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ON STRICT WHITNEY ARCS AND t-QUASI SELF-SIMILAR ARCS

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ABSTRACT. A connected compact subset E of $\mathbb{R}^N$ is said to be a strict Whitney set if there exists a real-valued C^1 function f on $\mathbb{R}^N$ with $\nabla f|_E \equiv 0$ such that f is constant on no non-empty relatively open subsets of E. We prove that each self-similar arc of Hausdorff dimension $s > 1$ in $\mathbb{R}^N$ is a strict Whitney set with criticality s. We also study a special kind of self-similar arcs, which we call "regular" self-similar arcs. We obtain necessary and sufficient conditions for a regular self-similar arc Λ to be a t-quasi-arc, and for the Hausdorff measure function on Λ to be a strict Whitney function. We prove that if a regular self-similar arc has "minimal corner angle" $\theta_{\min} > 0$, then it is a 1-quasi-arc and hence its Hausdorff measure function is a strict Whitney function. We provide an example of a one-parameter family of regular self-similar arcs with various features. For some values of the parameter τ, the Hausdorff measure function of the self-similar arc is a strict Whitney function on the arc, and hence the self-similar arc is an s-quasi-arc, where s is the Hausdorff dimension of the arc. For each $t_0 \geq 1$, there is a value of τ such that the corresponding self-similar arc is a t-quasi-arc for each $t > t_0$, but it is not a t_0-quasi-arc. For each $t_0 > 1$, there is a value of τ such that the corresponding self-similar arc is a t_0-quasi-arc, but it is a t-quasi-arc for no $t \in [1, t_0)$.

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1. Introduction

In fractal geometry, Morse–Sard theorem (see [5]) states that if $f \in C^k(\mathbb{R}^m, \mathbb{R}^N)$ with $k \geq \max(m-N+1, 1)$, then the set of critical values of f has zero Lebesgue measure in $\mathbb{R}^N$. However, Whitney in 1935 constructed a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ whose critical set is a fractal planar arc γ with Hausdorff dimension $\log 4/\log 3$, and whose set $f(\gamma)$ of critical values contains an interval and therefore has positive Lebesgue measure (see [9]). This is called Whitney phenomenon; it seems to contradict the Morse–Sard theorem. It is due to the fact that the arc γ is a fractal and f has lower smoothness. Such a set is called a Whitney set.

Definition 1.1. A connected set $E \subset \mathbb{R}^N$ is said to be a *Whitney set*, if there is a C^1 function $f : \mathbb{R}^N \to \mathbb{R}$ such that $\nabla f|_E \equiv 0$ but $f|_E$ is not constant. The function f is said to be a *Whitney function* for E, and its restriction $f|_E$ to E is said to be a *Whitney function* on E. If a Whitney function $f|_E$ on E is non-constant on each non-empty relatively open subset of E, then $f|_E$ is said to be a *strict Whitney function* on E, f is said to be a *strict Whitney function* for E, and the set E is said to be a *strict Whitney set*.

The following special case of the Whitney Extension Theorem [8] will be used.

LEMMA 1.2. *Suppose that $E \subset \mathbb{R}^N$ is compact and $f : E \to \mathbb{R}$ is a function. If for each $\varepsilon > 0$, there exists $\delta > 0$ such that for each pair of points $x, y \in E$ with $|x-y| < \delta$, one has $|f(x) - f(y)| \leq \varepsilon|x-y|$, then there is a C^1 extension $\tilde{f} : \mathbb{R}^N \to \mathbb{R}$ of f such that $\tilde{f}|_E = f$ and $\nabla \tilde{f}|_E \equiv 0$.*

Lemma 1.2 suggests the following definition.

Definition 1.3. A compact connected metric space A is said to be a *Whitney set* if there is a non-constant function $f : A \to \mathbb{R}$ such that $|f(x) - f(y)| = o(d_A(x,y))$ for $x, y \in A$.

By Lemma 1.2, for a compact connected subset A of $\mathbb{R}^N$, Definition 1.3 is consistent with Definition 1.1.

About Whitney sets, we know the following.

(a) For a set $E \subset \mathbb{R}^N$, if every pair of points in E are connected by a rectifiable arc lying in E, then E is not a Whitney set (Whyburn [10], 1929).

(b) For a continuous function $g : \mathbb{R} \to \mathbb{R}$, the graph G of g is not a Whitney set (Choquet [2], 1944).

Due to lack of work on critical sets with fractal feature, it is natural to ask how to characterize Whitney sets geometrically. Whitney posted this problem in his original paper [9]. The problem can be stated as follows.

Given a function f, how far from rectifiable must a closed connected set be to be a critical set for f on which f is not constant?

Definition 1.4 ([7])**.** Let Λ be an arc, a homeomorphic image of the interval $[0,1]$ in $\mathbb{R}^N$, and let Λ be a Whitney set. Then Λ is said to be a *monotone Whitney arc* if there is an increasing Whitney function f on Λ.

Xi and Wu ([11]) in 2003 gave an interesting example of a Whitney arc which is not a monotone Whitney arc.

Xi and Wu's Whitney arc γ mentioned above is not a strict Whitney set because it contains small line segments, and each Whitney function on γ must be constant on those line segments. It is not known whether there exists a strict Whitney arc which is not a monotone Whitney arc.

A mapping $f:(A,d_A)\to(B,d_B)$ between two metric spaces is said to be *non-expanding* if $d_B(f(x),f(y))\le d_A(x,y)$ for $x,y\in A$.

Wen and Xi obtained the following geometric characterization of Whitney sets (see [7, Theorem 1]).

A compact connected metric space A is a Whitney set if and only if there is a non-expanding mapping from A onto a monotone Whitney arc.

The "if" part is immediate because the "pull-back" of a Whitney function on the monotone Whitney arc is a Whitney function on A. The "only if" part can be seen as follows.

Suppose that f is a Whitney function on A with $f(A)=[0,1]$. Let $B=[0,1]$. For $0\le s<t\le 1$, set

$$\begin{aligned} d_B(s,t) &= \inf\left\{\sum_{i=0}^{n-1} D(t_i,t_{i+1}) : s=t_0<t_1<\cdots<t_n=t\right\},\\ D(s,t) &= \inf_{f(x)=s,\,f(y)=t} d_A(x,y). \end{aligned}$$

Then d_B extends in an obvious way to be a distance function on B. The metric space (B,d_B) is a monotone Whitney arc since the identity map $\tau : B\to[0,1]$ is a monotone Whitney function on B. Moreover, $f:A\to B$ is a non-expanding map. For details, see [7, p. 315].

Definition 1.5. Let Λ be an arc, a homeomorphic image of the interval $[0,1]$ in $\mathbb{R}^N$, and let $t\ge 1$. The arc Λ is said to be a *t-quasi-arc*, if there is a constant $\lambda>0$ such that

$$|\Lambda(x,y)|^t \le \lambda|x-y| \tag{1}$$

for each pair of points $x,y\in\Lambda$, where $|\Lambda(x,y)|$ is the diameter of the subarc $\Lambda(x,y)$ lying between x and y. A 1-quasi-arc is called a *quasi-arc*.

Note that (1) does not hold when $t<1$, because $|\Lambda(x,y)|\ge|x-y|$. One can see that if an arc Λ is a t_0-quasi-arc, then Λ will be a t-quasi-arc for all $t\ge t_0$. Therefore, each quasi-arc is a t-quasi-arc for each $t\ge 1$.

With the above definition of t-quasi-arcs, Norton (see [4]) obtained the following sufficient condition for an arc Λ to be a Whitney set: if Λ is a t-quasi-arc and if t is less than the Hausdorff dimension $\dim_H(\Lambda)$ of Λ, then Λ is a Whitney set.

Seeking for necessary conditions for a t-quasi-arc to be a Whitney set, Norton posed the following question (see [4]): is there an arc Λ and a C^1 function f critical but not constant on Λ such that for every subarc η of Λ on which f is not constant, η is a t-quasi-arc for no $t \in [1, \infty)$?

In [6], Wen and Xi gave an affirmative answer to the above question. They gave a Whitney function f on a self-similar arc Λ such that each subarc of Λ is a t-quasi-arc for no $t \in [1, \infty)$. In that paper, the function f is constant on some subarcs of Λ, which means that f is not a strict Whitney function on Λ. We are interested in finding a strict Whitney function on Λ.

In [4], Norton also considered the criticality of Whitney sets.

Definition 1.6. For a Whitney set E, the *Criticality* of E is defined to be

$$\begin{aligned}\mathrm{Cr}(E) = \sup\big\{r :\ &\text{there exists a non-constant function } f : E \to \mathbb{R}\\ &\text{and an } M > 0 \text{ such that } \big|f(x) - f(y)\big| \le M|x-y|^r\ \forall x, y \in E\big\}.\end{aligned}$$

If E is a Whitney set, then $1 \le \mathrm{Cr}(E) \le \dim_H(E)$ (see [4]). Recall that $\dim_H(E)$ is the Hausdorff dimension of E.

Wen and Xi worked on self-similar arcs in [6], and obtained that each self-similar arc of Hausdorff dimension greater than 1 is a Whitney set. In this paper, we obtain the following result.

THEOREM 1.7. *Let Λ be a self-similar arc of Hausdorff dimension $s > 1$. Then Λ is a strict Whitney set with criticality* $\mathrm{Cr}(\Lambda) = s$.

Theorem 1.7 improves the main result in [6] in two aspects. First, the constructed Whitney function is strictly monotone. Second, the involved Hölder component $\tilde{s}$ is arbitrarily close to the Hausdorff dimension s, hence it determines the criticality to be exactly s.

In Section 4, we define "Condition W_p" for a self-similar arc at the p-th vertex, and prove that for a self-similar arc Λ with $\ell + 1$ vertices, the Hausdorff measure function is a Whitney function on Λ if and only if Condition W_p is satisfied for $p = 1, \dots, \ell - 1$. We also define "Condition Q_p^t", and prove that a self-similar arc is a t-quasi-arc if and only if Condition Q_p^t is satisfied for all "inner vertices".

In order to have a better understanding of self-similar arcs, we introduce the notion of regular self-similar arcs in Section 5. Roughly speaking, a regular self-similar arc is a self-similar arc in $\mathbb{R}^2$ generated by a "basic figure" with certain properties. One classical example of regular self-similar arc is the Koch curve.

In Section 6, we further analyze Conditions W_p and Q_p^t for regular self-similar arcs, and reduce them to certain inequalities. We first prove that if the p-th corner angle $\theta_p > 0$ then Conditions W_p and Q_p^t (for each $t \geq 1$) are satisfied. Consequently, a regular self-similar arc with positive corner angles is necessarily a quasi-arc and its Hausdorff measure function is a strictly monotone Whitney function.

In case a corner angle is zero, Condition W_p and Condition Q_p^t are reduced to inequalities about specific parameters of the self-similar arc. By using these algebraic expressions, we could easily recognize t-quasi-arcs among regular self-similar arcs and determine whether the Hausdorff measure function on a regular self-similar arc is a Whitney function.

In the last section, we provide an example of a one-parameter family of regular self-similar arcs with various features. For some values of the parameter τ, the Hausdorff measure function on the self-similar arc is a strict Whitney function, and hence the self-similar arc is an s-quasi-arc, where s is the Hausdorff dimension of the arc. For each $t_0 \geq 1$, there is a value of τ such that the corresponding self-similar arc is a t-quasi-arc for each $t > t_0$, but it is not a t_0-quasi-arc. For each $t_0 > 1$, there is a value of τ such that the corresponding self-similar arc is a t_0-quasi-arc, but it is a t-quasi-arc for no $t \in [1, t_0)$.

In the construction of the above mentioned one-parameter family of self-similar arcs, a crucial step in the reasoning is that the self-similar arc is a t-quasi-arc if and only if the parameter τ has approximation property $J_{(t-1)\log(15/7)}$. See Definition 7.1 for the definition of approximation property J_a.

The significance of the given family of self-similar arcs lies in that it provides a method to produce various examples.

2. Self-similar arcs

A mapping $F : \mathbb{R}^N \to \mathbb{R}^N$ is said to be a contractive mapping if there exists $k \in [0, 1)$ such that $|F(x) - F(y)| \leq k|x - y|$ for all $x, y \in \mathbb{R}^N$.

A compact set $K \subset \mathbb{R}^N$ is said to be *invariant* with respect to a finite set $\mathcal{S} = \{S_1, \ldots, S_\ell\}$ of contractive mappings on K, if

$$K = \bigcup_{j=1}^{\ell} S_j(K).$$

In [3], Hutchinson gave the following theorem.

THEOREM 2.1. *Let $X = (X, d)$ be a complete metric space and let $\mathcal{S} = \{S_1, \ldots, S_\ell\}$ be a finite set of contractive mappings on X. Then there exists a unique closed bounded set K such that $K = \bigcup_{j=1}^{\ell} S_j(K)$. Furthermore, K is compact and is the closure of the set of fixed points $s_{j_1 \ldots j_p}$ of finite compositions $S_{j_1} \circ \cdots \circ S_{j_\ell}$ of members of $\mathcal{S}$.*

A mapping $S:\mathbb{R}^N\to\mathbb{R}^N$ is called a similitude if there is an $r>0$ such that

$$\bigl|S(x)-S(y)\bigr|=r|x-y|,\quad \text{for } x,y\in\mathbb{R}^N.$$

If $0<r<1$, we say that S is a contractive similitude.

Suppose that $\mathcal{S}:=\{S_1,\dots,S_\ell\}$ is a family of contractive similitudes with ratios $r_1,\dots,r_\ell$. Then there is a unique set E satisfying

$$E=\bigcup_{i=1}^{\ell}S_i(E).$$

The set E is called the self-similar set associated to $\mathcal{S}$.

Definition 2.2. The compact self-similar set Λ associated to a family of contractive similitudes $\mathcal{S}=\{S_i\}_{1\le i\le\ell}$ is called a *self-similar arc* if the following two conditions are satisfied:

(1) $S_i(\Lambda)\cap S_j(\Lambda)$ is a singleton for $|i-j|=1$;
(2) $S_i(\Lambda)\cap S_j(\Lambda)=\emptyset$ for $|i-j|>1$.

