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QUANTUM SEMIGROUPS GENERATED BY LOCALLY COMPACT SEMIGROUPS

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ABSTRACT. Let S be a subsemigroup of a second countable locally compact group G, such that $S^{-1}S = G$. We consider the C^*-algebra $C^*_\delta(S)$ generated by the operators of translation by all elements of S in $L^2(S)$. We show that this algebra admits a comultiplication which turns it into a compact quantum semigroup. The same is proved for the von Neumann algebra $\mathrm{VN}(S)$ generated by $C^*_\delta(S)$.

1. Introduction

The notion of a quantum semigroup, as a C^*- or von Neumann algebra with a comultiplication, appeared well before the term and before the notion of a locally compact quantum group. But it is especially these last years that substantial examples of quantum semigroups are considered; we would like to mention families of maps on finite quantum spaces [22], quantum semigroups of quantum partial permutations [2], quantum weakly almost periodic functionals [9], quantum Bohr compactifications [20], [21].

In this article, we construct a rather "classical" family of compact quantum semigroups, which are associated to sub-semigroups of locally compact groups. The interest of our objects is in fact that they provide natural examples of C^*-bialgebras which are co-commutative and are not however duals of functions algebras. Recall that the classical examples of quantum groups belong to one of the two following types: they are either function algebras,

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such as the algebra $C_0(G)$ of continuous functions vanishing at infinity on a locally compact group G, or their duals, such as the reduced group C^*-algebras $C^*_r(G)$. In the semigroup situation, one can go beyond this dichotomy.

If S is a discrete semigroup, then the algebra $C^*_\delta(S)$ which we consider coincides with the reduced semigroup C^*-algebra $C^*_r(S)$ which has been known since long ago [6], [7], [3], [26], [18]. If $S=G$ is a locally compact group, then $C^*_\delta(S)=C^*_\delta(G)$ is the C^*-algebra generated by all left translation operators in $B(L^2(G))$ [12], [4]. If G is moreover Abelian, then $C^*_\delta(G)$ equals to the algebra $C(\widehat{G}_d)$ of continuous functions on the dual of the discrete group G_d [12].

The new case considered in this paper concerns non-discrete nontrivial sub-semigroups of locally compact groups, and our objective is to show that their algebras admit a natural coalgebra structure. Let G be a second countable locally compact group, and let S be its sub-semigroup such that $S^{-1}S=G$. Set $H_S=\{f\in L^2(G):\operatorname{supp} f\subset S\}$; let E_S be the orthogonal projection of $L^2(G)$ onto H_S and let J_S be the right inverse of E_S, so that $E_SJ_S=\mathrm{Id}_{H_S}$. After the study of semigroup ideals in Section 2, in Section 3 we define $C^*_\delta(S)$ as the C^*-algebra generated in $B(H_S)$ by the operators $T_a=E_SL_aJ_S$ over all $a\in S$, where L_a is the operator of the left translation by a on $L^2(G)$.

The strong closure of $C^*_\delta(S)$ in $B(H_S)$ is denoted $\mathrm{VN}(S)$ and is said to be the semigroup von Neumann algebra. In the case $S=G$, this is the classical group von Neumann algebra, and in the case when the interior of S is dense in it, this equals to the von Neumann algebra generated by the reduced C^*-algebra $C^*_r(S)$ introduced by Muhly and Renault [16], see Section 4.

By defining it first on $\mathrm{VN}(S)$, in Section 6 we show that $C^*_\delta(S)$ admits a comultiplication Δ such that $\Delta(T_a)=T_a\otimes T_a$. To obtain the main result, we are using techniques of inductive limits and crossed products analogous to the constructions carried out in [14] and [8] for the discrete case. The proof is also based on the duality of semi-lattices in Section 6.1. The discrete Abelian case was studied in detail in [1].

The article concludes by a more explicit description of $C^*_\delta(S)$ in the Abelian case.

2. Semigroup ideals

Let G be a second countable locally compact group, S a closed subsemigroup of G containing the identity e of G and such that $G=S^{-1}S$. Denote by μ the left Haar measure on G. We suppose that $\mu(S)>0$, otherwise our definition would produce a trivially zero algebra; this implies immediately that S has a nonempty interior (apply [10, 20.17] with $A,B\subset S$ compact of positive measure). One can hold in mind a model example $S=[0,+\infty)$, $G=\mathbb{R}$.

For any subset $X \subset S$ and any $p \in S$, define the *translations in* S:

$$pX = \{pq \colon q \in X\}, \qquad p^{-1}X = \{q \in S \colon pq \in X\}. \tag{2.1}$$

Obviously, pS is a right ideal in S, $eS = e^{-1}S = S$ and $p^{-1}S = S$ for any $p \in S$. For the usual translations in G, we use the notation $g \underset{G}{\cdot} X = \{gh : h \in X\}$, so that $p^{-1}X = S \cap p^{-1} \underset{G}{\cdot} X$. It is also easy to see that $p(qX) = (pq)X$ and $p^{-1}(q^{-1}X) = (qp)^{-1}X$ for any $X \subset S$ and all $p, q \in S$. We will omit parentheses in the products of this type. Moreover, $p^{-1}pX = X$, but in general, the products pq^{-1} or $p^{-1}q$ should be viewed purely formally, and $pp^{-1}X$ might differ from X (see, for example, Lemma 2.2).

More precisely, denote by $\mathcal{F} = \mathcal{F}(S)$ the free monoid generated by S and $S^{-1} \setminus S$. Any element in $\mathcal{F}$ can be canonically reduced to a form of a finite word with alternating symbols in S and S^{-1}, replacing expressions of the type st, $s^{-1}t^{-1}$ with $s, t \in S$ by u, v^{-1} respectively, with $u = st$ and $v = ts$ in S. The monoid operation on $\mathcal{F}$ is concatenation of words combined with multiplication in S of neighbouring elements, for example,

$$\big(p^{-1}qr^{-1}\big) \cdot \big(a^{-1}bc^{-1}d\big) = p^{-1}q(ar)^{-1}bc^{-1}d.$$

The operation of taking inverse in G induces the operation $w \mapsto w^{-1}$ on the monoid $\mathcal{F}$, by $(p_1^{\pm 1} \cdots p_n^{\pm 1})^{-1} = p_n^{\mp 1} \cdots p_1^{\mp 1}$.

For every $w = p_1^{\pm 1} \cdots p_n^{\pm 1} \in \mathcal{F}$ and $X \subset S$, define by induction $wX = p_1^{\pm 1}(\cdots (p_n^{\pm 1}X) \cdots)$. If $X = S$, then wS is a right ideal in S. Define the family of all *constructible right ideals* in S [14]:

$$\mathcal{J} = \left\{ \bigcap_{i=1}^{n} w_i S \colon w_i \in \mathcal{F} \right\} \cup \{\emptyset\}.$$

Suppose that $w \in \mathcal{F}$ has the form $w = p_1^{-1}q_1p_2^{-1}q_2 \cdots p_n^{-1}q_n$ with $p_j, q_j \in S$, maybe with $p_1 = e$ or $q_n = e$. Then it follows from the definition that wS is the set of elements x satisfying

$$\begin{aligned} x &= p_1^{-1}q_1 \cdots p_n^{-1}q_nr_{n+1}, \\ &\quad \text{where } r_{n+1} \in S \quad \text{and} \\ r_k &= p_k^{-1}q_k \cdots p_n^{-1}q_nr_{n+1} \in S \quad \text{for all } k = 1, \ldots, n. \end{aligned} \tag{2.2}$$

Define a homomorphism $\mathcal{F} \to G$: $w \mapsto (w)_G$, by $(p^{\pm 1})_G = p^{\pm 1}$ for $p \in S$. We fix also an injection $\imath : G \hookrightarrow \mathcal{F}$ which might not be a homomorphism: for any element $g \in G$ we fix one of its representations $g = p^{-1}q$ and set $\imath(g) = p^{-1}q \in \mathcal{F}$. The notation gX, where $g \in G$ and $X \subset S$, is understood in the sense $gX = \imath(g)X$.

