Illinois Journal of Mathematics
Volume 61, Numbers 1–2, Spring and Summer 2017, Pages 81–96
S 0019-2082

BOUNDS ON THE NORM OF THE BACKWARD SHIFT AND RELATED OPERATORS IN HARDY AND BERGMAN SPACES

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ABSTRACT. We study bounds for the backward shift operator $f \mapsto (f(z) - f(0))/z$ and the related operator $f \mapsto f - f(0)$ on Hardy and Bergman spaces of analytic and harmonic functions. If u is a real valued harmonic function, we also find a sharp bound on $M_1(r, u - u(0))$ in terms of $\|u\|_{h^1}$, where M_1 is the integral mean with $p = 1$.

1. Introduction

For a space of continuous functions in the unit disc with bounded point evaluation, we can consider the operator $\mathcal{B}$ defined by $\mathcal{B}(f) = f - f(0)$. For a space of analytic functions in the unit disc, it also makes sense to consider the backward shift operator B defined by $B(f) = [f - f(0)]/z$. On H^2, the backward shift operator is the adjoint of the forward shift operator given by $Sf(z) = zf(z)$. Both S and B have been extensively studied in the literature.

In this article, we study the norms of these operators on various spaces. For Hardy spaces, both B and $\mathcal{B}$ have the same norm, and since $|f(0)| \le \|f\|$, it is clear that the norm is at most 2. However, we are not aware of any references in the literature that discuss this question further, beyond the observation that the norms of both B and $\mathcal{B}$ are exactly 1 for H^2. One can use the above facts and interpolation to show that $\|B\|_{H^p} \le 2^{(2-p)/p}$ if $1 \le p \le 2$ and $\|B\|_{H^p} \le 2^{(p-2)/p}$ for $2 \le p \le \infty$. However, this does not settle the question of whether the norm on H^1 and H^∞ is less than 2. We prove that $\|B\|_{H^\infty} = 2$ but that $\|B\|_{H^1} < 2$. In fact, we prove that $\|B\|_{H^1} \le 1.71$.

Received February 10, 2017; received in final form September 15, 2017.

Partial support for this work has been provided by the University of Alabama RGC-2015-22 grant.

2010 *Mathematics Subject Classification.* Primary 47B38. Secondary 30H10, 30H20.

We also study bounds for B and $\mathcal{B}$ on other spaces. Let A^p denote the Bergman space of the unit disc with normalized area measure and let $a^p_{\mathbb{R}}$ be the real harmonic Bergman space on the unit disc with normalized area measure. We show that $\|\mathcal{B}\|_{A^p} \le \|\mathcal{B}\|_{H^p}$. This also implies that $\|B\|_{A^p} \le 2\|\mathcal{B}\|_{H^p}$. We also show that $\|\mathcal{B}\|_{a^1_{\mathbb{R}}} \le 1.84$.

Lastly, we consider the operator $\mathcal{B}_r$ from the real valued harmonic Hardy space $h^1_{\mathbb{R}}$ onto $L^1(\partial\mathbb{D})$ defined by

$$u \mapsto u\big(re^{i\theta}\big) - u(0).$$

This is the operator that maps a harmonic function u in $h^1_{\mathbb{R}}$ to the restriction of $u - u(0)$ to the circle of radius r. We show that this operator has norm $2 - \frac{4}{\pi}\arccos(r)$. Furthermore, the maximum in the definition of norm is attained, for example, by the Poisson kernel.

It turns out that the question of bounds for B and $\mathcal{B}$ on subspaces of L^1 are related to questions about concentrations of functions. Consider for example the space $h^1_{\mathbb{R}}$. Let $f_n(z)$ be the function in $h^1_{\mathbb{R}}$ with boundary values given by

$$f_n\big(e^{i\theta}\big) = \begin{cases} \pi n & \text{if } -1/n \le \theta \le 1/n, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Then $f_n(0) = 1$ and $\|f_n(z) - f_n(0)\|_{h^1_{\mathbb{R}}} = 2 - (4/n)$. This shows that the norm of $\mathcal{B}$ on $h^1_{\mathbb{R}}$ is 2. Notice that the boundary functions of the f_n have most of their L^1 norm concentrated on sets of small measure. Moreover, their sign does not oscillate on these sets. Contrast these functions with the functions g_n with boundary values given by

$$g_n\big(e^{i\theta}\big) = \begin{cases} -\pi n & \text{if } -1/n \le \theta < 0, \\ \pi n & \text{if } 0 \le \theta \le 1/n, \\ 0 & \text{otherwise.} \end{cases}$$

The g_n also have boundary values concentrated on sets of small measure, but the sign of their boundary values oscillates, which allows $g_n(0)$ to be 0. Thus, $\|g(z) - g(0)\|_{h^1_{\mathbb{R}}} = 1$. Roughly speaking, for $\mathcal{B}f$ to have large norm, f should have most of its mass concentrated on a set of small measure, and its sign should not oscillate much.

For a related example, we define the Poisson kernel by

$$P_r\big(e^{it}\big) = \frac{1 - r^2}{1 - 2r\cos(t) + r^2}.$$

If $z = re^{it}$, then $P_r(t)$ is a harmonic function of z, which we may denote here by $P(z)$. It is not difficult to see that $P(0) = 0$ and $\|P - 1\|_{h^1_{\mathbb{R}}} = 2$. This is related to the fact that the Poisson kernel is a Poisson integral of a point mass, so that P is as concentrated as possible in some sense. This example also hints at that the fact that $\|\mathcal{B}\|_{H^1} < 2$, because of the fact that the analytic completion of the Poisson kernel is not in H^1. Thus, we might expect that there is some limit to the concentration of boundary values of H^1 functions. Related to this is the fact that it is not difficult to find an analytic function in H^1 that is large only on a set of small measure - for example $c_n(1+z)^n/2^n$, where c_n is chosen so that the function has norm 1. However, such functions oscillate in sign near the points where they are large.

In order to formalize the above observations, we prove two different theorems about concentration of of analytic functions on sets of small measure. Suppose $\|f\|_{H^1} = 1$. In the first theorem, we prove that if $A \subset \mathbb{T}$ and $\int_A \operatorname{Re} f(0)\, d\theta/(2\pi) > 1-\varepsilon$ for small enough ε, then $m(A)$ cannot be too small, where m is the arc length measure on the unit circle. The second says that if $\|f\|_{L^1(A)} > 1-\varepsilon$ for some small enough set A and for small enough ε, then $f(0)$ cannot be too large. We provide two proofs that $\|\mathcal{B}\|_{H^1} < 2$, where each proof uses one of the above theorems.

