THE HÖRMANDER MULTIPLIER THEOREM, I:
THE LINEAR CASE REVISITED

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ABSTRACT. We discuss $L^p(\mathbb{R}^n)$ boundedness for Fourier multiplier operators that satisfy the hypotheses of the Hörmander multiplier theorem in terms of an optimal condition that relates the distance $|\frac{1}{p} - \frac{1}{2}|$ to the smoothness s of the associated multiplier measured in some Sobolev norm. We provide new counterexamples to justify the optimality of the condition $|\frac{1}{p} - \frac{1}{2}| < \frac{s}{n}$ and we discuss the endpoint case $|\frac{1}{p} - \frac{1}{2}| = \frac{s}{n}$.

1. Introduction

To a bounded function σ on $\mathbb{R}^n$, we associate a linear multiplier operator

$$T_\sigma(f)(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)\sigma(\xi)e^{2\pi i x \cdot \xi}\,d\xi,$$

where f is a Schwartz function on $\mathbb{R}^n$ and $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x \cdot \xi}\,dx$ is its Fourier transform. The classical theorem of Mikhlin [12] (see also Stein [19]) states that if condition

$$(1) \qquad \qquad \left|\partial^\alpha \sigma(\xi)\right| \le C_\alpha |\xi|^{-|\alpha|}$$

holds for all multi-indices α with size $|\alpha| \le [n/2]+1$, then T_σ admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself for all $1 < p < \infty$.

Mikhlin's theorem was extended by Hörmander [10] to multipliers with fractional derivatives in some L^r space. To precisely describe this extension, let Δ be the Laplacian, let $(I - \Delta)^{s/2}$ denote the operator given on the Fourier

transform by multiplication by $(1 + 4\pi^2|\xi|^2)^{s/2}$ and for $s > 0$, and let L_s^r be the standard Sobolev space of all functions h on $\mathbb{R}^n$ with norm

$$\|h\|_{L_s^r} := \left\|(I - \Delta)^{s/2}h\right\|_{L^r} < \infty.$$

Let Ψ be a Schwartz function whose Fourier transform is supported in the annulus of the form $\{\xi : 1/2 < |\xi| < 2\}$ which satisfies $\sum_{j\in\mathbb{Z}} \widehat{\Psi}(2^{-j}\xi) = 1$ for all $\xi \neq 0$. Hörmander's extension of Mikhlin's theorem says that if $1 < r \leq 2$ and $s > n/r$, a bounded function σ satisfies

$$(2) \qquad\qquad \sup_{k\in\mathbb{Z}}\left\|\widehat{\Psi}\sigma(2^k\cdot)\right\|_{L_s^r} < \infty,$$

i.e., σ is uniformly (over all dyadic annuli) in the Sobolev space L_s^r, then T_σ admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself for all $1 < p < \infty$, and is also of weak type $(1,1)$. An endpoint result for this multiplier theorem involving a Besov space was given by Seeger [17]. The least number of derivatives imposed on the multiplier in Hörmander's condition (2) is when $r = 2$. In this case, under the assumption of $n/2 + \varepsilon$ derivatives in L^2 uniformly (over all dyadic annuli), we obtain boundedness of T_σ on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$. It is natural to ask whether L^p boundedness holds for some p if $s < n/2$.

Calderón and Torchinsky [2] used an interpolation technique to prove that if (2) holds, then the multiplier operator T_σ is bounded from $L^p(\mathbb{R}^n)$ to itself whenever p satisfies

$$(3) \qquad\qquad \left|\frac{1}{p} - \frac{1}{2}\right| < \frac{s}{n}$$

and

$$(4) \qquad\qquad \left|\frac{1}{p} - \frac{1}{2}\right| \leq \frac{1}{r}.$$

It is not hard to verify that if σ satisfies (2) and T_σ is bounded from $L^p(\mathbb{R}^n)$ to itself, then we must necessarily have $rs \geq n$; see Proposition 2.1. Thus, $\frac{1}{r} \leq \frac{s}{n}$ and comparing conditions (3) and (4) we notice that (4) restricts (3). On the other hand, if we only have conditions (2) and (3) for some r, s with $rs > n$, $r \in (1, \infty)$, $s \in (0, \infty)$, then one can find an r_o such that $|\frac{1}{p} - \frac{1}{2}| \leq \frac{1}{r_o} < \frac{s}{n}$ and $r_o < r$. In view of standard embeddings between Sobolev spaces[1] we obtain that