Let Λ be the self-similar arc associate to a family $\mathcal{S}=\{S_1,\dots,S_\ell\}$. The Hausdorff dimension s of Λ is determined by the equation

$$\sum_{j=1}^{\ell}r_j^s=1,$$

where $\{r_j\}_{j=1}^{\ell}$ are the contractive ratios of $\{S_j\}_{j=1}^{\ell}$ (see [3]). We say that a self-similar arc Λ is *non-trivial* if the Hausdorff dimension of Λ is $s>1$, *i.e.*, Λ is not a line segment.

Suppose that the non-trivial self-similar arc Λ is defined by a homeomorphism $h:[0,1]\to\Lambda$ so that $h(0)\in S_1(\Lambda)$. For $x,y\in\Lambda$, we say that x precedes y, and write $x\prec y$, if $h^{-1}(x)<h^{-1}(y)$. Then we define intervals on Λ, $[x,y]=\{z\in\Lambda:x\preceq z\preceq y\}$. Now on Λ, there are points $z_0\prec z_1\prec\cdots\prec z_\ell$ so that $S_j(\Lambda)=[z_{j-1},z_j]$. Set $\mathcal{S}^{(k)}=\{S_{j_1}\cdots S_{j_k}:1\le j_1,\dots,j_k\le\ell\}$ for $k\ge1$ and $\mathcal{S}^{(0)}=\{\mathrm{Id}\}$. Here $S_jS_i=S_j\circ S_i$, $S_j^k=S_j\cdots S_j$ (k times), etc.

By Definition 1.5, for $x,y\in\Lambda$, $\Lambda(x,y)$ is the subarc between x and y. Here, we denote by $[x,y]$ the subarc from x to y. So $\Lambda(x,y)=[x,y]$ or $\Lambda(x,y)=[y,x]$.

The sets $S_{j_1}\cdots S_{j_k}(\Lambda)$ are intervals on Λ overlapping only at end points. Thus there are points $z_j^{(k)}, j=1,\dots,\ell^k$, and a numbering $\{S_j^{(k)}:1\le j\le\ell^k\}$ of elements of $\mathcal{S}^{(k)}$ such that $z_0^{(k)}\prec z_1^{(k)}\prec\cdots\prec z_{\ell^k}^{(k)}$ and $S_j^{(k)}(\Lambda)=[z_{j-1}^{(k)},z_j^{(k)}]$. In other works, $S_j^{(k)}$ is the unique member of $\mathcal{S}^{(k)}$ which maps Λ to $[z_{j-1}^{(k)},z_j^{(k)}]$. If $k\le m$, we have

$$z_j^{(k)}=z_{j\ell^{m-k}}^{(m)}. \tag{2}$$

Note that $S_1^{(k)}$ is not necessarily equal to S_1^k, because S_1 may be "order reversing".

A similitude $S_{j_1}\cdots S_{j_k}$ is order-preserving if $S_{j_1}\cdots S_{j_k}(z_0)\prec S_{j_1}\cdots S_{j_k}(z_\ell)$; otherwise it is called order-reversing.

Let τ be the function on the collection of finite sequences $(j_1,\dots,j_k)$ of members of $\{1,\dots,\ell\}$ defined by

$$\tau(j_1,\dots,j_k)=\begin{cases}1, & \text{if } S_{j_1}\cdots S_{j_k} \text{ order-preserving,}\\ -1, & \text{if } S_{j_1}\cdots S_{j_k} \text{ order-reversing.}\end{cases}\tag{3}$$

The mapping $S_{j_1}S_{j_2}$ is order-preserving if S_{j_1} and S_{j_2} are both order-preserving or both order-reversing; it follows that $\tau(j_1,\dots,j_k)=\tau(j_1)\cdots\tau(j_k)$.

It would be more convenient for us if S_1 and S_ℓ are order-preserving. Of course, that is not the case in general. One might hope that when S_1 and S_ℓ are not both order-preserving, $S_1^{(k)}$ and $S_{\ell^k}^{(k)}$ could be made order-preserving by choosing k suitably. Unfortunately, that could not be achieved either. In other words, [6, Lemma 1] is incorrect.

Example 2.3. Suppose that S_1 is order-preserving, S_ℓ is order-reversing, and $k>1$. Since $S_{\ell^k}^{(k)}$ is the unique composition $S_{j_1}\cdots S_{j_k}$ such that $S_{j_1}\cdots S_{j_k}(\Lambda)\ni z_\ell$, and since $S_\ell(S_1^{k-1})(z_0)=z_\ell$, it follows that $S_{\ell^k}^{(k)}=S_\ell S_1^{k-1}$. Therefore, $S_{\ell^k}^{(k)}$ is order-reversing for each $k>1$.

By arguments similar to above, we obtain

$$\begin{aligned}
&S_1^{(k)}=S_1^k, && S_{\ell^k}^{(k)}=S_\ell^k, && \text{if } \tau(1)=1,\tau(\ell)=1,\\
&S_1^{(k)}=S_1^k, && S_{\ell^k}^{(k)}=S_\ell S_1^{k-1}, && \text{if } \tau(1)=1,\tau(\ell)=-1,\\
&S_1^{(k)}=S_1 S_\ell^{k-1}, && S_{\ell^k}^{(k)}=S_\ell^k, && \text{if } \tau(1)=-1,\tau(\ell)=1,\\
&S_1^{(2k)}=(S_1S_\ell)^k, && S_{\ell^k}^{(2k)}=(S_\ell S_1)^k, && \text{if } \tau(1)=-1,\tau(\ell)=-1,\\
&S_1^{(2k+1)}=(S_1S_\ell)^kS_1, && S_{\ell^k}^{(2k+1)}=(S_\ell S_1)^kS_\ell, && \text{if } \tau(1)=-1,\tau(\ell)=-1.
\end{aligned}\tag{4}$$

We now define a homeomorphism g from $[0,1]$ onto Λ, which has properties necessary for the proofs of several theorems. Set

$$\Gamma=\{z_j^{(k)}:k\ge 1, 0\le j\le \ell^k\},$$
$$Q=\left\{\frac{j}{\ell^k}:k\ge 1,0\le j\le \ell^k\right\}\subset[0,1].$$

Let $g:Q\to\Gamma$ be defined by

$$g\left(\frac{j}{\ell^k}\right)=z_j^{(k)}.\tag{5}$$

By (2), g is well defined. By its very definition, the function g is bijective and order-preserving, *that is*, $u<v$ implies that $g(u)\prec g(v)$. Since Q is dense in $[0,1]$ and Γ is dense in Λ, g extends to be a homeomorphism from $[0,1]$ onto Λ.

Suppose that $\alpha \in [0,1] \setminus Q$. Then we can uniquely split $[0,1]$ into (A,B) such that $a < \alpha < b$ for $a \in A$ and $b \in B$. Then the unique point on Λ which split Λ in $(g(A), g(B))$ is denoted by $g(\alpha)$. We have proved the following lemma.

LEMMA 2.4. *There is an order-preserving homeomorphism $g : [0,1] \to \Lambda$ such that for each $k \geq 1$ and each $j = 1, 2, \ldots, \ell^k$, we have*

$$g\left(\left[\frac{j-1}{\ell^k}, \frac{j}{\ell^k}\right]\right) = \left[z_{j-1}^{(k)}, z_j^{(k)}\right].$$

3. Proof of Theorem 1.7

LEMMA 3.1. *Let Λ be the self-similar arc associated to similitudes $S_1, \ldots, S_\ell$, let $s > 1$ be the Hausdorff dimension of Λ, let $\tilde{s} \in (1, s)$, and let $\{\varepsilon_k\}$ be a sequence of positive numbers with $\varepsilon_1 \leq 1$ and $\varepsilon_k \searrow 0$. Suppose that the ratios r_j of S_j satisfy*

$$r_1^{\tilde{s}} + r_\ell^{\tilde{s}} < \sum_{j=1}^{\ell} r_j^{\tilde{s}} - 1. \tag{6}$$

Then there exists a number $s' \in (\tilde{s}, s)$, a sequence $\{\tau_k\}$ of positive numbers with $\tau_k \searrow 0$, and a probability measure μ on Λ such that

(i) $\tau_k \leq \min(r_1^s, r_\ell^s, \varepsilon_k^{\tilde{s}})$;
(ii) $\mu([x,y]) > 0$ *if* $x, y \in \Lambda$ *and* $x \prec y$;
(iii) $\mu(S_{j_1} \cdots S_{j_k}(E)) \leq (1+\varepsilon_1)(r_{j_1} \cdots r_{j_k})^{s'} \mu(E)$ *for each Borel subset E of* Λ;
(iv) $\mu(S_{i_1} \cdots S_{i_k}(\Lambda)) = \tau_1 \cdots \tau_k$ *if $i_n = 1$ or ℓ for $n = 1, \ldots, k$.*

Proof. By (6), we have $\sum_{j=2}^{\ell-1} r_j^{\tilde{s}} > 1$. By the basic property, $\sum_{j=1}^{\ell} r_j^s = 1$, of the self-similar set Λ, we obtain that $\sum_{j=2}^{\ell-1} r_j^s < 1$. So we see that there is an $s' \in (\tilde{s}, s)$ such that

$$\sum_{j=2}^{\ell-1} r_j^{s'} = 1. \tag{7}$$

Let $r = \min_{2 \leq j \leq \ell-1} r_j$. Since $s' < s$, we can choose $\gamma > 0$ so that $s' + \gamma < s$ and $r^{-(\gamma\pi^2)/6} < 1 + \varepsilon_1$. Therefore, $s' < s' + \gamma/k^2 < s$ for any $k \in \mathbb{N}$. It is easy to see that when we raise s' to $s' + \gamma/k^2$ in (7), we have the sum

$$\sum_{j=2}^{\ell-1} r_j^{s'+\gamma/k^2} < 1, \quad \text{for each } k \in \mathbb{N}. \tag{8}$$

Now let τ_k' be such that

$$2\tau_k' + \sum_{j=2}^{\ell-1} r_j^{s'+\gamma/k^2} = 1. \tag{9}$$

Because of (9) and

$$1 = r_1^s + r_\ell^s + \sum_{j=2}^{\ell-1} r_j^s < r_1^s + r_\ell^s + \sum_{j=2}^{\ell-1} r_j^{s'+\gamma/k^2},$$

we have $0 < 2\tau_k' < r_1^s + r_\ell^s$. By (7) and (9), we know that

$$\lim_{k\to\infty} \tau_k' = \frac{1}{2} \lim_{k\to\infty} \left(1 - \sum_{j=2}^{\ell-1} r_j^{s'+\gamma/k^2}\right) = \frac{1}{2}\left(1 - \sum_{j=2}^{\ell-1} r_j^{s'}\right) = 0. \tag{10}$$

Let $\tau_k = \min(r_1^s, r_\ell^s, \varepsilon_k^{\tilde{s}}, \tau_k')$. Then

$$2\tau_k + \sum_{j=2}^{\ell-1} r_j^{s'+\gamma/k^2} \le 2\tau_k' + \sum_{j=2}^{\ell-1} r_j^{s'+\gamma/k^2} = 1,$$

and then we can choose $s_k \le s' + \gamma/k^2$ so that

$$2\tau_k + \sum_{j=2}^{\ell-1} r_j^{s_k} = 1. \tag{11}$$

By comparing (7) and (11), it follows that $s' < s_k$. Then $s' < s_k \le s' + \gamma/k^2$ and therefore $s_k \to s'$ as $k \to \infty$. Since we choose τ_k to be less than or equal to τ_k' and since (10), we know that $\tau_k \searrow 0$ as $k \to \infty$.

For $j = 1, \dots, \ell$, $k = 1, 2, \dots$, we define numbers r_{jk} by

$$r_{jk} = \begin{cases} r_j^{s_k}, & \text{if } j \ne 1, \ell, \\ \tau_k, & \text{if } j = 1 \text{ or } \ell. \end{cases}$$

Then we have

$$\sum_{j=1}^{\ell} r_{jk} = 1. \tag{12}$$

We now define a probability measure μ by

$$\mu(\Lambda) = 1, \qquad \mu\big(S_{j_1} \cdots S_{j_k}(\Lambda)\big) = r_{j_1,1} \cdots r_{j_k,k}. \tag{13}$$

Equality (12) implies that for $k = 1, 2, \dots,$

$$\mu\big(S_{j_1} \cdots S_{j_{k-1}}(\Lambda)\big) = \sum_{j=1}^{\ell} \mu\big(S_{j_1} \cdots S_{j_{k-1}} S_j(\Lambda)\big).$$

Thus the definition (13) is consistent.

Now (i), (ii), and (iv) are satisfied; it remains to prove (iii). It suffices to show that (iii) holds for $E = S_{i_1} \cdots S_{i_n}(\Lambda)$, *i.e.*, for arbitrary $j_1, \dots, j_k$, $i_1, \dots, i_n$,

$$\mu\big(S_{j_1} \cdots S_{j_k} S_{i_1} \cdots S_{i_n}(\Lambda)\big) \le (1+\varepsilon_1)(r_{j_1} \cdots r_{j_k})^{s'} \mu\big(S_{i_1} \cdots S_{i_n}(\Lambda)\big).$$

We know that if $2 \le j \le \ell - 1$, then

$$\frac{r_{j,k+m}}{r_{jm}} = \frac{r_j^{s_{k+m}}}{r_j^{s_m}} \le r^{-(s_m - s_{k+m})} \le r^{-(s_m - s')} < r^{-\gamma/m^2};$$

if $j = 1$ or ℓ, then

$$\frac{r_{j,k+m}}{r_{jm}} = \frac{\tau_{k+m}}{\tau_k} \le 1 < r^{-\gamma/m^2}.$$

Thus,

$$\frac{r_{j,k+m}}{r_{jm}} < r^{-\gamma/m^2}, \quad j = 1, 2, \ldots, \ell, m = 1, 2, \ldots.$$

On the other hand we have that $r_{jk} < r_j^{s'}$ for $j = 1, \ldots, \ell$, $k = 1, 2, \ldots$. Therefore,

$$\begin{aligned}\frac{\mu(S_{j_1} \cdots S_{j_k} S_{i_1} \cdots S_{i_n}(\Lambda))}{\mu(S_{i_1} \cdots S_{i_n}(\Lambda))} &= r_{j_1,1} \cdots r_{j_k,k} \frac{r_{i_1,(k+1)}}{r_{i_1,1}} \cdots \frac{r_{i_\ell,(k+n)}}{r_{i_n,n}} \\ &\le r_{j_1}^{s'} \cdots r_{j_k}^{s'} \prod_{m=1}^{\infty} r^{-\gamma/m^2} \\ &= (r_{j_1} \cdots r_{j_k})^{s'} r^{-\gamma\pi^2/6} \\ &< (1 + \varepsilon_1)(r_{j_1} \cdots r_{j_k})^{s'}. \qquad \square\end{aligned}$$

Proof of Theorem 1.7. Let $\tilde{s} \in (1, s)$ be given. We prove that there exists a function f on Λ, constant on no non-empty relatively open subsets of Λ, and a constant C such that $|f(x) - f(y)| \le C|x - y|^{\tilde{s}}$ for all $x, y \in \Lambda$.

