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ON THE INJECTIVE DIMENSION OF $\mathscr{F}$-FINITE MODULES AND HOLONOMIC $\mathscr{D}$-MODULES

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ABSTRACT. Let R be a regular local ring containing a field k of characteristic p and M be an $\mathscr{F}$-finite module. In this paper, we study the injective dimension of M. We prove that $\dim_R(M) - 1 \leq \text{inj.dim}_R(M)$. If $R = k[[x_1, \dots, x_n]]$ where k is a field of characteristic 0 we prove the analogous result for a class of holonomic $\mathscr{D}$-modules which contains local cohomology modules.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring with unit. If M is an R-module and $\mathrm{I} \subset R$ is an ideal, we denote the ith local cohomology of M with support in I by $H^i_{\mathrm{I}}(M)$.

In a remarkable paper, [7], Lyubeznik used $\mathscr{D}$-modules to prove if R is any regular ring containing a field of characteristic 0 and I is an ideal of R, then

(a) $H^i_{\mathfrak{m}}(H^i_{\mathrm{I}}(R))$ is injective for every maximal ideal $\mathfrak{m}$ of R.
(b) $\text{inj.dim}_R(H^i_{\mathrm{I}}(R)) \leq \dim_R(H^i_{\mathrm{I}}(R))$.
(c) For every maximal ideal $\mathfrak{m}$ of R the set of associated primes of $H^i_{\mathrm{I}}(R)$ contained in $\mathfrak{m}$ is finite.
(d) All the bass numbers of $H^i_{\mathrm{I}}(R)$ are finite.

Here $\text{inj.dim}_R(H^i_{\mathrm{I}}(R))$ stands for the injective dimension of $H^i_{\mathrm{I}}(R)$, $\dim_R(H^i_{\mathrm{I}}(R))$ denotes the dimension of the support of $H^i_{\mathrm{I}}(R)$ in $\operatorname{Spec}(R)$ and the jth Bass number of an R-module M with respect to a prime ideal $\mathfrak{p}$ is defined as $\mu^j(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} \operatorname{Ext}^j_{R_{\mathfrak{p}}}(k(\mathfrak{p}), M_{\mathfrak{p}})$ where $k(\mathfrak{p})$ is the residue field of $R_{\mathfrak{p}}$.

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The analogous results had proved earlier for regular local ring of positive characteristic by Huneke and Sharp [6], using the Frobenius functor.

Later Lyubeznik [8] developed the theory of $\mathscr{F}$-modules over regular ring of char $p > 0$ and proved the same results in char $p > 0$. The theory of $\mathscr{F}$-modules turned out to be very effective. For example, Lyubeznik and etc. [1] used $\mathscr{D}$-modules over $\mathbb{Z}$ and $\mathbb{Q}$ along with the theory of $\mathscr{F}$-modules to prove if R is a smooth $\mathbb{Z}$-algebra and I an ideal of R then the set of associated primes of local cohomology module $H^i_{\mathrm{I}}(R)$ is finite.

By Lyubeznik results, the injective dimension of $H^i_{\mathrm{I}}(R)$ is bounded by its dimension. More generally, if M is an $\mathscr{F}$-module over a regular ring of positive characteristic or is a $\mathscr{D}$-module over power series ring $k[[x_1,\dots,x_n]]$ where k is a field of char 0, then the injective dimension of M is bounded by its dimension, see [8, Theorem 1.4] and [7, Theorem 2.4(b)]. A question of Hellus [5] asks when $\operatorname{inj.dim}_R(H^i_{\mathrm{I}}(R)) = \dim_R(H^i_{\mathrm{I}}(R))$. He proved the equality $\operatorname{inj.dim}_R(H^i_{\mathrm{I}}(R)) = \dim_R(H^i_{\mathrm{I}}(R))$ for a regular local ring R which contains a field and cofinite local cohomology $H^i_{\mathrm{I}}(R)$, see [5, Corollary 2.6]. On the other hand, he presented counterexamples for this equality in which $\operatorname{inj.dim}_R(H^i_{\mathrm{I}}(R)) = 0$ but $\dim_R(H^i_{\mathrm{I}}(R)) = 1$, see [5, Example 2.9, 2.11]. Also for polynomial ring $R = k[x_1,\dots,x_n]$ with field k of characteristic zero, Puthenpurakal, [10, Corollary 1.2], proved $\operatorname{inj.dim}_R(H^i_{\mathrm{I}}(R)) = \dim_R(H^i_{\mathrm{I}}(R))$.

In this paper, motivated by these results, we attempt to obtain lower bound for the injective dimension of $\mathscr{F}$-modules and $\mathscr{D}$-modules. We succeed in this for a subclass of $\mathscr{F}$-modules called $\mathscr{F}$-finite and subclass of $\mathscr{D}$-modules which contains local cohomology modules. In fact we prove that

THEOREM 1.1 (Theorem 4.1). *Let $(R,\mathfrak{m})$ be a regular local ring which contains a field. Let I be an ideal of R. The following hold.*

(i) *Assume characteristic of R is $p > 0$ and M is an $\mathscr{F}$-finite module. Then* $\dim_R M - 1 \leq \operatorname{inj.dim}_R M$.
(ii) *Assume characteristic of R is 0 and $M = H^i_{\mathrm{I}}(R)_f$ for some $f \in R$. Then* $\dim_R M - 1 \leq \operatorname{inj.dim}_R M$.