$$(5) \qquad\qquad \sup_{k\in\mathbb{Z}}\left\|\widehat{\Psi}\sigma(2^k\cdot)\right\|_{L_s^{r_o}} \leq C\sup_{k\in\mathbb{Z}}\left\|\widehat{\Psi}\sigma(2^k\cdot)\right\|_{L_s^r} < \infty,$$

and thus we can deduce the boundedness of T_σ on $L^p(\mathbb{R}^n)$ by the aforementioned Calderón and Torchinsky [2] result using the space $L_s^{r_o}$. So assumption (4) is not necessary.

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[1] This could be proved via the Kato–Ponce inequality $\|FG\|_{L_s^q} \leq C\|F\|_{L_s^{q_1}}\|G\|_{L_s^{q_2}}$, $1/q = 1/q_1 + 1/q_2$ with $q = r_o$ and $q_1 = r$; see [11], [8].

In this note, we show that (3) is optimal in the sense that within the class of multipliers σ for which (2) holds, if T_σ maps L^p to L^p, then we must necessarily have $|\frac{1}{p} - \frac{1}{2}| \leq \frac{s}{n}$.

THEOREM 1.1. *Fix $1 < r < \infty$ and $0 < s \leq \frac{n}{2}$ such that $rs > n$. Assume that (2) holds. Then T_σ maps $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$ such that $|\frac{1}{p} - \frac{1}{2}| < \frac{s}{n}$. Moreover, if T_σ is bounded from $L^p(\mathbb{R}^n)$ to itself for all σ such that (2) holds, then we must necessarily have $|\frac{1}{p} - \frac{1}{2}| \leq \frac{s}{n}$.*

The proof of the positive direction in Theorem 1.1 is mostly folklore, and is omitted. We only mention that the theorem could be proved via the interpolation result of Connett and Schwartz [5] or directly via the interpolation technique of Calderón and Torchinsky [2]; on this see also the presentation in Carbery, Gasper, and Trebels [3]. In this note, we focus on certain counterexamples related to the optimality of the hypotheses $rs > n$ and $|\frac{1}{p} - \frac{1}{2}| < \frac{s}{n}$ of Theorem 1.1; these are in the spirit of the presentation of Chapter 4 in the book of Wolff [21].

On the line $|\frac{1}{p} - \frac{1}{2}| = \frac{1}{r}$ there are positive results for $1 < p < 2$ (see [16]) and for $p = 1$ by Seeger [17]. In Section 3, we discuss a direct way to relate the results in the cases $p = 1$ and $1 < p < 2$ via direct interpolation that yields the following result as a consequence of the main theorem in [17].

PROPOSITION 1.2. *Given $0 \leq s \leq \frac{n}{2}$, $1 < p < 2$ satisfy $|\frac{1}{p} - \frac{1}{2}| = \frac{s}{n}$, then we have*

$$\|T_\sigma\|_{L^p \to L^{p,2}} \leq C \sup_{k \in \mathbb{Z}} \left\| \sigma(2^k \cdot) \widehat{\Psi} \right\|_{B^{s,1}_{\frac{n}{s}}}.$$

Here $L^{p,2}$ denotes the Lorentz space of functions f for which $t^{1/p} f^*(t)$ lies in $L^2((0, \infty), \frac{dt}{t})$, where f^* is the nondecreasing rearrangement of f; for the definition of the Besov space $B^{s,1}_{n/s}$ see the last section. Other types of endpoint results involving L^p norms as opposed to $L^{p,2}$ norms were provided by Seeger [18]. We are also aware of a direct proof of Proposition 1.2 based on weighted L^2 inequalities as in Christ [4].

2. Necessary conditions

In this section, we discuss examples that reinforce the minimality of the conditions on the indices in Theorem 1.1. One way to see this is to use the multiplier $m_{a,b}(\xi) = \psi(\xi)|\xi|^{-b} e^{i|\xi|^a}$ where $a > 0$, $a \neq 1$, $b > 0$, and ψ is a smooth function which vanishes in a neighborhood of the origin and is equal to 1 for large $|\xi|$. One can verify that $m_{a,b}$ satisfies (2) for $s = b/a$ and $r > n/s$. But it is known that $T_{m_{a,b}}$ is bounded in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $|\frac{1}{p} - \frac{1}{2}| \leq \frac{b/a}{n}$ (see Hirschman [9, comments after Theorem 3c], Wainger [20, Part II], and Miyachi [13, Theorem 3]). Alternative examples were given in Miyachi and Tomita [14, Section 7].