Suppose that Λ is the self-similar arc associated to a family $\mathcal{S} := \{S_1, \ldots, S_\ell\}$ of contractive similitudes with ratios $r_1, \ldots, r_\ell$. Let $g : [0,1] \to \Lambda$ be the homeomorphism defined in Lemma 2.4.

Suppose that $S_j(\Lambda) = [z_{j-1}, z_j]$, $j = 1, \ldots, \ell$. For each $j = 1, \ldots, \ell - 1$, we consider sequences of points $\{\alpha_{jk}\}_{k=1}^\infty$ and $\{\beta_{jk}\}_{k=1}^\infty$ in Λ which are converging to z_j, where

$$\alpha_{jk} = g\left(\frac{j}{\ell} - \frac{1}{\ell^k}\right),$$
$$\beta_{jk} = g\left(\frac{j}{\ell} + \frac{1}{\ell^k}\right).$$

So $\alpha_{j1} = z_{j-1}$ and $\beta_{j1} = z_{j+1}$.

In the following, let $\mathrm{dis}(X, Y)$ denote the euclidean distance between the two sets X, Y. Set

$$\varepsilon_{kj} = \min\bigl\{1, \mathrm{dis}\bigl([z_{j-1}, \alpha_{j,k+1}], [z_j, z_{j+1}]\bigr), \mathrm{dis}\bigl([z_{j-1}, z_j], [\beta_{j,k+1}, z_{j+1}]\bigr)\bigr\},$$

and $\varepsilon_k = \min\{\varepsilon_{kj} : j = 1, \ldots, \ell - 1\}$.

Since $\tilde{s} < s$, we have $\sum_{j=1}^{\ell} r_j^{\tilde{s}} - 1 > 0$, hence (6) holds provided that r_1, r_ℓ are sufficiently small. Note that the quantity on the right-hand side of (6) becomes larger when $\mathcal{S}$ is replaced by $\mathcal{S}^{(k)}$. Therefore, replacing $\mathcal{S}$ by $\mathcal{S}^{(k)}$ if necessary, we assume that r_1 and r_ℓ are so small that (6) holds.

By Lemma 3.1, there is a probability measure μ on Λ with properties (i)–(iv) specified in the lemma.

Now we define a function $f : \Lambda \to \mathbb{R}$ by letting $f(x) = \mu([z_0, x])$. By (ii), if $x \prec y$, then $f(y) - f(x) = \mu[x, y]$, hence f is non-constant on each subarc of Λ. We shall show that there is a constant $C > 0$ such that

$$\big|f(x) - f(y)\big| \le C|x-y|^{\tilde{s}}, \quad \text{for } x, y \in \Lambda. \tag{14}$$

Consider two distinct points x, y in Λ. Let L be the diameter of Λ and let $R = \max r_j$. Set

$$W(x,y) = \big\{\kappa : \kappa \ge 0, x, y \in S_j^{(\kappa)}(\Lambda) \text{ for some } j, 1 \le j \le \ell^\kappa\big\}.$$

Then $0 \in W(x,y)$ and $W(x,y) \ne \emptyset$. When $\kappa > \log(|x-y|/L)/\log R$, the diameter of $S_j^{(\kappa)}(\Lambda)$ has estimate $\operatorname{diam}(S_j^{(\kappa)}(\Lambda)) \le R^\kappa L < |x-y|$, which implies that $\{x,y\} \not\subset S_j^{(\kappa)}(\Lambda)$, and hence $\kappa \notin W(x,y)$. Thus, $\kappa \le \log(|x-y|/L)/\log R$ for $\kappa \in W(x,y)$. Let $k = \max W(x,y)$. Then $x, y \in S_j^{(k)}(\Lambda)$ for some j with $1 \le j \le \ell^k$. Let $x', y' \in \Lambda$ be such that $x = S_j^{(k)}(x')$, $y = S_j^{(k)}(y')$. Without loss of generality, we assume that $x' \prec y'$. Choose integers d_1, d_2 so that $x' \in S_{d_1}(\Lambda)$ and $y' \in S_{d_2}(\Lambda)$. By the maximality of k, $d_1 < d_2$. We consider the following two cases.

Case 1. $S_{d_1}(\Lambda) \cap S_{d_2}(\Lambda) = \emptyset$.

By the definition of f and Lemma 3.1, there exists $s' \in (\tilde{s}, s)$ such that

$$\big|f(x) - f(y)\big| \le \mu\big(S_{j_1} \cdots S_{j_k}(\Lambda)\big) \le (1+\varepsilon_1)(r_{j_1} \cdots r_{j_k})^{s'} \mu(\Lambda).$$

Let δ be the least distance between two disjoint subarcs $S_i(\Lambda)$ and $S_j(\Lambda)$ with $1 \le i+1 < j \le \ell$. Then

$$\begin{aligned} |x-y| &\ge \operatorname{dis}\big(S_{j_1} \cdots S_{j_k} S_{d_1}(\Lambda), S_{j_1} \cdots S_{j_k} S_{d_2}(\Lambda)\big) \\ &= r_{j_1} \cdots r_{j_k} \operatorname{dis}\big(S_{d_1}(\Lambda), S_{d_2}(\Lambda)\big) \\ &\ge r_{j_1} \cdots r_{j_k} \delta. \end{aligned}$$

It follows that

$$\frac{|f(x) - f(y)|}{|x-y|^{\tilde{s}}} \le \frac{(1+\varepsilon_1)(r_{j_1} \cdots r_{j_k})^{s'}}{(r_{j_1} \cdots r_{j_k})^{\tilde{s}} \delta^{\tilde{s}}}.$$

Therefore,

$$\frac{|f(x) - f(y)|}{|x-y|^{\tilde{s}}} \le \delta^{-\tilde{s}}(1+\varepsilon_1). \tag{15}$$

Case 2. $S_{d_1}(\Lambda) \cap S_{d_2}(\Lambda) \ne \emptyset$.

In this case, we assume that $d_1 = p$ and $d_2 = p+1$. Then $x' \in [z_{p-1}, z_p] = S_p(\Lambda)$, $y' \in [z_p, z_{p+1}] = S_{p+1}(\Lambda)$. By the maximality of k, we have $x' \prec z_p \prec y'$. Set $z = S_{j_1} \cdots S_{j_k}(z_p)$. Let m be the least positive integer such that $x' \prec \alpha_{p,m+1}$. So $\alpha_{pm} \preceq x' \prec \alpha_{p,m+1}$. Similarly, let q be the unique positive integer so that $\beta_{p,q+1} \prec y' \preceq \beta_{pq}$. Then we have

$$\begin{aligned}|x-y| &\geq \operatorname{dis}\big(S_{j_1} \cdots S_{j_k}\big([\alpha_{p1}, \alpha_{p,m+1}]\big), S_{j_1} \cdots S_{j_k}\big([\beta_{p,q+1}, \beta_{p1}]\big)\big)\\ &= r_{j_1} \cdots r_{j_k} \operatorname{dis}\big([\alpha_{p1}, \alpha_{p,m+1}], [\beta_{p,q+1}, \beta_{p1}]\big)\\ &\geq r_{j_1} \cdots r_{j_k} \max(\varepsilon_m, \varepsilon_q).\end{aligned}$$

We also have

$$\begin{aligned}\big|f(x)-f(y)\big| &= \mu\big([x,z]\big) + \mu\big([z,y]\big)\\ &\leq \mu\big(S_{j_1} \cdots S_{j_k}\big([\alpha_{pm}, z_p]\big)\big) + \mu\big(S_{j_1} \cdots S_{j_k}\big([z_p, \beta_{pq}]\big)\big)\\ &\leq (1+\varepsilon_1)(r_{j_1} \cdots r_{j_k})^{s'}\big(\mu\big([\alpha_{pm}, z_p]\big) + \mu\big([z_p, \beta_{pq}]\big)\big)\\ &= (1+\varepsilon_1)(r_{j_1} \cdots r_{j_k})^{s'}(r_{i_1,1} \cdots r_{i_m,m} + r_{\iota_1,1} \cdots r_{\iota_q,q})\\ &\leq (1+\varepsilon_1)(r_{j_1} \cdots r_{j_k})^{s'}(r_{i_m,m} + r_{\iota_q,q}).\end{aligned}$$

Here $r_{i_1,1} = r_p^{s_1}$, $r_{\iota_1,1} = r_{p+1}^{s_1}$, $r_{i_j,j} = \tau_j$ for $j > 1$, and $r_{\iota_n,n} = \tau_n$ for $n > 1$. If $m > 1$, then $r_{i_m,m}/\varepsilon_m^{\tilde{s}} = \tau_m/\varepsilon_m^{\tilde{s}} \leq 1 \leq \varepsilon_1^{-s}$. Also, $r_{i_1,1}/\varepsilon_1^{\tilde{s}} \leq 1/\varepsilon_1^{\tilde{s}} \leq \varepsilon_1^{-s}$. In any case, $r_{i_m,m}/\varepsilon_m^{\tilde{s}} \leq \varepsilon_1^{-s}$. Similarly, $r_{\iota_q,q}/\varepsilon_q^{\tilde{s}} \leq \varepsilon_1^{-s}$. Therefore,

$$\begin{aligned}\frac{|f(x)-f(y)|}{|x-y|^{\tilde{s}}} &\leq \frac{(1+\varepsilon_1)(r_{j_1} \cdots r_{j_k})^{s'}(r_{i_m,m} + r_{\iota_q,q})}{(r_{j_1} \cdots r_{j_k})^{\tilde{s}} \max(\varepsilon_m, \varepsilon_q)^{\tilde{s}}}\\ &\leq 2\varepsilon_1^{-s}(1+\varepsilon_1).\end{aligned} \tag{16}$$

It follows from (15) and (16) that

$$\big|f(x)-f(y)\big| \leq C|x-y|^{\tilde{s}},$$

where $C = \max(2\varepsilon_1^{-s}, \delta^{-\tilde{s}})(1+\varepsilon_1)$. Since $\tilde{s} > 1$ is arbitrarily close to s, we see that the self-similar arc Λ is a strict Whitney set and $\operatorname{Cr}(\Lambda) = s$. ☐

4. Localization

In this section, we define "Condition W_p" for a self-similar arc at the p-th vertex, and prove that for a self-similar arc Λ with $\ell+1$ vertices, the Hausdorff measure function is a Whitney function on Λ if and only if Condition W_p is satisfied for $p = 1, \ldots, \ell-1$. We also define "Condition Q_p^t", and prove that a self-similar arc is a t-quasi-arc if and only if Condition Q_p^t is satisfied for all inner vertices.

Suppose that Λ is the self-similar arc associated to a family $\mathcal{S} := \{S_1, \ldots, S_\ell\}$ of contractive similitudes with ratios $r_1, \ldots, r_\ell$, and that the Hausdorff dimension of Λ is $s > 1$. Recall that for $x, y \in \Lambda$ with $x \prec y$, $[x,y]$ is the subarc of Λ from x to y. Let $H^s([x,y])$ be the s-dimensional Hausdorff measure of $[x,y]$, and let $f(x)=H^s([z_0,x])$, where z_0 is the "initial point" of Λ.

As in the proof of Theorem 1.7, for distinct points $x,y\in\Lambda$, let $W(x,y)$ denote the set of positive integers k such that $x,y\in S_j^{(k)}$ for some j with $1\le j\le \ell^k$.

Definition 4.1. Let Λ be a self-similar arc with $\ell+1$ vertices, and let $1\le p\le \ell-1$. The arc Λ is said to satisfy *Condition* W_p if

$$\bigl|f(x)-f(y)\bigr| = o\bigl(|x-y|\bigr) \quad \text{for } z_{p-1}\preceq x\preceq z_p\preceq y\preceq z_{p+1} \tag{17}$$

or, equivalently, if for each $\varepsilon>0$ there is a $\mu_p>0$ such that

$$\bigl|f(x)-f(y)\bigr|\le \varepsilon|x-y| \quad \text{whenever } z_{p-1}\preceq x\preceq z_p\preceq y\preceq z_{p+1} \text{ and } |x-y|<\mu_p.$$

PROPOSITION 4.2. *The function f has the property*

$$\bigl|f(x)-f(y)\bigr| = o\bigl(|x-y|\bigr), \quad x,y\in\Lambda \tag{18}$$

if and only if Λ satisfies Condition W_p for $p=1,\dots,\ell-1$.

Proof. The "only if" part is trivial.

Suppose that Λ satisfies Condition W_p for $p=1,\dots,\ell-1$. Let $\varepsilon>0$ be given. Let $\mu_p>0$ be the associated number in Definition 4.1, $p=1,\dots,\ell-1$. Set $\mu=\min\{\mu_1,\dots,\mu_{\ell-1}\}$ and $r=\max\{r_1,\dots,r_\ell\}$. Let δ_0 be the least distance between two disjoint subarcs $S_i(\Lambda)$ and $S_j(\Lambda)$ with $2\le i+1<j\le\ell$. Let

$$\delta_1=\left(\frac{\varepsilon\delta_0^s}{\beta}\right)^{1/(s-1)},$$
$$\delta_2=\left(\frac{\varepsilon\mu^s}{\beta}\right)^{1/(s-1)},$$
$$\delta=\min(\delta_1,\delta_2),$$

where s is the Hausdorff dimension of Λ and $\beta=H^s(\Lambda)$.