This manuscript is organized as follows. In Section 2, we recall some definitions and properties of $\mathscr{D}$-modules and $\mathscr{F}$-modules. Later, in Section 3, we discuss some lemmas and propositions which will help us in proving our main theorem. In Section 4, we prove our main theorem.

2. Preliminaries

Throughout this paper, we always assume that R is a regular local ring which contains a field. In this section, we review the theory of $\mathscr{D}$-modules and $\mathscr{F}$-modules and state two useful lemmas.

$\mathscr{D}$-modules. Let k be a field of characteristic 0 and let R denote the formal power series ring $k[[x_1,\dots,x_n]]$ in n variables over k. Let $\mathscr{D} = \mathscr{D}(R,k)$ denote

the subring of the k-vector space endomorphisms of R generated by R and the usual differential operators $\delta_1, \dots, \delta_n$, defined formally, so that $\delta_i f = \frac{\partial f}{\partial x_i}$. We simply say $\mathscr{D}$-modules for left $\mathscr{D}(R,k)$ modules. $\mathscr{D}(R,k)$ is left and right Noetherian [2, Lemma 3.1.6]. This implies that every finitely generated $\mathscr{D}$-module is Noetherian. The natural action of $\mathscr{D}(R,k)$ on R makes R as a $\mathscr{D}$-module. In addition if M is a $\mathscr{D}$-module and $S \subset R$ is a multiplicative system of elements, using the quotient rule, M_S carries a natural structure of $\mathscr{D}$-module. Let I be an ideal of R. The Čech complex on a generating set for I is a complex of $\mathscr{D}$-modules; it then follows that each local cohomology module $H^i_{\mathrm{I}}(R)$ is a $\mathscr{D}$-module.

We will use the following several times in this paper.

Remark 2.1. Adopt the above notations.

(a) Let M be a $\mathscr{D}$-module. Then $\operatorname{inj.dim}_R M \leq \dim_R M$ [7, Theorem 2.4(b)].
(b) Let M be a $\mathscr{D}$-module and I be an ideal of R. Then $H^i_{\mathrm{I}}(M)$ have a natural structure of $\mathscr{D}$-modules [7, Example 2.1(iv)]. In particular, $\Gamma_I(M)$ is a $\mathscr{D}$-submodule of M where Γ_I is the I-torsion functor.
(c) Let $\mathfrak{p}$ be a prime ideal of R and let $E_R(R/\mathfrak{p})$ denote the injective envelope of $R/\mathfrak{p}$. Assume $\operatorname{ht}_R(\mathfrak{p}) = d$. Recall that $E_R(R/\mathfrak{p}) = H^d_{\mathfrak{p}}(R)_{\mathfrak{p}}$. It follows that $E_R(R/\mathfrak{p})$ is a $\mathscr{D}$-module and the natural inclusion $H^d_{\mathfrak{p}}(R) \to E_R(R/\mathfrak{p})$ is $\mathscr{D}(R,k)$-linear.
(d) Let $(S, \mathfrak{m})$ be a regular local ring which contains a field of characteristic zero. We denote by $\hat{S}$ the completion of S with respect to the maximal ideal $\mathfrak{m}$. By Cohen structure theorem $\hat{S} = k[[x_1, \dots, x_n]]$ where k is a field of characteristic zero. Let $\mathfrak{p}$ be the prime ideal of S such that $\operatorname{ht}_S(\mathfrak{p}) = d$. Recall that $E_S(S/\mathfrak{p}) = H^d_{\mathfrak{p}}(S)_{\mathfrak{p}}$. Then $E_S(S/\mathfrak{p}) \otimes_S \hat{S} \cong H^d_{\mathfrak{p}\hat{S}}(\hat{S})_{\mathfrak{p}}$, see [3, Theorem 4.3.2]. Hence, $E_S(S/\mathfrak{p}) \otimes_S \hat{S}$ has a natural structure of $\mathscr{D}(\hat{S}, k)$-module.

There exists a remarkable class of finitely generated $\mathscr{D}$-modules, called holonomic $\mathscr{D}$-modules. See [2, Definition 7.12] for a definition of a holonomic $\mathscr{D}$-module.

Remark 2.2. Some of the properties of holonomic modules are as follows:

(a) R with its natural structure of $\mathscr{D}(R,k)$-module is holonomic [2, Theorem 3.3.2].
(b) If M is holonomic and $f \in R$, then M_f is holonomic [2, Theorem 3.4.1].
(c) Let M be a holonomic $\mathscr{D}$-module. Assume $\operatorname{Ass}_R M = \{\mathfrak{p}\}$ and M is $\mathfrak{p}$-torsion. Then there exists $h \in R \setminus \mathfrak{p}$ such that $\operatorname{Hom}_R(R/\mathfrak{p}, M)_h$ is finitely generated as an R_h-module [10, Proposition 2.3].
(d) The holonomic modules form an abelian subcategory of the category of $\mathscr{D}$-modules, which is closed under formation of submodules, quotient modules and extensions. (A proof of this is completely analogous to the proof of [2, Proposition 1.5.2].) So $H^i_{\mathrm{I}}(R)$ is a holonomic $\mathscr{D}$-module.