In this section, we provide yet new examples to indicate the necessity of the indices in Theorem 1.1. We are not sure as to whether boundedness into L^p, or even weak L^p, is valid in general under assumption (2) exactly on the critical line $|\frac{1}{p} - \frac{1}{2}| = \frac{s}{n}$.

PROPOSITION 2.1. *If for all $\sigma \in L^\infty(\mathbb{R}^n)$ such that*

$$\sup_{k \in \mathbb{Z}} \|\sigma(2^k \cdot) \widehat{\Psi}\|_{L_s^r(\mathbb{R}^n)} < \infty,$$

we have

(6) $$\|T_\sigma\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} \le C_p \sup_k \|\sigma(2^k \cdot)\widehat{\Psi}\|_{L_s^r(\mathbb{R}^n)} < \infty,$$

then we must necessarily have $rs \ge n$ and $|\frac{1}{p} - \frac{1}{2}| \le \frac{s}{n}$.

Proof. First, we prove the necessary condition $rs \ge n$. Let $\widehat{\zeta}$ be a smooth function supported in the ball $B(0, 1/10)$ in $\mathbb{R}^n$ and let $\widehat{\phi}$ be supported in the ball $B(0, 1/2)$ equal to 1 on $B(0, 1/5)$. Define $\widehat{f}(\xi) = \widehat{\zeta}(N(\xi - a))$ with $|a| = 1$, and $\sigma(\xi) = \widehat{\phi}(N(\xi - a))$. A direct calculation gives $\|f\|_{L^p(\mathbb{R}^n)} \approx N^{-n+n/p}$ and $\|\sigma\|_{L_s^r(\mathbb{R}^n)} \le CN^s N^{-n/r}$; for the last estimate see Lemma 2.2. Moreover, $T_\sigma(f)(x) = N^{-n}\zeta(x/N)e^{2\pi i x \cdot a}$. We thus obtain that $\|T_\sigma(f)\|_{L^p(\mathbb{R}^n)} \approx N^{-n+n/p}$. Then (6) yields the inequality $N^{-n+n/p} \le CN^s N^{-n/r}N^{-n+n/p}$, which forces $s - n/r \ge 0$ by letting N go to infinity. We note that the strict inequality in condition $rs > n$ in Theorem 1.1 is necessary as there exist unbounded functions in $L_{n/r}^r(\mathbb{R}^n)$, while Fourier multipliers are always bounded.

We now turn to the other necessary condition $|\frac{1}{p} - \frac{1}{2}| \le \frac{s}{n}$. By duality it suffices to prove the case when $1 < p \le 2$. We will prove our result by constructing an example. We consider the case $n = 1$ first while the higher dimensional case will be an easy generalization.

Let $\widehat{\psi}, \widehat{\varphi} \in \mathscr{C}_0^\infty(\mathbb{R})$ such that $0 \le \widehat{\varphi} \le \chi_{[-1/100,1/100]}$ and

$$\chi_{[-1/10,1/10]} \le \widehat{\psi} \le \chi_{[-1/2,1/2]}.$$

Therefore, $\widehat{\psi}\widehat{\varphi} = \widehat{\varphi}$. For a fixed large positive integer N, we define

(7) $$\widehat{f}_N(\xi) = \sum_{j=-N}^{N} \widehat{\varphi}(N\xi - j), \qquad \sigma_{N,t}(\xi) = \sum_{j \in J_N} a_j(t)\widehat{\psi}(N\xi - j),$$

where $J_N = \{j \in \mathbb{Z} : \frac{N}{2} \le |j| \le 2N\}$ and $t \in [0, 1]$. Here $\{a_j\}_{j=-\infty}^{\infty}$ is the sequence of Rademacher functions indexed by all integers.