Suppose that $x,y\in\Lambda$ with $0<|x-y|<\delta$. Let $k=\max W(x,y)$. Then $x,y\in S_j^{(k)}(\Lambda)$ for some j with $1\le j\le\ell^k$. Let $x',y'\in\Lambda$ be such that $x=S_j^{(k)}(x')$ and $y=S_j^{(k)}(y')$. Without loss of generality, we assume that $x'\prec y'$. There are integers $d_1\le d_2$ such that $x'\in S_{d_1}(\Lambda)$, $y'\in S_{d_2}(\Lambda)$. By the maximality of k, $d_1<d_2$. We consider the following three cases.

Case 1. $d_2-d_1>1$. Write $S_j^{(k)}=S_{j_1}\cdots S_{j_k}$. Then $(r_{j_1}\cdots r_{j_k})|x'-y'|=|x-y|<\delta_1$ implies that $(r_{j_1}\cdots r_{j_k})<\delta_1/\delta_0$. Thus

$$\begin{aligned}\frac{|f(x)-f(y)|}{|x-y|}&=(r_{j_1}\cdots r_{j_k})^{s-1}\frac{|f(x')-f(y')|}{|x'-y'|}\\&<\left(\frac{\delta_1}{\delta_0}\right)^{s-1}\frac{\beta}{\delta_0}=\varepsilon.\end{aligned}$$

Case 2. $d_2-d_1=1$ and $|x'-y'|<\mu$. For convenience, set $p=d_1, p+1=d_2$. Since $|x'-y'|<\mu_p$, Condition W_p tells us that

$$\begin{aligned}\frac{|f(x)-f(y)|}{|x-y|} &= (r_{j_1}\cdots r_{j_k})^{s-1}\frac{|f(x')-f(y')|}{|x'-y'|}\\ &< (r_{j_1}\cdots r_{j_k})^{s-1}\varepsilon \le \varepsilon.\end{aligned}$$

Case 3. $d_2-d_1=1$ and $|x'-y'|\ge\mu$. As above, set $p=d_1, p+1=d_2$. Since $(r_{j_1}\cdots r_{j_k})=|x-y|/|x'-y'|<\delta_2/\mu$, it follows that

$$\begin{aligned}\frac{|f(x)-f(y)|}{|x-y|} &= (r_{j_1}\cdots r_{j_k})^{s-1}\frac{|f(x')-f(y')|}{|x'-y'|}\\ &< \left(\frac{\delta_2}{\mu}\right)^{s-1}\frac{\beta}{\mu}=\varepsilon.\end{aligned}$$

Therefore, $|f(x)-f(y)|<\varepsilon|x-y|$ whenever $|x-y|<\delta$. The proof is complete. □

Let $t\ge 1$. Set $L(x,y)=|\Lambda(x,y)|^t/|x-y|$. Here $|\Lambda(x,y)|$ is the diameter of the subarc $\Lambda(x,y)$ of Λ between x and y. For $p=1,\dots,\ell-1$ we define Condition Q_p^t as follows.

Definition 4.3. A self-similar arc Λ is said to satisfy *Condition* Q_p^t, if there is a constant $C_p>0$ such that $|\Lambda(x,y)|^t\le C_p|x-y|$ when $z_{p-1}\preceq x\preceq z_p\preceq y\preceq z_{p+1}$.

Proposition 4.4. *A self-similar arc Λ is a t-quasi-arc if and only if Λ satisfies Condition Q_p^t for $p=1,\dots,\ell-1$.*

Proof. The "only if" part is trivial.

Suppose that Λ satisfies Condition Q_p^t for $p=1,\dots,\ell-1$. Let $C_1,\dots,C_{\ell-1}$ be the numbers in Definition 4.3 for the vertices $z_1,\dots,z_{\ell-1}$, and set $C=\max\{C_1,\dots,C_{\ell-1}\}$. Let $L=|\Lambda|$ denote the diameter of Λ. As in the proof of Proposition 4.2, let δ_0 denote the least distance between two disjoint subarcs $S_i(\Lambda)$ and $S_j(\Lambda)$, $0\le i<j\le\ell$. Set $M=\max(C,L^t/\delta_0)$.

Consider distinct points $x,y\in\Lambda$. Let $k=\max W(x,y)$. Then $x,y\in S_j^{(k)}(\Lambda)$ for some j with $1\le j\le\ell^k$. Let $x',y'\in\Lambda$ be such that $x=S_j^{(k)}(x')$ and $y=S_j^{(k)}(y')$. Without loss of generality, we assume that $x'\prec y'$. There are integers $d_1\le d_2$ such that $x'\in S_{d_1}(\Lambda)$, $y'\in S_{d_2}(\Lambda)$. By the maximality of k, $d_1<d_2$. We consider the following two cases.

Case 1. $d_2-d_1>1$. Write $S_j^{(k)}=S_{j_1}\cdots S_{j_k}$. Then $|\Lambda(x,y)|^t=(r_{j_1}\cdots r_{j_k})^t|\Lambda(x',y')|^t$ and $|x-y|=(r_{j_1}\cdots r_{j_k})|x'-y'|$. Therefore,

$$L(x,y)=(r_{j_1}\cdots r_{j_k})^{t-1}L\big(x',y'\big)\le(r_{j_1}\cdots r_{j_k})^{t-1}\frac{L^t}{\delta_0}\le\frac{L^t}{\delta_0}.$$

Case 2. $d_2 - d_1 = 1$. For convenience, set $p = d_1, p+1 = d_2$. Since $z_{p-1} \preceq x' \preceq z_p \preceq y' \preceq z_{p+1}$, we have

$$L(x,y) = (r_{j_1} \cdots r_{j_k})^{t-1} L\big(x', y'\big) \leq (r_{j_1} \cdots r_{j_k})^{t-1} C_p \leq C.$$

Therefore, $L(x,y) \leq M$ for distinct points $x, y \in \Lambda$. By Definition 1.5, Λ is a t-quasi-arc. $\square$

5. Regular self-similar arcs in $\mathbb{R}^2$

In this section, we study "regular" self-similar arcs. We identify the euclidean plane with the complex plane $\mathbb{C}$ and consider the similitudes on $\mathbb{C}$. It is an elementary fact that an orientation preserving similitude S is of the form $S(z) = az + b$, where $a, b \in \mathbb{C}$, while an orientation reversing similitude S has the form $S(z) = a\bar{z} + b$.

Let Ω be a polygon formed by a sequence of successive segments in the plane. Suppose that Ω has $\ell + 1$ vertices $\{A_0, A_1, \ldots, A_\ell\}$, and that the points A_1, $A_{\ell-1}$ lie on segment $\overline{A_0 A_\ell}$ and the point $A_{\ell-1}$ lies on segment $\overline{A_1 A_\ell}$. Suppose that there is a vertex A_q such that all vertices of the polygon belong to the set Π, which is defined to be the union of the point A_q, the segment $\overline{A_0 A_\ell}$, and the set Π_0, which is in turn defined to be the interior of triangle $A_0 A_\ell A_q$. Let Π_1 be the closure of Π_0. For $j = 1, \ldots, \ell$, there is a unique orientation preserving similitude S_j such that $S_j(A_0) = A_{j-1}$ and $S_j(A_\ell) = A_j$. We assume that the similitudes S_j are contractive, that the sets $S_j(\Pi_0)$ are pairwise disjoint, and that $S_j(A_q) \in \Pi_1$ for $j = 1, \ldots, \ell$. Finally, we assume that

$$S_j(\Pi_1) \cap \big(\overline{A_q A_0} \cup \overline{A_q A_\ell}\big) = \emptyset, \quad \text{if } j \neq 1, \ell, q, q+1. \tag{19}$$

If all the above conditions are satisfied, we say that Ω is a basic figure (see Figure 1), and Π_1 (and/or Triangle $A_0 A_\ell A_q$, which is the union of the three sides) is the corresponding basic triangle.

Let Ω be a basic figure with vertices $\{z_0, z_1, \ldots, z_\ell\}$ and let $\mathcal{S} = \{S_1, \ldots, S_\ell\}$ be the corresponding contractive similitudes for Ω. Let Λ be the self-similar set associated to $\mathcal{S}$, *that is*, Λ is the unique compact set such that $\Lambda = \bigcup_{i=1}^{\ell} S_i(\Lambda)$.

We now discuss under what conditions Λ is an arc. For convenience, we assume that $z_0 = 0$, $z_\ell = 1$.

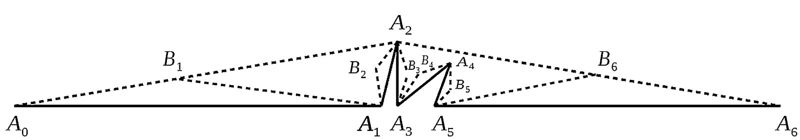

FIGURE 1. Basic figure.

Proposition 5.1. *The self-similar set Λ is an arc if and only if*

$$S_j(\Lambda) \cap S_{j+1}(\Lambda) = \{z_j\}, \quad j = 1, \dots, \ell - 1. \tag{20}$$

Proof. Suppose that Λ is an arc. Let $1 \le j \le \ell$. Then $S_j(\Lambda)$ is the subarc of Λ from z_{j-1} to z_j, and $S_{j+1}(\Lambda)$ is the subarc from z_j to z_{j+1}. Thus, (20) is satisfied.

Conversely, suppose that (20) holds. In order to prove that Λ is an arc, we only need to prove that there is a homeomorphism between $[0,1]$ and Λ.

Since S_j are orientation preserving contractive similitudes for Ω, we know that $S_j(z) = b_j z + z_{j-1}$, where $b_j = z_j - z_{j-1}$ for $j = 1, \dots, \ell$. Now each $x \in [0,1]$ has a unique expansion $x = \sum_{j=1}^{\infty} u_j/\ell^j$, where $u_j = 0, \dots, \ell - 1$. Recall that

$$Q := \left\{ \frac{j}{\ell^k} : k \ge 1, 0 \le j \le \ell^k \right\} \subset [0,1],$$

$$\Gamma := \left\{ z_j^{(k)} : k \ge 1, 0 \le j \le \ell^k \right\} \subset \Lambda.$$

The function $g : Q \to \Gamma$ defined in (5) now has the form

$$g\left(\frac{j}{\ell^k}\right) = z_j^{(k)} = S_j^{(k)}(1) = S_{u_1+1} S_{u_2+1} \cdots S_{u_{k-1}+1} S_{u_k+1}(1),$$

where

$$\frac{j}{\ell^k} = \sum_{i=1}^{k} \frac{u_i}{\ell^i}.$$

It is straightforward but somewhat tedious to verify that

$$g\left(\sum_{j=1}^{k} \frac{u_j}{\ell^j}\right) = \sum_{j=1}^{k} a_{j-1}(u_1, \dots, u_{j-1}) z_{u_j},$$

where

$$a_0 = 1, \qquad a_j(u_1, \dots, u_j) = \prod_{m=1}^{j} b_{u_m+1}, \quad j \ge 1.$$

By Lemma 2.4, the function g extends to be a homeomorphism $g : [0,1] \to \Lambda$, which is given by

$$g(x) = \sum_{j=1}^{\infty} a_{j-1}(x) z_{u_j},$$

where

$$a_0(x) = 1, \qquad a_j(x) = \prod_{m=1}^{j} b_{u_m+1}, \quad j \ge 1; \qquad x = \sum_{j=1}^{\infty} \frac{u_j}{\ell^j}.$$

The proposition has been proved. $\square$

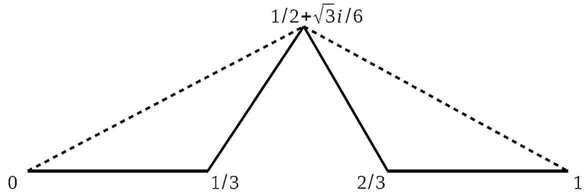

FIGURE 2. Basic figure of Koch curve.

Suppose that Ω is a basic figure with corresponding contractive, orientation preserving similitudes $\mathcal{S} = \{S_1, \dots, S_\ell\}$ and self-similar set Λ. If Λ is an arc, then Λ is a self-similar arc by Definition 2.2; in this case we say that Λ is the self-similar arc generated by the basic figure Ω. For example, the Koch curve is the self-similar arc generated by the basic figure which is the polygon with vertices $\{0, 1/3, 1/2 + \sqrt{3}i/6, 2/3, 1\}$ (see Figure 2), where we identify points on the complex plane with their complex number representations. Figure 1 gives us an example of a basic figure with 7 vertices. Triangle $A_2A_0A_6$ is the corresponding basic triangle.

Suppose that Λ is a self-similar arc generated by some basic figure and the associated similitudes $\mathcal{S} = \{S_1, \dots, S_\ell\}$ have contractive ratios $r_1, \dots, r_\ell$. The vertices of the generating basic figure are not collinear, which implies that $r_1 + \cdots + r_\ell > 1$. Since the Hausdorff dimension s of Λ is determined by the equation $r_1^s + \cdots + r_\ell^s = 1$, it follows that $s > 1$.

Let Ω be a basic figure with vertices $\{z_0, z_1, \dots, z_\ell\}$, and let $\triangle z_q z_0 z_\ell$ be the corresponding basic triangle. From now on, we have a standing assumption that

$$\Im \frac{z_q - z_0}{z_\ell - z_0} > 0,$$

which simplifies to $\Im z_q > 0$ when $z_0 = 0$ and $z_\ell = 1$. For $p = 1, \dots, \ell - 1$, set

$$\theta_p = \arg \frac{S_p(z_q) - z_p}{S_{p+1}(z_q) - z_p}, \quad 0 \le \theta_p < 2\pi.$$

Here θ_p is the argument of the fraction, so it is the angle between the two segments from the vertex z_p to $S_p(z_q)$ and $S_{p+1}(z_q)$, respectively. We call θ_p the *corner angle* at vertex z_p. Set $\theta_{\min} = \min\{\theta_1, \dots, \theta_{\ell-1}\}$.