(e) If M is holonomic, then $H^i_I(M)$ is holonomic [7, 2.2 d].
(f) If M is holonomic, all the Bass numbers of M are finite [7, Theorem 2.4(d)].
(g) If M is holonomic, the set of the associated primes of M is finite [7, Theorem 2.4(c)].

$\mathscr{F}$-modules. The notion of $\mathscr{F}$-modules was introduced by Lyubeznik in [8]. We collect some notations and preliminary results from [8]. Let R be a regular ring containing a field of characteristic $p>0$. Let R' be the additive group of R regarded as an R-bimodule with the usual left R-action and with the right R-action defined by $r'r = r^p r'$ for all $r \in R$, $r' \in R'$. For an R-module M, define $F(M) = R' \otimes_R M$; we view this as an R-module via the left R-module structure on R'.

An $\mathscr{F}_R$-module M is an R-module M with an R-module isomorphism $\theta : M \to F(M)$ which is called the structure morphism of M. We will abbreviate $\mathscr{F}_R$ to $\mathscr{F}$ for the sake of readability (if this causes no confusion). A homomorphism of $\mathscr{F}$-modules is an R-module homomorphism $f : M \to M'$ such that the following diagram commutes (where θ and θ' are the structure morphisms of M and M').

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\theta"] & M' \arrow[d, "\theta'"] \\ F(M) \arrow[r, "F(f)"] & F(M') \end{tikzcd}$$

It is not hard to see that the category of $\mathscr{F}$-modules is Abelian.

Remark 2.3. Some of the properties of $\mathscr{F}$-modules are as follows:

(a) The ring R has a natural $\mathscr{F}$-module structure [8, Example 1.2(a)].
(b) Let I be an ideal of R and M be an $\mathscr{F}$-module. Then an $\mathscr{F}$-module structure on an R-module M induces an $\mathscr{F}$-module structure on the local cohomology module $H^i_I(M)$. In particular, $\Gamma_I(M)$ is an $\mathscr{F}$-submodule of M [8, Example 1.2(b)].
(c) If M is an $\mathscr{F}$-module and $0 \to M \to E^\bullet$ is the minimal injective resolution of M in the category of R-modules, then each E^i acquires a structure of $\mathscr{F}$-module such that the resolution becomes a complex of $\mathscr{F}$-modules and $\mathscr{F}$-module homomorphisms [8, Example 1.2(b'')].
(d) Let M be an $\mathscr{F}$-module. Then $\operatorname{inj.dim}_R M \leq \dim_R M$ [8, Theorem 1.4].
(e) Let M be an $\mathscr{F}$-module and $S \subset R$ be a multiplicative set. Then M_S has a natural structure of $\mathscr{F}$-module such that the natural localization map $M \to M_S$ is the $\mathscr{F}$-module homomorphism [8, Proposition 1.3(b)].

There exists an important class of $\mathscr{F}$-modules, called $\mathscr{F}$-finite modules. See [8, Definition 2.1] for a definition of an $\mathscr{F}$-finite module.

REMARK 2.4. Some of the properties of $\mathscr{F}$-finite modules are as follows:

(a) The $\mathscr{F}$-finite modules form a full Abelian subcategory of the category of $\mathscr{F}$-modules which is closed under formation of submodules, quotient modules and extensions [8, Theorem 2.8].
(b) If M is an $\mathscr{F}$-finite module, then M_f is $\mathscr{F}$-finite, where $f \in R$ [8, Proposition 2.9(b)].
(c) If M is an $\mathscr{F}$-finite module and I is an ideal of R, then $H^i_I(M)$ with its induced $\mathscr{F}$-module structure is $\mathscr{F}$-finite [8, Proposition 2.10].
(d) All the Bass numbers of an $\mathscr{F}$-finite module M are finite [8, Theorem 2.11].
(e) The set of the associated primes of an $\mathscr{F}$-finite module M is finite [8, Theorem 2.12].
(f) If M is an $\mathscr{F}_R$-finite module, then $M_{\mathfrak{p}}$ is $\mathscr{F}_{R_{\mathfrak{p}}}$-finite, where $\mathfrak{p} \in \operatorname{Spec}(R)$ [8, Proposition 2.9(a)].

For the convenience of the reader, we state the following proved facts.

LEMMA 2.5. *Let R be a Noetherian local ring which has a finitely generated injective module. Then R is an Artinian ring.*

Proof. By [4, Theorem 3.1.17], $\operatorname{depth} R = 0$. Also well known proved conjecture of Bass implies that R is Cohen–Macaulay. Then $\dim R = 0$. □

LEMMA 2.6. *Let $R \to S$ be a faithfully flat map of Noetherian rings. Then an R module L is finitely generated if and only if $L \otimes_R S$ is finitely generated as an S-module.*

Proof. See [10, Proposition 3.3]. □

3. Preliminary lemmas

In this section, our objective is to prove Proposition 3.8 which will enable us to prove the main theorem in the next section. Let $(R, \mathfrak{m})$ be a local ring and M be an R-module. By $\operatorname{depth}_R(M)$, we mean the length of the maximal M-regular sequence in $\mathfrak{m}$.

