that in general there exists a global Kähler moduli space, and there are two large volume limit points: one corresponds to the crepant resolution of the orbifold singularity, and one corresponds to the orbifold singularity. They are usually derived equivalent, hence some information in the orbifold side determines the side of the crepant resolution. On the other hand, “The Crepant Transformation Conjecture” (CTC) of Y. Ruan in Gromov–Witten (GW) theory has been attracting a lot of interests, see [32], [31], [20], [14]. It is interesting to consider the general CTC conjecture for DT-theory. Also this gives us a chance to learn Bridgeland’s method of Hall algebra identities.

An orbifold flop of Calabi–Yau threefold DM stacks is given by the diagram:

$$(1) \quad \begin{array}{ccc} & \mathcal{Z} & \\ f \swarrow & & \searrow f' \\ \mathcal{X} & \overset{\phi}{\dashrightarrow} & \mathcal{X}' \\ \psi \searrow & & \swarrow \psi' \\ & Y & \end{array}$$

where

- (1)  $\mathcal{X}$  and  $\mathcal{X}'$  are smooth Calabi–Yau threefold DM stacks;
- (2)  $Y$  is a singular variety with only zero-dimensional singularities;
- (3) Both  $\psi$  and  $\psi'$  contract cyclic quotients of weighted projective lines  $\mathbb{P}(a_1, a_2)$ ,  $\mathbb{P}(b_1, b_2)$  respectively;
- (4)  $\mathcal{Z}$  is the common weighted blow-up along the exceptional locus.

REMARK 1.2. Actually the contraction map  $\psi : \mathcal{X} \rightarrow Y$  can be made to have one dimensional singularities as in [15]. One still can get some formulas on the DT type invariants, see [17]. Here we only focus on the case of  $Y$  with isolated singularities.

Similar to Abramovich–Chen in [1], [18], we prove that the derived categories of  $\mathcal{X}$  and  $\mathcal{X}'$  are equivalent for such orbifold flops using the idea of perverse point sheaves of Bridgeland. The equivalence

$$(2) \quad \Phi : D(\mathcal{X}) \rightarrow D(\mathcal{X}')$$

is given by the Fourier–Mukai transformation  $\Phi(-) = f'_*(f^*(-))$ . Moreover, the equivalence  $\Phi$  also sends perverse sheaves to perverse sheaves.

$$(3) \quad \Phi({}^q \text{Per}(\mathcal{X})) = {}^p \text{Per}(\mathcal{X}'),$$

where  $q = -(p + 1)$ .

Let  ${}^p \mathcal{A} := {}^p \mathcal{A}(\mathcal{X}) := {}^p \text{Per}(\mathcal{X})$ . We work on the Hall algebra  $H({}^p \mathcal{A})$  of  ${}^p \mathcal{A}$ . Let  $K(\mathcal{X})$  be the numerical  $K$ -group of  $\mathcal{X}$ , and

$$F_0 K(\mathcal{X}) \subset F_1 K(\mathcal{X}) \subset \cdots \subset K(\mathcal{X})$$

be the filtration by dimension of support, see Section 3.3.1 for more details.

Fixing a  $K$ -group class  $\alpha \in F_1 K(\mathcal{X})$ , let  $\text{Hilb}^\alpha(\mathcal{X})$  be the Hilbert scheme of substacks of  $\mathcal{X}$  with class  $\alpha$ . The DT-invariant is defined by

$$\text{DT}_\alpha(\mathcal{X}) = \chi(\text{Hilb}^\alpha(\mathcal{X}), \nu_H),$$

where  $\nu_H$  is the Behrend function in [4] of  $\text{Hilb}^\alpha(\mathcal{X})$ . Define the DT-partition function by

$$(4) \quad \text{DT}(\mathcal{X}) = \sum_{\alpha \in F_1 K(\mathcal{X})} \text{DT}_\alpha(\mathcal{X}) q^\alpha.$$

Similarly the notion of PT-stable pair for the threefold DM stack  $\mathcal{X}$  is very similar to Definition 1.1. A PT-stable pair  $[\mathcal{O}_{\mathcal{X}} \xrightarrow{s} F] \in D^b(\mathcal{X})$  is an object in the derived category such that  $F$  is a pure one-dimensional sheaf supported on curves in  $\mathcal{X}$  with topological data  $\beta$ , and the cokernel  $\text{Coker}(s)$  is zero-dimensional. Let  $\text{PT}^\beta(\mathcal{X})$  be the PT-moduli space of stable pairs with  $K$ -group class  $\beta$ . Then the PT-invariant is defined by:

$$\text{PT}_\beta(\mathcal{X}) = \chi(\text{PT}^\beta(\mathcal{X}), \nu_{\text{PT}}),$$

where  $\nu_{\text{PT}}$  is the Behrend function of  $\text{PT}^\beta(\mathcal{X})$ . The PT-partition function by

$$(5) \quad \text{PT}(\mathcal{X}) = \sum_{\beta \in F_1 K(\mathcal{X})} \text{PT}_\beta(\mathcal{X}) q^\beta.$$

Define the following DT-type partition functions:

$$\begin{aligned} \text{DT}_0(\mathcal{X}) &= \sum_{\alpha \in F_0 K(\mathcal{X})} \text{DT}_\alpha(\mathcal{X}) q^\alpha; \\ \text{DT}_{\text{exc}}(\mathcal{X}) &= \sum_{\substack{\alpha \in F_1 K(\mathcal{X}); \\ \psi_* \alpha = 0}} \text{DT}_\alpha(\mathcal{X}) q^\alpha; \\ \text{DT}_{\text{exc}}^\vee(\mathcal{X}) &= \sum_{\substack{\alpha \in F_1 K(\mathcal{X}); \\ \psi_* \alpha = 0}} \text{DT}_{-\alpha}(\mathcal{X}) q^\alpha. \end{aligned}$$

The main result in the paper is:

**THEOREM 1.3.** *Let  $\phi: \mathcal{X} \dashrightarrow \mathcal{X}'$  be an orbifold flop of Calabi–Yau threefold DM stacks. Assume the DT/PT-correspondence formula of A. Bayer [3]. Then we have*

$$\Phi_* \left( \text{DT}(\mathcal{X}) \cdot \frac{\text{DT}_{\text{exc}}^\vee(\mathcal{X})}{\text{DT}_0(\mathcal{X})} \right) = \text{DT}(\mathcal{X}') \cdot \frac{\text{DT}_{\text{exc}}^\vee(\mathcal{X}')}{\text{DT}_0(\mathcal{X}')},$$

where  $\Phi_*$  is understood as sending the data  $\alpha \in K(\mathcal{X})$  to  $\varphi(\alpha) \in K(\mathcal{X}')$ .

We prove Theorem 1.3 along the method of Bridgeland [8] and Calabrese [15] by working on the Hall algebra identities in  $H({}^p\mathcal{A})$ . One can define the perverse Hilbert scheme

$${}^p \text{Hilb}^\alpha(\mathcal{X}/Y)$$

which parametrizes the quotients  $\mathcal{O}_{\mathcal{X}} \rightarrow F$  in the category  ${}^p\mathcal{A}$  with fixed class  $[F] = \alpha$ , since the structure sheaf  $\mathcal{O}_{\mathcal{X}} \in {}^p\mathcal{A}$ . Then we define

$${}^p \text{DT}_\alpha(\mathcal{X}/Y) = \chi({}^p \text{Hilb}^\alpha(\mathcal{X}/Y), \nu_{pH}),$$

where  $\nu_{pH}$  is the Behrend function for  ${}^p \text{Hilb}^\alpha(\mathcal{X}/Y)$ . The partition function is defined by:

$${}^p \text{DT}(\mathcal{X}/Y) = \sum_{\alpha \in F_1 K(\mathcal{X})} {}^p \text{DT}_\alpha(\mathcal{X}/Y) q^\alpha.$$

We prove that

$${}^p\mathrm{DT}(\mathcal{X}/Y) = \mathrm{DT}(\mathcal{X}) \cdot \mathrm{PT}_{\mathrm{exc}}^{\vee}(\mathcal{X}),$$

where

$$\mathrm{PT}_{\mathrm{exc}}^{\vee}(\mathcal{X}) = \sum_{\beta \in F_1 K(\mathcal{X}), \psi_*(\beta)=0} \mathrm{PT}_{-\beta}(\mathcal{X})q^{\beta}.$$

We need a DT/PT-correspondence result for Calabi–Yau threefold stacks:

$$(6) \quad \mathrm{PT}_{\mathrm{exc}}^{\vee}(\mathcal{X}) = \frac{\mathrm{DT}_{\mathrm{exc}}^{\vee}(\mathcal{X})}{\mathrm{DT}_0(\mathcal{X})}.$$

This formula is announced by A. Bayer in [3]. Since the paper is still unavailable, we comment Bayer’s proof for (6) here. Bayer’s proof is a generalization of the DT/PT-correspondence of Bridgeland in [8] to the Calabi–Yau threefold DM stacks. He works on some Hall algebra identities in  $H({}^p\mathcal{A}_{\leq 1})$ . Since the DT and PT moduli spaces are objects in  $H({}^p\mathcal{A}_{\leq 1})$ , the main Hall algebra identity Proposition 6.5 in [8] holds for Calabi–Yau threefold DM stacks. Then Bayer derives Formula (6) from [8, Proposition 6.5] and the no-pole theorem used in [8, Theorem 6.3] and Proposition 5.14 in this paper.

Since for the orbifold flop  $\phi: \mathcal{X} \dashrightarrow \mathcal{X}'$ , the derived equivalence  $\Phi$  sends  ${}^q\mathcal{A}(\mathcal{X})$  to  ${}^p\mathcal{A}(\mathcal{X}')$ , where  $q = -(p+1)$ . The theorem follows.

**1.3. Relation to the crepant resolution conjecture.** The “Crepan Resolution Conjecture” for the DT-invariants was formulated by J. Bryan etc in [11] for Calabi–Yau orbifolds satisfying the Hard Lefschetz (HL) conditions. In [16], Calabrese proves part of the conjecture for Calabi–Yau threefold stacks satisfying the HL conditions, using similar method of Hall algebra identities in [15]. Note that Bryan and Steinberg [12] also prove partial result of the crepant resolution conjecture for the DT-invariants.

Our orbifold flops need not to satisfy the HL condition as required by J. Bryan etc. in [11]. There exists orbifold flops  $\mathcal{X} \dashrightarrow \mathcal{X}'$  of threefold Calabi–Yau stacks such that  $\mathcal{X}$  satisfies the HL condition, while  $\mathcal{X}'$  does not. The main result in Theorem 1.3 implies some information on the Donaldson–Thomas invariants for  $\mathcal{X}'$  from the ones for  $\mathcal{X}$ , see Section 6. We hope that our study of orbifold flop may shed more light on the crepant resolution conjecture.

**1.4. Comparison to the Gromov–Witten invariants under flops.**

DT-invariants have deep connections to Gromov–Witten (GW) invariants via the GW/DT-correspondence in [34], [35]. This conjecture has been proved in many cases, including toric threefolds in [36], and quintic threefolds in [38].

For two birational Calabi–Yau stacks, the crepant transformation conjecture (CTC) says that the partition functions of their GW invariants are related by the analytic continuation. Let  $\mathcal{X} \dashrightarrow \mathcal{X}'$  be a toric crepant birational transformation given by a toric wall crossing. In [20], the authors prove the genus

zero CTC. Using Givental’s quantization, in [19], Coates and Iritani solved the higher genus CTC. For such a toric flop, their derived categories are equivalent, and the kernel for the Fourier–Mukai transform is given by the common blow-up. In [20], the authors prove that the Fourier–Mukai transform matches the analytic continuation of the quantum connections for  $\mathcal{X}$  and  $\mathcal{X}'$ . Since applying twice the Fourier–Mukai transform, one gets the monodromy for the  $K$ -theory, and hence the monodromy of the derived category, the CTC implies that the monodromy given by the Fourier–Mukai transform is the same as the monodromy given by the quantum connections. More general orbifold flops are studied in [14].

Recall for the orbifold flop of Calabi–Yau threefold DM stacks, the Fourier–Mukai transform preserves the perverse sheaves for  $\mathcal{X}$  and  $\mathcal{X}'$ . Applying twice the Fourier–Mukai transform gives the Seidel–Thomas twist [40] for the derived category. It is interesting to study how the Fourier–Mukai transform can relate DT-invariants and GW-invariants together using the method in this paper and the calculation in [20].

**1.5. Outline.** The brief outline of the paper is as follows. We introduce the orbifold flops for Calabi–Yau threefold DM stacks in Section 2. In Section 3, we talk about the perverse sheaves on the Calabi–Yau threefold DM stacks and prove the derived equivalence for the orbifold flops. This generalizes the results as in [1] and [7]. We also define the counting invariants in the derived category and form the partition functions of the invariants. In Section 4, we review the motivic Hall algebra of Joyce [26] and Bridgeland [9], and define the integration map. We prove Theorem 1.3 in Section 5 using the method of Bridgeland and Calabrese on Hall algebra identities. Finally in Section 6, we talk about the HL condition for the orbifold flops.

## 2. Orbifold flop of three dimensional Calabi–Yau DM stacks

We define the orbifold flop for three dimensional Calabi–Yau DM stacks. We usually use calligraphic letter  $\mathcal{X}$  to represent a stack, and  $X$  for its coarse moduli space. For orbifold flops, we use  $Z, Z'$  to represent the cyclic quotients of exceptional weighted projective line stacks.

**2.1. The local construction.** In this section, we give the local construction of orbifold flop in three dimensional Calabi–Yau orbifolds or Deligne–Mumford (DM) stacks.

Fix  $\mathbf{a} = (a_0, a_1)$  and  $\mathbf{b} = (b_0, b_1)$  as positive integers. Let  $\mathbb{P}(a_0, a_1), \mathbb{P}(b_0, b_1)$  be the corresponding weighted projective stack lines. If  $\gcd(a_0, a_1, b_0, b_1) = d > 1$ , then our stacks  $\mathcal{X}, \mathcal{X}'$  below have global nontrivial gerbe structures, since  $\mathcal{X}$  and  $\mathcal{X}'$  have a global trivial  $\mu_d$  action. For simplicity, we make the assumption

that  $\gcd(a_0, a_1, b_0, b_1) = 1$ . To preserve the Calabi–Yau property we require  $a_0 + a_1 = b_0 + b_1$ . We will call such condition the *Calabi–Yau condition*.