We now consider which points of Λ lie on the sides $\overline{z_q z_0}$ and $\overline{z_q z_\ell}$ of the basic triangle. First, points $\{S_1^j(z_q)\}$ lie on $\overline{z_q z_0}$ and accumulate at z_0; while $\{S_\ell^j(z_q)\}$ lie on $\overline{z_q z_\ell}$ and accumulate at z_ℓ. For some basic figure, the side $\overline{z_q z_0}$ may contain more points of Λ. For example, for the Koch curve, $q = 2$, $\ell = 4$, and the points $\{S_2 S_4^j(z_2)\}$ lie on the side $\overline{z_2 z_0}$ and accumulate at z_2.

For the self-similar arc Λ, we define angles η_1, η_2 by

$$\eta_1 = \arg \frac{S_q(z_q) - z_q}{z_0 - z_q}, \qquad \eta_2 = \arg \frac{z_\ell - z_q}{S_{q+1}(z_q) - z_q}, \quad 0 \le \eta_1, \eta_2 < 2\pi. \tag{21}$$

(For the Koch curve $\eta_1 = \eta_2 = 0$.) Set

$$\eta_0 = \min(\eta_1, \eta_2), \qquad \xi := \theta_{\min} + \eta_0. \tag{22}$$

The angle ξ is said to be the *characteristic angle* of Λ and of the corresponding basic figure Ω.

For example, in Figure 1, the basic figure Ω has 7 vertices $\{A_0, A_1, A_2, A_3, A_4, A_5, A_6\}$ with A_0, A_1, A_5, A_6 collinear. We also have a family of contractive similitudes $\mathcal{S} = \{S_j : j = 1, \ldots, 6\}$, where

$$S_j(z) = \frac{A_j - A_{j-1}}{A_6 - A_0}(z - A_0) + A_{j-1}.$$

The triangle $\triangle A_2 A_0 A_6$ is the basic triangle, and its images under the similitudes are the smaller triangles: $\triangle B_1 A_0 A_1 = S_1(\triangle A_2 A_0 A_6)$, $\triangle B_2 A_1 A_2 = S_2(\triangle A_2 A_0 A_6)$, etc. Therefore, the corner angle $\theta_1 = \angle B_2 A_1 B_1$, θ_2 is the reflex angle $\angle B_3 A_2 B_2$, and θ_4 is the reflex angle $\angle B_5 A_4 B_4$, etc. The angles η_1, η_2 are $\eta_1 = \angle A_0 A_2 B_2$, $\eta_2 = \angle B_3 A_2 A_6$.

Definition 5.2. A *regular* self-similar arc is a self-similar arc generated by some basic figure with a positive characteristic angle.

As in Proposition 5.1, let Ω be a basic figure with vertices $\{z_0, z_1, \ldots, z_\ell\}$, where $z_0 = 0$, $z_\ell = 1$. We now express the condition (20) in terms of the corner angles and other parameters of Ω. Let Λ be the self-similar set generated by Ω.

We fix an index p, where $1 \le p \le \ell - 1$. We first consider the case where $\theta_p > 0$. Recall that Π_1 is the union of the basic triangle and its interior. Since $\theta_p > 0$, we see that $S_p(\Pi_1) \cap S_{p+1}(\Pi_1) = \{z_p\}$, and $z_p = S_p(z_\ell) = S_{p+1}(z_0)$. It follows that

$$S_p(\Lambda) \cap S_{p+1}(\Lambda) = \{z_p\}.$$

Therefore condition (20) holds for $j = p$ when $\theta_p > 0$.

Now we assume that $\theta_p = 0$. Let γ be the segment $S_p(\overline{z_q z_\ell})$, and let $\omega = S_{p+1}(\overline{z_q z_0})$. That $\theta_p = 0$ means that one of the two segments γ, ω is contained in the other. Since $S_p(\Lambda) \subset S_p(\Pi_1)$ and $S_{p+1}(\Lambda) \subset S_{p+1}(\Pi_1)$, we see that

$$S_p(\Lambda) \cap S_{p+1}(\Lambda) \subset S_p(\Pi_1) \cap S_{p+1}(\Pi_1) = \gamma \cap \omega.$$

For $j, k = 0, 1, 2, \ldots$, let

$$Z_j = S_p S_\ell^j(z_q),$$
$$W_k = S_{p+1} S_1^k(z_q).$$

Since $\eta_0+\theta_{\min}>0$, the assumption $\theta_p=0$ implies that $\eta_0=\min(\eta_1,\eta_2)>0$, hence $S_p(\Lambda)\cap(\gamma\setminus\{z_p\})=\{Z_j:j=1,2,\dots\}$ and $S_{p+1}(\Lambda)\cap(\omega\setminus\{z_p\})=\{W_k: k=1,2,\dots\}$. It follows that

$$\begin{aligned}\big(S_p(\Lambda)\cap S_{p+1}(\Lambda)\big)\setminus\{z_p\}&=\big(S_p(\Lambda)\cap\big(\gamma\setminus\{z_p\}\big)\big)\cap\big(S_{p+1}(\Lambda)\cap\big(\omega\setminus\{z_p\}\big)\big)\\&=\{Z_j:j=0,1,\dots\}\cap\{W_k:k=0,1,\dots\}.\end{aligned}$$

Therefore

$$S_p(\Lambda)\cap S_{p+1}(\Lambda)=\{z_p\}$$

if and only if

$$\{Z_j:j=0,1,2,\dots\}\cap\{W_k:k=0,1,2,\dots\}=\emptyset. \tag{23}$$

To summarize, we conclude that Λ is an arc if and only if for each p with $1\le p\le\ell-1$ and $\theta_p=0$, (23) holds.

Since $S_j(z)=b_jz+z_{j-1}$ and $b_j=z_j-z_{j-1}$, it follows that

$$\begin{aligned}Z_j&=z_p-b_pr_\ell^j(1-z_q),\\W_k&=z_p+b_{p+1}r_1^kz_q.\end{aligned}$$

Here $r_i=|b_i|$, for $i=1,\dots,\ell$. Set

$$\alpha=\big|r_p(1-z_q)\big|,\qquad\beta=|r_{p+1}z_q|, \tag{24}$$

$$\lambda=r_\ell,\qquad\mu=r_1,\qquad\iota=(Z_0-z_p)/|Z_0-z_p|. \tag{25}$$

Since $\theta_p=0$, we have

$$\frac{Z_j-z_p}{|Z_j-z_p|}=\frac{W_k-z_p}{|W_k-z_p|}=\iota,$$

hence

$$Z_j-z_p=\alpha\lambda^j\iota,\qquad W_k-z_p=\beta\mu^k\iota, \tag{26}$$

$$Z_j-W_k=\big(\alpha\lambda^j-\beta\mu^k\big)\iota. \tag{27}$$

Set

$$x=-\log\lambda,\qquad y=-\log\mu,\qquad u=\log(\alpha/\beta). \tag{28}$$

Then

$$u-jx+ky=\log\frac{\alpha\lambda^j}{\beta\mu^k}. \tag{29}$$

Therefore, $Z_j\neq W_k$ for $j,k=0,1,2,\dots$ if and only if $u-jx+ky\neq0$ for $j,k=0,1,2,\dots$.

As a conclusion of the above discussion, we have the following proposition.

PROPOSITION 5.3. *Let Ω be a basic figure with corner angles θ_p, $p=1,\dots,\ell-1$, and let Λ be the self-similar set generated by Ω. Then Λ is a regular self-similar arc if and only if for each p with $1\le p\le \ell-1$ and $\theta_p=0$ the following holds*:

$$u-jx+ky\neq 0, \quad \text{for } j,k=0,1,2,\dots.$$

6. Reduction of conditions W_p and Q_p^t

In this section, we assume that Λ is the regular self-similar arc generated by a basic figure Ω and $\{S_1,\dots,S_\ell\}$ are the corresponding contractive similitudes. Let r_j be the ratio of S_j for $j=1,\dots,\ell$. Recall that $|\Lambda(x,y)|$ is the diameter of the subarc $\Lambda(x,y)$ of Λ between x and y, and that $L=|\Lambda|$. Recall also that when $x\prec y$, $[x,y]$ denotes the subarc from x to y.

PROPOSITION 6.1. *Suppose that $1\le p\le \ell-1$ and that the corner angle $\theta_p>0$. Then Λ satisfies Condition W_p.*

Proof. Recall that the s-dimensional Hausdorff measure function $f:\Lambda\to\mathbb{R}$ is defined by

$$f(x)=H^s\big([z_0,x]\big),$$

where z_0 is the initial point of Λ. It is clear that f is non-constant on each subarc of Λ. We shall prove that there is a constant $M>0$ such that

$$\big|f(x)-f(y)\big|\le M|x-y|^s \quad \text{whenever } z_{p-1}\prec x\prec z_p\prec y\prec z_{p+1}, \tag{30}$$

which implies (17).

Suppose that $z_{p-1}\prec x\prec z_p\prec y\prec z_{p+1}$. Let $m\ge 0$ be the greatest integer such that $x\in S_pS_\ell^m(\Lambda)$. Then $x\in S_pS_\ell^m(\Lambda)$ and $x\notin S_pS_\ell^{m+1}(\Lambda)$. Upon setting $x'=S_pS_\ell^{-m}S_p^{-1}(x)$ we obtain that $x'\in S_p(\Lambda)\setminus S_pS_\ell(\Lambda)$. Let A denote the positive number

$$A:=\sup\big\{\big|f(w)-f(z_p)\big|/|w-z_p|^s : w\in S_p(\Lambda)\setminus S_pS_\ell(\Lambda)\big\}.$$

Since the similitude $S_pS_\ell^{-m}S_p^{-1}$ maps x, z_p to x', z_p, respectively, it follows that

$$\frac{|f(z_p)-f(x)|}{|z_p-x|^s}=\frac{|f(z_p)-f(x')|}{|z_p-x'|^s}\le A. \tag{31}$$

Similarly,

$$\frac{|f(y)-f(z_p)|}{|y-z_p|^s}\le B, \tag{32}$$

where

$$B:=\sup\big\{\big|f(w)-f(z_p)\big|/|w-z_p|^s : w\in S_{p+1}(\Lambda)\setminus S_{p+1}S_1(\Lambda)\big\}.$$

Let

$$\varrho_p := \arg \frac{z_{p+1} - z_p}{z_{p-1} - z_p}, \quad 0 < \varrho_p < 2\pi,$$

be the positive angle from line segment $\overline{z_p z_{p-1}}$ to line segment $\overline{z_p z_{p+1}}$. Set

$$\psi_p = \min(\theta_p, \varrho_p).$$

It follows from the law of sines that

$$|x - z_p| \le |x - y| \csc \psi_p, \qquad |y - z_p| \le |x - y| \csc \psi_p. \tag{33}$$

By (31), (32) and (33), we obtain that

$$\begin{aligned} \big|f(x) - f(y)\big| &= \big|f(x) - f(z_p)\big| + \big|f(y) - f(z_p)\big| \\ &\le A|x - z_p|^s + B|y - z_p|^s \\ &\le (A + B)\big(\csc^s \psi_p\big)|x - y|^s. \end{aligned}$$

Thus, (30) holds with $M = (A + B)(\csc^s \psi_p)$. □

PROPOSITION 6.2. *Suppose that $1 \le p \le \ell - 1$ and that the corner angle $\theta_p > 0$. Then Λ satisfies Condition Q_p^t for $t \ge 1$.*

Proof. Fix a number $t \ge 1$. We need to prove that there is a constant $M > 0$ such that

$$\big|\Lambda(x, y)\big|^t \le M|x - y| \quad \text{whenever } z_{p-1} \prec x \prec z_p \prec y \prec z_{p+1}. \tag{34}$$

Recall that $|\Lambda(x, y)|$ is the diameter of the subarc $\Lambda(x, y)$ of Λ between x and y, and that L is the diameter of Λ. Suppose that $x, y \in \Lambda$ satisfy $z_{p-1} \prec x \prec z_p \prec y \prec z_{p+1}$. Let $m \ge 0$ be the greatest integer such that $x \in S_p S_\ell^m(\Lambda)$. Then $x \in S_p S_\ell^m(\Lambda) \setminus S_p S_\ell^{m+1}(\Lambda)$. As in the proof of Proposition 6.1, the point $x' := S_p S_\ell^{-m} S_p^{-1}(x)$ satisfies $x' \in S_p(\Lambda) \setminus S_p S_\ell(\Lambda)$. Upon setting

$$\delta = \min\big(\mathrm{dis}\big(z_0, \Lambda \setminus S_1(\Lambda)\big), \mathrm{dis}\big(z_\ell, \Lambda \setminus S_\ell(\Lambda)\big)\big),$$

we obtain

$$\big|x' - z_p\big| \ge \mathrm{dis}\big(z_p, S_p(\Lambda) \setminus S_p S_\ell(\Lambda)\big) = r_p \,\mathrm{dis}\big(z_\ell, \Lambda \setminus S_\ell(\Lambda)\big) \ge r_p \delta.$$

Thus,

$$\begin{aligned} \frac{|\Lambda(x, z_p)|}{|x - z_p|^{1/t}} &= r_\ell^{m(1-1/t)} \frac{|\Lambda(x', z_p)|}{|x' - z_p|^{1/t}} \\ &\le \frac{|\Lambda(z_{p-1}, z_p)|}{r_p^{1/t} \delta^{1/t}} \\ &= L \delta^{-1/t} r_p^{1-1/t}. \end{aligned} \tag{35}$$

Similarly,

$$\frac{|\Lambda(y, z_p)|}{|y - z_p|^{1/t}} \le L \delta^{-1/t} r_{p+1}^{1-1/t}. \tag{36}$$

It follows from (33), (35) and (36) that

$$\begin{aligned}\big|\Lambda(x,y)\big| &\le \big|\Lambda(x,z_p)\big| + \big|\Lambda(y,z_p)\big| \\ &\le L\delta^{-1/t}\big(r_p^{1-1/t}|x-z_p|^{1/t} + r_{p+1}^{1-1/t}|y-z_p|^{1/t}\big) \\ &\le L\big(\delta^{-1}\csc\psi_p\big)^{1/t}\big(r_p^{1-1/t} + r_{p+1}^{1-1/t}\big)|x-y|^{1/t}.\end{aligned}$$

Therefore,

$$\big|\Lambda(x,y)\big|^t \le L^t\delta^{-1}(\csc\psi_p)\big(r_p^{1-1/t} + r_{p+1}^{1-1/t}\big)^t|x-y|,$$

and (34) has been proved. □

By Propositions 4.2, 4.4, 6.1 and 6.2 we have the following theorem.