2. Bounds for Hardy spaces

For a continuous function f in the unit disc $\mathbb{D}$, we define the pth integral mean of f at radius r by

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta\right)^{1/p}$$

if $0 < p < \infty$ and we define $M_\infty(r,f) = \sup_{0\le\theta<2\pi} |f(re^{i\theta})|$.

Denote by H^p the space of analytic functions in the unit disc such that $\|f\|_{H^p} = \sup_{0\le r<1} M_p(r,f) < \infty$. We let $h^p_{\mathbb{R}}$ be the space of real valued harmonic functions in the unit disc such that $\|f\|_{H^p} = \sup_{0\le r<1} M_p(r,f) < \infty$. Note that $M_p(r,f)$ is increasing for $0 < p \le \infty$ if f is analytic and for $1 \le p \le \infty$ if f is harmonic (see [2, Theorems 1.5 and 1.6]). Functions in H^1 have radial limits almost everywhere on the boundary of the unit disc, and they are uniquely determined by their boundary value functions. In fact, the norm of an H^1 function is equal to the L^1 norm of its boundary function. In contrast to this, $h^1_{\mathbb{R}}$ functions, even though they have radial limits almost everywhere, are not uniquely determined by their boundary values. However, they can be written as convolutions of Poisson kernels with finite Radon measures (see [2]).

We define A^p to be the subspace of L^p of the unit disc (with normalized area measure) consisting of analytic functions. Let $a^p_{\mathbb{R}}$ be the (real) subspace of L^p consisting of real valued harmonic functions.

In this section, we discuss bounds for the operators B and $\mathcal{B}$ on H^1 and also on H^∞. We begin with two theorems. The first roughly says that an analytic function in H^1 that has most of the mass of its boundary value function concentrated on a small set must show an appreciable degree of cancellation if the function is integrated over the set. The second theorem is similar, but deals instead with the integral of the function over all of the unit circle.

THEOREM 2.1. *Suppose that* $\|f\|_{H^1}=1$ *and that for some set* $A\subset\mathbb{T}$, *we have* $\int_A \operatorname{Re} f\,dt/2\pi \geq 1-\varepsilon$ *for* $\varepsilon<1/4$. *Then*

$$m(A)\geq \max_{0<\gamma<1}\frac{\log(\gamma+\varepsilon)-\log(1-2\varepsilon)}{\log\gamma}.$$

Proof. Let

$$F(z)=\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log\psi(t)\,dt\right\},$$

where

$$\psi(t)=\begin{cases}\gamma & \text{on } A,\\ 1 & \text{on } A^c\end{cases}$$

and $0<\gamma<1$. Then $F(0)$ is real and $\log(F(0))=\log(\gamma)\cdot m(A)$, and thus $F(0)=\gamma^{m(A)}$. Also, $|F(e^{i\theta})|=1$ a.e. if $e^{i\theta}\in A^c$, and $|F(e^{i\theta})|=\gamma$ a.e. if $e^{i\theta}\in A$ a.e. Now note that

$$\left|\int_0^{2\pi} f\left(e^{it}\right)\frac{dt}{2\pi}\right|\geq\left|\int_A \operatorname{Re} f\left(e^{it}\right)\frac{dt}{2\pi}\right|-\int_{A^c}\left|f\left(e^{it}\right)\right|\frac{dt}{2\pi}\geq(1-\varepsilon)-\varepsilon.$$

Thus, $|f(0)|\geq 1-2\varepsilon$ and

$$\left|F(0)\cdot f(0)\right|\geq\gamma^{m(A)}(1-2\varepsilon).$$

But also

$$\left|F(0)\cdot f(0)\right|=\left|\int F\cdot f\left(e^{i\theta}\right)\frac{dt}{2\pi}\right|\leq\gamma\int_A|f|\frac{dt}{2\pi}+\int_{A^c}|f|\frac{dt}{2\pi}\leq\gamma+\varepsilon.$$

Thus,

$$\gamma^{m(A)}(1-2\varepsilon)\leq\gamma+\varepsilon.$$

But this means

$$m(A)\geq\frac{\log(\gamma+\varepsilon)-\log(1-2\varepsilon)}{\log\gamma}.$$

To see the maximum of this quantity for $0<\gamma<1$ is attained, notice that the expression on the right of the above inequality approaches 0 as $\gamma\to0^+$ and approaches $-\infty$ as $\gamma\to1^-$, but is positive for $\gamma=\varepsilon$. □

In the next theorem, we let m denote normalized arc length measure.

THEOREM 2.2. *Suppose that* $f\in H^p$ *and that* $\|f\|_{H^p}=1$. *Furthermore, suppose that* $\|f\|_{L^1(E)}\geq1-\varepsilon$ *for some set* $E\subset\mathbb{T}$ *where* $m(E)\leq\delta$ *and* $0<\varepsilon,\delta<1/2$. *Then*

$$\left|f(0)\right|\leq\left(\frac{1-\varepsilon}{\delta}\right)^{\delta}\left(\frac{\varepsilon}{1-\delta}\right)^{1-\delta}.$$

Proof. Let $m(E)=\delta_0$ and $\|f\|_{L^1(E)}=1-\varepsilon_0$. Writing f as the product of an outer function and an inner function (see [2]) shows that it suffices to assume that f is outer. Then we may assume that

$$f(z)=\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log\psi(t)\,dt\right\}$$

for some nonnegative function $\psi\in L^p$ such that $\log\psi\in L^1$. Also $|f(e^{it})|=\psi(t)$ a.e. Note that $f(0)=\exp(\frac{1}{2\pi}\int_0^{2\pi}\log\psi(t)\,dt)$. Now, Jensen's inequality shows that

$$\begin{aligned}\exp\left(\int_E\log\psi(t)\frac{dt}{2\pi}\right)&=\exp\left(\int_E m(E)\log\psi(t)\frac{dt}{2\pi m(E)}\right)\\&=\left\{\exp\left(\int_E\log\psi(t)\frac{dt}{2\pi m(E)}\right)\right\}^{m(E)}\\&\le\left\{\int_E\psi(t)\frac{dt}{2\pi m(E)}\right\}^{m(E)}\\&=\left(\frac{1-\varepsilon_0}{\delta_0}\right)^{\delta_0}.\end{aligned}$$