LEMMA 2.1. *For any $w_1, w_2 \in \mathcal{F}$, we have $w_1w_2S \subset w_1S$.*

Proof. Follows immediately from the facts that $pS \subset S, p^{-1}S = S$ for any $p \in S$. ☐

LEMMA 2.2. *For any $w \in \mathcal{F}$, $wS = ww^{-1}S$.*

Proof. We can assume that w has the form $w = p_1^{-1}q_1p_2^{-1}q_2\cdots p_n^{-1}q_n$ with $p_j, q_j \in S$, maybe with $p_1 = e$ or $q_n = e$. Then every $x \in wS$ has the form (2.2) with $r_{n+1} \in S$ and $r_k = p_k^{-1}q_k\cdots p_n^{-1}q_nr_{n+1} \in S$ for all $k = 1,\dots,n$.

Now write $x = p_1^{-1}q_1\cdots p_n^{-1}q_nq_n^{-1}p_n\cdots q_1^{-1}p_1x$. In this product, $x \in S$ and $r_{k+1} = q_k^{-1}p_k\cdots q_1^{-1}p_1x \in S$ for $k = 1,\dots,n$, as well as $r_k = p_k^{-1}q_k\cdots p_n^{-1}q_nq_n^{-1}p_n\cdots q_1^{-1}p_1x \in S$ for $k = 1,\dots,n$. It follows that $x \in ww^{-1}S$, so $wS \subset ww^{-1}S$. The inverse inclusion follows from Lemma 2.1. ☐

LEMMA 2.3. *Let a word $w \in \mathcal{F}$ have the form $w = w_1w_2$, where $w_1, w_2 \in \mathcal{F}$. Then $wS = w_1S \cap (w_1)_Gw_2S$.*

Proof. Suppose $w_1 = p_1^{-1}q_1p_2^{-1}q_2\cdots p_i^{-1}q_i$, $w_2 = p_{i+1}^{-1}q_{i+1}p_{i+2}^{-1}q_{i+2}\cdots p_n^{-1}q_n$ with $p_j, q_j \in S$. Then every $x \in wS$ satisfies (2.2). This implies directly that $x \in w_1S$. If we denote $(w_1)_G = p^{-1}q$, $p, q \in S$, then in the notations (2.2) we have also $x = p^{-1}qr_{i+1}$ which implies that $x \in p^{-1}qw_2S = (w_1)_Gw_2S$.

Conversely, if $x \in w_1S \cap (w_1)_Gw_2S$, then

$$x = p_1^{-1}q_1p_2^{-1}q_2\cdots p_i^{-1}q_ir'_{i+1},$$
$$r'_{i+1} \in S, \quad r'_k = p_k^{-1}q_k\cdots p_i^{-1}q_ir'_i \in S \quad \text{for } k = 1,\dots,i,$$

and

$$x = p^{-1}qp_{i+1}^{-1}q_{i+1}p_{i+2}^{-1}q_{i+2}\cdots p_n^{-1}q_nr_{n+1},$$
$$x \in S, r_{n+1} \in S, \quad r_k = p_k^{-1}q_k\cdots p_n^{-1}q_nr_{n+1} \in S \quad \text{for } k = i+1,\dots,n.$$

By cancellation, it follows that $r'_{i+1} = p_{i+1}^{-1}q_{i+1}p_{i+2}^{-1}q_{i+2}\cdots p_n^{-1}q_nr_{n+1}$, thus in fact the condition (2.2) holds for x and $x \in wS$. ☐

COROLLARY 2.4. *For any $v, w \in \mathcal{F}$, we have $vS \cap wS = ww^{-1}vS$.*

Proof. Since $(ww^{-1})_G = e$, by Lemmas 2.2 and 2.3 we have

$$wS \cap vS = ww^{-1}S \cap vS = ww^{-1}S \cap \left(ww^{-1}\right)_G vS = ww^{-1}vS. \qquad ☐$$

It follows directly that

$$\mathcal{J} = \{wS | w \in \mathcal{F}\} \cup \{\emptyset\}.$$

DEFINITION 2.5. We will say that measurable subsets X, Y of G are equivalent and write $X \sim Y$ if $\mu(X\Delta Y) = 0$, where Δ denotes the symmetric difference. The equivalence class of X is denoted by $[X]$.

For any $X, X', Y, Y' \in \mathcal{J}$ and $p \in S$ the following holds.

(1) If $X \sim X'$ and $Y \sim Y'$, then $X \cap Y \sim X' \cap Y'$ and $X \cup Y \sim X' \cup Y'$.
(2) If $X \sim X'$, then $pX \sim pX'$ and $p^{-1}X \sim p^{-1}X'$.

We define, as usual, $[X]\cap[Y]=[X\cap Y]$, $[X]\cup[Y]=[X\cup Y]$, $p[X]=[pX]$, $p^{-1}[X]=[p^{-1}X]$, for any $X,Y\in\mathcal{J}$ and $p\in S$. We will work further with the set $\mathcal{J}'=\{[X]\colon X\in\mathcal{J}\}$.

The following notion was defined by X. Li in [14]. The constructible right ideals of the semigroup S are *independent*, if $X=\bigcup_{j=1}^{n}X_j$ for $X,X_1,\dots,X_n\in\mathcal{J}$, $n\in\mathbb{N}$ implies $X=X_j$ for some $1\le j\le n$. This notion is appropriate for a discrete semigroup S. In our case, this definition should be adjusted.

We say that the constructible right ideals of S are *measurably independent*, if

$$\begin{aligned}&X\sim\bigcup_{j=1}^{n}X_j\quad\text{for } X,X_1,\dots,X_n\in\mathcal{J}\text{ implies}\\&X\sim X_j\quad\text{for some } 1\le j\le n.\end{aligned}$$

Passing to $\mathcal{J}'$, we get the following reformulation of measurable independence.

$$\begin{aligned}&[X]=\bigcup_{j=1}^{n}[X_j]\quad\text{for } X,X_1,\dots,X_n\in\mathcal{J}\text{ implies}\\&[X]=[X_j]\quad\text{for some } 1\le j\le n.\end{aligned}$$

The notions of independence and measurable independence in fact do not coincide. We present a simple example of a semigroup which has certain non-independent constructible right ideals, and at the same time all its (constructible right) ideals are measurably independent.

Example 2.6. Let G be the group $\mathbb{R}_+\setminus\{0\}$ with respect to the usual multiplication, and consider the subsemigroup $S=\{1\}\cup\{2\}\cup[3;+\infty)$ in G. Computing the constructible right ideals $2S$, $4^{-1}3S$, $3^{-1}2S$ we get

$$\begin{aligned}2S&=\{2\}\cup\{4\}\cup[6;+\infty),\\4^{-1}3S&=[3;+\infty),\\3^{-1}2S&=\{2\}\cup[3;+\infty).\end{aligned}$$

Hence, we have non-independent ideals: $2S\cup 4^{-1}3S=3^{-1}2S$. At the same time, all the ideals are equivalent to the ideals of the type $[a;+\infty), a\ge 3$, and therefore are measurably independent.

The following is an example of a semigroup whose ideals are not measurably independent.

Example 2.7. Consider $S=\{0\}\cup[1;1.5]\cup[2;\infty)$ as a subsemigroup of the group $\mathbb{R}$ with respect to usual addition and the usual topology. Further compute the following ideals:

$$1 + S = \{1\} \cup [2; 2.5] \cup [3; \infty),$$
$$1.5 + S = \{1.5\} \cup [2.5; 3] \cup [3.5; \infty),$$
$$-1.5 + (1 + S) = \{1\} \cup \{1.5\} \cup [2; \infty).$$

We easily see that $-1.5 + (1+S) = (1+S) \cup (1.5+S)$, and the same is true for the equivalence classes of these ideals. Hence, the ideals of S are not independent and not measurably independent.

Both examples above are called *perforated semigroups*, since they are obtained from $\mathbb{R}_+$ by deleting some intervals.