LEMMA 3.1. *Let k be a field of characteristic zero and $R = k[[x_1, \dots, x_n]]$. Let $\mathfrak{p}$ be a prime ideal of R of height less than $n-1$. Then $E_R(R/\mathfrak{p})$ is not a holonomic $\mathscr{D}$-module.*

Proof. Suppose on the contrary $E_R(R/\mathfrak{p})$ is a holonomic $\mathscr{D}$-module. It is well known that $\Gamma_{\mathfrak{p}}(E_R(R/\mathfrak{p})) = E_R(R/\mathfrak{p})$ and $\operatorname{Ass}_R E(R/\mathfrak{p}) = \mathfrak{p}$. Then by Remark 2.2(c), there exists $h \in R \setminus \mathfrak{p}$ such that $\operatorname{Hom}_R(R/\mathfrak{p}, E_R(R/\mathfrak{p}))_h$ is a finitely generated R_h-module. Pick $\mathfrak{q} \in \operatorname{Spec}(R)$ which contains $\mathfrak{p}$ such that $\operatorname{ht}_R(\mathfrak{q}) = n-1$ and $h \notin \mathfrak{q}$. It follows that $M := \operatorname{Hom}_R(R/\mathfrak{p}, E_R(R/\mathfrak{p}))_{\mathfrak{q}}$ is a non-zero finitely generated $R_{\mathfrak{q}}$-module. On the other hand M is an injective $R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}$-module. Then, in view of Lemma 2.5, $R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}$ is an Artinian ring.

This contradicts with the fact that $R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}$ is a domain of dimension greater than one. $\square$

Let I be an ideal of a ring R. By $\min_R(\mathrm{I})$, we mean the set of all minimal prime ideals of I.

LEMMA 3.2. *Let $(R,\mathfrak{m})$ be a regular local ring of dimension n which contains a field of characteristic zero. Assume $P\in\operatorname{Spec}(R)$ such that $\operatorname{ht}_R(P)=d\le n-2$. Let $\hat{R}$ denote the completion of R with respect to the maximal ideal $\mathfrak{m}$. In view of Remark 2.1(d) $E_R(R/P)\otimes_R\hat{R}$ has a natural structure of $\mathscr{D}(\hat{R},k)$-module where k is a suitable coefficient field of $\hat{R}$. Then $E_R(R/P)\otimes_R\hat{R}$ is a non-holonomic $\mathscr{D}$-module.*

Proof. Recall that $E_R(R/P)\cong H^d_P(R)_P$ and $E_R(R/P)\otimes_R\hat{R}\cong H^d_P(R)_P\otimes_R\hat{R}\cong H^d_{P\hat{R}}(\hat{R})_P$. In view of Remark 2.1(d), $E_R(R/P)\otimes_R\hat{R}$ has a natural structure of $\mathscr{D}(\hat{R},k)$-module where k is a field of characteristic zero which is contained in $\hat{R}$. We simply say $E_R(R/P)\otimes_R\hat{R}$ is a $\mathscr{D}$-module. It is obvious that $\operatorname{ht}_{\hat{R}}(P\hat{R})=d$. Let $\min_{\hat{R}}(P\hat{R})=\{\mathfrak{q}_1,\dots,\mathfrak{q}_s\}$. There are infinitely many primes $\mathfrak{p}\in\operatorname{Spec}(R)$ such that $\operatorname{ht}_R(\mathfrak{p})=d+1$ and $P\subsetneqq\mathfrak{p}$, see [9, Theorem 31.2]. For such $\mathfrak{p}$, $\operatorname{ht}_{\hat{R}}(\mathfrak{p}\hat{R})=d+1$ and $\mathfrak{p}\hat{R}\cap R=\mathfrak{p}$. Thus without loss of generality, we can assume that $\operatorname{ht}_{\hat{R}}(\mathfrak{q}_1)=d$ and there are infinitely many primes $\mathfrak{q}\in\operatorname{Spec}(\hat{R})$ of height $d+1$ which contains $\mathfrak{q}_1$ and $\operatorname{ht}_R(\mathfrak{q}\cap R)=d+1$.

Suppose on the contrary that $H^d_{P\hat{R}}(\hat{R})_P$ is holonomic.

CLAIM 1. *$H^d_{\mathfrak{q}_1}(\hat{R})_P$ is holonomic.*

The composition of functors $\Gamma_{\mathfrak{q}_1}(-)=\Gamma_{\mathfrak{q}_1}(\Gamma_{P\hat{R}}(-))$ leads to the spectral sequence $E_2^{p,q}=H^p_{\mathfrak{q}_1}(H^q_{P\hat{R}}(\hat{R}))\Rightarrow H^{p+q}_{\mathfrak{q}_1}(\hat{R})$. It follows that $\Gamma_{\mathfrak{q}_1}(H^d_{P\hat{R}}(\hat{R}))=H^d_{\mathfrak{q}_1}(\hat{R})$. Hence $H^d_{\mathfrak{q}_1}(\hat{R})$ is the $\mathscr{D}$-submodule of $H^d_{P\hat{R}}(\hat{R})$. Therefore $H^d_{\mathfrak{q}_1}(\hat{R})_P$ is a holonomic $\mathscr{D}$-module, see Remark 2.2(d). This yields the claim.

CLAIM 2. $\operatorname{Ass}_{\hat{R}}(H^d_{\mathfrak{q}_1}(\hat{R})_P)=\mathfrak{q}_1$.