Recall that in [29] Kawamata defines the construction of so called “toric flops”. We briefly explain the construction here. Let  $\mathbb{C}^*$  acts on the affine variety  $\mathbb{A} = \mathbb{A}^4$  by:

$$(7) \quad \lambda(x_0, x_1, y_0, y_1) = (\lambda^{a_0} x_0, \lambda^{a_1} x_1, \lambda^{-b_0} y_0, \lambda^{-b_1} y_1).$$

Consider the following stack quotients:

$$\begin{aligned} \tilde{\mathcal{X}} &= [(\mathbb{A} \setminus \{x_0 = x_1 = 0\})/\mathbb{C}^*]; \\ \tilde{\mathcal{X}}' &= [(\mathbb{A} \setminus \{y_0 = y_1 = 0\})/\mathbb{C}^*]; \\ \tilde{\mathcal{Y}} &= [\mathbb{A}/\mathbb{C}^*]. \end{aligned}$$

Let  $\bar{Y}$  be the coarse moduli space of  $\tilde{\mathcal{Y}}$ , Then  $\bar{Y} = \text{Spec } R^{\mathbb{C}^*}$ , where  $R = \mathbb{C}[x_0, x_1, y_0, y_1]$ . There is a diagram of threefold flops with quotient singularities:

$$(8) \quad \begin{array}{ccc} & \tilde{\mathcal{Z}} & \\ f \swarrow & & \searrow f' \\ \tilde{\mathcal{X}} & & \tilde{\mathcal{X}}' \\ \psi \searrow & & \swarrow \psi' \\ & \bar{Y} & \end{array}$$

where  $\tilde{\mathcal{Z}}$  is the fibre product. From [29], the morphisms  $\psi, \psi'$  are birational contraction morphisms whose exceptional loci  $\tilde{\mathcal{Z}}, \tilde{\mathcal{Z}}'$  are isomorphic to the weighted projective stack lines  $\mathbb{P}(a_0, a_1)$  and  $\mathbb{P}(b_0, b_1)$ , respectively. Both  $\psi$  and  $\psi'$  contract the weighted projective stacks  $\tilde{\mathcal{Z}}, \tilde{\mathcal{Z}}'$  to a single point. The DM stack  $\tilde{\mathcal{Z}}$  is the common blow-up of  $\tilde{\mathcal{X}}, \tilde{\mathcal{X}}'$  along  $\tilde{\mathcal{Z}} \subset \tilde{\mathcal{X}}, \tilde{\mathcal{Z}}' \subset \tilde{\mathcal{X}}'$ , respectively.

If all the  $a_i$  and  $b_i$  are one, this is the local model of the famous Atiyah flop or the conifold flop. We are interested in three dimensional orbifolds, whose coarse moduli spaces are  $\mathbb{Q}$ -Gorenstein algebraic varieties with quotient singularities. If  $X$  is a Calabi–Yau threefold with terminal singularities, by Kollar [30], the flop  $X'$  of  $X$  and the contraction  $Y$  all have terminal singularities. The flopping curves are always  $\mathbb{P}^1/\mu_n$ , where  $\mu_n$  is a cyclic group of order  $n$  acting on  $\mathbb{P}^1$  by rotation. This is due to the fact that a terminal singularity inside  $Y$  is isolated, which is a hypersurface singularity, and is deformation equivalent to the quotient  $\mathbb{C}^3/\mu_n$  with action by  $(1, -1, r)$ , where  $(r, n) = 1$ . We put this construction into the toric picture of Kawamata.



DEFINITION 2.1. A local orbifold flop is given by the following diagram of stack quotients:

$$(9) \quad \begin{array}{ccc} & \mathcal{Z} = [\tilde{\mathcal{Z}}/\mu_n] & \\ f \swarrow & & \searrow f' \\ \mathcal{X} = [\tilde{\mathcal{X}}/\mu_n] & & \mathcal{X}' = [\tilde{\mathcal{X}}'/\mu_n] \\ \psi \searrow & & \swarrow \psi' \\ & \mathcal{Y} = [\overline{\mathcal{Y}}/\mu_n] & \end{array}$$

where  $\mu_n$  act on  $\overline{\mathcal{Y}}$  by  $\zeta(x_0, x_1, y_0, y_1) = (\zeta x_0, \zeta^{-1}x_1, \zeta^r y_0, y_1)$ , where  $(n, r) = 1$ ; and  $[\overline{\mathcal{Y}}/\mu_n]$  is the coarse moduli space of  $[\mathcal{Y}/\mu_n]$ . The morphism  $\psi, \psi'$  are birational contraction morphisms whose exceptional loci  $Z, Z'$  are isomorphic to the weighted projective lines  $\mathbb{P}(a_0, a_1)/\mu_n$  and  $\mathbb{P}(b_0, b_1)/\mu_n$ , respectively.

If  $a_0, a_1$  are coprime, then the weighted projective line  $\mathbb{P}(a_0, a_1)$  has only two singular points  $[1, 0]$  and  $[0, 1]$  and the quotient  $\mathbb{P}(a_0, a_1)/\mu_n$  is a toric orbifold in the sense of [6], [23]. The two singular points  $[1, 0]$  and  $[0, 1]$  will have local orbifold groups  $\mu_{a_0 n}$  and  $\mu_{a_1 n}$ .

If  $a_0, a_1$  are not coprime, then the weighted projective line  $\mathbb{P}(a_0, a_1)$  is a  $\mu_d$ -gerbe over  $\mathbb{P}(\frac{a_0}{d}, \frac{a_1}{d})$ , where  $d = \gcd(a_0, a_1)$ . The weighted projective line  $\mathbb{P}(\frac{a_0}{d}, \frac{a_1}{d})$  has two singular points  $[1, 0]$  and  $[0, 1]$  and the quotient  $\mathbb{P}(a_0, a_1)/\mu_n$  is a toric Deligne–Mumford stack in the sense of [6], [23]. The two singular points  $[1, 0]$  and  $[0, 1]$  will also have local orbifold groups  $\mu_{a_0 n}$  and  $\mu_{a_1 n}$ , but the local action on it and the normal bundle are quite different comparing to the previous case.

**2.2. Orbifold flop for threefold stacks.** In this section, we establish the general definition of flops of Calabi–Yau threefold stacks. Let  $X$  be a quasi-projective  $\mathbb{Q}$ -Gorenstein Calabi–Yau variety. The minimal positive integer  $m$  satisfying the condition that the saturation  $\omega_X^{[m]}$  is invertible, is called the canonical index of  $X$ . We denote by  $\mathcal{X}$  the covering Deligne–Mumford stack of  $X$ . As in [1, Definition 2.1.1], the stack  $\mathcal{X}$  is a quotient stack

$$\mathcal{X} = [P_X/\mathbb{C}^*],$$

where  $P_X = \text{Spec}_X(\bigoplus_{i \in \mathbb{Z}} \omega_X^{[i]})$ .

DEFINITION 2.2. We say that two smooth Calabi–Yau threefold DM stacks are related by an *orbifold flop*  $\phi: \mathcal{X} \dashrightarrow \mathcal{X}'$  if they fit into the following commutative diagram

$$\begin{array}{ccc}
 & E \subset \mathcal{Z} & \\
 f \swarrow & & \searrow f' \\
 Z \subset \mathcal{X} & & Z' \subset \mathcal{X}' \\
 \psi \searrow & & \swarrow \psi' \\
 & p \in Y &
 \end{array}$$

such that

- $Z \cong \mathbb{P}(\mathbf{a})/\mu_n$  and  $Z' \cong \mathbb{P}(\mathbf{b})/\mu_n$ , where  $\mathbb{P}(\mathbf{a}) = \mathbb{P}(a_0, a_1)$ , and  $\mathbb{P}(\mathbf{b}) = \mathbb{P}(b_0, b_1)$ ;
- the normal bundle  $N_Z$  is isomorphic to  $(\oplus_i \mathcal{O}_{\mathbb{P}(\mathbf{a})}(-b_i))/\mu_n$  and  $N_{Z'}$  is isomorphic to  $(\oplus_i \mathcal{O}_{\mathbb{P}(\mathbf{b})}(-a_i))/\mu_n$ ;
- $\psi$  and  $\psi'$  are birational contractions such that the exceptional loci  $Z$  and  $Z'$  map to the point  $p$ ;
- $\mathcal{Z}$  is the common blow-up of  $\mathcal{X}$  and  $\mathcal{X}'$  along  $Z$  and  $Z'$ , respectively.  $E = \mathbb{P}(\mathbf{a})/\mu_n \times \mathbb{P}(\mathbf{b})/\mu_n$  is the exceptional divisor.

REMARK 2.3. In the special case  $\mu_n = 1$ , we call  $\phi: \mathcal{X} \dashrightarrow \mathcal{X}'$  *orbifold flop* of type  $(\mathbf{a}, \mathbf{b})$ .

REMARK 2.4. The orbifold flop in Remark 2.3 is defined and studied in [14], where the authors consider the general quasimple orbifold flops for higher dimensional DM stacks. The flopping locus are the weighted projective stacks  $\mathbb{P}(a_1, \dots, a_r)$  and  $\mathbb{P}(b_0, \dots, b_r)$ , respectively.

**2.3. Hard Lefschetz condition.** Recall that a DM stack  $\mathcal{X}$  satisfies Hard Lefschetz (HL) condition, if the age for a local isotropy group element of any point  $x \in \mathcal{X}$  is equal to the age of its inverse.

PROPOSITION 2.5. *Let  $\phi: \mathcal{X} \dashrightarrow \mathcal{X}'$  be an orbifold flop of type  $(\mathbf{a}, \mathbf{b})$ . In order for both  $\mathcal{X}$  and  $\mathcal{X}'$  to satisfy the hard Lefschetz condition, it is necessary and sufficient that  $a_i = b_i$  for all  $i = 0, 1$  after reordering. We call this type of orbifold flop the HL orbifold flop.*

*Proof.* This is a three dimensional case of the more general quasi-simple orbifold flops defined in [14]. The result is a special case of a more general result there. We provide a proof here.

The sufficient condition is clear. Let us prove the necessary condition. Let  $I$  be the involution of the twisted sector corresponding to  $g \mapsto g^{-1}$ . Then the

orbifold is said to satisfy the *hard Lefschetz condition* if

$$\text{age}(x) = \text{age}(I(x))$$

for all  $x$  in the twisted sector.

Fix  $\mathbf{a}, \mathbf{b}$  satisfying the CY condition  $\sum a_i = \sum b_i$ . We may replace the stacks  $\mathcal{X}$  and  $\mathcal{X}'$  by their corresponding local models. A twisted sector of  $\mathcal{X}$  is determined by a pair of integers  $(d, k)$  satisfying the following condition:

$$(10) \quad d \text{ divides at least one } a_i, 0 < k < d \text{ and } \gcd(k, d) = 1.$$

Let  $[\mathbf{a} : d]$  be the subset of  $a_0, a_1$  such that  $d|a_i$ . The corresponding twisted sector is isomorphic to  $\bigoplus_{b_i: d|b_i} \mathcal{O}_{\mathbb{P}([\mathbf{a}:d])}(-b_i)$ . The normal bundle to this twisted sector is

$$\bigoplus_{d|a_i} \mathcal{O}(a_i) \oplus \bigoplus_{d|b_i} \mathcal{O}(-b_i).$$

The hard Lefschetz condition can be stated as an equality of ages for a group element and its inverse. Applying that to  $(d, k)$  and  $(d, d - k)$  yields the condition

$$\sum_{i=0}^1 \left\langle \frac{ka_i}{d} \right\rangle + \sum_{i=0}^1 \left\langle \frac{-kb_i}{d} \right\rangle = \sum_{i=0}^1 \left\langle \frac{-ka_i}{d} \right\rangle + \sum_{i=0}^1 \left\langle \frac{kb_i}{d} \right\rangle,$$

where  $\langle y \rangle$  denotes the fractional part of  $y$ . Rewrite the above equation:

$$\sum_{i=0}^1 \left( \left\langle \frac{ka_i}{d} \right\rangle - \left\langle \frac{-ka_i}{d} \right\rangle \right) = \sum_{i=0}^1 \left( \left\langle \frac{kb_i}{d} \right\rangle - \left\langle \frac{-kb_i}{d} \right\rangle \right).$$

Let  $\{y\}$  equal  $\langle y \rangle$  when  $y$  is not an integer and  $1/2$  when  $y$  is an integer. Then

$$\langle y \rangle - \langle -y \rangle = 2\{y\} - 1,$$

and the equation above becomes

$$(11) \quad \sum_{i=0}^1 2\{xa_i\} = \sum_{i=0}^1 2\{xb_i\},$$

where  $x = k/d$  satisfying condition (10) above. Note that for small positive  $x$  such that  $xa_i < 1$  (11) becomes

$$2 \sum_{i=1}^1 (xa_i - 1) = 2 \sum_{j=1}^1 (xb_j - 1).$$

Assume that  $a_0 \geq a_1$  and likewise for  $b_i$ . If  $a_0 > b_0$ , then choose  $x = 1/a_0$ , i.e.  $(d, k) = (a_0, 1)$  and  $\text{RHS} = \text{LHS} - 1$ . This contradicts the equation, so it must be that  $a_0 = b_0$ . The  $a_1 = b_1$  is a similar argument.  $\square$

### 3. Perverse coherent sheaves and the derived equivalence

**3.1. Perverse coherent sheaves.** Fix a smooth Calabi–Yau threefold DM stack  $\mathcal{X}$ , denote by  $\mathcal{A} := \text{Coh}(\mathcal{X})$  the Abelian category of coherent sheaves over  $\mathcal{X}$ . Let  $D(\mathcal{X}) := D(\mathcal{A}) = D(\text{Coh}(\mathcal{X}))$  be the derived category of coherent sheaves over  $\mathcal{X}$ . The Abelian category  $\text{Coh}(\mathcal{X})$  is the heart of the standard  $t$ -structure of  $D(\text{Coh}(\mathcal{X}))$ . Let  $D^b(\mathcal{X})$  be the bounded derived category of coherent sheaves over  $\mathcal{X}$ .