One can verify that $T_{N,t}(f_N) =: (\sigma_{N,t}f_N)^\vee = (\sum_{j \in J_N} a_j(t)\widehat{\varphi}(N\xi - j))^\vee$. Recall that Rademacher functions satisfy for any $p \in (0, \infty)$

$$c_p \left\|\sum_j a_j(t)A_j\right\|_{L^p([0,1])} \le \left(\sum_j |A_j|^2\right)^{1/2} \le C_p \left\|\sum_j a_j(t)A_j\right\|_{L^p([0,1])},$$

where c_p and C_p are constants. Therefore,

$$\left(\int_0^1 \|T_{N,t}(f_N)\|_{L^p(\mathbb{R})}^p \, dt\right)^{1/p}$$

$$= \left(\int_0^1 \int_{\mathbb{R}} \left| \sum_{j \in J_N} a_j(t) N^{-1}\varphi(N^{-1}x)e^{2\pi i xj/N} \right|^p dx\, dt\right)^{1/p}$$

$$\approx \left(\int_{\mathbb{R}} \left(\sum_{j \in J_N} |N^{-1}\varphi(N^{-1}x)e^{2\pi i xj/N}|^2 \right)^{p/2} dx \right)^{1/p}$$

$$\approx N^{-1}\left(\int_{\mathbb{R}} |N^{1/2}\varphi(N^{-1}x)|^p \, dx\right)^{1/p}$$

$$\approx N^{1/p-1/2}.$$

The Sobolev norm of $\sigma_{N,t}$ is given by the following lemma, proved in all dimensions. For $n \geq 1$ and $\vec{t} = (t_1, \ldots, t_n) \in [0,1]^n$ we define a function on $\mathbb{R}^n$ by

$$\sigma_{N,\vec{t}}(\xi_1, \ldots, \xi_n) = \sum_{\vec{j} \in J_N} a_{j_1}(t_1) \cdots a_{j_n}(t_n) \widehat{\varphi}(N\xi_1 - j_1) \cdots \widehat{\varphi}(N\xi_n - j_n),$$

where $J_N = \{\vec{j} = (j_1, \ldots, j_n) \in \mathbb{Z}^n : \frac{N}{2} \leq |j_k| \leq 2N, 1 \leq k \leq N\}$. This $\sigma_{N,\vec{t}}$ coincides with $\sigma_{N,t}$ when $n = 1$.

LEMMA 2.2. *We have that* $\|\sigma_{N,\vec{t}}\|_{L_s^r(\mathbb{R}^n)} \leq CN^s$.

Proof. It is easy to verify that $\|\sigma_{N,\vec{t}}\|_{L^r} \leq C$ and $\|\sigma_{N,\vec{t}}\|_{L_2^r} \leq CN^2$. The rest follows by interpolation between Sobolev spaces [1]. $\square$

We continue with the proof of Proposition 2.1 when $n = 1$. We note that $\widehat{f}_N$ has L^q norm bounded by a constant independent of N, which implies by the Young's inequality that $\|f_N\|_{L^q} \leq C$ with C independent of N when $2 \leq q \leq \infty$. We show in the following lemma that this property is valid for all $q \in (1, \infty]$.

LEMMA 2.3. *Let f_N be as in* (7) *and let $p \in (1, \infty]$. Then there is a constant C_p independent of N such that* $\|f_N\|_{L^p} \leq C_p$.

Proof. We note that $f_N = \sum_{j=-N}^{N} \frac{1}{N}\varphi(x/N)e^{2\pi i xj/N} = \frac{1}{N}\varphi(x/N)D_N(x/N)$, where D_N is the Dirichlet kernel, whose L^p-norm over $[0,1]$ is comparable to $N^{1/p'}$ when $p > 1$; see, for example, [6, Exercise 3.1.6]. Using this fact and that φ is a Schwartz function we obtain

$$\|f_N\|_{L^p(\mathbb{R})} = \left\| \frac{1}{N}\varphi\left(\frac{\cdot}{N}\right)D_N\left(\frac{\cdot}{N}\right) \right\|_{L^p(\mathbb{R})}$$

$$= \frac{1}{N}N^{1/p}\|\varphi D_N\|_{L^p(\mathbb{R})}$$

$$= N^{-1/p'} \left(\sum_{j=-\infty}^{\infty} \int_{j-1}^{j} |\varphi(x) D_N(x)|^p \, dx \right)^{1/p}$$

$$\leq CN^{-1/p'} \left(\sum_{j=-\infty}^{\infty} \frac{1}{(1+|j|)^M} \int_{j-1}^{j} |D_N(x)|^p \, dx \right)^{1/p}$$

$$\leq C_p N^{-1/p'} N^{1/p'} = C_p.$$

This proves the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In view of Lemma 2.3, we obtain the following inequalities

$$N^{\frac{1}{p}-\frac{1}{2}} \leq C \left(\int_0^1 \|T_{N,t}(f_N)\|_{L^p(\mathbb{R})}^p \, dt \right)^{\frac{1}{p}}$$

$$\leq CA\|f\|_p \left(\int_0^1 \|\sigma_{N,t}\|_{L_s^r}^p \, dt \right)^{\frac{1}{p}}$$

$$\leq CC_p A N^s.$$

Letting N go to infinity forces $1/p - 1/2 \leq s$.