A similar calculation for the set E^c shows that

$$\exp\left(\int_{E^c}\log\psi(t)\frac{dt}{2\pi}\right)\le\left(\frac{\varepsilon_0}{1-\delta_0}\right)^{1-\delta_0}.$$

Putting this together gives

$$f(0)=\exp\left(\int_{E\cup E^c}\log\psi(t)\frac{dt}{2\pi}\right)\le\left(\frac{1-\varepsilon_0}{\delta_0}\right)^{\delta_0}\left(\frac{\varepsilon_0}{1-\delta_0}\right)^{1-\delta_0}.$$

Since the function $(1-\varepsilon_0)^x\varepsilon_0^{1-x}$ is increasing, it follows that $(1-\varepsilon_0)^{\delta_0}\varepsilon_0^{1-\delta_0}\le(1-\varepsilon_0)^{\delta}\varepsilon_0^{1-\delta}$. Since the function $x(1-x)$ is increasing for $0<x<1/2$, we have

$$(1-\varepsilon_0)^\delta\varepsilon_0^{1-\delta}=(1-\varepsilon_0)^\delta\varepsilon_0^\delta\varepsilon_0^{1-2\delta}\le(1-\varepsilon)^\delta\varepsilon^\delta\varepsilon^{1-2\delta}=(1-\varepsilon)^\delta\varepsilon^{1-\delta}.$$

Since the function $x^x(1-x)^{1-x}$ is decreasing for $0\le x<1/2$, we have

$$\delta_0^{-\delta_0}(1-\delta_0)^{\delta_0-1}\le\delta^{-\delta}(1-\delta)^{\delta-1}$$

if $0<\delta_0\le\delta<1/2$. Putting this together gives the result. □

We now use Theorem 2.1 to bound $\|B\|_{H^1}$.

THEOREM 2.3. *The norm of B (and of $\mathcal{B}$) on the Hardy space H^1 is at most* 1.952396.

Proof. Let m denote Lebesgue measure divided by 2π on the unit circle. Suppose that $\|f\|_{H^1} = 1$. Without loss of generality, we may assume that $f(0) > 0$. Suppose that $\|f - f(0)\|_{H^1} > 2 - \alpha$ for some $0 < \alpha < 1/2$. We will show that this leads to a contradiction for small enough α. Note that $f(0) > 1 - \alpha$ and $|f(0)| \leq 1$.

Now consider $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$. Let $0 < \beta < 1/2$. Define $A = \{e^{i\theta} : u > \beta\}$ and $B = \{e^{i\theta} : u \leq \beta\}$. Now if $u(e^{i\theta}) \geq u(0)$ we have $|u(e^{i\theta}) - u(0)| \leq |u(e^{i\theta})|$ so $|f(e^{i\theta}) - f(0)| \leq |f(e^{i\theta})|$ since $f(0) = u(0)$ is real.

However if $\beta < u(e^{i\theta}) < u(0)$ we have $|f(e^{i\theta}) - f(0)| = |u(0) - u(e^{i\theta}) + iv(e^{i\theta})| \leq 1 - \beta + |f(e^{i\theta})|$. So if $e^{i\theta} \in A$ we have $|f(e^{i\theta}) - f(0)| \leq |f(e^{i\theta})| + 1 - \beta$. Thus

$$\int_A \big|f - f(0)\big|\, dm \leq \int_A |f|\, dm + (1-\beta)m(A).$$

And if $e^{i\theta} \in B$, we have $|f(e^{i\theta}) - f(0)| \leq |f(e^{i\theta})| + 1$. Thus

$$\int_B \big|f - f(0)\big|\, dm \leq \int_B |f|\, dm + m(B).$$

Therefore,

$$\begin{aligned} 2 - \alpha < \int \big|f - f(0)\big|\, dm &\leq \int |f|\, dm + (1-\beta)m(A) + m(B) \\ &= 1 + (1-\beta)m(A) + m(B). \end{aligned}$$

And therefore $(1-\beta)m(A) + m(B) > 1 - \alpha$. But $m(A) + m(B) = 1$ so $-\beta m(A) \geq -\alpha$ so

$$m(A) \leq \frac{\alpha}{\beta}.$$

But it is also clear that $\int_A u\, dm + \beta > \int u\, dm > 1 - \alpha$. By Theorem 2.1 we have

$$\alpha/\beta \geq \max_{0<\gamma<1} \frac{\log(\gamma + (\alpha+\beta)) - \log(1 - 2(\alpha+\beta))}{\log \gamma}$$

as long as $\alpha + \beta < 1/4$. However, this is false for $\alpha = 0.047604$ and $\beta = 0.127079$, as can be seen by taking $\gamma = 0.104634$. ☐

Similarly to the above theorem, we now use Theorem 2.2 to bound $\|B\|_{H^1}$. This yields a better bound than the previous result.

Theorem 2.4. *The norm of B (and of $\mathcal{B}$) on H^1 is at most* 1.7047.

Proof. Suppose that $\|f - f(0)\|_{H^1} > 2 - \alpha$ for $0 < \alpha < 1/2$. Then $|f(0)| > 1 - \alpha$ and we may assume without loss of generality that $f(0)$ is positive. Now consider $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$. Let $0 < \beta < 1/2$. Define $A = \{e^{i\theta} : u > \beta\}$

and $B = \{e^{i\theta} : u \le \beta\}$. By the reasoning in the proof of the previous theorem, $m(A) \le \alpha/\beta$ and $\int_A |f|\,dm \ge \int_A u\,dm > 1-\alpha-\beta$. So by Theorem 2.2,

$$1-\alpha < \bigl|f(0)\bigr| \le \left(\frac{1-(\alpha+\beta)}{\alpha/\beta}\right)^{\alpha/\beta} \left(\frac{\alpha+\beta}{1-\alpha/\beta}\right)^{1-(\alpha/\beta)}.$$

However, the above inequality is false if $\alpha = 0.295302$ and $\beta = 0.476286$. □

In contrast to the case with H^1, we show that the norm of the backward shift operator on H^∞ is exactly 2.