Let  $\phi : \mathcal{X} \dashrightarrow \mathcal{X}'$  be an orbifold flop of Calabi–Yau threefold stacks, i.e. there exists a commutative diagram in Definition 2.2

$$\begin{array}{ccc} Z \subset \mathcal{X} & & Z' \subset \mathcal{X}' \\ & \searrow \psi & \swarrow \psi' \\ & p \in Y & \end{array}$$

This orbifold flop satisfies the following properties:

- (1)  $\psi$  and  $\psi'$  are proper, birational and an isomorphism in codimension one;
- (2)  $Y$  is a projective variety and only has zero dimensional singular locus;
- (3) the dualising sheaf of  $Y$  is trivial, i.e.  $\omega_Y = \mathcal{O}_Y$ ;
- (4)  $R\psi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_Y$ ;  $R\psi'_* \mathcal{O}_{\mathcal{X}'} = \mathcal{O}_Y$ ;
- (5)  $\dim_{\mathbb{Q}} N^1(\mathcal{X}/Y)_{\mathbb{Q}} = 1$ , so is  $\dim_{\mathbb{Q}} N^1(\mathcal{X}'/Y)_{\mathbb{Q}}$ ,

where  $N^1(\mathcal{X}/Y)_{\mathbb{Q}} = N^1(\mathcal{X}/Y)_{\mathbb{Z}} \otimes \mathbb{Q}$  and  $N^1(\mathcal{X}/Y)_{\mathbb{Z}}$  is the group of divisors on  $\mathcal{X}$  modulo numerical equivalence over  $Y$ . Similar results hold for  $N^1(\mathcal{X}'/Y)_{\mathbb{Z}}$ . Note that  $N^1(\mathcal{X}/Y)_{\mathbb{Z}}$  is different from the numerical  $K$ -group  $F_1K(\mathcal{X})$ .

*Perverse  $t$ -structure on  $\mathcal{X}$ .* Let

$$\pi : \mathcal{X} \rightarrow X$$

be the map to its coarse moduli space, so that we have the following diagram:

$$\begin{array}{ccc} \mathcal{X} & & \\ \pi \downarrow & \searrow \psi & \\ X & \xrightarrow{\bar{\psi}} & Y \end{array}$$

As in [1, §4.2] there are two sub-categories of  $D^b(\mathcal{X})$ :

$$\begin{cases} B = \{L\pi^*C \in D^b(\mathcal{X}) \mid C \in D(X)\}; \\ C_2 = \{C \in D(\mathcal{X}) \mid R\pi_*C = 0\}. \end{cases}$$

The pair  $(B, C_2)$  gives a semiorthogonal decomposition on  $D(\mathcal{X})$ . On the category  $C_2$ , there is a standard  $t$ -structure which is induced from the standard  $t$ -structure on  $D(\mathcal{X})$ .

Recall from [7], for the map  $\bar{\psi} : X \rightarrow Y$ , there is a perverse  $t$ -structure  $t(-1)$  and the heart of this  $t$ -structure is denoted by  $\text{Per}^{-1}(X/Y)$ .

DEFINITION 3.1. The derived functor  $R\pi_*$  has right adjoint  $\pi^1$  and the left adjoint  $L\pi^*$ . Denote by  $t(p, 0)$  the  $t$ -structure obtained by gluing: the perverse  $t$ -structure  $t(p)$  on  $D^b(X)$ , and the standard  $t$ -structure on  $C_2$ . We denote by the heart of this  $t$ -structure by  $\text{Per}^p(\mathcal{X}/Y) := \text{Per}^{p,0}(\mathcal{X}/Y)$ . Usually we take  $p = -1, 0$  and we always denote by  $\text{Per}(\mathcal{X}/Y) := \text{Per}^{-1}(\mathcal{X}/Y)$ .

Recall that in [1, §4.2], the perverse sheaves are classified as follows: An object  $E$  in  $D(\mathcal{X})$  is a “perverse sheaf” that is,  $E \in \text{Per}(\mathcal{X}/Y)$  if:

- (1)  $R\pi_*E$  is a perverse sheaf for  $\bar{\psi}: X \rightarrow Y$ ;
- (2)  $\text{Hom}(E, C) = 0$  for all  $C$  in  $C_2^{>0}$  and  $\text{Hom}(D, E) = 0$  for all  $D$  in  $C_2^{<0}$ .

Then Lemma 4.2.1. of [1] classifies all perverse coherent sheaves.

LEMMA 3.2. *An object  $E \in D(\mathcal{X})$  is a perverse sheaf if and only if the following conditions are satisfied:*

- (1)  $H_i(E) = 0$  unless  $i = 0$  or  $1$ ;
- (2)  $R^1\psi_*H_0(E) = 0$  and  $R^0\psi_*H_1(E) = 0$ ;
- (3)  $\text{Hom}(\pi_*H_0(E), C) = 0$  for any sheaf  $C$  on  $X$  satisfying  $\bar{\psi}_*C = R^1\bar{\psi}_*C = 0$ ;
- (4)  $\text{Hom}(D, H_1(E)) = 0$  for any sheaf  $D$  in  $C_2$ .

Recall that in [7], [1], the perverse sheaves can be obtained by tilting a torsion pair. We say that an object  $E \in D(\mathcal{A})$  connects to  $C_2$ , denoted by  $E|C_2$  if  $E$  satisfies the conditions:  $\text{Hom}(E, C) = 0$  for all  $C$  in  $C_2^{>0}$  and  $\text{Hom}(D, E) = 0$  for all  $D$  in  $C_2^{<0}$ . Denoted by  $\text{Coh}(X)$  the Abelian category of coherent sheaves on  $X$ . Let

$$\mathcal{C} = \{E \in \text{Coh}(X) | R\bar{\psi}_*E = 0\}$$

and let

$$\begin{aligned} {}^0\mathcal{T} &= \{T \in \mathcal{A} | R^1\bar{\psi}_*(R\pi_*T) = 0; T|C_2\}; \\ {}^{-1}\mathcal{T} &= \{T \in \mathcal{A} | R^1\bar{\psi}_*(R\pi_*T) = 0, \text{Hom}(T, \mathcal{C}) = 0, T|C_2\}; \\ {}^0\mathcal{F} &= \{F \in \mathcal{A} | R^0\bar{\psi}_*(R\pi_*T) = 0; \text{Hom}(\mathcal{C}, F) = 0, F|C_2\}; \\ {}^{-1}\mathcal{F} &= \{F \in \mathcal{A} | R^0\bar{\psi}_*(R\pi_*T) = 0; F|C_2\}. \end{aligned}$$

Then  $({}^p\mathcal{T}, {}^p\mathcal{F})$  is a torsion pair on  $\mathcal{A}$  for  $p = -1, 0$  and a tilt of  $\mathcal{A}$  with respect to the torsion pair is the category of perverse coherent sheaves  ${}^p\mathcal{A} := \text{Per}^p(\mathcal{X}/Y)$ . Then every element  $E \in {}^p\mathcal{A}$  fits into the exact sequence:

$$(12) \quad F[1] \hookrightarrow E \twoheadrightarrow T$$

with  $F \in {}^p\mathcal{F}$  and  $T \in {}^p\mathcal{T}$ .

From Bridgeland [7] and Abramovich–Chen [1], the category of perverse sheaves forms a heart of  $t$ -structure on  $D(\mathcal{X})$ . Usually there are actually two perversities  $p = -1, 0$ .

**3.2. Derived equivalence.** Let  $\phi : \mathcal{X} \dashrightarrow \mathcal{X}'$  be an orbifold flop as in Definition 2.2, in this section we prove, following the method of [7], [1], that there is an equivalence between derived categories:

$$(13) \quad \Phi : D(\mathcal{X}) \rightarrow D(\mathcal{X}')$$

by the Fourier–Mukai transformation and

$$\Phi(\mathrm{Per}^{-1}(\mathcal{X}/Y)) = \mathrm{Per}^0(\mathcal{X}'/Y).$$

### 3.2.1. Perverse point ideal sheaves.

DEFINITION 3.3. A perverse ideal sheaf  $F \in {}^p\mathcal{A}$  is a sheaf such that it fits into the exact sequence

$$0 \rightarrow F \rightarrow \mathcal{O}_{\mathcal{X}} \rightarrow E \rightarrow 0$$

in  ${}^p\mathcal{A}$ . The object  $E$  is called the “perverse structure sheaf”. A perverse point sheaf is a perverse structure sheaf such that it is numerically equivalent to the structure sheaf of a point.

We have a similar proposition as in [1, Lemma 4.3.2].

PROPOSITION 3.4. *A perverse ideal sheaf is a sheaf. A sheaf  $F \in \mathrm{Coh}(\mathcal{X})$  is a perverse ideal sheaf if and only if it satisfies the following conditions:*

- (1)  $R\pi_*F$  is a perverse ideal sheaf of  $\bar{f} : X \rightarrow Y$ ;
- (2)  $\mathrm{Hom}(D, F) = 0$  for any sheaf  $D \in C_2$ .

Perverse point sheaves and perverse point-ideal sheaves are simple objects, which satisfy the following properties. Let  $E_1, E_2$  be two perverse point sheaves. Then

$$\mathrm{Hom}(E_1, E_2) = \begin{cases} 0, & E_1 \not\cong E_2; \\ \mathbb{C}, & E_1 \cong E_2. \end{cases}$$

Similarly let  $F_1, F_2$  be two perverse point-ideal sheaves. Then from [1]

$$(14) \quad \mathrm{Hom}(F_1, F_2) = \begin{cases} 0, & F_1 \not\cong F_2; \\ \mathbb{C}, & F_1 \cong F_2. \end{cases}$$

### 3.2.2. Moduli of perverse point sheaves. Let

$$\mathcal{M}(\mathcal{X}/Y) : \mathrm{Sch} \rightarrow \mathrm{Sets}$$

be the functor that sends a scheme  $S$  to the set of equivalence classes of families of perverse point sheaves parametrized by  $S$ . The functor  $\mathcal{M}(\mathcal{X}/Y)$  can be taken as the moduli functor of equivalence classes of perverse point-ideal sheaves. From (14), the automorphism groups of perverse point-ideal sheaves are  $\mathbb{C}^*$ . Then the moduli functor  $\mathcal{M}(\mathcal{X}/Y)$  is represented by a fine moduli space  $M(\mathcal{X}/Y)$ . As in [1, Lemma 5.1.1], the moduli space  $M(\mathcal{X}/Y)$  is separated.

As in [1, §4.2], let  $W \subset M(\mathcal{X}/Y)$  be the distinguished component which is birational to  $Y$ . We want to prove that  $W$  is isomorphic to the smooth DM stack  $\mathcal{X}'$  in the orbifold flop diagram:

$$\begin{array}{ccc} & \mathcal{Z} & \\ f \swarrow & & \searrow f' \\ \mathcal{X} & \text{-----} & \mathcal{X}' \end{array}$$

PROPOSITION 3.5. *There exists a birational morphism  $\mathcal{X}' \rightarrow W$ .*

*Proof.* We construct a family of perverse point sheaves over  $\mathcal{X}'$ . The candidate for such a family is  $\mathcal{Z}$ . But  $\mathcal{Z}$  in this case contains an extra embedded component and we take the reduction of  $\mathcal{Z}$  by removing this component.

It is sufficient to work on the local model in Diagram (9) of Definition 2.1. In this case  $\mathcal{Z} = \tilde{\mathcal{Z}}/\mu_n$ , where  $\tilde{\mathcal{Z}} = \mathcal{O}_{\mathbb{P}(\mathbf{a}) \times \mathbb{P}(\mathbf{b})}(-1, -1)$ . We work on the reduction  $\mathcal{Z}_{\text{red}}$  of  $\mathcal{Z}$ . We show that the structure sheaf  $\mathcal{O}_{\mathcal{Z}_{\text{red}}}$  is a family of perverse point sheaves over  $\mathcal{X}'$ . Let

$$\text{id} \times \pi : \mathcal{X}' \times_Y \mathcal{X} \rightarrow \mathcal{X}' \times_Y \mathcal{X}$$

be the natural morphism. We check that  $(\text{id} \times \pi)_* I_{\mathcal{Z}_{\text{red}}}$  is a perverse ideal sheaf. This is the Condition (1) in Proposition 3.4.

To check Condition (2) in Proposition 3.4, we need to prove that  $\text{Hom}(D, I_{\mathcal{Z}_{\text{red}}}) = 0$  for any  $D \in C_2$ . Here for  $C_2$  we mean the similar category applied to the DM stack  $\mathcal{Z}$ . We use the method in [1]. Let

$$p : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$$

be the finite morphism as in Definition 2.1, which taken as a base change. We argue that  $\text{Hom}(D, I_{\mathcal{Z}}) = 0$ . Let

$$p : \tilde{\mathcal{X}} \times_{\tilde{\mathcal{Y}}} \tilde{\mathcal{X}}' \rightarrow \tilde{\mathcal{X}} \times_Y \mathcal{X}' \hookrightarrow \tilde{\mathcal{X}} \times \mathcal{X}'$$

be the corresponding morphisms, where the first is finite, and the second is an embedding. Let the image be  $T$ . Then we have

$$0 \rightarrow I_T \rightarrow \mathcal{O}_{\tilde{\mathcal{X}} \times \mathcal{X}'} \rightarrow \mathcal{O}_T \rightarrow 0.$$

Let  $i : p \hookrightarrow \mathcal{X}'$  be a point. We prove that  $i^* I_T$  has torsion with support in pure dimension one and it can not have sections at the preimages of the stacky points of  $\mathcal{X}$  under  $p$ . So  $\text{Hom}(D, I_T) = 0$  for any  $D \in C_2$ .  $\square$

3.2.3. *The derived equivalence.* The relationship between derived categories is proved in [1] in the case of flops, and the proof works for orbifold flops.

Since  $W$  is the distinguished component of  $M(\mathcal{X}/Y)$ , the universal perverse point sheaf  $\mathcal{E}$  gives a diagram:

$$\begin{array}{ccc} D(W) & \xrightarrow{\Phi} & D(\mathcal{X}) \\ & \searrow & \swarrow \\ & D(Y) & \end{array}$$

To prove that  $W \cong \mathcal{X}'$  and  $\Phi$  is an equivalence, we already know from Proposition 3.5 there is a birational morphism  $\mathcal{X}' \rightarrow W$ , if  $W$  is smooth, then this birational morphism is an isomorphism.