Indeed let $m/s\in H^d_{\mathfrak{q}_1}(\hat{R})_P$ such that $m\in H^d_{\mathfrak{q}_1}(\hat{R})$ and $s\in R\setminus P$. If $r\in\hat{R}$ such that $r.m/s=0$, then there exists $r'\in R\setminus P\subseteq\hat{R}\setminus\mathfrak{q}_1$ such that $r'rm=0$. Keep in mind that $\operatorname{Ass}_{\hat{R}}(H^d_{\mathfrak{q}_1}(\hat{R}))=\mathfrak{q}_1$. So $r'r\in\mathfrak{q}_1$ and thus $r\in\mathfrak{q}_1$. This yields the claim.

Also $\Gamma_{\mathfrak{q}_1}(H^d_{\mathfrak{q}_1}(\hat{R})_P)=H^d_{\mathfrak{q}_1}(\hat{R})_P$. Then by Remark 2.2(c), there exists $h\in\hat{R}\setminus\mathfrak{q}_1$ such that $\operatorname{Hom}_{\hat{R}}(\frac{\hat{R}}{\mathfrak{q}_1\hat{R}},H^d_{\mathfrak{q}_1}(\hat{R})_P)_h$ is a finitely generated $\hat{R}_h$-module. Since $\mathfrak{q}_i\nsubseteq\mathfrak{q}_1$ for all $2\le i\le s$, we can pick $t_i\in\mathfrak{q}_i\setminus\mathfrak{q}_1$ for all $2\le i\le s$. Thus $t=t_2\dots t_sh\notin\mathfrak{q}_1$. Note that the set of minimal prime ideals of the ideal generated by t and $\mathfrak{q}_1$ is finite. Then by assumption on choosing $\mathfrak{q}_1$, we can pick $\mathfrak{q}\in\operatorname{Spec}(\hat{R})$ of height $d+1$ which contains $\mathfrak{q}_1$ and $t\notin\mathfrak{q}$ such that $\operatorname{ht}_R(\mathfrak{q}\cap R)=d+1$.

Thus $\operatorname{Hom}_{\hat{R}_\mathfrak{q}}(\frac{\hat{R}_\mathfrak{q}}{\mathfrak{q}_1\hat{R}_\mathfrak{q}}, (H^d_{\mathfrak{q}_1}(\hat{R})_P)_\mathfrak{q})$ is a finitely generated $\hat{R}_\mathfrak{q}$-module. Since $\min_{\hat{R}_\mathfrak{q}}(P\hat{R}_\mathfrak{q}) = \mathfrak{q}_1\hat{R}_\mathfrak{q}$, then $H^d_{\mathfrak{q}_1\hat{R}_\mathfrak{q}}(\hat{R}_\mathfrak{q}) = H^d_{P\hat{R}_\mathfrak{q}}(\hat{R}_\mathfrak{q})$. Also $\frac{\hat{R}_\mathfrak{q}}{P\hat{R}_\mathfrak{q}}$ has a filtration of $\hat{R}_\mathfrak{q}$-modules such that quotients of it are isomorph to $\frac{\hat{R}_\mathfrak{q}}{\mathfrak{q}_1\hat{R}_\mathfrak{q}}$ or $\frac{\hat{R}_\mathfrak{q}}{\mathfrak{q}\hat{R}_\mathfrak{q}}$, as $\hat{R}_\mathfrak{q}$-module. Thus $\operatorname{Hom}_{\hat{R}_\mathfrak{q}}(\frac{\hat{R}_\mathfrak{q}}{P\hat{R}_\mathfrak{q}}, (H^d_{P\hat{R}}\hat{R}_P)_\mathfrak{q})$ is a finitely generated $\hat{R}_\mathfrak{q}$-module.

Look at the faithfully flat map $R_{\mathfrak{q}\cap R} \to \hat{R}_\mathfrak{q}$. We have following isomorphisms:

$$\begin{aligned}
&\operatorname{Hom}_{R_{\mathfrak{q}\cap R}}\left(\frac{R_{\mathfrak{q}\cap R}}{PR_{\mathfrak{q}\cap R}}, \big(H^d_P(R)_P\big)_{\mathfrak{q}\cap R}\right) \otimes_{R_{\mathfrak{q}\cap R}} \hat{R}_\mathfrak{q} \\
&\quad\cong \operatorname{Hom}_{\hat{R}_\mathfrak{q}}\left(\frac{R_{\mathfrak{q}\cap R}}{PR_{\mathfrak{q}\cap R}} \otimes_{R_{\mathfrak{q}\cap R}} \hat{R}_\mathfrak{q}, \big(H^d_P(R)_P\big)_{\mathfrak{q}\cap R} \otimes_{R_{\mathfrak{q}\cap R}} \hat{R}_\mathfrak{q}\right) \\
&\quad\cong \operatorname{Hom}_{\hat{R}_\mathfrak{q}}\big((R/P \otimes_R R_{\mathfrak{q}\cap R}) \otimes_{R_{\mathfrak{q}\cap R}} \hat{R}_\mathfrak{q}, \\
&\qquad \big(\big(H^d_P(R) \otimes_R R_P\big) \otimes_R R_{\mathfrak{q}\cap R}\big) \otimes_{R_{\mathfrak{q}\cap R}} \hat{R}_\mathfrak{q}\big) \\
&\quad\cong \operatorname{Hom}_{\hat{R}_\mathfrak{q}}\big(R/P \otimes_R \hat{R}_\mathfrak{q}, \big(H^d_P(R) \otimes_R R_P\big) \otimes_R \hat{R}_\mathfrak{q}\big) \\
&\quad\cong \operatorname{Hom}_{\hat{R}_\mathfrak{q}}\big(R/P \otimes_R (\hat{R} \otimes_{\hat{R}} \hat{R}_\mathfrak{q}), \big(H^d_P(R) \otimes_R R_P\big) \otimes_R (\hat{R} \otimes_{\hat{R}} \hat{R}_\mathfrak{q})\big) \\
&\quad\cong \operatorname{Hom}_{\hat{R}_\mathfrak{q}}\left(\frac{\hat{R}_\mathfrak{q}}{P\hat{R}_\mathfrak{q}}, \big(H^d_{P\hat{R}}(\hat{R})_P\big)_\mathfrak{q}\right).
\end{aligned}$$