THEOREM 2.5. *The norm of B (and also of $\mathcal{B}$) on H^∞ is* 2.

Proof. If we let $\phi_a(z) = \frac{a-z}{1-\overline{a}z}$ for $0 < a < 1$, then $\phi_a(1) = -1$ and $\phi_a(0) = a$ and $\|\phi_a\|_\infty = 1$. But by the continuity of these functions in $\overline{D}$, we have

$$\|B\phi_a\|_\infty = \|\phi_a - a\|_\infty \ge \bigl|\phi_a(1) - a\bigr| = 1 + a.$$

Since this can be made as close to 2 as we like by letting $a \to 1$, we see that $\|B\| = 2$. □

We remark that a proof could also be given by an argument involving conformal maps.

3. Bergman spaces

We first prove a theorem relating the norm of $\mathcal{B}$ on Bergman spaces to its norm on Hardy spaces.

THEOREM 3.1. *Suppose that the norm of the operator $\mathcal{B}$ is equal to K on H^p. Let μ be a radial weight such that $\mu(\mathbb{D}) < \infty$. Then the norm of $\mathcal{B}$ is at most K on the Bergman space $A^p(\mu)$.*

Proof. Let $\mu\,dA = \widetilde{\mu}(r)2r\,dr\,d\theta$. Note that $M_p(r, f - f(0)) \le K M_p(r, f)$ by the Hardy space bound applied to the dilation $f_r(z)$. Thus,

$$\begin{aligned}\bigl\|f - f(0)\bigr\|^p_{A^p(\mu)} &= \int_0^1 M_p^p\bigl(r, f - f(0)\bigr)2r\,d\widetilde{\mu}(r) \le K^p \int_0^1 M_p^p(r, f)2r\,d\widetilde{\mu}(r)\\ &= K^p \|f\|^p_{A^p(\mu)}.\end{aligned}$$

□

We now prove a similar theorem to the one above, but for B. This theorem is slightly more difficult to prove since dividing by z can increase the norm of a Bergman space function.

THEOREM 3.2. *Suppose that the norm of the operator B is equal to K on H^1. Let μ be a finite radial weight that is increasing. Then the norm of B is at most $2K$ on the Bergman space $A^1(\mu)$.*

Proof. This theorem follows immediately from Theorem 3.1 and the following lemma. □

LEMMA 3.3. *Let μ be an increasing radial measure. Then* $\|zf\|_{A^1(\mu)} \geq (1/2)\|f\|_{A^1(\mu)}$.

Proof. Let $\mu\, dA = \widetilde{\mu}(r)2r\, dr\, d\theta$. Note that

$$\begin{aligned}\int_0^1 & M_1(r,f)2r\widetilde{\mu}(r)\, dr \\ &= \int_0^{1/2} M_1(r,f)2r + M_1(1-r,f)2(1-r)\widetilde{\mu}(1-r)\, dr \\ &= \int_0^{1/2} M_1(r,f)4(1/2)r + M_1(1-r,f)4(1/2)(1-r)\widetilde{\mu}(1-r)\, dr.\end{aligned}$$

Now note that $M_1(r,f) \leq M_1(1-r,f)$ and $\widetilde{\mu}(r) \leq \widetilde{\mu}(1-r)$ for $0 \leq r \leq 1/2$ to see that the last displayed expression is at most

$$\begin{aligned}\int_0^{1/2} & M_1(r,f)4r^2\widetilde{\mu}(r) + M_1(1-r,f)4(1-r)^2\widetilde{\mu}(1-r)\, dr \\ &= 2\int_0^1 M_1(r,zf)2r\widetilde{\mu}(r)\, dr.\end{aligned}$$

□

Using a different method, we now establish a bound for $\mathcal{B}$ on the real harmonic Bergman space $a^1_{\mathbb{R}}$.

THEOREM 3.4. *The norm of the backward shift on the real harmonic Bergman space $a^1_{\mathbb{R}}$ is at most* 1.835. *In fact, the same estimate holds on any subspace X of L^1 with the property that $u \in X$ implies that $|u(re^{i\theta})| \leq (1-r)^{-2}$ and the property that the average value of any $u \in X$ on circles centered at the origin is constant.*

Proof. Suppose that there is a $u \in a^1_{\mathbb{R}}$ with $\|u\| = 1$ and $\|u - u(0)\| > 2 - \alpha$, where $0 < \alpha < 1/2$. This implies that $u(0) > 1 - \alpha$. Without loss of generality, assume $u(0) > 0$. Choose β such that $0 < \beta < 1/2$ and define $A = \{z : u > \beta\}$ and $B = \{z : u \leq \beta\}$.

Now, we have that $|u(z)| \leq (1-r)^{-2}$ (see [3, Chapter 1, Theorem 1]). Let $A_r = A \cap \{z : |z| = r\}$ and define B_r similarly.

Let m denote normalized area measure and let m_r denote Hausdorff 1-measure on the circle of radius r divided by $2\pi r$. Since

$$\int_{A_r} u\, dm_r + \beta > \int u\, dm_r = u(0) > 1 - \alpha$$

we have $\int_{A_r} u\, dm_r > 1 - \alpha - \beta$.

Notice that $\int_{B_r} u\, dm_r \leq m_r(B_r)\beta = (1 - m_r(A_r))\beta$ and $\int_{A_r} u\, dm_r \leq m_r(A_r)/(1-r)^2$. Since $\int_{A_r} u\, dm_r + \int_{B_r} u\, dm_r = u(0)$, we have that

$$m_r(A_r) \geq \frac{u(0) - \beta}{(1-r)^{-2} - \beta},$$

and thus

$$m_r(A_r) \geq \frac{1-\alpha-\beta}{(1-r)^{-2}-\beta}.$$

Therefore

$$m(A) = \int_0^1 m(A_r) 2r\, dr \geq \frac{1-\alpha-\beta}{\beta}\left(\frac{1}{2\sqrt{\beta}}\ln\left(\frac{1+\sqrt{\beta}}{1-\sqrt{\beta}}\right) - 1\right).$$

However,

$$\int_B \big|u - u(0)\big|\, dm \leq \int_B |u|\, dm + \int_B u(0)\, dm = \int_B |u|\, dm + m(B)u(0).$$