For the functor induced by the universal perverse point sheaf  $\Phi : D(W) \rightarrow D(\mathcal{X})$ , as in [10],  $\Phi$  has a left adjoint  $\Psi$  and the composition  $\Psi \circ \Phi$  is defined by a sheaf  $Q$  on  $W \times W$ , which is supported in  $W \times_Y W$ . Then the proof in Sections 6 and 7 in [10, §6–7] go through to prove that  $Q$  vanishes outside the diagonal. The argument of [10, §6], steps 5–6 shows that  $W$  is smooth, and  $\Phi$  is an equivalence sending perverse sheaves to perverse sheaves. Since the argument is the same as in [10, §6–7]. We omit the details.

EXAMPLE 3.6. Instead of proving the tedious construction as in §3 of [1], and §6, §7 of [10], we give an example of orbifold flop. Let

$$\phi : \mathcal{X} = \mathbb{P}_{\mathbb{P}(2,2)}(\mathcal{O}(-1) \oplus \mathcal{X}(-3)) \dashrightarrow \mathcal{X}' = \mathbb{P}_{\mathbb{P}(1,3)}(\mathcal{O}(-2) \oplus \mathcal{X}(-2))$$

be an example of the local model. For notational reason let  $\mathcal{C} = \mathbb{P}(2,2)$  and  $\mathcal{C}' = \mathbb{P}(1,3)$  be the exceptional locus of  $\mathcal{X}$  and  $\mathcal{X}'$  respectively, which are contracted to the singular point  $P \in Y$ . Geometrically we can construct the flop  $\mathcal{X}'$  as follows. We can do weighted blow-up of  $\mathcal{X}$  along the exceptional locus  $\mathcal{C}$  and then blowing-down another exceptional curve to get  $\mathcal{X}'$ .

From [7], [1], let  $y \in \mathcal{C}$  be a point such that  $\bar{y} \in \mathcal{C}' := \mathbb{P}^1$  is its image in the maps between coarse moduli spaces:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\pi} & \mathcal{C}' \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{\pi} & \mathcal{X}' \end{array}$$

Then we have an exact sequence on  $X$ :

$$(15) \quad 0 \rightarrow \mathcal{O}_{\mathcal{C}}(-1) \rightarrow \mathcal{O}_{\mathcal{C}} \rightarrow \mathcal{O}_{\bar{y}} \rightarrow 0,$$

pulling back to  $\mathcal{X}$  we have the following exact sequence:

$$(16) \quad 0 \rightarrow \mathcal{O}_{\mathcal{C}}(-2) \rightarrow \mathcal{O}_{\mathcal{C}} \rightarrow \mathcal{O}_{\bar{y}} \rightarrow 0.$$

As in [1] and [7], the coherent sheaves  $\mathcal{O}_{\mathcal{C}}(-1)$ ,  $\mathcal{O}_{\mathcal{C}}(-2)$  are not perverse, hence the exact sequences (15) and (16) do not define exact sequences in  $\text{Per}(\mathcal{X}/Y)$ .



But the shifted ones  $\mathcal{O}_{\mathcal{C}}(-1)[1]$ ,  $\mathcal{O}_{\mathcal{C}}(-2)[1]$  are perverse sheaves, and we have:

$$(17) \quad 0 \rightarrow \mathcal{O}_{\mathcal{C}} \rightarrow \mathcal{O}_y \rightarrow \mathcal{O}_{\mathcal{C}}(-2)[1] \rightarrow 0.$$

This makes  $\mathcal{O}_y$  is not stable in  $\text{Per}(\mathcal{X}/Y)$ . So the flopping  $\mathcal{X}' \rightarrow Y$  means that we can replace the exceptional curve  $\mathcal{C}$  by  $\mathcal{C}'$  so that it parameterizes the extension

$$(18) \quad 0 \rightarrow \mathcal{O}_{\mathcal{C}}(-2)[1] \rightarrow \mathcal{E} \rightarrow \mathcal{O}_{\mathcal{C}} \rightarrow 0,$$

which is stable in  $\text{Per}(\mathcal{X}/Y)$ . The moduli stack of perverse point sheaves  $W = M(\mathcal{X}/Y)$  parameterizes perverse point sheaves  $\mathcal{E}$  on  $\mathcal{X}$ . Geometrically  $\mathcal{X}'$  is obtained by replacing  $\mathcal{C}$  parameterizing the exact sequence (17) by  $\mathcal{C}'$  parameterizing the exact sequence (18).

### 3.3. Moduli of perverse ideal sheaves.

3.3.1. *K-theory class.* Let  $\mathcal{X}$  be the smooth Calabi–Yau DM stack and  $K_0(\mathcal{X})$  the Grothendieck group of  $K$ -theory with compact support. Recall that in [11], two  $F_1, F_2 \in K_0(\mathcal{X})$  are numerically equivalent, that is,

$$F_1 \sim_{\text{num}} F_2$$

if

$$\chi(E \otimes F_1) = \chi(E \otimes F_2)$$

for all locally free sheaves  $E$  on  $\mathcal{X}$ . Recall that there is a Chern character map

$$\widetilde{\text{Ch}}: K_0(\mathcal{X}) \rightarrow H_{\text{CR}}^*(\mathcal{X})$$

from the  $K$ -group of  $\mathcal{X}$  to the Chen-Ruan cohomology of  $\mathcal{X}$ , such that

$$\chi(F) = \int_{I\mathcal{X}} \widetilde{\text{Ch}}(F) \cdot \widetilde{Td}(\mathcal{X}).$$

Let

$$K(\mathcal{X}) := K_0(\mathcal{X}) / \sim_{\text{num}}.$$

There is a natural filtration

$$F_0(K(\mathcal{X})) \subset F_1(K(\mathcal{X})) \subset \dots \subset K(\mathcal{X})$$

which is given by the dimension of the support of coherent sheaves.

3.3.2. *Hilbert scheme of sub-stacks.* Let  $\alpha \in K(\mathcal{X})$ . We define  $\text{Hilb}^\alpha(\mathcal{X})$  to be the moduli stack of closed sub-stacks  $\mathcal{Z} \subset \mathcal{X}$  having  $[\mathcal{O}_{\mathcal{Z}}] = \alpha$ . From [11], [37],  $\text{Hilb}^\alpha(\mathcal{X})$  is represented by a scheme which we still denote by  $\text{Hilb}^\alpha(\mathcal{X})$ . Let  $\mathcal{I}_{\mathcal{Z}}$  be the ideal sheaf of  $\mathcal{Z}$  in  $\mathcal{O}_{\mathcal{X}}$ , then we can take  $\text{Hilb}^\alpha(\mathcal{X})$  to be the moduli space of ideal sheaves  $\mathcal{I}_{\mathcal{Z}}$  with  $[\mathcal{O}_{\mathcal{Z}}] = \alpha$ . Since the associated substack of  $\mathcal{I}_{\mathcal{Z}}$  is  $\mathcal{Z} \subset \mathcal{X}$ , there is a bijection between the points in  $\text{Hilb}^\alpha(\mathcal{X})$  and the moduli space of ideal sheaves. So these are the same schemes. In the case that  $\mathcal{X}$  is a smooth scheme, this is the original DT-moduli space, see [41], [34].

3.3.3. *Stable pairs.* For the Calabi–Yau threefold stack  $\mathcal{X}$ , generalizing the definition of Pandharipande–Thomas [39], a stable pair  $[\mathcal{O}_{\mathcal{X}} \xrightarrow{s} F]$  is an object in  $D^b(\mathcal{X})$ , such that

- (1)  $\dim \text{Supp}(F) \leq 1$  and  $F$  is pure;
- (2)  $\text{Coker}(s)$  is zero dimensional.

The stable pairs lie in the heart of a  $t$ -structure constructed in [8]. As in [8], let

$$\mathcal{P} := \text{Coh}_0(\mathcal{X}) \subset \mathcal{A} := \text{Coh}(\mathcal{X})$$

be the sub-category consisting of sheaves supported on dimension zero. Let

$$\mathcal{Q} = \{E \in \mathcal{A} \mid \text{Hom}(P, E) = 0 \text{ for } P \in \mathcal{P}\}.$$

Then  $(\mathcal{P}, \mathcal{Q})$  is a torsion pair:

- (1) if  $P \in \mathcal{P}$  and  $Q \in \mathcal{Q}$ , then  $\text{Hom}_{\mathcal{A}}(P, Q) = 0$ ;
- (2) Every  $E \in \mathcal{A}$  fits into a short exact sequence

$$0 \rightarrow P \rightarrow E \rightarrow Q \rightarrow 0$$

with  $P \in \mathcal{P}$  and  $Q \in \mathcal{Q}$ .

A new  $t$ -structure on  $D(\mathcal{X}) = D(\mathcal{A})$  is defined by tilting the standard  $t$ -structure, see Section 2.2 of [8], or [21]. The heart  $\mathcal{A}^\#$  of this new  $t$ -structure is given by:

$$\mathcal{A}^\# = \{E \in D(\mathcal{A}) \mid H_0(E) \in \mathcal{Q}, H_1(E) \in \mathcal{P}, H_i(E) = 0 \text{ for } i \notin \{0, 1\}\}.$$

We have  $\mathcal{Q} = \mathcal{A} \cap \mathcal{A}^\#$  and  $\mathcal{O}_{\mathcal{X}} \in \mathcal{A}^\#$ . Bridgeland [8] proves the following result:

**PROPOSITION 3.7.** *A stable pair  $[\mathcal{O}_{\mathcal{X}} \xrightarrow{s} F]$  is an epimorphism  $\mathcal{O}_{\mathcal{X}} \twoheadrightarrow F$  in  $\mathcal{A}^\#$  with  $\dim \text{Supp}(F) \leq 1$  and  $F \in \mathcal{Q}$ .*

Fixing  $[\mathcal{O}_F] = \beta \in K(\mathcal{X})$ , let  $\text{PT}^\beta(\mathcal{X})$  be the moduli stack of stable pairs, parameterizing the objects  $[\mathcal{O}_{\mathcal{X}} \xrightarrow{s} F]$  satisfying the conditions in the definition. From [2], [3], it is represented by a scheme  $\text{PT}^\beta(\mathcal{X})$ .

### 3.3.4. *DT-type invariants.*

**DEFINITION 3.8.** The DT-invariant of  $\mathcal{X}$  of class  $\alpha \in K(\mathcal{X})$  is defined by the weighted Euler characteristic

$$\text{DT}_\alpha(\mathcal{X}) = \chi(\text{Hilb}^\alpha(\mathcal{X}), \nu_H),$$

where

$$\nu_H : \text{Hilb}^\alpha(\mathcal{X}) \rightarrow \mathbb{Z}$$

is the Behrend function of [4]. Similarly, the PT-invariant of  $\mathcal{X}$  of class  $\beta \in K(\mathcal{X})$  is defined by the weighted Euler characteristic

$$\text{PT}_\beta(\mathcal{X}) = \chi(\text{PT}^\beta(\mathcal{X}), \nu_{\text{PT}}),$$

where

$$\nu_{\text{PT}} : \text{PT}^\beta(\mathcal{X}) \rightarrow \mathbb{Z}$$

is the Behrend function of  $\text{PT}^\beta(\mathcal{X})$ .

REMARK 3.9. Both  $\text{Hilb}^\alpha(\mathcal{X})$  and  $\text{PT}^\beta(\mathcal{X})$  have symmetric obstruction theories in the sense of Behrend [4]. If  $\mathcal{X}$  is compact, then the invariants defined by virtual fundamental class are the same as weighted Euler characteristic of Behrend, see Theorem 4.18 of [4].

3.3.5. *Partition function.* Define the *DT*-partition function by

$$(19) \quad \text{DT}(\mathcal{X}) = \sum_{\alpha \in F_1 K(\mathcal{X})} \text{DT}_\alpha(\mathcal{X}) q^\alpha$$

and the *PT*-partition function by

$$(20) \quad \text{PT}(\mathcal{X}) = \sum_{\beta \in F_1 K(\mathcal{X})} \text{PT}_\beta(\mathcal{X}) q^\beta.$$

The degree zero *DT*-partition function is defined by

$$(21) \quad \text{DT}_0(\mathcal{X}) = \sum_{\alpha \in F_0 K(\mathcal{X})} \text{DT}_\alpha(\mathcal{X}) q^\alpha$$

and the reduced *DT*-partition function by

$$(22) \quad \text{DT}'(\mathcal{X}) = \frac{\text{DT}(\mathcal{X})}{\text{DT}_0(\mathcal{X})}.$$

#### 4. The motivic Hall algebra

In this section, we review the definition and construction of the motivic Hall algebra of Joyce and Bridgeland in [26], [9]. Then we review the integration map from the motivic Hall algebra to the ring of functions of the quantum torus.

**4.1. Motivic Hall algebra.** We briefly review the notion of motivic Hall algebra in [9], more details can be found in [9], [26].

DEFINITION 4.1. The Grothendieck ring of stacks  $K(\text{St}/\mathbb{C})$  is defined to be the  $\mathbb{C}$ -vector space spanned by isomorphism classes of Artin stacks of finite type over  $\mathbb{C}$  with affine stabilizers, modulo the relations:

(1) for every pair of stacks  $\mathcal{X}_1$  and  $\mathcal{X}_2$  a relation:

$$[\mathcal{X}_1 \sqcup \mathcal{X}_2] = [\mathcal{X}_1] + [\mathcal{X}_2];$$

(2) for any geometric bijection  $f : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ ,  $[\mathcal{X}_1] = [\mathcal{X}_2]$ ;

(3) for any Zariski fibrations  $p_i : \mathcal{X}_i \rightarrow \mathcal{Y}$  with the same fibers,  $[\mathcal{X}_1] = [\mathcal{X}_2]$ .