Therefore, by virtue of Lemma 2.6, $\operatorname{Hom}_{R_{\mathfrak{q}\cap R}}(\frac{R_{\mathfrak{q}\cap R}}{PR_{\mathfrak{q}\cap R}}, E_R(R/P)_{\mathfrak{q}\cap R}) \cong \operatorname{Hom}_{R_{\mathfrak{q}\cap R}}(\frac{R_{\mathfrak{q}\cap R}}{PR_{\mathfrak{q}\cap R}}, (H^d_P(R)_P)_{\mathfrak{q}\cap R})$ is a non-zero finitely generated $R_{\mathfrak{q}\cap R}$-module. So, by Lemma 2.5, $\frac{R_{\mathfrak{q}\cap R}}{PR_{\mathfrak{q}\cap R}}$ is an Artinian ring. Again, it is a contradiction because $\frac{R_{\mathfrak{q}\cap R}}{PR_{\mathfrak{q}\cap R}}$ is a domain of dimension greater than one. □

Next, we want to establish analogous result such Lemma 3.1 for characteristic $p > 0$. To show this we need some lemmas.

LEMMA 3.3. *Let R be a regular local ring which contains a field and I be an ideal of R. Let* $\operatorname{inj.dim}_R(H^i_\mathrm{I}(R)) = \dim_R(H^i_\mathrm{I}(R)) = c$. *If* $\mu^c(\mathfrak{p}, H^i_\mathrm{I}(R)) \neq 0$ *for* $\mathfrak{p} \in \operatorname{Spec}(R)$, *then $\mathfrak{p}$ is a maximal ideal of R.*

Proof. Let $\dim(R) = n$. We suppose on the contrary $\operatorname{ht}_R \mathfrak{p} \leq n-1$. Thus, $\dim_{R_\mathfrak{p}}(H^i_\mathrm{I}(R))_\mathfrak{p} \leq c-1$. Since $\mu^c(\mathfrak{p}, H^i_\mathrm{I}(R)) \neq 0$, we deduce that $\operatorname{inj.dim}_{R_\mathfrak{p}}(H^i_\mathrm{I}(R)_\mathfrak{p}) = c$. But this is impossible because in view of [7, Theorem 3.4(b)] and [8, Theorem 1.4], we must have $\operatorname{inj.dim}_{R_\mathfrak{p}}(H^i_\mathrm{I}(R)_\mathfrak{p}) \leq \dim_{R_\mathfrak{p}}(H^i_\mathrm{I}(R))_\mathfrak{p}$. □

LEMMA 3.4. *Let $(R, \mathfrak{m})$ be a local ring of dimension n. Let $\hat{R}$ denote the completion of R with respect to the maximal ideal $\mathfrak{m}$. Let M be an R-module. Then* $\dim_R(M) = \dim_{\hat{R}}(M \otimes_R \hat{R})$.

Proof. Let $\dim_R(M) = d$. There exists $\mathfrak{p} \in \operatorname{Supp}_R(M)$ such that $d = \dim R/\mathfrak{p} = \dim \hat{R}/\mathfrak{p}\hat{R}$. Thus there exists $\mathfrak{q} \in \operatorname{Spec}(\hat{R})$ such that $\mathfrak{q}$ is minimal over $\mathfrak{p}\hat{R}$ and $\dim \hat{R}/\mathfrak{q}\hat{R} = d$. We show that $\mathfrak{q} \in \operatorname{Supp}_{\hat{R}}(M \otimes_R \hat{R})$ and so $\dim_{\hat{R}}(M \otimes_R \hat{R}) \geq d$. It is clear that $\mathfrak{q} \cap R = \mathfrak{p}$. Hence, the natural map $R_\mathfrak{p} \to \hat{R}_\mathfrak{q}$ is faithfully flat. Thus,

$$(M \otimes_R \hat{R}) \otimes_{\hat{R}} \hat{R}_\mathfrak{q} \cong M \otimes_R \hat{R}_\mathfrak{q} \cong M \otimes_R (R_\mathfrak{p} \otimes_{R_\mathfrak{p}} \hat{R}_\mathfrak{q}) \cong (M \otimes_R R_\mathfrak{p}) \otimes_{R_\mathfrak{p}} \hat{R}_\mathfrak{q}.$$

So $(M \otimes_R \hat{R})_\mathfrak{q} \neq 0$ as desired.