THEOREM 6.3. *Let Λ be a regular self-similar arc and let $s = \dim_H(\Lambda)$. If $\theta_{\min} := \min\{\theta_p : p = 1, \ldots, \ell-1\} > 0$, then Λ is a t-quasi-arc for each $t \ge 1$ and the s-dimensional Hausdorff measure function is a Whitney function on Λ.*

When the minimal corner angle $\theta_{\min} = 0$, the analysis of Hausdorff measure function on Λ is more complicated. We now consider the case where $\theta_p = 0$ for some $1 \le p \le \ell - 1$. As before, we assume that the three vertices of the basic triangle of the basic figure Ω under consideration are $z_0 = 0$, $z_\ell = 1$, and z_q with $\Im z_q > 0$.

Let $g : [0,1] \to \Lambda$ be the homeomorphism in the proof of Proposition 5.1. Note that $g(p/\ell) = z_p$ for $p = 0, \ldots, \ell$. Since $\theta_q > 0$ and $\theta_p = 0$, we see that $p \ne q$.

Recall that $b_p = z_{p+1} - z_p$. In Section 5, we constructed two sequences of points on the self-similar arc Λ,

$$\begin{aligned}Z_j &= S_pS_\ell^j(z_q) = z_p - b_pr_\ell^j(1-z_q), \\ W_k &= S_{p+1}S_1^k(z_q) = z_p + b_{p+1}r_1^kz_q.\end{aligned}$$

Set

$$\begin{aligned}a &= r_p^sH^s\big([z_q,z_\ell]\big), \\ b &= r_{p+1}^sH^s\big([z_0,z_q]\big), \\ c &= r_p\big|[z_q,z_\ell]\big|, \\ d &= r_{p+1}\big|[z_0,z_q]\big|,\end{aligned} \tag{37}$$

where $|[z_p,z_\ell]|$ denotes the diameter of the subarc of Λ from z_p to z_ℓ. Then we have

$$\begin{aligned}H^s\big([Z_j,z_p]\big) &= a\lambda^{sj}, \qquad H^s\big([z_p,W_j]\big) = b\mu^{sj}, \\ \big|[Z_j,z_p]\big| &= c\lambda^j, \qquad \big|[z_p,W_j]\big| = d\mu^j,\end{aligned}$$

where λ, μ are defined by (25).

As usual, let $\mathbb{Z}$ denote the set of integers, let $\mathbb{N} = \{1,2,\ldots\}$ be the set of natural numbers, and let $\mathbb{Z}_+ = \{0,1,2,\ldots,\}$.

LEMMA 6.4. *Suppose that Λ is a regular self-similar arc with $\theta_p = 0$ for some $1 \le p \le \ell - 1$. Then there exists a constant $\Upsilon > 0$ such that if $j, k \in \mathbb{Z}_+$ and if*

$$\begin{aligned} &x \in \Lambda(Z_j, Z_{j'}), \qquad \left|j - j'\right| = 1, \qquad |x - Z_j| \le |x - Z_{j'}|, \\ &y \in \Lambda(W_k, W_{k'}), \qquad \left|k - k'\right| = 1, \qquad |y - W_k| \le |y - W_{k'}|, \end{aligned} \tag{38}$$

then

$$|Z_j - W_k| \le \Upsilon |x - y|.$$

Proof. Since $\theta_{\min} = 0$, the angles η_0, η_1 and η_2 defined by (21) and (22) are positive. Let Θ denote the line containing the points $\{Z_j\}$ and $\{W_k\}$. Since $\eta_1, \eta_2 > 0$ and since (19) holds, it follows that the subarc $\Lambda(Z_0, Z_1)$ intersects Θ at exactly two points Z_0, Z_1. The line Θ divides the plane into two half planes. Let us denote by H_1 the closed half plane which contains $\Lambda(Z_0, Z_1)$. The other closed half plane is denoted by H_2. When $x \in \Lambda(Z_0, Z_1)$ is sufficiently close to Z_0, the law of sines provides an estimate

$$|z - Z_0| < (\csc \eta_0)|z - x|, \quad z \in H_2.$$

It follows that there exists a constant $C > 0$ such that

$$|z - Z_0| \le C|z - x|, \quad \text{if } z \in H_2, x \in \Lambda(Z_0, Z_1), |x - Z_0| \le |x - Z_1|.$$

Similarly, there is a $C > 0$ so that

$$|z - Z_1| \le C|z - x|, \quad \text{if } z \in H_2, x \in \Lambda(Z_0, Z_1), |x - Z_0| \ge |x - Z_1|.$$

It follows that for some constant $C > 0$, we have

$$|z - Z_j| < C|z - x| \tag{39}$$

whenever

$$\begin{aligned} &z \in H_2, \qquad j \in \{0, 1\}, \qquad j' = 1 - j, \\ &x \in \Lambda(Z_0, Z_1), \qquad |x - Z_j| \le |x - Z_{j'}|. \end{aligned}$$

Since $\Lambda(Z_j, Z_{j+1}) = S_p S_\ell^j S_p^{-1}(\Lambda(Z_0, Z_1))$ and since Θ and H are invariant under the similitude $S_p S_\ell^j S_p^{-1}$, it follows that for the same constant C, (39) holds whenever

$$\begin{aligned} &z \in H_2, \qquad j \in \mathbb{N}, \qquad \left|j - j'\right| = 1, \\ &x \in \Lambda(Z_j, Z_{j'}), \qquad |x - Z_j| \le |x - Z_{j'}|. \end{aligned}$$

Similarly, there exists a constant $C' > 0$ such that

$$|z - W_k| \le C'|z - y|,$$

whenever

$$\begin{aligned} &z \in H_1, \qquad k \in \mathbb{N}, \qquad \left|k - k'\right| = 1, \\ &y \in \Lambda(W_k, W_{k'}), \qquad |y - W_k| \le |y - W_{k'}|. \end{aligned}$$

Now suppose that $j,k\in\mathbb{N}$ and (38) holds. Then

$$|Z_j-W_k|\le C|x-W_k|\le CC'|x-y|.$$

The proof is complete. □

PROPOSITION 6.5. *Suppose that* $1\le p\le \ell-1$ *and* $\theta_p=0$. *Then* Λ *satisfies Condition* W_p *if and only if*

$$(\lambda^j+\mu^k)^s=o\big(|\alpha\lambda^j-\beta\mu^k|\big),\quad j,k=0,1,2,\dots. \tag{40}$$

Proof. Suppose that Λ satisfies Condition W_p. By Definition 4.1, we know that (17) holds for all Z_j and W_k, *i.e.*,

$$H^s\big([Z_j,W_k]\big)=\big|f(Z_j)-f(W_k)\big|=o\big(|Z_j-W_k|\big). \tag{41}$$

By (37), we have

$$H^s\big([Z_j,W_k]\big)=a\lambda^{sj}+b\mu^{sk},$$

which, together with (27) and (41), implies that

$$a\lambda^{sj}+b\mu^{sk}=o\big(|\alpha\lambda^j-\beta\mu^k|\big). \tag{42}$$

The Hölder inequality tells us that

$$(\lambda^j+\mu^k)^s\le F\big(a\lambda^{sj}+b\mu^{sk}\big), \tag{43}$$

where $F=(a^{-1/(s-1)}+b^{-1/(s-1)})^{s-1}$. Now (40) is a consequence of (42) and (43).

Conversely, suppose that (40) holds. Let $x,y\in\Lambda$ be such that $z_{p-1}\prec x\prec z_p\prec y\prec z_{p+1}$. Let m be the least positive integer such that $x\prec Z_m$, so $x\in[Z_{m-1},Z_m]$. If $|x-Z_{m-1}|\le|x-Z_m|$ let $j=m-1$ and $j'=m$; otherwise, let $j=m, j'=m-1$. In either case we have

$$x\in\Lambda(Z_j,Z_{j'}),\qquad |x-Z_j|\le|x-Z_{j'}|,\qquad |j-j'|=1. \tag{44}$$

Similarly, there are integers $k,\ k'\ge 0$ such that

$$y\in\Lambda(W_k,W_{k'}),\qquad |y-W_k|\le|y-W_{k'}|,\qquad |k-k'|=1. \tag{45}$$

By Lemma 6.4, we have

$$|Z_j-W_k|\le\Upsilon|x-y|,$$

which, together with (40), implies that

$$(\lambda^j+\mu^k)^s=o\big(|x-y|\big). \tag{46}$$

Setting $Z_{-1}=Z_0$ and $W_{-1}=W_0$, we have

$$\begin{aligned}H^s\big([x,y]\big)&\le H^s\big([Z_{j-1},W_{k-1}]\big)\\&\le a\lambda^{s(j-1)}+b\mu^{s(k-1)}\\&\le\big(a\lambda^{-s}+b\mu^{-s}\big)\big(\lambda^{sj}+\mu^{sk}\big)\\&\le\big(a\lambda^{-s}+b\mu^{-s}\big)\big(\lambda^j+\mu^k\big)^s.\end{aligned}$$

The last inequality and (46) imply that

$$H^s([x,y]) = o(|x-y|).$$

This completes the proof. ☐

PROPOSITION 6.6. *Suppose that* $t \geq 1$, $1 \leq p \leq \ell - 1$, *and* $\theta_p = 0$. *Then* Λ *satisfies Condition* Q_p^t *if and only if there exists a constant* $C > 0$ *such that*

$$(\lambda^j + \mu^k)^t \leq C|\alpha\lambda^j - \beta\mu^k|, \quad j,k = 0,1,2,\ldots. \tag{47}$$

Proof. Suppose that Λ satisfies Condition Q_p^t. Then there exists a constant $C_p > 0$ such that

$$|[Z_j, W_k]|^t \leq C_p|Z_j - W_k|, \quad j,k = 0,1,\ldots. \tag{48}$$

We have the following estimate

$$\min(c,d)(\lambda^j + \mu^k) \leq c\lambda^j + d\mu^k = |[Z_j, z_p]| + |[z_p, W_k]| \leq 2|[Z_j, W_k]|,$$

which, together with (27) and (48), implies that

$$(\lambda^j + \mu^k)^t \leq C'|[Z_j, W_k]|^t \leq C|\alpha\lambda^j - \beta\mu^k|,$$

where $C = C'C_p = \{2/\min(c,d)\}^t C_p$. Thus (47) holds.

Conversely, suppose that there exists a constant $C > 0$ such that (47) holds. Let $x, y \in \Lambda$ be such that $z_{p-1} \prec x \prec z_p \prec y \prec z_{p+1}$. We need to prove that there exists constant $M > 0$ such that

$$|[x,y]|^t \leq M|x-y|.$$

As in the proof of previous proposition, there exist integers j, j', k, k' such that (44) and (45) hold. By Lemma 6.4, we have

$$|Z_j - W_k| \leq \Upsilon|x-y|,$$

which, together with (27) and (47), implies that

$$(\lambda^j + \mu^k)^t \leq C\Upsilon|x-y|. \tag{49}$$

Now

$$\begin{aligned}
|[x,y]|^t &\leq |[Z_{j-1}, W_{k-1}]|^t \\
&\leq (|[Z_{j-1}, z_p]| + |[z_p, W_{k-1}]|)^t \\
&\leq (c\lambda^{j-1} + d\mu^{k-1})^t \\
&\leq (c\lambda^{-1} + d\mu^{-1})^t (\lambda^j + \mu^k)^t.
\end{aligned}$$

The last inequality and (49) imply that

$$|[x,y]|^t \leq M|x-y|,$$

where $M = (c\lambda^{-1} + d\mu^{-1})^t C\Upsilon$. ☐

PROPOSITION 6.7. *Let Λ be a regular self-similar arc and let $s=\dim_H(\Lambda)$. If the s-dimensional Hausdorff measure function f is a Whitney function on Λ, then Λ is an s-quasi-arc. If Λ is a t-quasi-arc for some t with $s>t\geq 1$, then f is a Whitney function on Λ.*

Proof. By Theorem 6.3 and Proposition 6.6, the self-similar arc Λ is a t-quasi arc if and only if for each p with $\theta_p=0$, one has

$$(\lambda^j+\mu^k)^t = O(|\alpha\lambda^j-\beta\mu^k|), \quad \text{where } j,k=0,1,2,\ldots. \tag{50}$$

By Theorem 6.3 and Proposition 6.5, the s-dimensional Hausdorff measure function on Λ is a Whitney function if and only if for each p with $\theta_p=0$, one has

$$(\lambda^j+\mu^k)^s = o(|\alpha\lambda^j-\beta\mu^k|), \quad \text{where } j,k=0,1,2,\ldots. \tag{51}$$

The proposition follows because (51) implies (50) when $t=s$, and because (51) follows from (50) when $1\leq t<s$. $\square$

The second part of Proposition 6.7 is contained in the result of Norton [4] mentioned in the introduction of this paper.

By Proposition 6.6, Condition Q_p^t is reduced to an inequality (47). In the following proposition, it is further reduced to an inequality of a certain form which is more convenient for determining whether a self-similar arc is a t-quasi-arc and which is directly related to the degree to which a number u is approximated by numbers of the form $jx-ky$, where x, y are fixed positive numbers and j, k are non-negative integers.

Recall that λ, μ, x, y, u are defined by (25) and (28).