Let  $[\mathbb{A}^1] = \mathbb{L}$  be the Lefschetz motive. If  $S$  is a stack of finite type over  $\mathbb{C}$ , we define the relative Grothendieck ring of stacks  $K(\text{St}/S)$  as follows:

DEFINITION 4.2. The relative Grothendieck ring of stacks  $K(\text{St}/\mathbb{C})$  is defined to be the  $\mathbb{C}$ -vector space spanned by isomorphism classes of morphisms

$$[\mathcal{X} \xrightarrow{f} S],$$

with  $\mathcal{X}$  an Artin stack over  $S$  of finite type with affine stabilizers, modulo the following relations:

(1) for every pair of stacks  $\mathcal{X}_1$  and  $\mathcal{X}_2$  a relation:

$$[\mathcal{X}_1 \sqcup \mathcal{X}_2 \xrightarrow{f_1 \sqcup f_2} S] = [\mathcal{X}_1 \xrightarrow{f_1} S] + [\mathcal{X}_2 \xrightarrow{f_2} S];$$

(2) for any diagram:

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{g} & \mathcal{X}_2 \\ & \searrow f_1 & \swarrow f_2 \\ & S & \end{array}$$

where  $g$  is a geometric bijection, then  $[\mathcal{X}_1 \xrightarrow{f_1} S] = [\mathcal{X}_2 \xrightarrow{f_2} S]$ ;

(3) for any pair of Zariski fibrations

$$\mathcal{X}_1 \xrightarrow{h_1} \mathcal{Y}; \mathcal{X}_2 \xrightarrow{h_2} \mathcal{Y}$$

with the same fibers, and  $g: \mathcal{Y} \rightarrow S$ , a relation

$$[\mathcal{X}_1 \xrightarrow{g \circ h_1} S] = [\mathcal{X}_2 \xrightarrow{g \circ h_2} S].$$

The motivic Hall algebra in [26] and [9] is defined as follows. Let  $\mathcal{M}$  be the moduli stack of coherent sheaves on  $\mathcal{X}$ . It is an algebraic stack, locally of finite type over  $\mathbb{C}$ , see [28]. The motivic Hall algebra is the vector space

$$H(\mathcal{A}) = K(\text{St}/\mathcal{M})$$

equipped with a non-commutative product given by the rule:

$$[\mathcal{X}_1 \xrightarrow{f_1} \mathcal{M}] \star [\mathcal{X}_2 \xrightarrow{f_2} \mathcal{M}] = [\mathcal{Z} \xrightarrow{b \circ h} \mathcal{M}],$$

where  $h$  is defined by the following Cartesian square:

$$\begin{array}{ccccc} \mathcal{Z} & \xrightarrow{h} & \mathcal{M}^{(2)} & \xrightarrow{b} & \mathcal{M} \\ \downarrow & & \downarrow (a_1, a_2) & & \\ \mathcal{X}_1 \times \mathcal{X}_2 & \xrightarrow{f_1 \times f_2} & \mathcal{M} \times \mathcal{M} & & \end{array}$$

with  $\mathcal{M}^{(2)}$  the stack of short exact sequences in  $\mathcal{A}$ , and the maps  $a_1, a_2, b$  send a short exact sequence

$$0 \rightarrow A_1 \rightarrow B \rightarrow A_2 \rightarrow 0$$

to sheaves  $A_1, A_2$ , and  $B$ , respectively. Then  $H(\mathcal{A})$  is an algebra over  $K(\text{St}/\mathbb{C})$ .

**4.2. The integration map.** Recall that in Section 3 of [9], there exists maps of commutative rings:

$$K(\text{Sch}/\mathbb{C}) \rightarrow K(\text{Sch}/\mathbb{C})[\mathbb{L}^{-1}] \rightarrow K(\text{St}/\mathbb{C}),$$

where  $K(\text{Sch}/\mathbb{C})$  is the Grothendieck ring of schemes of finite type over  $\mathbb{C}$ . Since  $H(\mathcal{A})$  is an algebra over  $K(\text{St}/\mathbb{C})$ , define a  $K(\text{Sch}/\mathbb{C})[\mathbb{L}^{-1}]$ -module

$$H_{\text{reg}}(\mathcal{A}) \subset H(\mathcal{A})$$

to be the span of classes of maps  $[X \xrightarrow{f} \mathcal{M}]$  with  $X$  a scheme. An element of  $H(\mathcal{A})$  is regular if it lies in  $H_{\text{reg}}(\mathcal{A})$ . The following is Theorem 5.1 of [9].

**THEOREM 4.3.** *The sub-module of regular elements of  $H(\mathcal{A})$  is closed under the convolution product:*

$$H_{\text{reg}}(\mathcal{A}) \star H_{\text{reg}}(\mathcal{A}) \subset H_{\text{reg}}(\mathcal{A})$$

and is a  $K(\text{Sch}/\mathbb{C})[\mathbb{L}^{-1}]$ -algebra. Moreover, the quotient

$$H_{\text{sc}}(\mathcal{A}) = H_{\text{reg}}(\mathcal{A})/(\mathbb{L} - 1)H_{\text{reg}}(\mathcal{A})$$

is a commutative  $K(\text{Sch}/\mathbb{C})$ -algebra.

The algebra  $H_{\text{sc}}(\mathcal{A})$  is called the semi-classical Hall algebra. In [9], Bridgeland also defines a Poisson bracket on  $H(\mathcal{A})$  by:

$$\{f, g\} = \frac{f \star g - g \star f}{\mathbb{L} - 1}.$$

This bracket preserves the subalgebra  $H_{\text{reg}}(\mathcal{A})$ .

Let  $\Delta \subset F_1 K(\mathcal{X})$  be the effective cone of  $F_1 K(\mathcal{X})$ , that is, the collection of elements of the form  $[E]$ , where  $E$  is a one-dimensional sheaf. Define

$$\mathbb{C}[\Delta] = \bigoplus_{\alpha \in \Delta} \mathbb{C} \cdot x^\alpha$$

to be the ring generated by symbols  $x^\alpha$  for  $\alpha \in \Delta$ , with product defined by:

$$x^\alpha \star x^\beta = (-1)^{\chi(\alpha, \beta)} \cdot x^{\alpha + \beta}.$$

The ring is commutative since the Euler form is skew-symmetric. The Poisson bracket is given by:

$$\{x^\alpha, x^\beta\} = (-1)^{\chi(\alpha, \beta)} \cdot \chi(\alpha, \beta) \cdot x^{\alpha + \beta}.$$

The following theorem is proved in Section 5.2 of [9].

**THEOREM 4.4** ([9, Theorem 5.2]). *Let  $\nu : \mathcal{M} \rightarrow \mathbb{Z}$  be the locally constructible Behrend function. Then there is a Poisson algebra homomorphism:*

$$(23) \quad I : H_{\text{sc}}(\mathcal{A}) \rightarrow \mathbb{C}[\Delta]$$

such that

$$I([\mathcal{Z} \xrightarrow{f} \mathcal{M}_\alpha]) = \chi(\mathcal{Z}, f^\star \nu) \cdot x^\alpha.$$

REMARK 4.5. The proof of Theorem 4.4 relies on the Behrend function identities in §10 of [28], which was originally proved for coherent sheaves by Joyce–Song [28]. These identities were recently proved by V. Bussi [13] using algebraic methods and also works in characteristic  $p$ , see [24] for another method using Berkovich spaces. In [25] we will generalize the integration map to the motivic level of the Behrend functions.

*Integration map for  $H({}^p\mathcal{A})$ .* For the Abelian category of perverse coherent sheaves  ${}^p\mathcal{A}$ , we have a similar definition  $H({}^p\mathcal{A})$ , the motivic Hall algebra of  ${}^p\mathcal{A}$ . The semi-classical Hall algebra  $H_{\text{sc}}({}^p\mathcal{A})$  can be similarly defined. Let  ${}^p\mathcal{M}$  be the moduli stack of objects in the category  ${}^p\mathcal{A}$ . There is an integration map

$$(24) \quad I : H_{\text{sc}}({}^p\mathcal{A}) \rightarrow \mathbb{C}[\Delta]$$

such that

$$I([\mathcal{Z} \xrightarrow{f} {}^p\mathcal{M}_\alpha]) = \chi(\mathcal{Z}, f^*\nu) \cdot x^\alpha.$$

Here  $\nu : {}^p\mathcal{M} \rightarrow \mathbb{Z}$  is the Behrend function of  ${}^p\mathcal{M}$ .

To prove that  $I$  is a Poisson algebra homomorphism, we require the Joyce–Song formula for the Behrend function identities similar to Section 10 of [28], [13]. Since the elements in  ${}^p\mathcal{A}$  are semi-Schur, i.e. for any  $E \in {}^p\mathcal{A}$ ,  $\text{Ext}^i(E, E) = 0$  for  $i < 0$ , in [24] the author proves the Joyce–Song formula for the Behrend function identities using Berkovich spaces. Thus, the integration map  $I$  is a Poisson algebra homomorphism and the proof is the same as in [9, Theorem 5.2].

## 5. DT-invariants identities under flops

In this section, we study the Hall algebra identities, following [8] and [15], and prove the main result.

**5.1. Infinite-type Hall algebras.** In this section, we enlarge the definition of the Hall algebra, as in Section 4.2 of [8] and [15]. For the stack  $\mathcal{M}$ , define infinite-type Grothendieck group  $L(\text{St}_\infty/S)$  by the symbols  $[\mathcal{X} \rightarrow S]$ , but with  $\mathcal{X}$  only assumed to be locally of finite type over  $S$ . Then we need to drop the relation (1) in Definition 4.2. The infinite-type Hall algebra is then

$$\begin{aligned} H_\infty(\mathcal{A}) &= L(\text{St}_\infty/\mathcal{M}), \\ H_\infty({}^p\mathcal{A}) &= L(\text{St}_\infty/{}^p\mathcal{M}). \end{aligned}$$

REMARK 5.1. By working on infinite-type Hall algebra, we may not have integration map  $I$  in (23) and (24). We will have such an integration map  $I$  in the Laurent Hall algebra  $H_\Lambda \subset H_\infty$ , and  $H(\mathcal{A}) \subset H_\Lambda$ .

**5.2. Perverse Hilbert scheme.** Let  $\mathcal{A}_{\leq 1} \subset \mathcal{A}$  be the full sub-category consisting of sheaves with support of  $\dim \leq 1$ . Similarly,  ${}^p\mathcal{A}_{\leq 1} \subset {}^p\mathcal{A}$  is the full sub-category consisting of perverse sheaves with support of  $\dim \leq 1$ . Let  $H_\infty(\mathcal{A}_{\leq 1})$  ( $H_\infty({}^p\mathcal{A}_{\leq 1})$ ) be the corresponding sub-Hall algebra.

The first element in our formula is

$$\mathcal{H}_{\leq 1} \in H_\infty(\mathcal{A}_{\leq 1}),$$

the Hilbert scheme of  $\mathcal{X}$ , which parameterizes quotients

$$\mathcal{O}_{\mathcal{X}} \rightarrow F$$

in  $\mathcal{A}_{\leq 1}$ . Let  $\mathcal{M}_{\leq 1} \subset \mathcal{M}$  be the moduli stack of coherent sheaves with support  $\dim \leq 1$ . Then  $\mathcal{H}_{\leq 1}$  is given by the morphism  $\text{Hilb}_{\leq 1}(\mathcal{X}) \rightarrow \mathcal{M}_{\leq 1}$ .

REMARK 5.2. If  $\mathcal{O}_{\mathcal{X}} \rightarrow E$  is a quotient in  $\mathcal{A}_{\leq 1}$ , then  $\mathcal{O}_{\mathcal{X}} \in {}^p\mathcal{T}$ . This is because  $E \in {}^p\mathcal{T}$ , and the quotient of an element in  ${}^p\mathcal{T}$  is in  ${}^p\mathcal{T}$ . So the morphism

$$\text{Hilb}_{\leq 1}(\mathcal{X}) \rightarrow \mathcal{M}_{\leq 1}$$

factors through the element  ${}^p\mathcal{T}_{\leq 1}$ , which is represented by  $[{}^p\mathcal{T} \rightarrow \mathcal{M}_{\leq 1}]$ . Hence  $\mathcal{H}_{\leq 1} \in H_\infty({}^p\mathcal{A}_{\leq 1})$ , since  ${}^p\mathcal{T}_{\leq 1} \in {}^p\mathcal{M}_{\leq 1}$ .

**5.3. Framed coherent sheaves.** Let  $\mathcal{B} \subset \mathcal{A}$  be a sub-category. We denote by  $\mathbb{1}_{\mathcal{B}}$  the element of  $H_\infty(\mathcal{A})$  represented by the inclusion of stacks  $\mathcal{B} \subset \mathcal{M}$ , when it is an open immersion. (Similar for  $\mathcal{A}_{\leq 1}$  and  ${}^p\mathcal{A}_{\leq 1}$ .)

Following Section 2.3 of [8], we define  $\mathcal{M}_{\leq 1}^{\mathcal{O}}$ , the stack of framed coherent sheaves, which parametrizes coherent sheaves with a fixed section  $\mathcal{O}_{\mathcal{X}} \rightarrow E$ . Then  $\text{Hilb}_{\leq 1}(\mathcal{X})$  is an open subscheme of  $\mathcal{M}_{\leq 1}^{\mathcal{O}}$  by considering a surjective section  $[\mathcal{O}_{\mathcal{X}} \rightarrow E] \in \text{Hilb}_{\leq 1}(\mathcal{X})$ . We also have a forgetful morphism:

$$\mathcal{M}_{\leq 1}^{\mathcal{O}} \rightarrow \mathcal{M}_{\leq 1}$$

by taking  $[\mathcal{O}_{\mathcal{X}} \rightarrow E]$  to  $E \in \mathcal{M}_{\leq 1}$ . Given any open substack  $\mathcal{B} \subset \mathcal{M}_{\leq 1}$ , we have a Cartesian diagram:

$$(25) \quad \begin{array}{ccc} \mathcal{B}^{\mathcal{O}} & \longrightarrow & \mathcal{M}_{\leq 1}^{\mathcal{O}} \\ \downarrow & & \downarrow \\ \mathcal{B} & \longrightarrow & \mathcal{M}_{\leq 1} \end{array}$$

and  $\mathbb{1}_{\mathcal{B}}^{\mathcal{O}} \in H_\infty(\mathcal{A}_{\leq 1})$ .