On the other hand let $\dim_{\hat{R}}(M \otimes_R \hat{R}) = c$. Thus, there exists $\mathfrak{q} \in \operatorname{Supp}_{\hat{R}}(M \otimes_R \hat{R})$ such that $\dim \hat{R}/\mathfrak{q}\hat{R} = c$. Let $\mathfrak{q} \cap R = \mathfrak{p}$. Thus, $\dim R/\mathfrak{p} = \dim \hat{R}/\mathfrak{p}\hat{R} \geq \dim \hat{R}/\mathfrak{q} = c$. So we only need to show that $\mathfrak{p} \in \operatorname{Supp}(M)$. It is obvious by the isomorphism $(M \otimes_R \hat{R})_\mathfrak{q} \cong (M \otimes_R R_\mathfrak{p}) \otimes_{R_\mathfrak{p}} \hat{R}_\mathfrak{q}$. □

PROPOSITION 3.5. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension n containing a field and I be an ideal of R such that $\operatorname{ht}_R(\mathrm{I}) = d$. Then $\operatorname{inj.dim}_R(H^d_\mathrm{I}(R)) = \dim_R(H^d_\mathrm{I}(R))$.*

Proof. Assume $\operatorname{ht}_R(\mathrm{I}) = d$. Let $\min_R(I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\} \cup \{\mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$ such that $\operatorname{ht}_R(\mathfrak{p}_i) = d$ and $\operatorname{ht}_R(\mathfrak{q}_i) > d$. Set $\mathrm{I}' := \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s$ and $\mathrm{I}'' = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$. We have the Mayer–Vietoris sequence

$$H^d_{\mathrm{I}'+\mathrm{I}''}(R) \to H^d_{\mathrm{I}'}(R) \oplus H^d_{\mathrm{I}''}(R) \to H^d_\mathrm{I}(R) \to H^{d+1}_{\mathrm{I}'+\mathrm{I}''}(R).$$

Since $H^d_{\mathrm{I}'+\mathrm{I}''}(R) = H^{d+1}_{\mathrm{I}'+\mathrm{I}''}(R) = H^d_{\mathrm{I}''}(R) = 0$ we deduce that $H^d_\mathrm{I}(R) \cong H^d_{I'}(R)$. Thus without loss of generality, we can assume that all minimal prime ideals of I have height d.

There exists the spectral sequence $H^i_\mathfrak{m}(H^j_\mathrm{I}(R)) \Rightarrow H^{i+j}_\mathfrak{m}(R)$. By using Hartshorne–Lichtenbaum theorem, we easily see that $\operatorname{inj.dim}_R(H^i_\mathrm{I}(R)) \leq \dim_R(H^i_\mathrm{I}(R)) \leq n - (i+1)$ for all $i > d$. So on the line $y + x = n$ of the spectral sequence $H^i_\mathfrak{m}(H^j_\mathrm{I}(R)) \Rightarrow H^{i+j}_\mathfrak{m}(R)$, we have $H^{n-i}_\mathfrak{m}(H^i_\mathrm{I}(R)) = 0$ for all $i > d$. By the definition of the spectral sequence $H^i_\mathfrak{m}(H^j_\mathrm{I}(R)) \Rightarrow H^{i+j}_\mathfrak{m}(R)$ there exists a filtration

$$0 \subseteq \cdots \subseteq F^t H_n \subseteq F^{t-1} H_n \subseteq \cdots \subseteq F^s H_n = H^n_\mathfrak{m}(R)$$

of $H^n_\mathfrak{m}(R)$ such that $E^{i,n-i}_\infty \cong \frac{F^i H_n}{F^{i+1} H_n}$. Since $E^{n-d-i,d+i}_\infty = 0$ for all $i \geq 1$ then $E^{n-d,d}_\infty \cong H^n_\mathfrak{m}(R)$. Note that $E^{n-d,d}_\infty$ is the quotient of $H^{n-d}_\mathfrak{m}(H^d_\mathrm{I}(R))$. Then $H^{n-d}_\mathfrak{m}(H^d_\mathrm{I}(R))$ must be non-zero. It implies that $\dim_R(H^d_\mathrm{I}(R)) = n - d \leq \operatorname{inj.dim}_R(H^d_\mathrm{I}(R))$. □

LEMMA 3.6. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension n which contains a field of characteristic $p > 0$. Let $\mathfrak{p}$ be a prime ideal of R such that $\operatorname{ht}_R \mathfrak{p} = d < n - 1$. Then $E_R(R/\mathfrak{p}) \cong H^d_\mathfrak{p}(R)_\mathfrak{p}$ with natural $\mathscr{F}$-module structure is not $\mathscr{F}$-finite.*

Proof. Note that $E_R(R/\mathfrak{p}) \cong H^d_{\mathfrak{p}}(R)_{\mathfrak{p}}$ and by Remark 2.3(e), $E_R(R/\mathfrak{p})$ has a natural $\mathscr{F}$-module structure.