If $z \in A$ we have $|u(z) - u(0)| \leq |u(z)| + u(0) - 2\beta$. Therefore,

$$\int_A \big|u - u(0)\big|\, dm \leq \int_A |u|\, dm + m(A)u(0) - 2m(A)\beta$$

and thus

$$\int \big|u - u(0)\big|\, dm \leq \int |u|\, dm + u(0) - 2m(A)\beta.$$

This implies that

$$\int \big|u - u(0)\big| - 1 - u(0) \leq -2m(A)\beta,$$

so

$$m(A) \leq \frac{1}{2\beta}\big(1 + u(0) - \big\|u - u(0)\big\|\big) \leq \frac{\alpha}{2\beta}.$$

Therefore

$$\frac{1-\alpha-\beta}{\beta}\left(\frac{1}{2\sqrt{\beta}}\ln\left(\frac{1+\sqrt{\beta}}{1-\sqrt{\beta}}\right) - 1\right) \leq \frac{\alpha}{2\beta}.$$

Choosing $\beta = 0.506$ and $\alpha = 0.165$ gives a contradiction. □

4. The norm of the operator $\mathcal{B}_r$

Suppose that f and g are functions defined on the interval $[a, b]$. By the convolution of f and g, we mean the function

$$f * g(x) = \frac{1}{b-a}\int_a^b \widetilde{f}(y)\widetilde{g}(x-y)\, dy,$$

where $\widetilde{f}$ and $\widetilde{g}$ are the periodic extensions of f and g to the real line.

We define the operator $\mathcal{B}_r : h^1_{\mathbb{R}} \to h^1_{\mathbb{R}}$ to be the operator $f \mapsto (\mathcal{B}f)_r$, where $(\mathcal{B}f)_r(z) = (\mathcal{B}f)(rz)$. Equivalently $\mathcal{B}_r$ can be thought of as the operator obtained by applying $\mathcal{B}$ and then restricting the function obtained to the circle centered at the origin with radius r. In this section, we investigate the norm of $\mathcal{B}_r$ and find that the Poisson kernel is a solution to the problem of finding a function $f \in h^1_{\mathbb{R}}$ of norm 1 such that $\|\mathcal{B}_r f\|$ is as large as possible.

In order to proceed, we need to prove several lemmas. The first is elementary but is surprisingly useful.

LEMMA 4.1. *Let f be a real function with average μ and let ν be a finite measure. Then*

$$\int |f-\mu|\,d\nu = 2\int_{\{x:f>\mu\}} (f-\mu)\,d\nu.$$

Proof. Note that $\int_{\{x:f=\mu\}}(f-\mu)\,d\nu = 0$. Since $\int (f-\mu)\,d\nu = 0$, this implies that $\int_{\{x:f>\mu\}}(f-\mu)\,d\nu = \int_{\{x:f<\mu\}} |f-\mu|\,d\nu$. □

The next theorem basically deals with maximizing $\int_{\{x:f>\mu\}} |f-\mu|$, where f is itself an average of rearrangements of some other function. However, we present the theorem in a discreet form which is easier to prove.

THEOREM 4.2. *Suppose we are given an $m\times n$ matrix A. Let μ be a fixed number, and let $C_j = \sum_{k=1}^m a_{kj}$. Define $D_j = \max(C_j-\mu, 0)$, and let $D = \sum_{j=1}^n D_j$. Suppose that $D_j \ge D_k$ but $a_{ij} \le a_{ik}$. Then we do not decrease D by interchanging a_{ij} and a_{ik}.*

Proof. We may assume without loss of generality that $D_1 \ge D_2$, but that $a_{11} \le a_{12}$. Let A' be the matrix formed by interchanging a_{11} and a_{12}. We claim that $D' \ge D$.

If $D=0$, then we are done. If $D_1>0$ and $D_2=0$, then since $D_1' \ge D_1$ and $D_2' \ge 0$, we are done. Suppose then that $D_1>0$ and $D_2>0$. If $D_2'>0$, then also $D_1'>0$ so $D_1+D_2 = D_1'+D_2' = C_1+C_2-2\mu$, so we are done. However, suppose that $D_2>0$ but $D_2'=0$. Let B_1 the the sum of the rest of the entries in column 1, and B_2 be the sum of the rest of the entries in column 2. Then $D_1+D_2 = a_{11}+a_{12}+B_1+B_2-2\mu$ and $D_1'+D_2' = a_{12}+B_1-\mu$. So we will have $D_1+D_2 \le D_1'+D_2'$ if $a_{11}+B_2 \le \mu$. But this is true since $D_2'=0$. □

We now have the following lemma, which is a continuous form of the above theorem. We omit the proof, since we prove a more general version in Lemma 4.5. Note that $P*1$ is the integral of P, and similarly for $P*f*1$ and $P*f$.

LEMMA 4.3. *Suppose that P and f are nonnegative integrable functions on $[a,b]$ and that $\|f\|_1 = 1$, where $\|\cdot\|_1$ denotes the L^1 norm with normalized Lebesgue measure. Then $\|P*f - P*f*1\|_1 \le \|P - P*1\|_1$.*

COROLLARY 4.4. *Suppose that u is a nonnegative harmonic function in the unit disc and that $0\le r<1$. Then*

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \left|u\left(re^{i\theta}\right)-u(0)\right|d\theta &\le u(0)\frac{1}{2\pi}\int_0^{2\pi}\left|P_r\left(e^{i\theta}\right)-1\right|d\theta\\ &= u(0)\left(2-\frac{4}{\pi}\arccos(r)\right),\end{aligned}$$

where P_r is a Poisson kernel.

Note that this corollary holds for $r=1$ if we replace $\frac{1}{2\pi}\int_0^{2\pi}|u(e^{i\theta})-u(0)|\,d\theta$ by $\|u-u(0)\|_{h^1_{\mathbb{R}}}$.

Proof. The rightmost equality is proved in Theorem 4.7. Also note that the inequality holds for $r=0$ trivially. For other values of r, if u is the real part of a function in H^1, then we can write $u(re^{i\theta})=P_\theta(re^{i\cdot})*f(e^{i\theta})$, where f is the boundary value function of u. The result then follows from the above lemma by letting $P=P_r$.