Similarly if  ${}^p\mathcal{B} \subset {}^p\mathcal{M}_{\leq 1}$  is an open stack, then we have similar diagram as in (25) and an element  $\mathbb{1}_{{}^p\mathcal{B}}^{\mathcal{O}} \in H_\infty({}^p\mathcal{A}_{\leq 1})$ .

Finally, let  ${}^p\text{Hilb}_{\leq 1}(\mathcal{X}/Y)$  be the “perverse Hilbert scheme” parametrizing quotients of  $\mathcal{O}_{\mathcal{X}}$  in  ${}^p\mathcal{A}_{\leq 1}$ . Then we have an element  ${}^p\mathcal{H}_{\leq 1} \in H_\infty({}^p\mathcal{A}_{\leq 1})$ .

**5.4. Hall algebra identities.** When we restrict to the subcategories  $\mathcal{A}_{\leq 1}$  and  ${}^p\mathcal{A}_{\leq 1}$ , the following identity

$${}^p\mathcal{F} = {}^p\mathcal{F}_{\leq 1}$$

is true from the definitions in Section 3.1 since  $\psi$  and  $\psi'$  are isomorphisms in codimension one. We prove several Hall algebra identities in this section, following the method in [15], [8].

THEOREM 5.3. *We have:*

$${}^p\mathcal{H}_{\leq 1} \star \mathbb{1}_{{}^p\mathcal{F}[1]} = \mathbb{1}_{{}^p\mathcal{F}[1]} \star \mathcal{H}_{\leq 1}.$$

*Proof.* First, let us analyze both sides of the equality. The left-hand side (LHS) is represented by a stack  $\mathfrak{M}_L$ , parameterizing diagrams:

$$\begin{array}{c} \mathcal{O}_{\mathcal{X}} \\ \downarrow \\ P_1 \hookrightarrow E \twoheadrightarrow P_2 \end{array}$$

where all objects are in  ${}^p\mathcal{A}_{\leq 1}$ , the bottom sequence is exact in  ${}^p\mathcal{A}_{\leq 1}$ ,  $\mathcal{O}_{\mathcal{X}} \twoheadrightarrow P_1$  is surjective in  ${}^p\mathcal{A}_{\leq 1}$ , and  $P_2 \in {}^p\mathcal{F}[1]$ .

The right-hand side (RHS) is represented by a stack  $\mathfrak{M}_R$ , parameterizing diagrams:

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{X}} & & \mathcal{O}_{\mathcal{X}} \\ \downarrow & & \downarrow \text{sur} \\ F[1] \hookrightarrow E & \twoheadrightarrow & T \end{array}$$

where the horizontal sequence

$$F[1] \hookrightarrow E \twoheadrightarrow T$$

is an exact sequence in  ${}^p\mathcal{A}_{\leq 1}$ , and  $F \in {}^p\mathcal{F}$ ,  $T \in {}^p\mathcal{T}_{\leq 1}$ . Moreover  $\mathcal{O}_{\mathcal{X}} \twoheadrightarrow T$  is surjective in  $\mathcal{A}_{\leq 1}$ , and has perverse cokernel lying in  ${}^p\mathcal{F}[1]$ . Actually given a perverse coherent sheaf  $E \in {}^p\mathcal{A}_{\leq 1}$ , there exists a unique exact sequence above.

As in Section 3.3 of [15], we construct the following diagram:

$$(26) \quad \begin{array}{ccccc} \mathfrak{M}_L & & \mathfrak{M} & & \mathfrak{M}_R \\ & \searrow f_L & & \searrow g & \\ & & \mathfrak{M}' & & \mathfrak{N} \\ & & \swarrow f' & & \swarrow g_R \end{array}$$

such that the maps are either geometric bijections or Zaraski fibrations.



We first define the stack  $\mathfrak{M}'$ , which parametrizes the diagrams of the form:

$$\begin{array}{c} \mathcal{O}_{\mathcal{X}} \\ \varphi \downarrow \\ E \end{array}$$

such that  ${}^p\text{Coker}(\varphi) \in {}^p\mathcal{F}[1]$ . By Lemma 3.2 of [15], this is equivalent to  $\text{Cone}(\varphi) \in D^{\leq 1}(\mathcal{X})$ , which is open. So  $\mathfrak{M}'$  is an open substack of the stack of framed perverse sheaves  ${}^p\mathcal{M}_{\leq 1}^{\mathcal{O}}$ .

The first lemma is:

LEMMA 5.4. *There is a map  $f_L : \mathfrak{M}_L \rightarrow \mathfrak{M}'$  induced by the composition*

$$\mathcal{O}_{\mathcal{X}} \rightarrow P_1 \hookrightarrow E,$$

*which is a geometric bijection.*

*Proof.* The map  $f_L : \mathfrak{M}_L \rightarrow \mathfrak{M}'$  is an equivalence on  $\mathbb{C}$ -points. As we see later,  ${}^p\mathcal{H}_{\leq 1}$ ,  $\mathbb{1}_{{}^p\mathcal{F}}$  are all Laurent elements in the Hall algebra  $H_{\infty}({}^p\mathcal{A}_{\leq 1})$ . So for any  $\alpha \in K(\mathcal{X})$ ,  $\mathfrak{M}_{L,\alpha} \rightarrow \mathfrak{M}'_{\alpha}$  is of finite type.  $\square$

Secondly, we define the stack  $\mathfrak{M}$ , which parametrizes the diagrams of the form:

$$\begin{array}{ccccc} & & \mathcal{O}_{\mathcal{X}} & & \\ & & \downarrow \phi & & \\ F[1] & \hookrightarrow & E & \twoheadrightarrow & T \end{array}$$

where the horizontal sequence is a short exact sequence of perverse sheaves and  $F \in {}^p\mathcal{F}$ ,  $T \in {}^p\mathcal{T}_{\leq 1}$ , and  ${}^p\text{Coker}(\phi) \in {}^p\mathcal{F}[1]$ . The stack  $\mathfrak{M}$  can be understood as a fibre product:

$$\begin{array}{ccc} \mathfrak{M} & \longrightarrow & \mathfrak{M}' \\ \downarrow & & \downarrow \\ \mathcal{Z} & \longrightarrow & {}^p\mathcal{M}_{\leq 1} \end{array}$$

where  $\mathcal{Z}$  is the element  $\mathbb{1}_{{}^p\mathcal{F}[1]} \star \mathbb{1}_{{}^p\mathcal{T}_{\leq 1}}$ .

LEMMA 5.5. *The morphism  $f' : \mathfrak{M} \rightarrow \mathfrak{M}'$  defined by forgetting the bottom exact sequence is a geometric bijection.*

*Proof.* As in Proposition 3.4 of [15], considering the following diagram:

$$\begin{array}{ccccc} \mathcal{Z} & \longrightarrow & {}^p\mathcal{M}_{\leq 1}^{(2)} & \xrightarrow{b} & {}^p\mathcal{M}_{\leq 1} \\ \downarrow & & \downarrow & & \\ {}^p\mathcal{F}[1] \times {}^p\mathcal{T}_{\leq 1} & \longrightarrow & {}^p\mathcal{M}_{\leq 1} \times {}^p\mathcal{M}_{\leq 1} & & \end{array}$$

where the bottom is an open immersion and  $b$  is of finite type. The morphism  $\mathcal{Z} \rightarrow {}^p\mathcal{M}_{\leq 1}$  induces an equivalence on  $\mathbb{C}$ -points since  $({}^p\mathcal{F}[1], {}^p\mathcal{T})$  is a torsion pair in  ${}^p\mathcal{A}$ . As  $\mathfrak{M} \rightarrow \mathfrak{M}'$  is a base change, it is a geometric bijection.  $\square$

So to prove the main identity, we need to prove that

$$[\mathfrak{M}] = [\mathfrak{M}_R] \in H_\infty({}^p\mathcal{A}).$$

In Diagram (26), we are only left to define the stack  $\mathfrak{N}$ . The stack  $\mathfrak{N}$  is defined as the moduli stack of the following diagrams:

$$\begin{array}{ccc} & & \mathcal{O}_{\mathcal{X}} \\ & & \downarrow \text{sur} \\ F[1] \hookrightarrow E & \twoheadrightarrow & T \end{array}$$

where the bottom exact sequence lies in  ${}^p\mathcal{A}$  and  $F \in {}^p\mathcal{F}$ ,  $T \in {}^p\mathcal{T}_{\leq 1}$ . Moreover the morphism  $\mathcal{O}_{\mathcal{X}} \rightarrow T$  is surjective in  $\mathcal{A}$  and has perverse cokernel in  ${}^p\mathcal{F}[1]$ . There exist two maps

$$l : \mathfrak{M} \rightarrow \mathfrak{N}$$

which is given by

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{X}} & \mapsto & \mathcal{O}_{\mathcal{X}} \\ \downarrow \phi & & \downarrow \text{sur} \\ F[1] \hookrightarrow E & \twoheadrightarrow T, & F[1] \hookrightarrow E \twoheadrightarrow T \end{array}$$

and

$$r : \mathfrak{M}_R \rightarrow \mathfrak{N}$$

which is given by:

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{X}} & & \mathcal{O}_{\mathcal{X}} \\ \downarrow & & \downarrow \text{sur} \\ F[1] \hookrightarrow E & \twoheadrightarrow T, & F[1] \hookrightarrow E \twoheadrightarrow T \end{array}$$

**PROPOSITION 5.6.** *The maps  $l$  and  $r$  are two Zaraski fibrations with the same fibers.*

*Proof.* First over a perverse sheaf  $E$ , we have

$$F[1] \hookrightarrow E \twoheadrightarrow T$$

in  ${}^p\mathcal{A}_{\leq 1}$ , and  $F \in {}^p\mathcal{F}, T \in {}^p\mathcal{T}_{\leq 1}$ . So over an element

$$\begin{array}{ccc} & & \mathcal{O}_{\mathcal{X}} \\ & & \downarrow \text{sur} \\ F[1] \hookrightarrow E & \twoheadrightarrow & T \end{array}$$

in  $\mathfrak{N}$ , the fiber of  $r$  is  $\mathrm{Hom}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}, F[1])$  and the fiber of  $l$  is: the lifts

$$\mathcal{O}_{\mathcal{X}} \longrightarrow E$$

which has perverse cokernel  ${}^p\mathrm{Coker}(\varphi) \in {}^p\mathcal{F}[1]$ .

Since  ${}^p\mathcal{A}_{\leq 1}$  is an Abelian category, the following is an exact sequence

$$0 \rightarrow \mathrm{Hom}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}, F[1]) \rightarrow \mathrm{Hom}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}, E) \rightarrow \mathrm{Hom}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}, T) \rightarrow 0.$$

For a map  $\varphi: \mathcal{O}_{\mathcal{X}} \rightarrow T$ , all lifts of  $\varphi$  by  $\mathcal{O}_{\mathcal{X}} \rightarrow E$  are in bijection with  $\mathrm{Hom}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}, F[1])$ . Then to finish the proof, we have to show that any lift of  $\mathcal{O}_{\mathcal{X}} \rightarrow T$  is one  $\mathcal{O}_{\mathcal{X}} \rightarrow F[1]$  such that the perverse cokernel is in  ${}^p\mathcal{F}[1]$ .

Let  $\varphi: \mathcal{O}_{\mathcal{X}} \rightarrow T$  be a map with  $T \in {}^p\mathcal{T}_{\leq 1}$  and  ${}^p\mathrm{Coker}(\varphi) \in {}^p\mathcal{F}[1]$ . Let  $\delta: \mathcal{O}_{\mathcal{X}} \rightarrow E$  be a lift such that

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{O}_{\mathcal{X}} & \xrightarrow{=} & \mathcal{O}_{\mathcal{X}} \\ \downarrow & & \downarrow \delta & & \downarrow \varphi \\ F[1] & \hookrightarrow & E & \twoheadrightarrow & T \end{array}$$

is an exact-sequence diagram. Hence, we have an exact sequence on cokernels:

$$F[1] \rightarrow {}^p\mathrm{Coker}(\delta) \rightarrow {}^p\mathrm{Coker}(\varphi) \rightarrow 0.$$

So from Lemma 1.5 of [15],  ${}^p\mathrm{Coker}(\delta) \in {}^p\mathcal{F}[1]$ . This construction works for families and we are done.  $\square$

Hence from Lemmas 5.4, 5.5, Proposition 5.6,

$$[\mathfrak{M}_L] = [\mathfrak{M}_R],$$

hence the theorem.  $\square$

**5.5. PT-type invariants identities.** Recall that in Section 3.3.3, we define a torsion pair  $(\mathcal{P}, \mathcal{Q})$  on  $\mathcal{A}$ , where

$$\mathcal{P} = \{\text{coherent sheaves supported on dimension zero}\}$$

and  $\mathcal{Q}$  is the right orthogonal of  $\mathcal{P}$ . Recall that the tilt of  $\mathcal{A}$  is given by  $\mathcal{A}^{\#}$ .

The scheme  $\mathrm{Hilb}_{\leq 1}^{\#}(\mathcal{X})$  parameterizes quotients  $\mathcal{O}_{\mathcal{X}} \twoheadrightarrow F$  in  $\mathcal{A}^{\#}$  supported on dimension  $\leq 1$ . So we have an element  $\mathcal{H}_{\leq 1}^{\#} \in H_{\infty}(\mathcal{A}_{\leq 1})$  which gives rise to the PT-stable pair invariants.