3. The semigroup C^*-algebras

Further on we will assume that the constructible right ideals of S are measurably independent. It is exactly this property which will guarantee that our comultiplication is well defined.

Consider the Hilbert space $L^2(G)$ with respect to μ. For any measurable subset $X \subset G$ set $H_X = \{f \in L^2(G) : \operatorname{ess\,supp} f \subset X\}$; this subspace is isomorphic to $L^2(X, \mu)$. Let $I_X \in L^2(G)$ be the characteristic function of X, and let E_X be the orthogonal projection of $L^2(G)$ onto H_X, which is just the multiplication by I_X. Let $L\colon G \to B(L^2(G))$ be the left regular representation of G, that is, for any $a, b \in G, f \in L^2(G)$

$$(L_a f)(b) = f\big(a^{-1}b\big). \tag{3.1}$$

We define the left regular representation $T\colon S \to B(H_S)$ of the semigroup S analogously to L. For any $a, b \in S, f \in H_S$ we set

$$(T_a f)(b) = f\big(a^{-1}b\big), \tag{3.2}$$

so that $T_a = E_S L_a E_S$; then

$$\big(T_a^* f\big)(b) = f(ab). \tag{3.3}$$

One can verify that T_a is an isometry, $T_a^* T_a = I$, and that for any $f \in H_S$ and $a, b \in S$ we have

$$\big(T_a T_a^* f\big)(b) = I_S\big(a^{-1}b\big) f(b).$$

Clearly, $a^{-1}b \in S$ if and only if $b \in aS$, where aS is a constructible right ideal defined in the previous section. Hence, the projection $T_a T_a^*$ is an operator of multiplication by I_{aS}. The map $T\colon S \to B(H_S)$ is obviously a representation of S.

DEFINITION 3.1. Let $C^*_\delta(S)$ be the C^*-subalgebra in $B(H_S)$ generated by the operators $\{T_a : a \in S\}$. Denote by $\mathrm{VN}(S)$ the strong operator closure of $C^*_\delta(S)$ in $B(H_S)$ and call it the *semigroup von Neumann algebra* of S.

If $S=G$, then $C^*_\delta(S)=C^*_\delta(G)$ is the C^*-algebra generated by all left translation operators in $B(L^2(G))$ [12], [4]. If S is discrete, then $C^*_\delta(S)=C^*_r(S)$ is the reduced semigroup C^*-algebra [18].

A finite product of the generators T_a, T_b^* for any $a,b\in S$ is called *a monomial.* We will denote also T_a^* by $T_{a^{-1}}$, which does not create confusion in the case $a^{-1}\in S$. Generally, for every $w=p_1^{\pm 1}\cdots p_n^{\pm 1}\in\mathcal{F}$ we denote $T_w=T_{p_1^{\pm 1}}\cdots T_{p_n^{\pm 1}}$, and clearly every monomial has this form.

LEMMA 3.2. *For any monomial T_w, function $f\in H_S$ and $x\in G$ we have*

$$(T_wf)(x)=I_{wS}(x)\cdot f\big(\big(w^{-1}\big)_G x\big). \tag{3.4}$$

Proof. Let k be the length of the word w. For $k=1$, either $w=a$ or $w=a^{-1}$ with some $a\in S$. If $w=a$ then for $f\in H_S$ we have $f(a^{-1}x)=f(a^{-1}x)I_S(a^{-1}x)=I_{aS}(x)f(a^{-1}x)$, thus the expressions (3.2) and (3.4) are equal. If $w=a^{-1}$, then, due to the fact that $a^{-1}S=S$, the formula (3.3) implies (3.4).

Suppose (3.4) is proved for $k\le n$ and $w=vw'$ is a word in $\mathcal{F}$ with the length $k+1$, where the length of v and w' is 1 and k, respectively. First assume that $v=a\in S$ and denote $g=T_{w'}f$. Then for any $x\in G$ we have

$$\begin{aligned}(T_wf)(x)&=(T_aT_{w'}f)(x)=(T_ag)(x)=g\big(a^{-1}x\big)\\&=(T_{w'}f)\big(a^{-1}x\big)=I_{w'S}\big(a^{-1}x\big)f\big(\big(w'^{-1}\big)_Ga^{-1}x\big)\\&=I_{aw'S}(x)f\Big(\Big(\big(aw'\big)^{-1}\Big)_Gx\Big)=I_{wS}(x)\cdot f\big(\big(w^{-1}\big)_Gx\big).\end{aligned}$$

Now assume that $v=a^{-1}\in S^{-1}$. Then for any $x\in G$ we have

$$\begin{aligned}(T_wf)(x)&=\big(T_a^*T_{w'}f\big)(x)=\big(T_a^*g\big)(x)\\&=I_S(x)g(ax)\\&=I_S(x)(T_{w'}f)(ax)=I_S(x)I_{w'S}(ax)f\big(\big(w'^{-1}\big)_Gax\big).\end{aligned}$$

Note that $x\in S$ and $ax\in w'S$ if and only if $x\in a^{-1}w'S$. Thus,

$$(T_wf)(x)=I_{a^{-1}w'S}(x)f\Big(\Big(\big(a^{-1}w'\big)^{-1}\Big)_Gx\Big)=I_{wS}(x)\cdot f\big(\big(w^{-1}\big)_Gx\big).$$

The formula (3.4) follows. □

LEMMA 3.3. *The C^*-algebra $C^*_\delta(S)$ is isomorphic to the C^*-subalgebra in $B(L^2(G))$ generated as a closed* linear space *by*

$$E_{wS}L_{(w)_G}E_S,\quad w\in\mathcal{F}, \tag{3.5}$$

and equivalently by

$$E_{wS}L_{(w)_G},\quad w\in\mathcal{F}. \tag{3.6}$$

Proof. It follows directly from (3.4) that $T_wE_S=E_{wS}L_{(w)_G}E_S$ for every $w\in\mathcal{F}$. At the same time, $E_STE_S=TE_S$ for every $T\in C^*_\delta(S)$. Thus, the mapping $T\mapsto TE_S=E_STE_S$ is a*-homomorphism from $C^*_\delta(S)$ to $B(L^2(G))$, and its image is generated exactly by the operators (3.6). Moreover, this mapping is clearly isometric and thus it is an isomorphism.

To arrive at the second description, one calculates that $L_g E_S = E_{g \underset{G}{\cdot} S} L_g$ for every $g \in G$. Thus,

$$E_{wS} L_{(w)_G} E_S = E_{wS \cap ((w)_G \underset{G}{\cdot} S)} L_{(w)_G}.$$

By definition, $wS \subset (w)_G \underset{G}{\cdot} S$, and (3.6) follows. □

The formula (3.4) shows that $T_w = 0$ if and only if $\mu(wS) = 0$, that is, $wS \sim \emptyset$. For a non-zero monomial T_w define its *index* by $(w)_G \in G$. We have $\operatorname{ind} T_w^* = (w)_G^{-1}$ and $\operatorname{ind}(T_v T_w) = (v)_G (w)_G$, if $T_v T_w \neq 0$. Recall that $E_X \in B(L^2(G))$ is the operator of multiplication by I_X.

COROLLARY 3.4. *A non-zero monomial T_w in $C_\delta^*(S)$ is an orthogonal projection if and only if* $\operatorname{ind} T_w = e$. *In this case,* $T_w = E_{wS}$.

Proof. Let T_w be an orthogonal projection. Then $(ww)_G = (w)_G^2 = (w)_G$ and $w_G^{-1} = w_G$. Hence, $(w)_G = \operatorname{ind} T_w = e$.

Suppose that $\operatorname{ind} T_w = e$. Then due to Lemma 3.2, $T_w = E_{wS}$ which is an orthogonal projection. □

LEMMA 3.5. *Every projection E_X for $X \in \mathcal{J}$ is contained in $C_\delta^*(S)$ and equals $T_{ww^{-1}}$ for some $w \in \mathcal{F}$.*

Proof. By Corollary 2.4, $X = wS$ for some $w \in \mathcal{F}$. Due to Corollary 3.4, if $(w)_G = e$ then $E_{wS} \in C_\delta^*(S)$.