PROPOSITION 6.8. *Suppose that $1\leq p\leq \ell-1$ and $\theta_p=0$. Then Λ satisfies Condition Q_p^t if and only if there exists a constant $M>0$ such that*

$$e^{-j(t-1)x}\leq M|u-jx+ky|, \quad j,k=0,1,2,\ldots. \tag{52}$$

Proof. By Proposition 6.6, we only need to show that there exists a constant $C>0$ such that (47) holds if and only if there exists a constant $M>0$ such that (52) holds.

Suppose that there exists no constant C such that (47) holds. Then there are increasing sequences $\{j_n\}$ and $\{k_n\}$ of positive integers such that

$$\lim_{n\to\infty}\frac{|\alpha\lambda^{j_n}-\beta\mu^{k_n}|}{(\lambda^{j_n}+\mu^{k_n})^t}=0. \tag{53}$$

Since $(\lambda^{j_n}+\mu^{k_n})\geq(\lambda^{j_n}+\mu^{k_n})^t$ when n is large enough, we see that

$$\lim_{n\to\infty}\frac{|\alpha\lambda^{j_n}-\beta\mu^{k_n}|}{\lambda^{j_n}+\mu^{k_n}}\leq \lim_{n\to\infty}\frac{|\alpha\lambda^{j_n}-\beta\mu^{k_n}|}{(\lambda^{j_n}+\mu^{k_n})^t}=0. \tag{54}$$

This implies that when n is sufficiently large, the quotient $|\alpha\lambda^{j_n} - \beta\mu^{k_n}|/(\lambda^{j_n} + \mu^{k_n})$ does not exceed $\alpha/2$, hence we have

$$\alpha\lambda^{j_n} \le \left|\alpha\lambda^{j_n} - \beta\mu^{k_n}\right| + \beta\mu^{k_n} \le \frac{\alpha}{2}\left(\lambda^{j_n} + \mu^{k_n}\right) + \beta\mu^{k_n},$$

$$\frac{\alpha}{2}\lambda^{j_n} \le \left(\frac{\alpha}{2} + \beta\right)\mu^{k_n}.$$

It follows that there is a constant Q_1 such that $\lambda^{j_n} \le Q_1^{-1}\mu^{k_n}$ for all n. Similarly, there is a constant Q_2 such that $\mu^{k_n} \le Q_2\lambda^{j_n}$ for all n. Therefore,

$$Q_1\lambda^{j_n} \le \mu^{k_n} \le Q_2\lambda^{j_n}. \tag{55}$$

Now

$$\beta\frac{(\alpha/\beta)\lambda^{j_n}\mu^{-k_n} - 1}{\lambda^{(t-1)j_n}} = \frac{\alpha\lambda^{j_n} - \beta\mu^{k_n}}{(\lambda^{j_n} + \mu^{k_n})^t}\left(\frac{\lambda^{j_n} + \mu^{k_n}}{\lambda^{j_n}}\right)^t \frac{\lambda^{j_n}}{\mu^{k_n}}. \tag{56}$$

The first factor on the right-hand side of (56) tends to 0 as $n \to \infty$ by (53), while the second and third factors are bounded above and below because of (55). It follows that

$$\lim_{n\to\infty} \frac{(\alpha/\beta)\lambda^{j_n}\mu^{-k_n} - 1}{\lambda^{(t-1)j_n}} = 0. \tag{57}$$

Since the denominator in (57) is ≤ 1, it follows that the numerator tends to 0 as $n \to \infty$. By (57) and the equality $\lim_{w\to 1}[(\log w)/(w-1)] = 1$, we have

$$\lim_{n\to\infty} \frac{\log((\alpha/\beta)\lambda^{j_n}\mu^{-k_n})}{\lambda^{(t-1)j_n}} = 0. \tag{58}$$

Substituting $x = -\log\lambda$, $y = -\log\mu$, and $u = \log(\alpha/\beta)$ into (58) yields that

$$\lim_{n\to\infty} \frac{u - j_n x + k_n y}{e^{-j_n(t-1)x}} = 0. \tag{59}$$

Thus, there exists no M such that (52) holds.

Conversely, suppose that there exists no M such that (52) holds. Then there are increasing sequences $\{j_n\}$ and $\{k_n\}$ of positive integers such that (59) holds, hence the equivalent equalities (58) and (57) hold. Since the numerator in (57) tends to 0 as $n \to \infty$, it follows that $1/2 < (\alpha/\beta)\lambda^{j_n}\mu^{-k_n} < 2$ for n large enough, which implies (55). Then (53) follows from (55) and (57). Therefore there exists no C such that (47) holds. ☐

Proposition 6.9. *Suppose that $1 \le p \le \ell - 1$ and $\theta_p = 0$. Then Λ satisfies Condition Q_p^1 if and only if x/y is rational.*

Proof. By Proposition 6.8, Λ satisfies Condition Q_p^1 if and only if the inequality (52) holds when $t = 1$.

Suppose that $\tau := x/y$ is rational. Then the set

$$\Pi := \{j\tau - k : j, k \in \mathbb{N}\}$$

is discrete. By Proposition 5.3, the distance from the point u/y to Π is positive. It follows that $|u-jx+ky|>\delta$ for some $\delta>0$. Thus, the inequality (52) holds with $M=1/\delta$.

Suppose that $\tau=x/y$ is irrational. We show that the set Π is dense in $\mathbb{R}$, which implies that (52) does not hold with $t=1$ and that Λ does not satisfy Condition Q_p^1. Let $c\in\mathbb{R}$ and $\varepsilon>0$. There exist $j_0,k_0\in\mathbb{Z}$ such that

$$\big|(j_0\tau-k_0)-c\big|<\varepsilon/2. \tag{60}$$

By Dirichlet's Approximation theorem (see, *e.g.*, [1, p. 143]), there are positive integers $j',k'>\max(|k_0|,|j_0|,2\tau/\varepsilon)$ such that

$$\left|\frac{j'}{k'}-\frac{1}{\tau}\right|<\frac{1}{k'^2}<\frac{\varepsilon}{2k'\tau},$$

hence

$$\big|j'\tau-k'\big|<\varepsilon/2. \tag{61}$$

Set $j=j_0+j'$ and $k=k_0+k'$. Then $j,k\in\mathbb{N}$. It follows from (60) and (61) that $|(j\tau-k)-c|<\varepsilon$. Therefore, Π is dense in $\mathbb{R}$. ☐

7. A one-parameter family of self-similar arcs

In this section, we construct and examine a one-parameter family of regular self-similar arcs with $\theta_p=0$ for some fixed p. For different values of the parameter τ, the corresponding regular self-similar arcs have various features. It turns out that the self-similar arc satisfies Condition Q_p^t if and only if the number τ satisfies a certain "approximation property" $J_{(t-1)\zeta}$, where $\zeta=\ln(15/7)$. We now define approximation property J_a, $a>0$, of irrational numbers.

Definition 7.1. Let $a>0$. An irrational number τ is said to have *approximation property* J_a if

$$\exists C>0,\quad |\tau-k/j|\ge Cj^{-1}e^{-aj},\quad \forall k\in\mathbb{Z},j\in\mathbb{N}. \tag{62}$$

It follows directly from the definition that if τ has approximation property J_{a_0} then τ has approximation property J_a for each $a>a_0$. By Liouville's Approximation Theorem (see, *e.g.*, [1, p. 146]), each algebraic irrational number τ satisfies $|\tau-k/j|>Cj^{-m}$, where m is the degree of the irreducible polynomial with integer coefficients of which τ is a root, hence τ has approximation property J_a for each $a>0$.

Theorem 7.2. *Let $a_0>0$ and let $\nu\in\mathbb{N}$. There exists a transcendental number τ with $1<\tau<1+2^{-\nu}$ such that τ has approximation property J_{a_0}, but τ has approximation property J_a for no $a\in(0,a_0)$.*

Proof. Define a number τ by

$$\begin{aligned} &\tau = 1 + 2^{-n_1} + 2^{-n_2} + \cdots, \\ &n_1 = m_1, \qquad n_2 = m_1 + m_2, \qquad n_3 = m_1 + m_2 + m_3, \ldots, \\ &m_1 = \max(8, \nu + 1), \\ &m_{i+1} = \max\big(\lceil 2^{n_i} a_0/\log 2\rceil - n_i, n_i + 2\big) \quad \text{for } i \geq 1, \end{aligned} \tag{63}$$

where $\lceil\cdot\rceil$ is the ceiling function, *i.e.*, $\lceil u\rceil$ is the least integer greater than or equal to u. Since $m_i \to \infty$ as $i \to \infty$, we see that

$$\tau = 1 + 2^{-m_1}\big(1 + 2^{-m_2} + 2^{-m_2-m_3} + \cdots\big) < 1 + 2^{-m_1+1} \leq 1 + 2^{-\nu}.$$

For $i \geq 1$, set $j_i = 2^{n_i}$ and $k_i = j_i\tau_i$, where

$$\tau_i = 1 + 2^{-n_1} + 2^{-n_2} + \cdots + 2^{-n_i}. \tag{64}$$

Then k_i is an integer, and

$$j_i\tau = k_i + \big(2^{-m_{i+1}} + 2^{-m_{i+1}-m_{i+2}} + \cdots\big).$$

It follows that

$$0 < j_i\tau - k_i < 2^{-m_{i+1}+1}. \tag{65}$$

From the definition of m_{i+1} in (63), we see that there is an $i_0 \in \mathbb{N}$ such that for $i \geq i_0$ we have

$$2^{m_{i+1}-1} < 2^{-n_i}\exp\big(2^{n_i}a_0\big) \leq 2^{m_{i+1}}. \tag{66}$$

Combining inequalities (65) and (66), we obtain that

$$0 < j_i\tau - k_i < 2j_ie^{-a_0j_i}.$$

Consider a fixed number $a \in (0, a_0)$. The above inequality tells us that for $i \geq i_0$,

$$\frac{|\tau - k_i/j_i|}{j_i^{-1}e^{-aj_i}} < 2j_ie^{(a-a_0)j_i}.$$

Since $a - a_0 < 0$ and hence the right-hand side of the above inequality tends to 0 as i approaches ∞, we see that (62) does not hold. Thus τ does not have approximation property J_a.

Now we assume that $j \geq 2^{n_{i_0}}$ and k is an arbitrary integer. Then there is an $i \geq i_0$ such that $2^{n_i} \leq j < 2^{n_{i+1}}$. By (66), the integers n_i and n_{i+1} satisfy

$$2^{n_{i+1}-1} < \exp\big(2^{n_i}a_0\big) \leq 2^{n_{i+1}}. \tag{67}$$

In order to obtain a lower bound for $|j\tau - k|$, we write

$$j\tau - k = (j\tau_{i+1} - k) + j(\tau - \tau_{i+1}). \tag{68}$$

Recall that τ_i is defined by (64). Since $2^{n_{i+1}}\tau_{i+1}$ is an odd integer, and since j is not a multiple of $2^{n_{i+1}}$, we see that $2^{n_{i+1}}j\tau_{i+1}$ is not a multiple of $2^{n_{i+1}}$. It follows that $|2^{n_{i+1}}j\tau_{i+1} - 2^{n_{i+1}}k| \geq 1$, and therefore

$$|j\tau_{i+1} - k| \geq 2^{-n_{i+1}}. \tag{69}$$

For the second term on the right-hand side of (68) we have

$$\begin{aligned} j(\tau - \tau_{i+1}) &< 2^{n_{i+1}}\big(2^{-n_{i+2}} + 2^{-n_{i+3}} + \cdots\big) \\ &= 2^{-m_{i+2}} + 2^{-m_{i+2}-m_{i+3}} + \cdots \\ &< 2^{-m_{i+2}+1}. \end{aligned}$$

Since $m_{i+2} \geq n_{i+1} + 2$, by the definition of m_i, the right-hand side of the last inequality is $\leq 2^{-n_{i+1}-1}$. Thus,

$$0 < j(\tau - \tau_{i+1}) < 2^{-n_{i+1}-1}. \tag{70}$$

Now inequalities (68), (69) and (70) tell us that

$$|j\tau - k| \geq 2^{-n_{i+1}-1}. \tag{71}$$

From (71) and (67), we obtain that

$$\begin{aligned} |j\tau - k| &\geq 2^{-n_{i+1}-1} \\ &> (1/4)\exp\big(-2^{n_i}a_0\big) \\ &\geq (1/4)e^{-a_0 j}. \end{aligned}$$

Therefore, the inequality in (62), with a replaced by a_0, holds with $C = 1/4$ as long as $j \geq 2^{n_{i_0}}$. This implies that τ has approximation property J_{a_0}.

Finally, since τ does not have approximation property J_a for $a < a_0$, it cannot be an algebraic number. Thus, τ is a transcendental number. $\square$

THEOREM 7.3. *Let $a_0 > 0$ and let $\nu \in \mathbb{N}$. There exists a transcendental number τ with $1 < \tau < 1 + 2^{-\nu}$ such that τ has approximation property J_a for each $a > a_0$, but τ does not have approximation property J_{a_0}.*

Proof. Define a number τ by

$$\begin{aligned} &\tau = 1 + 2^{-n_1} + 2^{-n_2} + \cdots, \\ &n_1 = m_1, \qquad n_2 = m_1 + m_2, \qquad n_3 = m_1 + m_2 + m_3, \ldots, \\ &m_1 = \max(8, \nu + 1), \qquad m_{i+1} = n_i + \max\big(\lceil 2^{n_i}a_0/\log 2\rceil, 2\big) \quad \text{for } i \geq 1. \end{aligned} \tag{72}$$

Then τ satisfies $1 < \tau < 1 + 2^{-\nu}$. As in the previous theorem, τ is a transcendental number because we shall show that τ does not have property J_{a_0}.

Setting $j_i = 2^{n_i}$ and $k_i = j_i\tau_i$, we obtain that

$$|j_i\tau - k_i| < 2^{-m_{i+1}+1}. \tag{73}$$

By the definition of m_{i+1}, there is an $i_0 \in \mathbb{N}$ such that for $i \geq i_0$ we have

$$2^{m_{i+1}-1} < 2^{n_i}\exp\big(2^{n_i}a_0\big) \leq 2^{m_{i+1}}. \tag{74}$$

We then combine (73) and (74) to obtain

$$|j_i\tau - k_i| < 2j_i^{-1}e^{-a_0 j_i},$$

which implies that for $i \geq i_0$,

$$\frac{|\tau - k_i/j_i|}{j_i^{-1}e^{-a_0 j_i}} < 2j_i^{-1}.$$

Thus, τ does not have approximation property J_{a_0}.