If u is not the real part of an H^1 function, then u is still in $h^1_{\mathbb{R}}$ since it is nonnegative (see [2, Theorem 1.1]). Let $0<s<1$. Define u_s by $u_s(z)=u(sz)$. Then u_s is the real part of an H^1 function, since it is actually continuous in $\mathbb{D}$. So for fixed r, the above inequality is true for u_s. (Note that $u_s(0)=u(0)$). If we let $s\to 1$, we get the result for u, since $u(rse^{it})\to u(re^{it})$ as $s\to 1$ uniformly for $t\in[0,2\pi)$. □

We must now deal with functions that are allowed to be negative.

LEMMA 4.5. *Suppose that P is a nonnegative integrable function on $[\alpha,\beta]$ and f is an integrable function such that $\|f\|_1=1$, where $\|\cdot\|_1$ denotes the L^1 norm with normalized Lebesgue measure. Let P^* denote the decreasing rearrangement of P, so that $P^*(t)=\inf\{x : m(\{y\in[\alpha,\beta] : P(y)>x\})\le t\}$. Then $\|P*f-P*f*1\|_1\le\|Q-Q*1\|_1$ where Q is some function of the form $aP^*(x)-bP^*(\alpha+\beta-x)$, where $a+b=1$.*

Proof. Without loss of generality, we may assume that $\alpha=0$ and $\beta=1$.

We may assume that P and f are continuous (and thus uniformly continuous), since these functions are dense in $L^1([0,1])$. Let $\varepsilon>0$ and assume without loss of generality that $\varepsilon<1$. Let $M=\|f\|_\infty+\|P\|_\infty+1$.

Approximate P and f by the step functions

$$\widetilde{P}=\sum_{k=0}^{n-1}c_k\chi_{[k/n,(k+1)/n)}\quad\text{and}$$
$$\widetilde{f}=\sum_{k=0}^{n-1}d_k\chi_{[k/n,(k+1)/n)}$$

respectively, so that

$$\int_{k/n}^{(k+1)/n}\widetilde{f}\,dx=\int_{k/n}^{(k+1)/n}f\,dx\quad\text{and}\quad\int_{k/n}^{(k+1)/n}\widetilde{P}\,dx=\int_{k/n}^{(k+1)/n}P\,dx$$

for each $0\le k\le n-1$. We may choose n large enough so that $|f(x)-f(y)|<\varepsilon$ if $|x-y|\le 1/n$ and $\|P-\widetilde{P}\|_\infty<\varepsilon$ and $\|f-\widetilde{f}\|_\infty<\varepsilon$. Thus, $\|P*f-\widetilde{P}*\widetilde{f}\|_\infty<M\varepsilon$. Also $\|P^*-\widetilde{P}^*\|_\infty<\varepsilon$ because decreasing rearrangement decreases the L^∞ distance between two functions.

For k not in $[0,n-1]$, define c_k and d_k so the sequences $\{c_k\}_{k\in\mathbb{Z}}$ and $\{d_k\}_{k\in\mathbb{Z}}$ are periodic with period n. Let A be the matrix with (i,j) entry

$a_{i,j} = c_{j-i} d_i$. Then the column sums C_j of A are equal to $n\widetilde{P} * \widetilde{f}(j/n)$. Define $D_j = \max(C_j - n, 0)$ and let D be the sum of the D_j.

Now, let A' be the matrix A with rows rearranged so that they are in decreasing order. Define C'_j and D'_j similarly. Then $C'_j = n\widetilde{Q}(j/n)$, where $\widetilde{Q}$ has the form $a\widetilde{P}^*(x) - b\widetilde{P}^*(1-x)$, where $\widetilde{P}^*$ is the decreasing rearrangement of $\widetilde{P}$ and $a+b=1$. Define $Q(x) = aP^*(x) + bP^*(1-x)$. Then $\|Q - \widetilde{Q}\| \le \varepsilon$.

Because $|c_j - c_{j+1}| < \varepsilon$ and $|d_j - d_{j+1}| < \varepsilon$ we have $|\widetilde{Q}(j/n+\delta) - \widetilde{Q}(j/n)| < \varepsilon$ and $|\widetilde{P} * \widetilde{f}(j/n+\delta) - \widetilde{P} * \widetilde{f}(j/n)| < M\varepsilon$ for $|\delta| < 1/n$.

Let $\mu = a - b$. Thus

$$\left|\max\big(\widetilde{P} * \widetilde{f}(j/n) - \mu, 0\big) - \max\big(\widetilde{P} * \widetilde{f}(j/n+\delta) - \mu, 0\big)\right| < M\varepsilon \quad \text{and}$$
$$\left|\max\big(\widetilde{Q}(j/n) - \mu, 0\big) - \max\big(\widetilde{Q}(j/n+\delta) - \mu, 0\big)\right| < \varepsilon.$$

Integrating each of these expressions over $\delta \in [0, 1/n]$ and summing from $j = 0$ to $n-1$ gives that

$$\left| D/n - \int_{\{x : \widetilde{P} * \widetilde{f}(x) > \mu\}} \widetilde{P} * \widetilde{f} - \mu \, dx \right| < \varepsilon$$

and

$$\left| D'/n - \int_{\{x : \widetilde{Q}(x) > \mu\}} \widetilde{Q}(x) - \mu \, dx \right| < M\varepsilon.$$

Theorem 4.2 with $\mu = a - b$ implies that $D \le D'$. Thus

$$\int_{\{x : \widetilde{P} * \widetilde{f}(x) > \mu\}} \widetilde{P} * \widetilde{f} - \mu \, dx \le (M+1)\varepsilon + \int_{\{x : \widetilde{Q}(x) > \mu\}} \widetilde{Q}(x) - \mu \, dx$$

and therefore

$$\int_{\{x : P * f(x) > \mu\}} P * f - \mu \, dx \le 2(M+1)\varepsilon + \int_{\{x : Q(x) > \mu\}} Q(x) - \mu \, dx.$$

Since this is true for any ε, we must have

$$\int_{\{x : P * f(x) > \mu\}} P * f - \mu \, dx \le \int_{\{x : Q(x) > \mu\}} Q(x) - \mu \, dx.$$

The result now follows from Lemma 4.1. □

It is useful to have a condition under which we can conclude that $Q(x) = P^*(x)$. The following theorem provides such a condition.