Suppose w is an arbitrary element in $\mathcal{F}$. By Lemma 2.2, $wS = ww^{-1}S$ and $E_{wS} = E_{ww^{-1}S}$. Since $(ww^{-1})_G = e$, by Corollary 3.4 we have that $E_{wS} = T_{ww^{-1}} \in C_\delta^*(S)$. □

Xin Li [14, Definition 2.2] defined the full semigroup C^*-algebra of a discrete semigroup as generated not only by isometries associated to the points of S, but also by projections corresponding to its constructible ideals. The aim of this construction is to obtain a smaller algebra than just generated by the isometries, so it stays reasonable at least in the case of a commutative semigroup. See [14] for a longer discussion. Taking into account the topology on S, we adjust the definition of [14], using the family $\mathcal{J}'$ instead of $\mathcal{J}$. If S is discrete, the new definition coincides with the old one. Consider a family of isometries $\{v_p | p \in S\}$ and a family of projections $\{e_X | X \in \mathcal{J}'\}$ satisfying the following relations for any $p, q \in S$, and $X, Y \in \mathcal{J}'$:

$$v_{pq} = v_p v_q, \qquad v_p e_X v_p^* = e_{pX}, \tag{3.7}$$

$$e_S = 1, \qquad e_\emptyset = 0, \qquad e_{X \cap Y} = e_X e_Y. \tag{3.8}$$

Definition 3.6. The universal C^*-algebra $C^*(S)$ of the semigroup S is the universal C^*-algebra generated by $\{v_p|p\in S\}\cup\{e_X|X\in\mathcal{J}'\}$ with the relations above. Since the relations between T_p and E_X in $B(H_S)$ are the same, this algebra is nonzero. Denote by $D(S)$ the commutative C^*-algebra generated by the family of projections $\{e_X|X\in\mathcal{J}'\}$ in $C^*(S)$.

Lemma 3.7. *There exists a surjective $*$-homomorphism $\lambda\colon C^*(S)\to C^*_\delta(S)$ such that $\lambda(v_p)=T_p$, $\lambda(e_{[X]})=E_X$. It extends to a normal $*$-homomorphism $\lambda: C^*(S)^{**}\to \mathrm{VN}(S)$. Both maps will be called the left regular representations, of $C^*(S)$ and $C^*(S)^{**}$ respectively.*

Proof. Clearly, $E_X=E_Y$ if and only if $X\sim Y$. Hence, the map $\lambda\colon e_{[X]}\mapsto E_X$ is well defined. Due to the definition of $\cap$ on $\mathcal{J}'$, λ is a semigroup isomorphism between the semigroups $\{e_{[X]}\colon [X]\in\mathcal{J}'\}$ and $\{E_X\colon X\in\mathcal{J}\}$. One can easily verify that the operators T_p and E_X satisfy the equations (3.7) and (3.8) for all $p\in S, X\in\mathcal{J}$. The universality of $C^*(S)$ implies the existence of the homomorphism λ. Its extension exists by the universality property of $C^*(S)^{**}$. □

Denote by $D_\delta(S)$ the C^*-subalgebra in $C^*_\delta(S)$ generated by monomials with index equal to e. By Corollary 3.4 and Lemma 3.5 $D_\delta(S)$ is generated by projections $\{E_X|X\in\mathcal{J}\}$, and is obviously commutative.

Lemma 3.8. *The algebras $D(S)$ and $D_\delta(S)$ are isomorphic.*

Proof. By definition, $\lambda(D(S))$ contains the generators of $D_\delta(S)$. Applying Lemma 2.20 in [14] and using the measurable independence of constructible right ideals in S we obtain injectivity of $\lambda|_{D(S)}$. □

There exists a natural action of the semigroup S on the C^*-algebra $D_\delta(S)$.

$$\tau_p(A)=T_pAT_p^*,\quad p\in S, A\in D_\delta(S). \tag{3.9}$$

Using the formula (3.4), we obtain for $A=E_X$, $X\in\mathcal{J}$:

$$\tau_p(E_X)=E_{pX}. \tag{3.10}$$

4. Comparison with reduced semigroup C^*-algebras

In the case when S has a dense interior (in addition to our assumptions), there exists a construction of the reduced C^*-algebra $C^*_r(S)$, see [16] and a more general construction in [23], [19]. The connection between $C^*_r(S)$ and $C^*_\delta(S)$ precisely corresponds to the case of C^*-algebras of locally compact groups. In particular, as we show further, the von Neumann closures of the two coincide.

We recall the construction of $C^*_r(S)$ of [16], [23] according to the symmetric case $G=S^{-1}S$ adopted here. Similar to the subspace $L^2(S)\subset L^2(G)$, we can consider $L^1(S) \subset L^1(G)$, which is in addition a Banach algebra with respect to convolution. For any $f \in L^1(S)$ define an operator $V_f \in B(L^2(S))$:

$$V_f = \int f(a) T_a \, d\mu(a). \tag{4.1}$$

One can easily verify that $V_f \xi = f * \xi$ for $\xi \in L^2(S)$ and

$$V_f^* = \int \overline{f(a)} T_a^* \, d\mu(a).$$

The reduced C^*-algebra of S is the C^*-subalgebra of $B(L^2(S))$ generated by operators V_f over all $f \in L^1(S)$.

PROPOSITION 4.1. *Let S be a closed subsemigroup with dense interior of a locally compact group G such that $G = S^{-1}S$. Then* $\mathrm{VN}(S)$ *equals to the von Neumann closure of $C_r^*(S)$ inside $B(L^2(S))$.*

Proof. It is sufficient to show that the commutants of $C_r^*(S)$ and $C_\delta^*(S)$ coincide. If $A \in B(L^2(S))$ commutes with T_a and T_a^* for every $a \in S$, then due to (4.1) A commutes with every V_f and V_f^* for every $f \in L^1(S)$.

To prove the reverse inclusion, take ϕ_i to be the approximate identity of $L^1(G)$ lying in $L^1(S)$ which exists by the assumption that the interior of S is dense in it (see also [19, Lemma 2.3] for more details on its construction). We can assume also that every ϕ_i has compact support so that $\phi_i \in L^2(S)$. For any $\xi \in L^2(G)$, we have then $\phi_i * \xi \to \xi$ in the L^2-norm. Denote $\phi_{i,a} = T_a \phi_i$, where $a \in S$; one verifies that $\phi_{i,a} \in L^1(S)$ and $V_{\phi_{i,a}} \xi = T_a(\phi_i * \xi)$ for $\xi \in L^2(S)$. Moreover, $T_a \phi_i \in L^1(S)$ so that $V_{\phi_{i,a}} \in C_r^*(S)$.

For any A in the commutant of $C_r^*(S)$ and any $\xi \in L^2(S)$,

$$AT_a \xi = AT_a \lim \phi_i * \xi = \lim AV_{\phi_{i,a}} \xi = \lim T_a(\phi_i * A\xi) = T_a A\xi,$$

so that the two commutants coincide. □

EXAMPLE 4.2. There are semigroups to which the construction of [16], [23] is not applicable while ours is. Let $C \subset [0,1]$ be the middle-fifth Cantor set (of positive measure). Let S be the additive semigroup

$$S = \{0\} \cup \{2\} \cup \bigcup_{n=0}^{\infty} \left(2 + 2^{-2n-1} + 2^{-2n-1} C\right) \cup [4, +\infty).$$

Then the interior of S is not dense in it so S does not satisfy the assumptions of [16], [23]. However its ideals are measurably independent, and other assumptions made in Section 2 are also satisfied.

5. The semigroup C^*-algebra and crossed products

In what follows, we establish a connection between $C^*_\delta(S)$, $\mathrm{VN}(S)$ on one hand and the C^*- and von Neumann group crossed products by the group G on the other hand. The proof repeats almost verbatim that of Lemma 4.2 of [8] and is based on Theorem 2.1 of [13]. The difference with the mentioned references lies in the topology of the action of G in the outcoming dynamical system.