We now consider the higher dimensional case. Let $F_N(\vec{x}) = f_N(x_1) \cdots f_N(x_n)$, where f_N is as in (7). It follows from Lemmas 2.2 and 2.3 that $\|F_N\|_{L^p} \leq C$ and $\|\sigma_N\|_{L_s^r} \leq CN^s$. A calculation similar to the one dimensional case shows that $\|T_N(F_N)\|_{L^p} \approx N^{(1/p-1/2)n}$, thus letting $N \to \infty$ we obtain that $|1/p - 1/2| \leq s/n$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Related examples were given by Olevskii [15] who showed among other things that $L_{n/2}^2 \cap L^\infty$ is not contained in the space of L^p Fourier multipliers for $p \neq 2$.

3. The endpoint case $|\frac{1}{p} - \frac{1}{2}| = \frac{s}{n}$

In this section, we discuss an interpolation theorem applicable in the critical case $|\frac{1}{p} - \frac{1}{2}| = \frac{s}{n}$. We introduce the Besov space norm

$$\|h\|_{B_{p,q}^s} := \left(\sum_{j \geq 1} \|2^{js} \Delta_j h\|_{L^p}^q \right)^{\frac{1}{q}} + \|S_0 h\|_{L^p},$$

where Δ_j are the Littlewood–Paley operators and S_0 is an averaging operator that satisfy $S_0 + \sum_{j=1}^{\infty} \Delta_j = I$. We assume that for $j \geq 1$, Δ_j have spectra supported in the annuli $2^j \leq |\xi| \leq 2^{j+2}$, while S_0 has spectrum inside the ball $B(0,2)$.

We recall the following result of Seeger [17]

$$(8) \qquad\qquad \|T_\sigma\|_{H^1 \to L^{1,2}} \leq C \sup_{k \in \mathbb{Z}} \|\sigma(2^k \cdot) \widehat{\Psi}\|_{B_2^{\frac{n}{2},1}}$$

concerning the endpoint case $p = 1$. We also have the trivial estimate

$$(9) \qquad \|T_\sigma\|_{L^2 \to L^2} = \|T_\sigma\|_{L^2 \to L^{2,2}} \le C \sup_{k \in \mathbb{Z}} \|\sigma(2^k \cdot)\widehat{\Psi}\|_{B^{0,1}_\infty}.$$

In this section, we derive the intermediate estimate contained in Seeger [16]:

$$(10) \qquad \|T_\sigma\|_{L^p \to L^{p,2}} \le C \sup_{k \in \mathbb{Z}} \|\sigma(2^k \cdot)\widehat{\Psi}\|_{B^{\frac{n}{s},1}_{\frac{n}{s}}}$$

for $|\frac{1}{p} - \frac{1}{2}| = \frac{s}{n}$, $1 < p < 2$, and $0 \le s \le \frac{n}{2}$. We deduce (10) from the following theorem.

THEOREM 3.1. *Fix* $1 < r_0, r_1 \le \infty$, $1 < p_0, p_1 < \infty$, $0 \le s_0, s_1 < \infty$. *Let* $\widehat{\Psi}$ *be supported in the annulus* $1/2 \le |\xi| \le 2$ *on* $\mathbb{R}^n$ *and satisfy*

$$\sum_{j \in \mathbb{Z}} \widehat{\Psi}(2^{-j}\xi) = 1, \quad \xi \ne 0.$$

Assume that for $k \in \{0, 1\}$ *we have*

$$(11) \qquad \|T_\sigma(f)\|_{L^{p_k},2} \le K_k \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot)\widehat{\Psi}\|_{B^{s_k,1}_{r_k}} \|f\|_{L^{p_k}}$$

for all $f \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ *and* σ *which make the right hand side finite. For* $0 < \theta < 1$ *define*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \qquad s = (1-\theta)s_0 + \theta s_1.$$