THEOREM 4.6. *Suppose that P is a continuous function on $[\alpha, \beta]$ and that P is nonnegative with average* 1 *and decreasing. Let $Q(x) = aP(x) - bP(\beta + \alpha - x)$ for some nonnegative real numbers a and b such that $a + b = 1$. Then there is a c between α and β such that $aP(c) - bP(\beta + \alpha - c) = a - b$. If for some such c,*

$$\int_{\alpha}^{\alpha+c} P \, dx + \int_{\beta-c}^{\beta} P \, dx \ge 2c,$$

then $\|P - P*1\|_1 \geq \|Q - Q*1\|_1$, *where* $\|\cdot\|_1$ *denotes the* L^1 *norm with normalized Lebesgue measure.*

Proof. We will suppose without loss of generality that $\alpha = 0$ and $\beta = 1$. Also we may suppose without loss of generality that $a \geq b$, since if $Q(x) = aP(x) - bP(1-x)$ and $\widehat{Q}(x) = bP(x) - aP(1-x)$, then $\widehat{Q}(x) = -Q(1-x)$, so $\|Q - Q*1\|_1 = \|\widehat{Q} - \widehat{Q}*1\|_1$. The average of $aP(x) - bP(1-x)$ is $a - b = 1 - 2b$. Let c be such that $aP(c) - bP(1-c) = 1 - 2b$. (Such a c exists by the integral mean value theorem.)

Suppose that $f(x)$ is nonnegative between 0 and c, and that ν is nonnegative. Let $A(\nu, f; c)$ denote the area below f and above ν and between 0 and c. Let $\widetilde{A}(\nu, f; c)$ be 0 if $f(c) \leq \nu$ and be $f(c) - \nu$ if $f(c) \geq \nu$. Then $A(\nu, f; c) = \int_0^c \widetilde{A}(\nu, f; x)\, dx$.

Let $\mu = a - b$. Note that between 0 and c, the function Q always has a value of at least μ, so $A(\mu, Q; c) = \int_0^c Q(x) - \mu\, dx$. Similarly, we have that $A(\mu, P; c) = \int_0^c P(x) - \mu\, dx$. Also $P(x) = Q(x) + bP(x) + bP(1-x)$. Thus $A(\mu, P; c) = A(\mu, Q; c) + b\int_0^c P\, dx + b\int_{1-c}^1 P\, dx$.

If $P(x) \geq 1$ observe that $P(x) - 1 = P(x) - \mu - 2b$ since $\mu = 1 - 2b$. So in this case $\widetilde{A}(1, P; c) = \widetilde{A}(\mu, P; c) - 2b$. If however $P(x) \leq 1$ then $\widetilde{A}(1, P; c) = 0$ and $\widetilde{A}(\mu, P; c) \leq 2b$ since $1 - \mu = 2b$. Thus in either case $\widetilde{A}(1, P; c) \geq \widetilde{A}(\mu, P; c) - 2b$. This implies that $A(1, P; c) \geq A(\mu, P; c) - 2bc$. And thus

$$A(1, P; c) \geq A(\mu, Q; c) + b\int_0^c P\, dx + b\int_{1-c}^1 P\, dx - 2bc.$$

So we will have $A(1, P; c) \geq A(\mu, Q; c)$ if

$$\int_0^c P\, dx + \int_{1-c}^1 P\, dx \geq 2c.$$

In this case, by Lemma 4.1,

$$\int_0^1 \big|P(x) - 1\big|\, dx = 2A(1, P; 1) \geq 2A(1, P; c) \geq 2A(\mu, Q; c).$$

Since $Q(c) = \mu$ and $Q(x) \leq \mu$ for $x > c$, we have that $2A(\mu, Q; c) = 2A(\mu, Q; 1) = \int_0^1 |Q(x) - \mu|\, dx$. □

We are now able to prove the following theorem.

THEOREM 4.7. *Suppose that* $0 \leq r < 1$ *and* $u \in h^1_{\mathbb{R}}$. *Then*

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \big|u(re^{i\theta}) - u(0)\big|\, d\theta &\leq \inf_{a\in\mathbb{R}} \|u - a\|_{h^1_{\mathbb{R}}} \frac{1}{2\pi}\int_0^{2\pi} \big|P_r(e^{i\theta}) - 1\big|\, d\theta \\ &= \inf_{a\in\mathbb{R}} \|u - a\|_{h^1_{\mathbb{R}}} \left(2 - \frac{4}{\pi}\arccos(r)\right),\end{aligned}$$

where P_r *is a Poisson kernel.*

Note that the theorem holds for $r=1$ if we replace $\frac{1}{2\pi}\int_0^{2\pi}|u(e^{i\theta})-u(0)|\,d\theta$ by $\|u-u(0)\|_{h^1_{\mathbb{R}}}$.

Proof. First note that it suffices to prove the statement for $a=0$, since $u(re^{i\theta})-u(0)=\widetilde{u}(re^{i\theta})-\widetilde{u}(0)$ if $\widetilde{u}=u-a$.

First assume that u is the real part of an H^1 function, and let f be its boundary value function. Let $P(\theta)$ be the Poisson kernel $P_r(e^{i\theta})$ restricted to $0\le\theta\le\pi$. Let $a,b\ge 0$ and $a+b=1$.

Let c be the number between 0 and π such that $aP(c)-bP(\pi-c)=a-b$. Such a c exists by the integral mean value theorem. Note that c is unique and a continuous function of a by the fact that $aP(c)-bP(\pi-c)$ is strictly decreasing and the implicit function theorem. Note that $P(\pi/2)=(1-r^2)/(1+r^2)<1$. If $a=1/2$, then $c=\pi/2$ since then $aP(\pi/2)-bP(\pi-(\pi/2))=0$. However, for $0\le a<1/2$, the number c is strictly less than $\pi/2$, because

$$aP(\pi/2)-bP\big(\pi-(\pi/2)\big)=(a-b)P(\pi/2)<(a-b).$$

Let

$$\alpha=\arg\big[\big(e^{ic}-r\big)/(1-r)\big]$$

and

$$\beta=\arg\big[(-1-r)/\big(e^{i(\pi-c)}-r\big)\big]=\arg\big[\big(e^{ic}+r\big)/(1+r)\big].$$