Recall that a semigroup is called *right reversible* if every pair of non-empty left ideals has a non-empty intersection (see [13], [5]). The following theorem by Ore (for discrete semigroups) can be found in [5].

Theorem 5.1. *A cancellative semigroup S can be embedded into a group G such that $G = S^{-1}S$ if and only if it is right reversible. And in this case, G is a unique up to isomorphism group generated by S.*

By our assumptions, S is embedded into G so that $G = S^{-1}S$; by Ore's theorem, S is right reversible. This allows us to define a preorder on S: set $p \le q$ if $qp^{-1} \in S$, or equivalently if $q \in Sp$. By Ore's theorem, for any $p, q \in S$ the left ideals $\{xp\colon x \in S\}$, $\{yq\colon y \in S\}$ have a non-empty intersection. Hence, S is upwards directed with respect to this preorder: for any $p, q \in S$ there exists $r \in S$ such that $q \le r$ and $p \le r$. In the case when $S \cap S^{-1} \ne \{e\}$, this might not be an order; in fact, $p \le q$ and $q \le p$ if $p^{-1}q \in S \cap S^{-1}$.

Consider the directed system of C^*-algebras $\mathcal{D}_p$ indexed by $p \in S$, where every $\mathcal{D}_p = D_\delta(S)$. For $p, q \in S$ such that $p \le q$ we have $qp^{-1} \in S$ and the action (3.9) generates a $*$-homomorphism $\tau_{qp^{-1}} : \mathcal{D}_p \to \mathcal{D}_q$:

$$\tau_{qp^{-1}}(A) = T_{qp^{-1}} A T^*_{qp^{-1}}$$

which acts on the generating projections as a translation, see (3.10). Clearly, $\tau_{qp^{-1}} = \tau_{qr^{-1}} \tau_{rp^{-1}}$ for $p \le r \le q$. Let $D^{(\infty)}_\delta(S)$ denote the C^*-inductive limit of the directed system $\{\mathcal{D}_p, \tau_{qp^{-1}}\}$.

Recall the notation $q^{-1} \underset{G}{\cdot} X = \{q^{-1}x\colon x \in X\} \subset G$ for $q \in S$ and $X \in \mathcal{J}$.

Lemma 5.2. *The C^*-algebra $D^{(\infty)}_\delta(S)$ is isomorphic to*

$$D_G = C^*\big(\{E_{q^{-1} \underset{G}{\cdot} X} : q \in S, X \in \mathcal{J}\}\big) \subset B\big(L^2(G)\big).$$

Proof. By definition, $D_\delta(S) \subset B(H_S)$. Recall that we denote $J_S : H_S \to L^2(G)$ the canonical imbedding; denote by $\pi\colon D_\delta(S) \to B(L^2(G))$ the lifting $\pi(A) = J_S A E_S$.

For any $p \in S$, the map

$$\phi_p(A) = L_p^* \pi(A) L_p, \quad A \in D_\delta(S),$$

is a $*$-homomorphism $\phi_p\colon D_\delta(S) \to D_G$, such that $\phi_p(E_X) = E_{p^{-1} \underset{G}{\cdot} X}$ for all $X \in \mathcal{J}$.

Then for $p \leq q$ and $X \in \mathcal{J}$ we have:

$$\phi_q \tau_{qp^{-1}}(E_X) = L_q^* E_{qp^{-1}X} L_q = E_{q^{-1} \underset{G}{\cdot} (qp^{-1}X)}.$$

Since $qp^{-1} \in S$, we have $qp^{-1}X = (qp^{-1}) \underset{G}{\cdot} X$, and we can continue the previous formula as

$$\phi_q \tau_{qp^{-1}}(E_X) = E_{p^{-1} \underset{G}{\cdot} X} = \phi_p(E_X).$$

So the maps ϕ_p are compatible with $\tau_{qp^{-1}}$, and there exists a limit map $\Phi\colon D_\delta^{(\infty)}(S) \to D_G$, such that $\Phi((A_p)_{p\in S}) = \phi_p(A_p)$, $p \in S$. The homomorphisms ϕ_p are injective since π is obviously injective and L_p is a unitary operator. It follows that Φ is also injective.

To prove surjectivity of Φ, it suffices to show that for any $q_1, \ldots, q_n \in S$, $X_1, \ldots, X_n \in \mathcal{J}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ we have

$$\sum_i \lambda_i E_{q_i^{-1} \underset{G}{\cdot} X_i} \in \Phi\big(D_\delta^{(\infty)}(S)\big).$$

Since the system $\{\mathcal{A}_p, \tau_{qp^{-1}}\}$ is upwards directed, there exists $s \in S$ such that $q_i \leq s$, $i = 1, 2, \ldots, n$, and this implies that $sq_i^{-1}X_i \in \mathcal{J}$ and $q_i^{-1} \underset{G}{\cdot} X_i = s^{-1} \underset{G}{\cdot} (sq_i^{-1}X_i) \in \phi_s(D_\delta(S))$. Hence,

$$\sum_i \lambda_i E_{q_i^{-1} \underset{G}{\cdot} X_i} \in \phi_s\big(D_\delta(S)\big)$$

and we obtain

$$D_G = \overline{\bigcup_{p\in S} \phi_p\big(D_\delta(S)\big)}.$$

Therefore, Φ is surjective and we get the isomorphism $D_\delta^{(\infty)}(S) \cong D_G$. □

We identify further the C^*-algebra $D_\delta^{(\infty)}(S)$ with D_G and in this way consider it to be a subalgebra of $B(L^2(G))$.

On $B(L^2(G))$, we have the adjoint action of G generated by the left regular representation: $\alpha_g(A) = L_g A L_g^*$, $g \in G$, $A \in B(L^2(G))$. On an operator of multiplication M_f by a function $f \in L^\infty(G)$ it acts by translation: $\alpha_g M_f = M_{L_g f}$, and in particular, $\alpha_g(E_X) = E_{g \underset{G}{\cdot} X}$ for $X \in \mathcal{J}$.

Let us show that D_G is invariant under this action. For $g \in G$, $q \in S$, $X \in \mathcal{J}$

$$\alpha_g(E_{q^{-1} \underset{G}{\cdot} X}) = E_{(gq^{-1}) \underset{G}{\cdot} X}. \tag{5.1}$$

Since $G = S^{-1}S$, we can write $gq^{-1} = t^{-1}s$ with some $s, t \in S$. Then $s \underset{G}{\cdot} X = s \cdot X \in \mathcal{J}$, and $E_{(gq^{-1}) \underset{G}{\cdot} X} = E_{t^{-1} \underset{G}{\cdot} (sX)} \in D_G$.

Moreover, since conjugation is strong operator continuous, D''_G is also invariant under α. One can easily verify that the action α is point-strong continuous on D_G. (D''_G, G, α) is thus a von Neumann dynamical system, and by definition the pair (Id, L) is a covariant representation of this system.