Then there is a constant $C_* = C_*(r_0, r_1, s_0, s_1, p_0, p_1, p, n)$ *such that for all* f *in* $\mathscr{C}_0^\infty(\mathbb{R}^n)$ *we have*

$$(12) \qquad \|T_\sigma(f)\|_{L^{p,2}(\mathbb{R}^n)} \le C_* K_0^{1-\theta} K_1^\theta \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot)\widehat{\Psi}\|_{B^{s,1}_r} \|f\|_{L^p(\mathbb{R}^n)}.$$

Moreover, conclusion (12) *also holds under the assumption that* $p_0 = 1$ *and* (11) *is substituted (only for* $k = 0$*) by*

$$(13) \qquad \|T_\sigma(f)\|_{L^{1,2}} \le K_0 \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot)\widehat{\Psi}\|_{B^{s_0,1}_{r_0}} \|f\|_{H^1}$$

for all $f \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ *with vanishing integral.*

Proof. Let $\widehat{\Phi}(\xi) = \sum_{j \le 0} \widehat{\Psi}(2^{-j}\xi)$ and $\widehat{\Phi}(0) = 1$; then $\widehat{\Phi}$ is supported in $|\xi| \le 2$. Fix a bounded function σ. For an integer k define the dilation of σ^k by setting $\sigma^k(\xi) = \sigma(2^k\xi)$. For z in the closed unit strip, we introduce linear functions

$$L(z) = \frac{r}{r_0}(1-z) + \frac{r}{r_1}z, \qquad M(z) = s - (1-z)s_0 - zs_1$$

and when $j \ge 1$ introduce Littlewood–Paley operators $\Delta_j(g) = g * \Psi_{2^{-j}}$, $\widetilde{\Delta}_j(g) = g * \widetilde{\Psi}_{2^{-j}}$, where $\widetilde{\Psi}$ is a Schwartz function whose Fourier transform is supported in an annulus only slightly larger than $1/2 \le |\xi| \le 2$ and equals 1 on the support of $\widehat{\Psi}$. We also define $\Delta_0(g) = g * \Phi$ and $\widetilde{\Delta}_0(g) = g * \widetilde{\Phi}$, where

the Fourier transform of $\widetilde{\Phi}$ is supported in $|\xi| \leq 4$ and equals 1 on the support of $\widehat{\Phi}$. Then define:

$$
\begin{aligned}
\sigma_z = \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} 2^{jM(z)} \left(c_j^k\right)^{1-L(z)} \\
\times \widetilde{\Delta}_j \left(\left|\Delta_j\left(\sigma^k \widehat{\Psi}\right)\right|^{L(z)} e^{i\mathrm{Arg}(\Delta_j(\sigma^k \widehat{\Psi}))}\right) \left(2^{-k} \cdot\right) \widehat{\widetilde{\Psi}} \left(2^{-k} \cdot\right),
\end{aligned}
$$

where

$$
c_j^k = \left\|\Delta_j\left(\sigma^k \widehat{\Psi}\right)\right\|_{L^r} \left(\sup_{\mu \in \mathbb{Z}} \sum_{l \geq 0} 2^{ls} \left\|\Delta_l\left(\sigma^\mu \widehat{\Psi}\right)\right\|_{L^r}\right)^{-1}.
$$

Next, we estimate

$$
\text{(14)} \qquad \sup_{\mu \in \mathbb{Z}} \sum_{l \geq 0} 2^{ls_0} \left\|\Delta_l\left(\sigma_{it}^\mu \widehat{\Psi}\right)\right\|_{L^{r_0}}.
$$

We notice that for a given $\mu \in \mathbb{Z}$, in the sum defining σ_{it}^μ, only finitely many terms in k appear, the ones with $k = \mu, \mu+1, \mu-1$. For simplicity, we only consider the term with $k = \mu$, since the other ones are similar. This part of (14) is estimated by

$$
\begin{aligned}
\text{(15)} \qquad \sup_{\mu \in \mathbb{Z}} \sum_{l \geq 0} \sum_{j=0}^{\infty} 2^{ls_0} 2^{j(s-s_0)} \\
\times \left|c_j^\mu\right|^{1-\frac{r}{r_0}} \left\|\widetilde{\Delta}_l\left(\widetilde{\Delta}_j\left(\left|\Delta_j\left(\sigma^\mu \widehat{\Psi}\right)\right|^{L(it)} e^{i\mathrm{Arg}(\Delta_j(\sigma^\mu \widehat{\Psi}))}\right) \widehat{\Psi} \widehat{\widetilde{\Psi}}\right)\right\|_{L^{r_0}}.
\end{aligned}
$$