Let  $\mathcal{Q}_{\leq 1}$  be the stack parameterizing objects in  $\mathcal{Q}_{\leq 1} \subset \mathcal{M}_{\leq 1}$ . Then there exists an element  $\mathbb{1}_{\mathcal{Q}_{\leq 1}} \in H_{\infty}(\mathcal{A}_{\leq 1})$ . Its framed version is denoted by  $\mathbb{1}_{\mathcal{Q}_{\leq 1}}^{\mathcal{O}}$ , parameterizing

$$\{\mathcal{O}_{\mathcal{X}} \rightarrow F\}$$

for  $F \in \mathcal{Q}_{\leq 1}$ . Similar to Section 4.5 of [8], we have the following Hall algebra identity:

$$(27) \quad \mathbb{1}_{\mathcal{Q}_{\leq 1}}^{\mathcal{O}} = \mathcal{H}_{\leq 1}^{\#} \star \mathbb{1}_{\mathcal{Q}_{\leq 1}}.$$

*Restriction to the exceptional locus.* Following Calabrese [15], we define the following:

$$\begin{cases} \mathcal{Q}_{\text{exc}} = \{Q \in \mathcal{Q}_{\leq 1} \mid \dim \text{Supp } R\psi_* Q = 0\}; \\ {}^p\mathcal{A}_{\text{exc}} = \{E \in {}^p\mathcal{A}_{\leq 1} \mid \dim \text{Supp } R\psi_* E = 0\}; \\ {}^p\mathcal{T}_{\text{exc}} = {}^p\mathcal{T} \cap {}^p\mathcal{A}_{\text{exc}}; \\ {}^p\mathcal{T}_\bullet = {}^p\mathcal{T}_{\text{exc}} \cap \mathcal{Q}_{\text{exc}}, \end{cases}$$

where  $\psi : \mathcal{X} \rightarrow Y$  is the contraction map. Hence inside  $\text{Hilb}_{\leq 1}^\#(\mathcal{X})$ , there is an open subscheme  $\text{Hilb}_{\text{exc}}^\#(\mathcal{X})$ , parameterizing quotients  $\mathcal{O}_{\mathcal{X}} \rightarrow F$  in  $\mathcal{A}_{\leq 1}^\#$  such that  $F \in {}^p\mathcal{T}_\bullet$ . Its Hall algebra element is denoted by  $\mathcal{H}_{\text{exc}}^\# \in H_\infty(\mathcal{A}_{\leq 1})$ .

PROPOSITION 5.7. *We have the following identity in  $H_\infty(\mathcal{A}_{\leq 1})$ :*

$$\mathbb{1}_{{}^p\mathcal{T}_\bullet} = \mathcal{H}_{\text{exc}}^\# \star \mathbb{1}_{{}^p\mathcal{T}_\bullet}.$$

*Proof.* First if we have a morphism  $\mathcal{O}_{\mathcal{X}} \rightarrow T$  in  $\mathcal{A}^\#$  with  $T \in {}^p\mathcal{T}_\bullet$ , then we have a sequence  $\mathcal{O}_{\mathcal{X}} \rightarrow I \rightarrow T$  in  $\mathcal{A}^\#$ , where  $I$  is the image in  $T$ . From Lemma 2.3 of [8],  $I$  is a sheaf, so  $\mathcal{O}_{\mathcal{X}} \twoheadrightarrow I$  has cokernel  $P \in \mathcal{P}$ . Considering

$$I \rightarrow T \rightarrow Q,$$

where  $Q$  is the quotient. The short exact sequence  $I \hookrightarrow T \twoheadrightarrow Q$  lies in  $\mathcal{A}$ ,  $Q \in {}^p\mathcal{T}$  since it is a quotient of  $T$ , and  $Q \in \mathcal{Q}$  since it is an object in  $\mathcal{A}^\#$ . Also  $R\psi_* Q$  supports on dimension zero since  $R\psi_* T$  is. So  $Q \in {}^p\mathcal{T}$ .

Conversely, let  $\mathcal{O}_{\mathcal{X}} \rightarrow I$  be an element in  $\text{Hilb}_{\text{exc}}^\#$ , where it is an epimorphism in  $\mathcal{A}^\#$ . Let

$$I \hookrightarrow T \twoheadrightarrow Q$$

be an exact sequence of coherent sheaves, with  $I \in \mathcal{Q}_{\text{exc}}$ ,  $Q \in {}^p\mathcal{T}_\bullet$ . So  $T \in {}^p\mathcal{T}_\bullet$ . Also  $T \in \mathcal{Q}_{\text{exc}}$ , so we need to prove  $I \in {}^p\mathcal{T}$ .

Considering the exact sequence

$$\mathcal{O}_{\mathcal{X}} \rightarrow I \twoheadrightarrow P,$$

with  $P$  supported in dimension zero. Let

$$I \twoheadrightarrow F$$

be the projection to the torsion free part of  $I$  with respect to  $({}^p\mathcal{T}, {}^p\mathcal{F})$ . Then the morphism

$$\mathcal{O}_{\mathcal{X}} \rightarrow I \twoheadrightarrow F$$

is zero, since  $F \in {}^p\mathcal{T}$  has no sections. Thus there exists a morphism

$$P \rightarrow F$$

such that

$$I \twoheadrightarrow P \rightarrow F = I \twoheadrightarrow F.$$

But  $P$  is a skyscraper sheaf, which implies that  $P \rightarrow F = 0$ . So  $I \rightarrow F = 0$ , which implies that  $F = 0$  and  $I \in {}^p\mathcal{T}$ . The RHS and LHS are given by the following correspondence:

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{X}} & \longmapsto & \mathcal{O}_{\mathcal{X}} \longrightarrow T \\ \downarrow & & \\ I \hookrightarrow T & \twoheadrightarrow & Q, \end{array}$$

which is a bijection on  $\mathbb{C}$ -points. □

**5.6. Duality functor.** We briefly recall the duality functor

$$(28) \quad \mathbb{D} : D^b(\mathcal{X}) \rightarrow D^b(\mathcal{X})$$

defined by:

$$E \mapsto R\mathrm{Hom}_{\mathcal{X}}(E, \mathcal{O}_{\mathcal{X}})[2].$$

This duality functor satisfies the following property:

$$(29) \quad \mathbb{D}({}^q\mathcal{T}_{\bullet}) = {}^p\mathcal{F},$$

where  $q = -(p + 1)$ .

REMARK 5.8. Since  $\mathcal{X}$  is a smooth Calabi–Yau threefold stack, the proof of (29) is very similar to Lemma 3.7 of [15]. We omit the details.

Let  $\mathbb{D}' := \mathbb{D}[1]$  be the functor of  $\mathbb{D}$  shifted by one.

PROPOSITION 5.9. *We have:*

$$\begin{aligned} \mathbb{D}'(\mathbb{1}_{q\mathcal{T}_{\bullet}}) &= \mathbb{1}_{p\mathcal{F}[1]}; \\ \mathbb{D}'(\mathbb{1}_{q\mathcal{T}_{\bullet}}^{\mathcal{O}}) &= \mathbb{1}_{p\mathcal{F}[1]}^{\mathcal{O}}, \end{aligned}$$

where  $\mathbb{1}_{q\mathcal{T}_{\bullet}}, \mathbb{1}_{p\mathcal{F}[1]}$  are elements in  $H_{\infty}(\mathcal{A}_{\leq 1})$  given by the stacks  ${}^q\mathfrak{T}_{\bullet}, {}^p\mathfrak{F} \in \mathcal{M}_{\leq 1}$ ; and  $\mathbb{1}_{q\mathcal{T}_{\bullet}}^{\mathcal{O}}, \mathbb{1}_{p\mathcal{F}[1]}^{\mathcal{O}}$  are elements in  $H_{\infty}({}^p\mathcal{A}_{\leq 1})$  given by the stacks  ${}^q\mathfrak{T}_{\bullet}^{\mathcal{O}}, {}^p\mathfrak{F}[1]^{\mathcal{O}} \in \mathcal{M}_{\leq 1}^{\mathcal{O}}$ .

*Proof.* Proof is very similar to Proposition 3.8 of [15]. □

PROPOSITION 5.10. *The formula in Theorem 5.3 is given by:*

$${}^p\mathcal{H}_{\leq 1} \star \mathbb{1}_{p\mathcal{F}[1]} = \mathbb{1}_{p\mathcal{F}[1]} \star \mathbb{D}'(\mathcal{H}_{\mathrm{exc}}^{\#}) \star \mathcal{H}_{\leq 1}.$$

*Proof.* The formula in Theorem 5.3 is:

$${}^p\mathcal{H}_{\leq 1} \star \mathbb{1}_{p\mathcal{F}[1]} = \mathbb{1}_{p\mathcal{F}[1]}^{\mathcal{O}} \star \mathcal{H}_{\leq 1}.$$

From Proposition 5.9,

$$\mathbb{1}_{p\mathcal{F}[1]}^{\mathcal{O}} = \mathbb{D}'(\mathbb{1}_{q\mathcal{T}_{\bullet}}^{\mathcal{O}}) = \mathbb{D}'(\mathcal{H}_{\leq 1}^{\#} \star \mathbb{1}_{q\mathcal{T}_{\bullet}}) = \mathbb{1}_{p\mathcal{F}[1]} \star \mathbb{D}'(\mathcal{H}_{\mathrm{exc}}^{\#}). \quad \square$$

**5.7. Laurent elements and a complete Hall algebra.** As in [8] and [15], we need to introduce Laurent elements in the numerical Grothendieck group  $K(\mathcal{X})$ . The reason to do this is that the infinite-type Hall algebra  $H_\infty(\mathcal{A}_{\leq 1})$  is too big to support an integration map and we have to work on spaces of locally finite type.

Recall that for the contraction  $\psi: \mathcal{X} \rightarrow Y$ , we have

$$N_1(\mathcal{X}/Y) \hookrightarrow N_1(\mathcal{X}) \rightarrow N_1(Y).$$

So we have

$$N_1(\mathcal{X}) = N_1(\mathcal{X}/Y) \oplus N_1(Y).$$

We can index elements in  $N_{\leq 1}(\mathcal{X}) = N_1(\mathcal{X}) \oplus N_0(\mathcal{X}) = N_1(Y) \oplus N_1(\mathcal{X}/Y) \oplus N_0(\mathcal{X})$  by  $(\gamma, \delta, n)$ . Recall that we have a Chern character map:

$$[E] \in F_1 K(\mathcal{X}) \mapsto (\text{Ch}_2(E), \text{Ch}_3(E)) \in N_1(\mathcal{X}) \oplus N_0(\mathcal{X}).$$

Let  ${}^p\Delta \subset F_1 K({}^p\mathcal{A}) \cong N_1(\mathcal{X}) \oplus N_0(\mathcal{X})$  be the image of the Chern character map of  ${}^p\mathcal{A}_{\leq 1}$ . Then the Hall algebra  $H({}^p\mathcal{A}_{\leq 1})$  is graded by  ${}^p\Delta$ . Let  $\mathcal{C} \subset N_1(\mathcal{X}/Y)$  be the effective curve classes in  $\mathcal{X}$  contracted by  $\psi$ .

**DEFINITION 5.11.** Let  $L \subset {}^p\Delta$  be a subset. We call  $L$  to be *Laurent* if the following conditions hold:

- (1) for any  $\gamma$ , there exists an  $n(\gamma, L)$  such that for all  $\delta, n$ , with  $(\gamma, \delta, n) \in L$ , we have  $n \geq n(\gamma, L)$ ;
- (2) for all  $\gamma, n$ , there exists a  $\delta(\gamma, n, L) \in \mathcal{C}$ , such that for all  $\delta$  with  $(\gamma, \delta, n) \in L$  one has  $\delta \leq \delta(\gamma, n, L)$ .

Let  $\Lambda$  be the set of all Laurent subsets of  ${}^p\Delta$ . The set  $\Lambda$  satisfies the following properties as in Lemma 3.10 of [15]:

- (1) If  $L_1, L_2 \in \Lambda$ , then  $L_1 + L_2 \in \Lambda$ ;
- (2) If  $\alpha \in {}^p\Delta$  and  $L_1, L_2 \in \Lambda$ , then there exist only finitely many decompositions  $\alpha = \alpha_1 + \alpha_2$  with  $\alpha_i \in L_i$ .

The  $\Lambda$ -completion  $H({}^p\mathcal{A}_{\leq 1})_\Lambda$ . Recall the algebra:

$$\mathbb{C}_\sigma[{}^p\Delta] = \bigoplus_{\alpha \in {}^p\Delta} x^\alpha.$$

The integration map is given by:

$$I: H_{\text{sc}}({}^p\mathcal{A}_{\leq 1}) \rightarrow \mathbb{C}_\sigma[{}^p\Delta].$$

For any  ${}^p\Delta$ -graded associative algebra  $R$ , the  $\Lambda$ -completion  $R_\Lambda$  is defined to be the vector space of formal series:

$$\sum_{(\gamma, \delta, n)} x_{(\gamma, \delta, n)}$$

with  $x_{(\gamma,\delta,n)} \in R_{x_{(\gamma,\delta,n)}}$ , and  $x_{(\gamma,\delta,n)} = 0$  outside a Laurent subset. The product is defined by:

$$x \cdot y = \sum_{\alpha \in {}^p\Delta} \sum_{\alpha_1 + \alpha_2 = \alpha} x_{\alpha_1} \cdot y_{\alpha_2}.$$

Then the integration map  $I : H_{\text{sc}}({}^p\mathcal{A}_{\leq 1}) \rightarrow \mathbb{C}_\sigma[{}^p\Delta]$  induces a morphism on the completions:

$$I_\Lambda : H_{\text{sc}}({}^p\mathcal{A}_{\leq 1})_\Lambda \rightarrow \mathbb{C}_\sigma[{}^p\Delta]_\Lambda.$$

*Elements in  $H({}^p\mathcal{A}_{\leq 1})_\Lambda$ .* Let  $\mathfrak{S}$  be an algebraic stack of locally of finite type over  $\mathbb{C}$ , such that  $[\mathfrak{S} \rightarrow {}^p\mathcal{M}_{\leq 1}]$  is a map to  ${}^p\mathcal{M}_{\leq 1}$ . For  $\alpha \in {}^p\Delta$ , the preimage of  ${}^p\mathcal{M}_\alpha$  is denoted by  $\mathfrak{S}_\alpha$ . The element

$$[\mathfrak{S} \rightarrow {}^p\mathcal{M}_{\leq 1}] \in H_\infty({}^p\mathcal{A}_{\leq 1})$$

is *Laurent* if  $\mathfrak{S}_\alpha$  is a stack of finite type for all  $\alpha \in {}^p\Delta$ , and  $\mathfrak{S}_\alpha$  is empty for  $\alpha$  outside a Laurent subset.

Then following results are due to Calabrese in [15].

PROPOSITION 5.12. *The elements*

$$\mathbb{1}_{{}^p\mathcal{F}[1]}, \quad \mathbb{1}_{{}^p\mathcal{F}[1]}^\mathcal{O}, \quad {}^p\mathcal{H}_{\leq 1}, \quad \mathcal{H}_{\leq 1}$$

are all *Laurent*.