LEMMA 5.3. *The crossed product $D''_G \rtimes_\alpha G$ of the commutative von Neumann algebra D''_G and the group G by the action α, is isomorphic to the von Neumann algebra $\mathcal{M} = \{E_X, L_g \colon X \in S^{-1} \cdot \mathcal{J}, g \in G\}''$.*

Proof. By definition (see [24, Definition X.1.6]), the crossed product $D''_G \rtimes_\alpha G$ is the von Neumann algebra generated by $\{\pi(A) : A \in D''_G\} \cup \{\tilde{L}_g : g \in G\} \subset B(L^2(G, L^2(G))) \simeq B(L^2(G \times G))$, where

$$\big(\pi(A)\xi\big)(s,t) = \big(\alpha_s^{-1}(A)\xi(s,\cdot)\big)(t),$$
$$(\tilde{L}_g\xi)(s,t) = \xi\big(g^{-1}s, t\big)$$

for any $\xi \in L^2(G, L^2(G))$, $g, s, t \in G$, $X \in S^{-1} \cdot \mathcal{J}$. Since every $A \in D''_G$ is a multiplication operator M_f with some $f \in L^\infty(G)$, we can write more precisely

$$\big(\pi(M_f)\xi\big)(s,t) = L_{s^{-1}} f(t)\xi(s,t) = f(st)\xi(s,t).$$

On $L^2(G \times G)$, define a unitary operator W (with δ_G being the modular function of G):

$$W\xi(s,t) = \delta_G(t)^{\frac{1}{2}}\xi(st, t),$$

which has the adjoint $W^*\xi(s,t) = \delta_G(t)^{-1/2}\xi(st^{-1}, t)$. One verifies that

$$\big(W(M_f \otimes 1)W^*\xi\big)(s,t) = f(st)\xi(s,t) = \big(\pi(M_f)\xi\big)(s,t),$$
$$W(L_g \otimes 1)W^* = \tilde{L}_g,$$

so that $D''_G \rtimes_\alpha G = \{W(A \otimes 1)W^* : A \in D''_G \cup \{L_g : g \in G\}\}''$. It is easy to see that this algebra is isomorphic to $\{D''_G \cup \{L_g : g \in G\}\}''$, and as a consequence, to $\mathcal{M}$. □

LEMMA 5.4. *The algebra $\mathcal{M}$ is equal to the strong operator closure of the linear space generated by the operators $E_{q^{-1} \underset{G}{\cdot} X} L_g$ with $q \in S$, $X \in \mathcal{J}$, $g \in G$.*

Proof. The statement follows from a direct calculation with $q, p \in S$, $X, Y \in \mathcal{J}$, $g, h \in G$:

$$\begin{aligned} E_{q^{-1} \cdot X} L_g E_{p^{-1} \cdot Y} L_h &= E_{q^{-1} \cdot X}\big(L_g E_{p^{-1} \cdot Y} L_g^*\big) L_g L_h \\ &= E_{q^{-1} \cdot X} E_{(gp^{-1}) \cdot Y} L_{gh} \\ &= E_{(q^{-1} \cdot X) \cap ((gp^{-1}) \cdot Y)} L_{gh}. \end{aligned}$$

Represent gp^{-1} as $gp^{-1} = s^{-1}t$, $s, t \in S$. There exists $r \in S$ such that $r \in Sq \cap Ss$; then $rq^{-1}, rs^{-1} \in S$ and $(q^{-1} \cdot X) \cap ((gp^{-1}) \cdot Y) = r^{-1} \cdot Z$ with

$$Z = \big(rq^{-1} \cdot X\big) \cap \big(rs^{-1}t \cdot Y\big) = \big(rq^{-1}X\big) \cap \big(rs^{-1}tY\big) \in \mathcal{J}.$$

This shows that the linear space in question is closed under multiplication, which proves the lemma. □

THEOREM 5.5. *The algebra* $\mathrm{VN}(S)$ *is isomorphic to the corner subalgebra* $E_S\mathcal{M}E_S$ *of* $\mathcal{M}$.

Proof. In fact, $E_S\mathcal{M}E_S$ is just the strong operator closure of the C^*-algebra $\mathcal{A}$ described in Lemma 3.3, as we will now show. By Lemma 5.4, $E_S\mathcal{M}E_S$ is the strong operator closure of the linear space generated by $E_SE_{q^{-1}\underset{G}{\cdot}X}L_gE_S$ with $q\in S$, $X\in\mathcal{J}$, $g\in G$. Such an operator can be written in another form:

$$\begin{aligned}E_SE_{q^{-1}\underset{G}{\cdot}X}L_gE_S &= E_{S\cap(q^{-1}\cdot X)}L_gE_SL_g^*L_g = E_{q^{-1}X}E_{g\cdot S}L_g\\ &= E_{q^{-1}X}E_SE_{g\cdot S}L_g = E_{q^{-1}X}E_{S\cap g\cdot S}L_g.\end{aligned}$$

Let $g=a^{-1}b$ with $a,b\in S$. Then $S\cap g\underset{G}{\cdot}S = S\cap(a^{-1}\underset{G}{\cdot}(bS)) = a^{-1}bS$. By Lemma 3.3, $E_{a^{-1}bS}L_{a^{-1}b}\in\mathcal{A}$. Next, if $\Psi: C_\delta^*(S)\to\mathcal{A}$ denotes the isomorphism in Lemma 3.3, then by Lemma 3.5 $E_{q^{-1}X}=\Psi(T_{ww^{-1}})$ for some $w\in\mathcal{F}$ which depends on qX. Thus, $E_{q^{-1}X}E_{a^{-1}bS}L_{a^{-1}b}\in\mathcal{A}$, so that $E_S\mathcal{M}E_S\subset\mathcal{A}''$. From the other side, $\mathcal{A}$ is generated as a C^*-algebra by the operators $E_SL_{a^{-1}b}E_S=\Psi(T_{a^{-1}b})$, $a,b\in S$ which are contained in $E_S\mathcal{M}E_S$; this shows that $E_S\mathcal{M}E_S=\mathcal{A}''$, which proves the theorem. □

6. The universal and reduced compact quantum semigroups

A compact quantum semigroup is a pair (A,Δ), where A is a unital C^*-algebra and $\Delta\colon A\to A\otimes_{\min}A$ is a unital $*$-homomorphism which is coassociative, i.e.

$$(\mathrm{id}\otimes\Delta)\Delta = (\Delta\otimes\mathrm{id})\Delta.$$

A Hopf von Neumann algebra is a pair (B,Δ), where B is a von Neumann algebra and $\Delta\colon B\to B\mathbin{\overline{\otimes}}B$ is a normal unital coassociative $*$-homomorphism. The homomorphism Δ in both cases is called *a comultiplication*. See [25] for details.

Consider the C^*-subalgebra $\mathcal{A}$ in $C^*(S)\otimes_{\min}C^*(S)$ generated by the elements

$$\{v_p\otimes v_p, e_X\otimes e_X : p\in S, X\in\mathcal{J}'\}.$$

Clearly, these elements satisfy relations (3.7), (3.8). The universal property of $C^*(S)$ implies the existence of a unital $*$-homomorphism $\Delta_u\colon C^*(S)\to\mathcal{A}$, such that

$$\Delta_u(v_p)=v_p\otimes v_p, \qquad \Delta_u(e_X)=e_X\otimes e_X.$$

The map Δ_u admits a restriction $\Delta_u|_{D(S)}: D(S)\to D(S)\otimes_{\min}D(S)$ which is also a unital $*$-homomorphism.

The pair $\mathbb{Q}(S)=(C^*(S),\Delta_u)$ is a compact quantum semigroup [1]. We call the algebra $C^*(S)$ with this structure the *universal algebra of functions on the compact quantum semigroup* $\mathbb{Q}(S)$ associated with the semigroup S.

6.1. On duality of semilattices and their C^*-algebras.

Definition 6.1. Let E be a semilattice, that is, a commutative semigroup of idempotents. A *character* on E is a semigroup morphism $\xi\colon E\to\{0,1\}$. The set of all characters on E is denoted $\hat{E}$; it forms a compact 0-dimensional semilattice with the pointwise multiplication and the topology of pointwise convergence. By Theorem 3.9 Chapter II of [11], the functor $E\mapsto\hat{E}$ is a duality functor between the category of discrete semilattices and the category of compact 0-dimensional semilattices. In particular, the map $\eta_E\colon E\to\hat{\hat{E}}$ defined by $\eta_E(s)(c)=c(s)$ is an isomorphism.

Proposition 6.2. *Let $E\subset B(H)$ be a set of linearly independent commuting projections closed under multiplication and containing 1_H. Then $C^*(E)$ is $*$-isomorphic to $C(\hat{E})$, where $\hat{E}$ is the dual semilattice of E. Under this isomorphism ϕ for every $e\in E$, $\chi\in\hat{E}$, $\phi(e)(\chi)=\chi(e)$.*

Proof. Note that E is a discrete (in the norm topology) semilattice. Denote $A=C^*(E)$. By Gelfand–Naimark theorem, the commutative C^*-algebra A is $*$-isomorphic to $C(\Omega)$, where $\Omega=\hat{A}$ is the space of characters on A with the topology of pointwise convergence, which is compact and Hausdorff. Obviously, every $\chi\in\hat{A}$ is a character on E. Let us show that every $\chi\in\hat{E}$ extends to a continuous character on A.