Now $\frac{1}{2\pi}\int_\gamma^\delta P(x)\,dx$ is equal to the harmonic measure of the arc of the unit circle $[e^{i\gamma},e^{i\delta}]$ at the point r, which equals $\phi/\pi-(\delta-\gamma)/(2\pi)$, where ϕ is the angle subtended at z by the arc (see [4, Chapter 1, Exercise 1]). Thus, $\frac{1}{2\pi}\int_0^c P(x)\,dx=\alpha/\pi-c/(2\pi)$. Also $\frac{1}{2\pi}\int_{\pi-c}^{\pi}P(x)\,dx=\beta/\pi-c/(2\pi)$. So the sum of the last two integrals is $(\alpha+\beta-c)/\pi$. We will show that this is at least $2c/2\pi$. To do so, we need $\alpha+\beta\ge 2c$. But

$$\frac{e^{ic}-r}{1-r}\,\frac{e^{ic}+r}{1+r}=\frac{e^{2ic}-r^2}{1-r^2}.$$

The argument of the last expression is measure of the angle with vertex r^2 and endpoints 0 and $2c$, which is at least $2c$ if $r>0$ and $0\le c\le\pi/2$. But we have shown above that $0\le c\le\pi/2$. So we always have $\int_0^c P(x)\,dx+\int_{\pi-c}^{\pi}P(x)\,dx\ge 2c$.

Let $\|\cdot\|$ denote the L^1 norm with normalized Lebesgue measure. Apply Theorem 4.6 to see that $\|Q-Q*1\|\le\|P-P*1\|$ for any Q of the form $aP(x)-bP(\pi-x)$ where $a,b\ge 0$ and $a+b=1$. If $\widetilde{P}(x)$ is defined on $[0,2\pi]$ by $\widetilde{P}(x)=P(x/2)$, and similarly for $\widetilde{Q}$, then this implies that $\|\widetilde{Q}-\widetilde{Q}*1\|\le\|\widetilde{P}-\widetilde{P}*1\|$.

Now let P^* be the decreasing rearrangement of $P_r(e^{i\theta})$, thought of as a function of θ, where $0\le\theta\le 2\pi$. Notice that Lemma 4.5 shows that $\|P_r*f-P_r*f*1\|\le\|\widehat{Q}-\widehat{Q}*1\|$, where $\widehat{Q}$ is some function of the form $aP_r^*(x)-bP_r^*(2\pi-x)$ and f is any continuous function on $[0,2\pi]$ that has norm 1. But for $0 \le x \le 2\pi$ one has $P_r^*(x) = P_r(x/2) = \widetilde{P}(x)$ since P_r is symmetric about 0. And thus we have shown above that $\|\widehat{Q} - \widehat{Q} * 1\|$ is at most $\|\widetilde{P} - \widetilde{P} * 1\|$. But $\widetilde{P} * 1 = P_r * 1 = 1$, where P_r is considered as a function on $[0, 2\pi]$. And also $P_r - 1$ is equimeasurable with $\widetilde{P} - 1$. And thus $\|P_r * f - P_r * f * 1\| \le \|P_r - P_r * 1\|$.

Now notice that $P_r(\theta) = 1$ if $\theta = \pm \arccos(r)$. Now

$$\frac{1}{2\pi} \int_{-\arccos(r)}^{\arccos(r)} P_r(\theta)\, d\theta = \alpha/\pi - 2\arccos(r)/(2\pi),$$

where α is the measure of the angle between $e^{-i\arccos(r)}$, r, and $e^{i\arccos(r)}$. But the measure of the angle is π. Thus the integral is $1 - \arccos(r)/\pi$. The (normalized) integral of P_r over the complementary interval is thus $\arccos(r)/\pi$.

Thus

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|P_r(\theta) - 1\right| d\theta &= \left[\left(1 - \frac{\arccos(r)}{\pi}\right) - \frac{\arccos r}{\pi}\right] \\ &\quad + \left[-\frac{\arccos(r)}{\pi} + \left(1 - \frac{\arccos(r)}{\pi}\right)\right] \\ &= 2 - \frac{4}{\pi}\arccos(r). \end{aligned}$$

This proves the result if u is the real part of an H^1 function.

Now suppose that u is not the real part of an H^1 function. As before, let u_s be defined by $u_s(z) = u(sz)$ for $0 < s < 1$. Then $\|u_s\|_{h^1_{\mathbb{R}}} \le \|u\|_{h^1_{\mathbb{R}}}$ since the M_1 integral means increase for harmonic functions (see [2]). So

$$\frac{1}{2\pi} \int_0^{2\pi} \left|u_s\left(re^{i\theta}\right) - u(0)\right| d\theta \le \|u\|_{h^1_{\mathbb{R}}} \left(2 - \frac{4}{\pi}\arccos(r)\right).$$

Letting $s \to 1$ gives the result. □

Corollary 4.8. *The value of* $\|\mathcal{B}\|_{h^1_{\mathbb{R}} \to a^1_{\mathbb{R}}} = 1$.

Proof. We have that

$$\|\mathcal{B}\|_{h^1_{\mathbb{R}} \to a^1_{\mathbb{R}}} \le \int_0^1 \|\mathcal{B}_r\| 2r\, dr = \int_0^1 \left(2 - \frac{4}{\pi}\arccos(r)\right) 2r\, dr = 1.$$

This is attained for the Poisson kernel, though of as the function $re^{i\theta} \mapsto P_r(e^{i\theta})$ defined in the unit disc. □

It would be interesting to find analogues of Theorem 4.7 for other values of p, and also to extend the result to complex valued harmonic functions. It would also be interesting to study similar questions for analytic functions instead of for complex valued harmonic functions.

We mention one partial result in this direction. If $\|f\|_{H^\infty} \le 1$, then $M_\infty(r, f - f(0)) \le 1$ for $r \le 1/3$. To see this, let $f(z) = \sum_{n=0}^\infty a_n z^n$ and let

$g(z) = \sum_{n=0}^{\infty} |a_n| z^n$. Bohr's theorem [1] says that $g(r) \le 1$ for $r \le 1/3$. Thus, if $|z| = r \le 1/3$ we have

$$\bigl|f(z) - f(0)\bigr| = \left|\sum_{n=1}^{\infty} a_n z^n\right| \le \sum_{n=1}^{\infty} |a_n| r^n \le \sum_{n=0}^{\infty} |a_n| r^n \le 1.$$

Acknowledgments. Thanks to the referee for pointing out a simpler proof of Theorem 2.5. Thanks to Dmitry Khavinson for alerting the author to the existence of Bohr's theorem.

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