Using Lemma 3.2 (stated and proved below), we obtain that (15) is bounded by

$$
\begin{aligned}
\text{(16)} \qquad \sup_{\mu \in \mathbb{Z}} \sum_{l \geq 0} \sum_{j=0}^{\infty} 2^{ls_0} 2^{j(s-s_0)} \\
\times \left|c_j^\mu\right|^{1-\frac{r}{r_0}} C_M 2^{-2\left|1-\frac{1}{r_0}\right| \max(j,l)M} \left\|\left|\Delta_j\left(\sigma^\mu \widehat{\Psi}\right)\right|^{\frac{r}{r_0}}\right\|_{L^{r_0}}.
\end{aligned}
$$

But the sum over l in (16) is bounded by $C_M 2^{js_0} 2^{-j2\left|1-\frac{1}{r_0}\right|M} \leq C_M 2^{js_0}$ for M sufficiently large, and consequently (16) is bounded by

$$
\begin{aligned}
\text{(17)} \qquad C_M \sup_{\mu \in \mathbb{Z}} \sum_{j=0}^{\infty} 2^{j(s-s_0)} 2^{js_0} \left|c_j^\mu\right|^{1-\frac{r}{r_0}} \left\|\Delta_j\left(\sigma^\mu \widehat{\Psi}\right)\right\|_{L^r}^{\frac{r}{r_0}} \\
\leq C_M \left(\sup_{\mu \in \mathbb{Z}} \sum_{j \geq 0} 2^{js} \left\|\Delta_j\left(\sigma^\mu \widehat{\Psi}\right)\right\|_{L^r}\right)^{\frac{r}{r_0}}
\end{aligned}
$$

by the choice of c_j^μ. Likewise, we obtain a similar estimate for the point $1+it$. We summarize these two estimates as follows: for $m = 0, 1$ and $\mathrm{Re}\, z = m$ we

have

$$(18) \qquad \sup_{\mu \in \mathbb{Z}} \sum_{l \geq 0} 2^{ls_m} \big\| \Delta_l \big(\sigma_z^\mu \widehat{\Psi} \big) \big\|_{L^{r_m}} \leq C_M \bigg(\sup_{k \in \mathbb{Z}} \sum_{j \geq 0} 2^{js} \big\| \Delta_j \big(\sigma^k \widehat{\Psi} \big) \big\|_{L^r} \bigg)^{\frac{r}{r_m}}.$$

Now consider the analytic family of operators T_z associated with σ_z defined by $f \mapsto T_{\sigma_z}(f)$. When $\operatorname{Re} z = 0$, T_z maps $L^{p_0,2}$ to L^{p_0} if $p_0 > 1$ and H^1 to $L^{1,2}$ if $p_0 = 1$ with constant B_0 and when $\operatorname{Re} z = 1$, T_z maps $L^{p_1,2}$ to L^{p_1} with constant B_1, where

$$B_m = C_M K_i \bigg(\sup_{k \in \mathbb{Z}} \sum_{j \geq 0} 2^{js} \big\| \Delta_j \big(\sigma^k \widehat{\Psi} \big) \big\|_{L^r} \bigg)^{\frac{r}{r_m}}, \quad m = 0, 1.$$

We now interpolate using Theorem 1.1 (with $m = 1$) in [7]. We obtain

$$\big\| T_{\sigma_\theta}(f) \big\|_{(L^{p_0,2})^{1-\theta}(L^{p_1,2})^\theta} \leq C(p_0, p_1, p) B_0^{1-\theta} B_1^\theta \|f\|_{(L^{p_0}, L^{p_1})_\theta}.$$

Noting that $(L^{p_0,2})^{1-\theta}(L^{p_1,2})^\theta = L^{p,2}$ and $(L^{p_0}, L^{p_1})_\theta = L^p$ (even when $p_0 = 1$, in which case L^{p_0} is replaced by H^1), we obtain the claimed assertion. $\qquad \square$