Let $a > a_0$. We now prove that τ has approximation property J_a. Choose an integer $i_1 \geq i_0$ such that whenever $i \geq i_1$, the following inequality holds:

$$2^{-2n_i-2}\exp\bigl(-2^{n_i}a_0\bigr) > \exp\bigl(-2^{n_i}a\bigr). \tag{75}$$

Assume that $i \geq i_1$, $2^{n_i} \leq j < 2^{n_{i+1}}$, and k is an arbitrary integer. Similar to the previous proof, we have

$$2^{n_{i+1}-1} < 2^{2n_i}\exp\bigl(2^{n_i}a_0\bigr) \leq 2^{n_{i+1}}, \tag{76}$$

and

$$|j\tau - k| \geq 2^{-n_{i+1}-1}. \tag{77}$$

From (75), (76) and (77), we obtain that

$$\begin{aligned}|j\tau - k| &\geq 2^{-n_{i+1}-1}\\ &> 2^{-2n_i-2}\exp\bigl(-2^{n_i}a_0\bigr)\\ &> \exp\bigl(-2^{n_i}a\bigr)\\ &\geq e^{-aj}.\end{aligned}$$

Therefore, the inequality in (62) holds with $C = 1$ as long as $j \geq 2^{n_{i_1}}$. This implies that τ has approximation property J_a. $\square$

THEOREM 7.4. *Let $\nu \in \mathbb{N}$. Then there exists a transcendental number τ with $1 < \tau < 1 + 2^{-\nu}$ such that τ has approximation property J_a for each $a > 0$.*

Proof. Define a number τ by

$$\begin{gathered}\tau = 1 + 2^{-n_1} + 2^{-n_2} + \cdots,\\ n_1 = m_1, \qquad n_2 = m_1 + m_2, \qquad n_3 = m_1 + m_2 + m_3, \ldots,\\ m_1 = \max(8, 2\nu), \qquad m_{i+1} = 2^{n_i/2} \quad \text{for } i \geq 1.\end{gathered} \tag{78}$$

Then $1 < \tau < 1 + 2^{-\nu}$, as in the previous theorem. It is clear that for each i, m_i is an integer, and $m_{i+1} > n_i + 2$, which will be needed later.

Setting $j_i = 2^{n_i}$ and $k_i = j_i\tau_i$, we obtain that

$$|j_i\tau - k_i| < 2^{-m_{i+1}+1} = 2^{-\sqrt{j_i}+1}. \tag{79}$$

It follows that for each positive integer n,

$$\lim_{i\to\infty}\frac{|j_i\tau - k_i|}{j_i^{-n}} = 0.$$

By Liouville's Approximation theorem, τ must be a transcendental number.

Let $a>0$. We now prove that τ has approximation property J_a. Choose i_0 so that when $i \geq i_0$, we have

$$2^{-n_{i+1}-1} = 2^{-n_i - 2^{n_i/2} - 1} > \exp\bigl(-2^{n_i} a\bigr).$$

Assume that $i \geq i_0$, $2^{n_i} \leq j < 2^{n_{i+1}}$, and k is an arbitrary integer. Similar to the previous proof, since $m_{i+1} > n_i + 2$, we see that

$$|j\tau - k| \geq 2^{-n_{i+1}-1}.$$

It follows that

$$|j\tau - k| > \exp\bigl(-2^{n_i} a\bigr) \geq e^{-aj}.$$

Therefore, τ has approximation property J_a. □

THEOREM 7.5. *Let $\nu \in \mathbb{N}$. Then there exists a transcendental number τ with $1 < \tau < 1 + 2^{-\nu}$ such that τ has approximation property J_a for no $a > 0$.*

Proof. Define a number τ by

$$\begin{aligned} &\tau = 1 + 2^{-n_1} + 2^{-n_2} + \cdots, \\ &n_1 = m_1, \qquad n_2 = m_1 + m_2, \qquad n_3 = m_1 + m_2 + m_3, \ldots, \\ &m_1 = \max(8, \nu+1), \qquad m_{i+1} = 2^{2^{n_i}} \quad \text{for } i \geq 1. \end{aligned} \tag{80}$$

Then $1 < \tau < 1 + 2^{-\nu}$, as in the previous theorem.

Setting $j_i = 2^{n_i}$ and $k_i = j_i \tau_i$, we obtain, as in the proof of the previous theorem, that

$$|j_i \tau - k_i| < 2^{-m_{i+1}+1} = 2^{-j_i^2+1}. \tag{81}$$

Consider a fixed number $a > 0$. Then (81) implies that

$$\lim_{i\to\infty} \frac{|j_i\tau - k_i|}{e^{-aj_i}} = 0,$$

and hence τ does not have approximation property J_a. By Liouville's Approximation theorem, τ is necessarily a transcendental number. □

Now we construct a one-parameter family of regular self-similar arcs. We start by constructing a family of basic figures depending on a parameter τ with $1 < \tau < 1.001$.

For a fixed τ with $1 < \tau < 1.001$, the corresponding basic figure is as in Figure 3. The points B, D, F lie on segment $\overline{AG}$, and the magnitudes of the

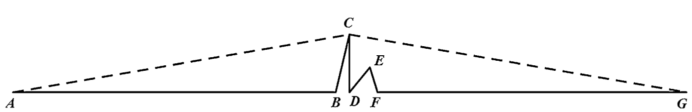

FIGURE 3. Basic figure with a zero corner angle.

segments are $\overline{AG} = 1$, $\overline{AD} = 1/2$, $\overline{AB} = (7/15)^{1/\tau}$, $\overline{FG} = 7/15$. The magnitudes of the angles are $\angle CAG = \angle CGA = \pi/18$, $\angle CDE = \pi/9$. The position of point E is determined by $\overline{DE} = (7/15)^{1/\nu}\overline{CD}$, where

$$\nu = \begin{cases} \tau, & \text{if } \tau \text{ is irrational,} \\ \tau - \frac{\tau-1}{\sqrt{2}}, & \text{if } \tau \text{ is rational.} \end{cases} \tag{82}$$

Note that ν is always irrational and $1 < \nu < 1.001$.

Let E' be the projection of E on $\overline{AG}$. Then

$$\begin{aligned} \overline{DE'} &= (1/2)\tan(\pi/18)(7/15)^{1/\tau}\sin(\pi/9) \\ &< (1/2)(7/15)^{1000/1001}\big(1-\cos(\pi/9)\big) \\ &< 1/30. \end{aligned}$$

Thus E' is between D and F. It follows that $\angle EFG > \pi/2$ and E is in the interior of triangle CAG. Therefore, polygon $ABCDEFG$ is a basic figure with basic triangle $\triangle CAG$.

We denote polygon $ABCDEFG$ by Ω_τ, and the corresponding self-similar set by Λ_τ. The corner angles satisfy $\theta_j > 0$ for $j \neq 3$ and $\theta_3 = 0$. It is clear that $\eta_0 > 0$. By Proposition 5.3, in order to show that Λ_τ is a regular self-similar arc, it suffices to verify

$$u \notin \Sigma := \{jx - ky : j, k \in \mathbb{N}\},$$

where

$$x = \zeta, \qquad y = \tau^{-1}\zeta, \qquad \zeta = \log(15/7), \qquad u = \log\frac{\overline{CD}\cdot\overline{CG}}{\overline{DE}\cdot\overline{AG}} = \nu^{-1}\zeta. \tag{83}$$

If τ is rational, then $\nu = \tau - (\tau-1)/\sqrt{2}$ is irrational, and for $j, k = 0, 1, 2, \ldots$,

$$u - (jx - ky) = \zeta\big(\nu^{-1} - (j - k/\tau)\big) \neq 0.$$

If τ is irrational, then $\nu = \tau$, and for $j, k = 0, 1, 2, \ldots$,

$$u - (jx - ky) = \tau^{-1}\zeta j\big(-\tau + (k+1)/j\big) \neq 0.$$

Therefore $u \notin \Sigma$ and Λ_τ is a regular self-similar arc.

When τ is rational, x/y is rational, hence Λ_τ is a 1-quasi-arc. We now consider the case where τ is irrational. In this case, we have $\nu = \tau$ and $u = y$. By Proposition 6.8, Λ_τ is a t-quasi-arc if and only if there is a constant $M_{\tau,t} > 0$ such that

$$e^{-j(t-1)x} \leq M_{\tau,t}|u - jx + ky|. \tag{84}$$

In light of (83), inequality (84) is reduced to

$$\tau(M_{\tau,t}\zeta)^{-1}j^{-1}e^{-j(t-1)\zeta} < \big|\tau - (k+1)/j\big|,$$

which is equivalent to

$$C_{\tau,t}j^{-1}e^{-j(t-1)\zeta} < |\tau - k/j|.$$

Therefore, for $t>1$, Λ_τ is a t-quasi-arc if and only if τ has approximation property $J_{(t-1)\zeta}$.

We summarize the above discussion as follows.

Example 7.6. For $1<\tau<1.001$, let Λ_τ be the regular self-similar arc generated by the basic figure Ω_τ in Figure 3, where

$$\overline{AG}=1,\qquad \overline{AD}=1/2,\qquad \overline{AB}=(7/15)^{1/\tau},\qquad \overline{FG}=7/15,$$
$$\angle CAG=\angle CGA=\pi/18,\qquad \angle CDE=\pi/9,\qquad \overline{DE}=(7/15)^{1/\nu}\overline{CD},$$

where ν is defined by (82). For $t>1$, Λ_τ is a t-quasi-arc if and only if τ has approximation property $J_{(t-1)\zeta}$.

(1) For a fixed $t_0>1$, by Theorem 7.2, there is a transcendental number $\tau\in(1,1.001)$ such that τ has approximation property $J_{(t_0-1)\zeta}$, but τ has approximation property $J_{(t-1)\zeta}$ for no $t\in(1,t_0)$. For such a τ, Λ_τ is a t_0-quasi-arc, but Λ_τ is a t-quasi-arc for no $t<t_0$.
(2) For a fixed $t_0>1$, by Theorem 7.3, there is a transcendental number $\tau\in(1,1.001)$ such that τ does not have approximation property $J_{(t_0-1)\zeta}$, but τ has approximation property $J_{(t-1)\zeta}$ for each $t>t_0$. For such a τ, Λ_τ is a t-quasi-arc for each $t>t_0$, but Λ_τ is not a t_0-quasi-arc.
(3) By Theorem 7.4, there is a transcendental number $\tau\in(1,1.001)$ such that τ has approximation property $J_{(t-1)\zeta}$ for each $t>1$. For such a τ, Λ_τ is a t-quasi-arc for each $t>1$. Since $x/y=\tau$, which is an irrational number, it follows from Theorem 6.9 that Λ_τ is not a 1-quasi-arc.
(4) By Theorem 7.5, there is a transcendental number $\tau\in(1,1.001)$ such that τ has approximation property $J_{(t-1)\zeta}$ for no $t>1$. For such a τ, Λ_τ is a t-quasi-arc for no $t>1$.

As a consequence of the example, we obtain the following theorem.

Theorem 7.7.

(1) *There exists a regular self-similar arc Λ with $\theta_{\min}=0$ such that Λ is t-quasi-arc for no $t\geq 1$.*
(2) *Let $t_0>1$. Then there exists a regular self-similar arc Λ with $\theta_{\min}=0$ such that Λ is a t_0-quasi-arc, but Λ is a t-quasi-arc for no $1\leq t<t_0$.*
(3) *Let $t_0\geq 1$. Then there exists a regular self-similar arc Λ with $\theta_{\min}=0$ such that Λ is a t-quasi-arc for each $t>t_0$, but Λ is not a t_0-quasi-arc.*

Remark. When $t_0=1$, the third part of Theorem 7.7 says that there exists a regular self-similar arc Λ with $\theta_{\min}=0$ which is an t-quasi-arc for each $t\in(1,s)$, where s is the Hausdorff dimension of Λ. Then by Theorem 6.7, the Hausdorff measure function f is a Whitney function on Λ.

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References

[1] T. M. Apostol, *Modular functions and Dirichlet series in number theory*, 2nd ed., Springer-Verlag, New York, 1990. MR 1027834

[2] G. Choquet, *L'isometrie des ensembles dans ses rapports avec la théorie du contact et la théorie de la measure*, Math. Bucharest **20** (1944), 29–64. MR 0012317

[3] J. E. Hutchinson, *Fractals and self similarity*, Indiana Univ. Math. J. **30** (1981), 713–747. MR 0625600

[4] A. Norton, *Functions not constant on fractal quasi-arcs of critical points*, Proc. Amer. Math. Soc. **106** (1989), 397–405. MR 0969524

[5] A. Sard, *The measure of critical values of differentiable maps*, Bull. Amer. Math. Soc. **48** (1942), 883–890. MR 0007523

[6] Z.-Y. Wen and L.-F. Xi, *Relations among Whitney sets, self-similar arcs and quasi-arcs*, Israel J. Math. **136** (2003), 251–267. MR 1998112

[7] Z.-Y. Wen and L.-F. Xi, *The geometry of Whitney's critical sets*, Israel J. Math. **174** (2009), 303–348. MR 2581221

[8] H. Whitney, *Analytic extensions of differentiable functions defined on closed sets*, Trans. Amer. Math. Soc. **36** (1934), 63–89. MR 1501735

[9] H. Whitney, *A function not constant on a connected set of critical points*, Duke Math. J. **1** (1935), 514–517. MR 1545896

[10] W. M. Whyburn, *Non-isolated critical points of functions*, Bull. Amer. Math. Soc. **35** (1929), 701–708. MR 1561794

[11] L.-F. Xi and M. Wu, *A note on Whitney's critical sets*, Prog. Nat. Sci. **13** (2003), 152–156. MR 1976882

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