Since E is linearly independent, we can extend χ by linearity to its linear span $B=\operatorname{lin}E\subset C^*(E)$. Let $t=\sum_{j=1}^n\lambda_jp_j\in B$, where $p_j\in E$, $\lambda_j\in\mathbb{C}$. Set $J_0=\{j:\chi(p_j)=0\}$, $J_1=\{j:\chi(p_j)=1\}$. Set $X=\bigvee\{p_j:j\in J_0\}$ to be the union of projections. Since X is a linear combination of p_j, $j\in J_0$ and their products, we have $\chi(X)=0$.

Next, set $Y=\prod_{j\in J_1}p_j$; we have $\chi(Y)=1$. Since $\chi(YX)=0$, we have $Y\neq YX=XY$, what means that $Y(H)\not\subset X(H)$. Pick $v\in Y(H)\cap(X(H))^\perp$ with $\|v\|=1$. Then

$$\begin{aligned}\left\|\sum_{j=1}^n\lambda_jp_j\right\|&\geq\left\|\sum_{j=1}^n\lambda_jp_jv\right\|=\left\|\sum_{j\in J_1}\lambda_jp_jv\right\|\\&=\left\|\sum_{j\in J_1}\lambda_jv\right\|=\left|\sum_{j\in J_1}\lambda_j\right|=\left|\chi\left(\sum_{j=1}^n\lambda_jp_j\right)\right|.\end{aligned}$$

This implies that χ has norm 1 on the linear span of E, so it can be extended to its closure by continuity.

Furthermore, the topologies on $\hat{E}$ and $\hat{A}$ are both defined by pointwise convergence, on E and on A, respectively. The bijection defined above is thus continuous in the direction $\hat{A}\to\hat{E}$; both spaces being Hausdorff and compact, they are in fact homeomorphic. □

Proposition 6.3. *Let E be a set of linearly independent commuting projections on a separable Hilbert space H, closed under multiplication and containing 1_H, and $A = C^*(E)$. Then there exists a positive measure μ on $\hat{E}$, such that A'' is $*$-isomorphic to $L^\infty(\hat{E},\mu) \subset B(L^2(\hat{E},\mu))$.*

Proof. By Lemma 4.4.1 in [17], there exists a separating vector $v \in H$ for the Abelian von Neumann algebra A''. Analogously to the proof of Theorem 4.4.4 of [17], projecting H onto its subspace $H_v = [A''v]$ we get that the vector v is cyclic for A'' restricted to H_v. Since v is separating for A'', A'' restricted to H_v is $*$-isomorphic to A''.

Denote $\phi\colon A \to C(\Omega)$ the $*$-isomorphism from Proposition 6.2, where $\Omega = \hat{E}$. In what follows we reproduce the proof of the Theorem 4.4.3 of [17]. Define a positive linear functional τ on $C(\Omega)$: $\tau(f) = \langle \phi^{-1}(f)v, v\rangle$. Applying the Riesz–Markov theorem we obtain a positive measure μ on Ω realizing the functional τ.

Let π be the composition of ϕ and the $*$-representation of $C(\Omega)$ by multiplication operators on $L^2(\Omega,\mu)$. The map $u\colon Av \to C(\Omega) \subset L^2(\Omega,\mu)$, $av \mapsto \phi(a)$ is linear and isometric, and hence can be extended to a unitary u from H_v onto $L^2(\Omega,\mu)$. In fact, $\pi(a) = uau^{-1}$. Since $\pi(A)$ is strongly dense in $L^\infty(\Omega,\mu)$, we get $uA''u^{-1} = L^\infty(\Omega,\mu)$. □

Proposition 6.4. *Let $E \subset B(H)$ be a set of linearly independent commuting projections on a separable Hilbert space H, closed under multiplication and containing 1_H, and $A = C^*(E)$. Then there exists a comultiplication on A'', such that $\Delta(e) = e \otimes e$ for every $e \in E$.*

Proof. By Proposition 6.3, A'' (resp. $A''\overline{\otimes}A''$) is $*$-isomorphic to $L^\infty(\hat{E},\mu)$ (resp. $L^\infty(\hat{E}\times\hat{E},\mu\times\mu)$). By Theorem 3.9 Chapter II of [11], the set $\hat{E}$ of characters on E is a compact zero-dimensional semilattice with the (jointly continuous) pointwise product. As in the group case, this product gives rise to a coproduct on $L^\infty(\hat{E},\mu)$ by the formula:

$$\Delta(f)(x,y) = f(xy)$$

Since elements of E are characters on $\hat{E}$, we have for every $e \in E$ and $x, y \in \hat{E}$:

$$\Delta(e)(x,y) = e(xy) = e(x)e(y) = (e\otimes e)(x,y).$$ □

Remark 6.5. The product on $\hat{E}$ is the pointwise product of characters. But this operation is not the pointwise product when the elements of $\hat{E}$ are considered as characters on $C^*(E)$. Namely, for any $\chi_1, \chi_2 \in \hat{E}$, $\lambda_i \in \mathbb{C}$, $e_i \in E$, $1 \le i \le n$ we have:

$$\chi_1\chi_2\left(\sum_{i=1}^n \lambda_i e_i\right) = \sum_{i=1}^n \lambda_i \chi_1(e_i)\chi_2(e_i)$$

6.2. The quantum semigroup associated to S.

THEOREM 6.6. *There exists a comultiplication* $\Delta\colon \mathrm{VN}(S) \to \mathrm{VN}(S)\overline{\otimes}\,\mathrm{VN}(S)$, *and its restriction* $\Delta\colon C^*_\delta(S) \to C^*_\delta(S)\otimes_{\min} C^*_\delta(S)$, *with which* $(\mathrm{VN}(S), \Delta)$ *is a Hopf von Neumann algebra and* $\mathbb{Q}(S) = (C^*_\delta(S), \Delta)$ *is a compact quantum semigroup.*

Proof. Recall that we suppose G to be second countable, so that $L^2(G)$ is separable. We use the Theorem 5.5 to identify $\mathrm{VN}(S)$ with the corner $E_S(D''_G \rtimes_\tau G)E_S$ inside $B(L^2(G))$. Due to Proposition 6.4, there exists a coproduct on D''_G, defined on generators by

$$\Delta(E_{q^{-1} \underset{G}{\cdot} X}) = E_{q^{-1} \underset{G}{\cdot} X} \otimes E_{q^{-1} \underset{G}{\cdot} X}$$

for $q \in S, X \in \mathcal{J}$. One can easily see that Δ commutes with the action of G on D''_G defined in (5.1). Consequently, Δ gives rise to a comultiplication on $D''_G \rtimes_\tau G$, which we also denote by Δ. Due to the fact that $E_S \in D''_G$, using Lemma 5.5 we obtain the required comultiplication Δ on $\mathrm{VN}(S)$.

Since $\Delta(E_{q^{-1} \underset{G}{\cdot} X}) \in C^*_\delta(S)\otimes C^*_\delta(S)$ for every generator $E_{q^{-1} \underset{G}{\cdot} X}$ with $q \in S$, $X \in \mathcal{J}$, the map Δ restricts to a comultiplication on $C^*_\delta(S)$. □

REMARK 6.7. The bialgebras $C^*(S)$ and $C^*_\delta(S)$ are co-commutative, as for example the group C^*-algebra $C^*(G)$ of G. But their dual algebras, unlike the Fourier–Stieltjes algebra $B(G) = C^*(G)^*$, cannot be viewed as function algebras on S or even on G. It is possible that $\phi, \psi \in (C^*_\delta(S))^*$ are nonequal but have the same values on T_a and T^*_a for all $a \in S$.