*Proof.* The proof of the result is very similar to Propositions 3.13, 3.14, and 3.15 of [15].

The Laurentness of  $\mathbb{1}_{{}^p\mathcal{F}[1]}$ ,  $\mathbb{1}_{{}^p\mathcal{F}[1]}^\mathcal{O}$  is from the fact that once fixing numerical data  $(\gamma, \delta, n)$ , Riemann–Roch tells us that the subset  $\alpha$  is bounded. That the element  ${}^p\mathcal{H}_{\leq 1}$  is Laurent comes from a detail analysis that once we fix  $\gamma, n$ , varying  $\delta$  then the corresponding perverse Hilbert scheme is of finite type. The case of  $\mathcal{H}_{\leq 1}$  is from the Hall algebra identity:

$${}^p\mathcal{H}_{\leq 1} \star \mathbb{1}_{{}^p\mathcal{F}[1]} = \mathbb{1}_{{}^p\mathcal{F}[1]}^\mathcal{O} \star \mathcal{H}_{\leq 1}$$

in Theorem 5.3. □

*Duality functor revisited.* Recall the duality functor in (28), and the shifted duality functor  $\mathbb{D}' = \mathbb{D}[1]$ . By Proposition 5.9,

$$(30) \quad T \in {}^q\mathcal{T}_\bullet \mapsto \mathbb{D}'(T) \in {}^p\mathcal{F},$$

where  $T$  has numerical data  $(0, \delta, n)$ , while  $\mathbb{D}'(T)$  has numerical data  $(0, -\delta, n)$ .

**5.8. Proof of the main results.** First, we have the following Hall algebra identity from Proposition 5.10:

$$(31) \quad {}^p\mathcal{H}_{\leq 1} \star \mathbb{1}_{p\mathcal{F}[1]} = \mathbb{1}_{p\mathcal{F}[1]} \star \mathbb{D}'(\mathcal{H}_{\text{exc}}^\#) \star \mathcal{H}_{\leq 1}.$$

We need to cancel  $\mathbb{1}_{p\mathcal{F}[1]}$  in (31). The elements  ${}^p\mathcal{H}_{\leq 1}$ ,  $\mathcal{H}_{\text{exc}}^\#$ ,  $\mathcal{H}_{\leq 1}$  are all regular in  $H_{\text{sc}}({}^p\mathcal{A}_{\leq 1})$ , but  $\mathbb{1}_{p\mathcal{F}[1]}$  is not. To overcome this difficulty, we use Joyce's stability result, as done by Bridgeland [8] and [15]. Recall that elements in  ${}^p\mathcal{F}[1]$  will have numerical data  $(0, \delta, n)$ , for  $n \geq 0$ . We need the fact

$$(\mathbb{L} - 1) \cdot \log(\mathbb{1}_{p\mathcal{F}[1]}) \in H_{\text{reg}}({}^p\mathcal{A}_{\leq 1}),$$

which can be done by introducing stability condition on the objects that have numerical data  $(0, \delta, n)$ . This means that we work in the category  ${}^p\mathcal{A}_{\text{exc}}$ . Define a stability condition  $\mu$  by:

$$(0, \delta, n) \mapsto \begin{cases} 1, & \delta \geq 0; \\ 2, & \delta < 0. \end{cases}$$

The stability condition  $\mu$  is a weak stability condition in sense of Definition 3.5 of [28].

**LEMMA 5.13.** *The set of  $\mu$ -semistable objects of slope  $\mu = 2$  is  ${}^p\mathcal{F}[1]$ , and the set of  $\mu$ -semistable objects of slope  $\mu = 1$  is  ${}^p\mathcal{T}_{\text{exc}}$ .*

*Proof.* An object  $P$  is said to be semistable if for all proper subobjects  $P' \subset P$  we have  $\mu(P') \leq \mu(P/P')$ . If  $P$  is any semistable object, we have the torsion and torsion-free exact sequence:

$$F[1] \hookrightarrow P \twoheadrightarrow T,$$

where  $F \in {}^p\mathcal{F}$ ,  $T \in {}^p\mathcal{T}_{\leq 1}$ . If  $F[1] \neq 0$  and  $T \neq 0$ , then  $2 = \mu(F[1]) \leq \mu(T) = 1$  which is impossible. So it must be torsion or torsion free.  $\square$

As in [15, Proposition 3.18], the stability condition  $\mu$  is permissible in sense of [27, Definition 4.7]. The following result is Theorem 6.3, Corollary 6.4 in [8], Proposition 3.20 of [15]:

**PROPOSITION 5.14.** *In the complete Hall algebra  $H({}^p\mathcal{A}_{\leq 1})_\Lambda$ , we have:*

$$\mathbb{1}_{p\mathcal{F}[1]} = \exp(\epsilon),$$

with  $\eta = (\mathbb{L} - 1) \cdot \epsilon \in H_{\text{reg}}({}^p\mathcal{A}_{\leq 1})_\Lambda$  a regular element. Here the element  $\epsilon$  is  $\log(\mathbb{1}_{p\mathcal{F}[1]})$ . The automorphism:

$$\text{Ad}_{\mathbb{1}_{p\mathcal{F}[1]}} : H({}^p\mathcal{A}_{\leq 1})_\Lambda \rightarrow H({}^p\mathcal{A}_{\leq 1})_\Lambda$$

preserves regular elements and the induced Poisson automorphism of  $H_{\text{sc}}({}^p\mathcal{A}_{\leq 1})_\Lambda$  is given by:

$$\text{Ad}_{\mathbb{1}_{p\mathcal{F}[1]}} = \exp\{\eta, -\}.$$



THEOREM 5.15. *We have:*

$${}^p \text{DT}(\mathcal{X}/Y) = I(\mathbb{D}'(\mathcal{H}_{\text{exc}}^\#)) \cdot \text{DT}(\mathcal{X}).$$

*Proof.* From the Hall algebra identity

$${}^p \mathcal{H}_{\leq 1} \star \mathbb{1}_{p\mathcal{F}[1]} = \mathbb{1}_{p\mathcal{F}[1]} \star \mathbb{D}'(\mathcal{H}_{\text{exc}}^\#) \star \mathcal{H}_{\leq 1}$$

in (31) and Proposition 5.14, we have the equation:

$${}^p \mathcal{H}_{\leq 1} = \mathbb{D}'(\mathcal{H}_{\text{exc}}^\#) \cdot \exp\{\eta, -\} \cdot \mathcal{H}_{\leq 1}.$$

So when applying the integration map and note that the Poisson bracket is trivial when applying the integration map we have:

$$I_\Lambda({}^p \mathcal{H}_{\leq 1}) = I_\Lambda(\mathbb{D}'(\mathcal{H}_{\text{exc}}^\#)) \cdot I_\Lambda(\mathcal{H}_{\leq 1}).$$

Hence the result follows, due to  $I_\Lambda(\mathcal{H}_{\leq 1}) = \text{DT}(\mathcal{X})$ . □

COROLLARY 5.16. *We have:*

$${}^p \text{DT}(\mathcal{X}/Y) = \frac{\text{DT}_{\text{exc}}^\vee(\mathcal{X})}{\text{DT}_0(\mathcal{X})} \cdot \text{DT}(\mathcal{X}).$$

*Proof.* We need to use A. Bayer’s DT/PT-correspondence for Calabi–Yau orbifolds in [3]. In [3], Bayer proves that

$$\text{DT}'(\mathcal{X}) = \text{PT}(\mathcal{X}).$$

Hence,

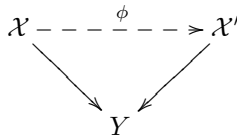
$$\text{DT}'_{\text{exc}}(\mathcal{X}) = \text{PT}_{\text{exc}}(\mathcal{X}).$$

Since  $I_\Lambda(\mathcal{H}_{\text{exc}}^\#) = \text{PT}_{\text{exc}}(\mathcal{X})$ ,

$$I_\Lambda(\mathbb{D}'(\mathcal{H}_{\text{exc}}^\#)) = \text{PT}_{\text{exc}}^\vee(\mathcal{X}).$$

The result follows since  $\text{PT}_{\text{exc}}^\vee(\mathcal{X}) = \text{DT}'_{\text{exc}}{}^\vee(\mathcal{X}) = \frac{\text{DT}_{\text{exc}}^\vee(\mathcal{X})}{\text{DT}_0(\mathcal{X})}$ . □

*Proof of Theorem 1.3.* For an orbifold flop



we have an equivalence:

$$\Phi : D(\mathcal{X}) \rightarrow D(\mathcal{X}')$$

which is given by the Fourier–Mukai transformation. Moreover,

$$\Phi(\text{Per}(\mathcal{X}/Y)) = \text{Per}(\mathcal{X}'/Y).$$

Then on the Hall algebra  $H_{\text{sc}}({}^p \mathcal{A}_{\leq 1})$ , we have

$$\Phi({}^p \mathcal{H}_{\leq 1}(\mathcal{X})) = {}^p \mathcal{H}_{\leq 1}(\mathcal{X}').$$

So  $\Phi_\star({}^p \text{DT}(\mathcal{X}/Y)) = {}^p \text{DT}(\mathcal{X}'/Y)$ . □

## 6. Discussion on the Hard Lefschetz condition

In this section, we give a short discussion on the Hard Lefschetz (HL) condition for orbifold flops.

Proposition 2.5 tells us that an orbifold flop  $\phi : \mathcal{X} \dashrightarrow \mathcal{X}'$  of type  $(a_0, a_1; b_0, b_1)$  satisfies the HL condition if and only if  $a_i = b_i$  for  $i = 0, 1$ . Our result in Theorem 1.3 may help compute the DT-invariants for  $\mathcal{X}$  that does not satisfy the HL condition.

**COROLLARY 6.1.** *Let  $\phi : \mathcal{X} \dashrightarrow \mathcal{X}'$  be an orbifold flop of type  $(\mathbf{a}, \mathbf{b})$  satisfying the HL condition. Then*

$$\Phi(\mathrm{DT}(\mathcal{X}) \cdot \mathrm{DT}_{\mathrm{exc}}^{\vee}(\mathcal{X})) = \mathrm{DT}(\mathcal{X}') \cdot \mathrm{DT}_{\mathrm{exc}}^{\vee}(\mathcal{X}').$$

*Proof.* Theorem 1.3 gives the formula:

$$\Phi_{\star} \left( \mathrm{DT}(\mathcal{X}) \cdot \frac{\mathrm{DT}_{\mathrm{exc}}^{\vee}(\mathcal{X})}{\mathrm{DT}_0(\mathcal{X})} \right) = \mathrm{DT}(\mathcal{X}') \cdot \frac{\mathrm{DT}_{\mathrm{exc}}^{\vee}(\mathcal{X}')}{\mathrm{DT}_0(\mathcal{X}')}.$$

We need to show that  $\Phi_{\star}(\mathrm{DT}_0(\mathcal{X})) = \mathrm{DT}_0(\mathcal{X}')$ .

If  $\phi$  satisfies the HL condition, then from Proposition 2.5  $a_0 = b_0$ ,  $a_1 = b_1$ . Hence the flopping locus are all weighted projective stacks  $\mathbb{P}(a_0, a_1)$ , and  $\mathcal{X}, \mathcal{X}'$  are isomorphic beyond the flopping locus. The degree zero Donaldson–Thomas invariants of  $\mathcal{X}, \mathcal{X}'$  are the weighted Euler characteristic of the Hilbert scheme of points on the threefold DM stacks  $\mathcal{X}$  and  $\mathcal{X}'$ . So it is sufficient to consider the local model case

$$\begin{aligned} \mathcal{X} &= \mathcal{O}_{\mathbb{P}(a_0, a_1)}(-a_0) \oplus \mathcal{O}_{\mathbb{P}(a_0, a_1)}(-a_1); \\ \mathcal{X}' &= \mathcal{O}_{\mathbb{P}(a_0, a_1)}(-a_0) \oplus \mathcal{O}_{\mathbb{P}(a_0, a_1)}(-a_1). \end{aligned}$$

Since  $\mathcal{X}$  and  $\mathcal{X}'$  are the same DM stacks, the degree zero Donaldson–Thomas invariants for both  $\mathcal{X}$  and  $\mathcal{X}'$  are the same. So  $\Phi_{\star}(\mathrm{DT}_0(\mathcal{X})) = \mathrm{DT}_0(\mathcal{X}')$  and the corollary follows.  $\square$

We discuss the case of the local picture of an orbifold flop of type  $(a_0, a_1; b_0, b_1)$  with  $\sum_i a_i = \sum_i b_i$ . Consider the diagram (8),  $\phi : \tilde{\mathcal{X}} \dashrightarrow \tilde{\mathcal{X}'}$  is an orbifold flop of type  $(a_0, a_1; b_0, b_1)$ . Both  $\tilde{\mathcal{X}}$  and  $\tilde{\mathcal{X}'}$  are Calabi–Yau threefold stacks with  $A_n$ -singularities. Bryan, Cadman and Young [11] study the DT-invariants of such Calabi–Yau threefold stacks satisfying the HL condition by the method of orbifold topological vertex. In some cases, they derive a nice formula for the DT-partition functions.

Our main result implies that using DT-invariants of Calabi–Yau threefold stacks with the HL condition, we may get DT-partition function for Calabi–Yau threefold stacks without the HL conditions.

EXAMPLE 6.2. Let  $(a_1, a_2; b_1, b_2) = (2, 2; 1, 3)$ . Consider

$$\mathcal{X} = \mathcal{O}_{\mathbb{P}(2,2)}(-1) \oplus \mathcal{O}_{\mathbb{P}(2,2)}(-3) \dashrightarrow \mathcal{X}' = \mathcal{O}_{\mathbb{P}(1,3)}(-2) \oplus \mathcal{O}_{\mathbb{P}(1,3)}(-2)$$

which is an orbifold flop such that  $\mathcal{X}$  satisfies HL condition, but  $\mathcal{X}'$  does not. The stack  $\mathcal{X}$  is a local  $B\mu_2$ -gerbe over  $\mathbb{P}^1$ , and the DT-partition function for  $\mathcal{X}$  was calculated in [11, §4.4]. Our main result Theorem 1.3 implies the relationship between the DT-partition function for  $\mathcal{X}$  and the DT-partition function for  $\mathcal{X}'$ .

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