LEMMA 3.2. *Using the notation of Theorem* 3.1, *for any* $M > 0$ *there is a constant* C_M (*also depending on the dimension* n, *on* Ψ, *and* $\widetilde{\Psi}$) *such that for any* $1 \leq q \leq \infty$ *we have*

$$(19) \qquad \big\| \widetilde{\Delta}_l \big(\widetilde{\Delta}_j(g) \widehat{\Psi} \widehat{\widetilde{\Psi}} \big) \big\|_{L^q} \leq C_M 2^{-2(1-\frac{1}{q}) \max(j,l) M} \|g\|_{L^q}$$

for all $l, j > 0$. *We also have that for any* $M > n$ *there is a constant* C_M *such that*

$$(20) \qquad \big\| \widetilde{\Delta}_l \big(\widetilde{\Delta}_j(g) \widehat{\Psi} \widehat{\widetilde{\Psi}} \big) \big\|_{L^1} \leq C_M 2^{-\max(j,l)(M-n)} \|g\|_{H^1}.$$

Proof. The claimed estimate is obviously true when $q = 1$. So we prove it for $q = 2$ and derive (19) as a consequence of classical Riesz–Thorin interpolation theorem. Examining the Fourier transform of the operator in (19), matters reduce to computing the L^∞ norm of the function

$$(21) \qquad \widehat{\widetilde{\Psi}}\big(2^{-j}\xi\big) \int_{\mathbb{R}^n} \widehat{\widetilde{\Psi}}\big(2^{-l}(\xi - \eta)\big) \phi(\eta) \, d\eta,$$

where $\phi(\eta) = \Psi * \widetilde{\Psi}$ is a Schwartz function. Since the integral is over the set $|\xi - \cdot| \approx 2^l$, we estimate the absolute value of (21) by $C_M[\sup\{(1 + |\eta|)^{-M} : |\xi - \eta| \approx 2^l\}]$ where $|\xi| \approx 2^j$. Notice that if $l > j + 10$, then $|\eta| \approx 2^l$, while if $j > l + 10$, then $|\eta| \approx 2^j$. These estimates yield the proof of (19).

We now turn our attention to (20). Using Fourier inversion, we write

$$\widetilde{\Delta}_l \big(\widetilde{\Delta}_j(g) \widehat{\phi} \big)(x) = \int_{\mathbb{R}^n} \widehat{g}(\eta) \widehat{\widetilde{\Psi}}\big(2^{-l}\eta\big) \int_{\mathbb{R}^n} \widehat{\widetilde{\Psi}}\big(2^{-j}\xi\big) \phi(\xi - \eta) e^{2\pi i x \cdot \xi} \, d\xi \, d\eta.$$

We integrate by parts in the inner integral with respect to the operator $(I - \Delta_\xi)^N$ to obtain that the preceding expression is equal to

$$\sum_{\beta+\gamma=2N} \frac{C_{\beta,\gamma}}{(1+4\pi^2|x|^2)^N}$$
$$\times \int_{\mathbb{R}^n} \widehat{g}(\eta)\overline{\widehat{\widetilde{\Psi}}}\big(2^{-l}\eta\big) \int_{\mathbb{R}^n} 2^{-j|\beta|}\big(\partial^\beta \overline{\widehat{\widetilde{\Psi}}}\big)\big(2^{-j}\xi\big)\big(\partial^\gamma \phi\big)(\xi-\eta)e^{2\pi ix\cdot\xi}\, d\xi\, d\eta.$$

Since for $g \in H^1$ we have $|\widehat{g}(\xi)| \leq c\|g\|_{H^1}$ for all ξ and we deduce the estimate

$$\big|\widetilde{\Delta}_l\big(\widetilde{\Delta}_j(g)\widehat{\phi}\big)(x)\big| \leq \frac{C_M\|g\|_{H^1}}{(1+4\pi^2|x|^2)^N}2^{ln}\sup_{|\eta|\approx 2^l}\int_{|\xi|\approx 2^j}\frac{d\xi}{(1+|\xi-\eta|)^{2M}}$$

for $M > n$. We easily derive from this estimate the validity of (20). Note that in the case $j=0$ the notation $|\xi| \approx 2^j$ should be interpreted as $|\xi| \lesssim 2$; likewise when $l=0$. $\qquad\square$

Proposition 1.2 is a consequence of Theorem 3.1 with initial estimates (8) and (9).

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