More specifically, consider for example $G = \mathbb{Z}$, $S = \mathbb{Z}_+$. Let $\delta_k \in \ell^2(\mathbb{Z})$ denote the indicator function of $k \in \mathbb{Z}$. Set $\phi_k(T) = \langle T\delta_k, \delta_k\rangle$, $k \in \mathbb{Z}$. Then $\phi_k(T_a) = \phi_k(T^*_a) = \delta_0(a)$ for all $k \in \mathbb{Z}$, $a \in \mathbb{Z}_+$, but $\phi_k(T_aT^*_a) = I_{\mathbb{Z}_+}(k - a)$ while $\delta_0(T_aT^*_a) = \delta_0(a)$.

In what follows we use the commutativity assumption just to guarantee that S, which is supposed to be right-reversible, is also left-reversible.

PROPOSITION 6.8. *Let S be Abelian. Then $\mu(X) > 0$ for any constructible right ideal X of S.*

Proof. Clearly, any ideal of the form pS equals Sp and is not empty. Due to the Theorem 5.1, the intersection of any pair of non-empty ideals is non-empty. For any non-empty ideal X in S and $p \in S$ we have

$$p^{-1}X = \{x \in S\colon px \in X\} = p^{-1} \underset{G}{\cdot} (pS \cap X).$$

Therefore, $p^{-1}X$ is non-empty for any non-empty ideal X in S, and so is pX. Hence, every constructible right ideal of S is non-empty.

Let U be any open subset in S with $\mu(U) > 0$. Then for any non-empty ideal X in S taking $p \in X$ we obtain $pU \subset X$, hence, $\mu(X) > 0$. □

REMARK 6.9. If S is Abelian, then there exists a natural short exact sequence connecting $C^*_\delta(S)$ with $C^*_\delta(G)$ described below.

Consider the commutator ideal K in $C^*_\delta(S)$, that is, the ideal generated by $\{[A,B]=AB-BA\colon A,B\in C^*_\delta(S)\}$. Among others, K contains the operators

$$T_aT_a^*-T_a^*T_a=E_{aS}-E_S$$

for all $a\in S$. For any $X\in\mathcal{J}$, $a\in S$ we have

$$T_a^*(E_X-E_S)T_a=E_{a^{-1}X}-E_S,$$
$$T_a(E_X-E_S)T_a^*=E_{aX}-E_{aS},$$

which allows to show by induction that $E_X-E_S\in K$ for every $X\in\mathcal{J}\setminus\{\emptyset\}$. Consequently, in $C^*_\delta(S)/K$ we have the equivalence classes $[E_X]=[E_S]=1$ for all $X\in\mathcal{J}\setminus\{\emptyset\}$.

By Lemma 3.2, $T_w=E_{wS}T_{(w)_G}$ for every $w\in\mathcal{F}$, and it follows that $[T_w]=[E_ST_{(w)_G}]$ in $C^*_\delta(S)/K$.

Due to Lemma 3.3, $C^*_\delta(S)$ is the closed linear space generated by operators of the form E_XT_g with $X=wS\in\mathcal{J}, g\in G$, $(w)_G=g$ (where T_g is understood as $T_g=T_{a^{-1}b}$ with any representation $g=a^{-1}b$). The discussion above implies that $C^*_\delta(S)/K$ is generated as a linear space by the classes $[E_ST_g]=[T_g]$, $g\in G$.

Denote $\hat{L}_g=[T_g]$. For all $g_1,g_2\in G$ one sees that $\hat{L}_{g_1}\hat{L}_{g_2}=\hat{L}_{g_1g_2}$, and every $\hat{L}_g$ is unitary since $\hat{L}_g^*=\hat{L}_{g^{-1}}$.

Let us show that $\|\sum_{k=1}^n c_k\hat{L}_{g_k}\|_{C^*_\delta(S)/K}=\|\sum c_kL_{g_k}\|_{B(L^2(G))}$ for all $c_k\in\mathbb{C}$, $g_k\in G$. We have (by Lemma 3.3)

$$\left\|\sum_{k=1}^n c_k\hat{L}_{g_k}\right\|_{C^*_\delta(S)/K}\le\left\|\sum c_kT_{g_k}\right\|_{C^*_\delta(S)}\le\left\|\sum c_kL_{g_k}\right\|_{B(L^2(G))}.$$

From the other side, the fact that S has non-empty interior implies that the norm of $T=\sum_{k=1}^n c_kL_{g_k}$ is attained on H_S. Indeed, for every $f\in L^2(G)$ and every $\varepsilon>0$ there is a compact set $K\subset G$ such that $\|f-E_Kf\|<\varepsilon$; there exists (see [15], or alternatively this can be shown directly) $g\in G$ such that $gK\subset S$, so that $E_SE_{gK}=E_{gK}$. Set $h=L_gf$. We have $E_{gK}h=L_g(E_Kf)$, and

$$\begin{aligned}\|E_Sh-h\|&\le\big\|E_S(h-E_{gK}h)\big\|+\|E_SE_{gK}h-h\|\\&\le 2\|h-E_{gK}h\|=2\|f-E_Kf\|<2\varepsilon.\end{aligned}$$

If f is chosen so that $\|f\|=1$ and $\|Tf\|>\|T\|-\varepsilon$, then $\|h\|=1$,

$$\|Th\|=\|L_gTf\|=\|Tf\|>\|T\|-\varepsilon,$$

and at the same time $\|TE_Sh-Th\|\le 2\|T\|\varepsilon$, which implies $\|TE_Sh\|>\|T\|-(1+2\|T\|)\varepsilon$. This proves the statement.

Thus, we have an isomorphism $C^*_\delta(S)/K \simeq C^*_\delta(G)$ and the short exact sequence:

$$0 \to K \to C^*_\delta(S) \to C^*_\delta(G) \to 0. \tag{6.1}$$

EXAMPLE 6.10. Let us calculate the algebra $C^*_\delta(\mathbb{R}_+)$; we specify that in our notation $\mathbb{R}_+ = [0, +\infty)$.

By our assumptions, $G = \mathbb{R}$. It is immediate that

$$\mathcal{J} = \big\{[t, +\infty) : t \in \mathbb{R}_+\big\} \cup \{\emptyset\}.$$

Below, denote $E_{[t,+\infty)} = E_t$, $t \in \mathbb{R}$. In the multiplicative notation used throughout the paper, one checks that $a^{-1}bS = [\max(0, b-a), +\infty)$ for $a, b \geq 0$, and in general $wS = [t, +\infty)$ with $t \geq \max(0, (w)_{\mathbb{R}})$ for every $w \in \mathcal{F}$. Conversely, if $g \in \mathbb{R}$ and $t \geq \max(0, g)$, then $g = t - s$ with $s \geq 0$, and $[t, +\infty) = ts^{-1}S$. According to Lemma 3.3, we have thus

$$C^*_\delta(\mathbb{R}_+) \simeq \overline{\operatorname{lin}}\{E_t L_g : t \in \mathbb{R}_+, g \in \mathbb{R}, t \geq g\}.$$

It is worth noting that

$$D(\mathbb{R}_+) = C^*(E_t : t \in \mathbb{R}_+) \subset B\big(L^2(\mathbb{R}_+)\big).$$

This algebra can be also described as the space of functions supported in $\mathbb{R}_+$ and such that $\lim_{t\to t_0-0} f(t)$ exists for every $t_0 \in (0, +\infty]$ and $f(t_0) = \lim_{t\to t_0+0} f(t)$ for every $t_0 \in [0, +\infty)$. This is the uniform closure of the algebra of piecewise continuous functions, and is sometimes called by the same name.

The short exact sequence (6.1) in this case is written as

$$0 \to K \to C^*_\delta(\mathbb{R}_+) \to C^*_\delta(\mathbb{R}) \to 0.$$

The commutator ideal K in $C^*_\delta(\mathbb{R}_+)$ has the following form:

$$K = \overline{\operatorname{lin}}\{E_{[a;b)} L_g : a, b \in \mathbb{R}_+, g \in \mathbb{R}, b \geq a \geq g\}.$$

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