First, assume that $\mathrm{ht}_R(\mathfrak{p}) = n-2$. By virtue of Proposition 3.5, $\mathrm{inj.dim}_R\, H^{n-2}_{\mathfrak{p}}(R) = 2$. Consider the following minimal injective resolution of $H^{n-2}_{\mathfrak{p}}(R)$.

$$0 \to H^{n-2}_{\mathfrak{p}}(R) \to E_R(R/\mathfrak{p}) \to E^1 \to E^2 \to 0.$$

By Remark 2.3(c), this is a complex of $\mathscr{F}$-modules and $\mathscr{F}$-homomorphisms. In view of Lemma 3.3 and Remark 2.4(d), $E^2 \cong E_R(R/\mathfrak{m})^s$ where s is a positive integer. Suppose on the contrary $E_R(R/\mathfrak{p})$ is $\mathscr{F}$-finite. Then following Remark 2.4(a), E^1 must be $\mathscr{F}$-finite. There exist infinitely many primes $\mathfrak{q} \in \mathrm{Spec}(R)$ which $\mathfrak{p} \subset \mathfrak{q}$ and $\mathrm{ht}_R(\mathfrak{q}) = n-1$. For all such $\mathfrak{q} \in \mathrm{Spec}(R)$, in view of Proposition 3.5, $\mathrm{inj.dim}_{R_{\mathfrak{q}}}\, H^{n-2}_{\mathfrak{p}R_{\mathfrak{q}}}(R_{\mathfrak{q}}) = 1$ and considering Lemma 3.3 we have $\mu^1(\mathfrak{q}, H^{n-2}_{\mathfrak{p}}(R)) > 0$. So we reach to a contradiction in view of Remark 2.4(d), (e).

For the convenience of the reader, we bring a different proof of the fact that $\mu^1(\mathfrak{q}, H^{n-2}_{\mathfrak{p}}(R)) > 0$ suggested by the referee. Suppose $\mathfrak{q} \supseteq \mathfrak{p}$ such that $\mathrm{ht}_R(\mathfrak{q}) = n-1$. Claim $E^1_{\mathfrak{q}} \neq 0$.

Suppose if possible $E^1_{\mathfrak{q}} = 0$. We have $H^{n-2}_{\mathfrak{p}R_{\mathfrak{q}}}(R_{\mathfrak{q}})$ is an injective $R_{\mathfrak{q}}$-module. Choose g such that $(\mathfrak{p}R_{\mathfrak{q}}, g)$ is $\mathfrak{q}R_{\mathfrak{q}}$-primary. By using the standard long-exact sequence of local cohomology modules and Hartshorne–Lichtenbaum theorem, we have an exact sequence

$$0 \to H^{n-2}_{\mathfrak{p}R_{\mathfrak{q}}}(R_{\mathfrak{q}}) \to \left(H^{n-2}_{\mathfrak{p}R_{\mathfrak{q}}}(R_{\mathfrak{q}})\right)_g \to H^{n-1}_{\mathfrak{q}R_{\mathfrak{q}}}(R_{\mathfrak{q}}) \to 0.$$

As $H^{n-2}_{\mathfrak{p}R_{\mathfrak{q}}}(R_{\mathfrak{q}})$ is an injective $R_{\mathfrak{q}}$-module we get that $\mathfrak{q}R_{\mathfrak{q}} \in \mathrm{Ass}_{R_{\mathfrak{q}}}(H^{n-2}_{\mathfrak{p}R_{\mathfrak{q}}}(R_{\mathfrak{q}}))_g$ which is a contradiction.

Now suppose $\mathrm{ht}_R(\mathfrak{p}) = n-3$. Let $\mathfrak{q} \in \mathrm{Spec}(R)$ such that $\mathrm{ht}_R(\mathfrak{q}) = n-1$ and $\mathfrak{p} \subset \mathfrak{q}$. Suppose on the contrary that $E_R(R/\mathfrak{p})$ is $\mathscr{F}$-finite. Thus $E_R(R/\mathfrak{p})_{\mathfrak{q}}$ is $\mathscr{F}_{R_{\mathfrak{q}}}$-finite by Remark 2.4(f). This contradicts with the first step of the proof.

By applying this argument for a finite step, we prove the lemma. $\square$

REMARK 3.7. (i) Adopt the above notations of Lemma 3.6. Let $\mathfrak{p}$ be a prime ideal of R such that $\mathrm{ht}(\mathfrak{p}) \geq n-1$. Then it is easy to see that $E_R(R/\mathfrak{p})$ is $\mathscr{F}$-finite. Indeed if $\mathfrak{p} = \mathfrak{m}$ then $E_R(R/\mathfrak{m}) = H^n_{\mathfrak{m}}(R)$. Otherwise let

$$0 \to H^{n-1}_{\mathfrak{p}}(R) \to E_R(R/\mathfrak{p}) \to E^1 \to 0$$

be the minimal injective resolution of $H^{n-1}_{\mathfrak{p}}(R)$. In view of Lemma 3.3 and Remark 2.4(d), $E^1 \cong E_R(R/\mathfrak{m})^s$ where s is a positive integer. Thus by Remark 2.4(a), $E_R(R/\mathfrak{p})$ is $\mathscr{F}$-finite.

(ii) Let $R = k[[x_1, \ldots, x_n]]$ and characteristic of k is 0. Let $\mathfrak{p}$ be a prime ideal of R such that $\mathrm{ht}(\mathfrak{p}) \geq n-1$. As (i) one can easily see that $E_R(R/\mathfrak{p})$ is holonomic.

Let M be a finitely generated module over a Cohen–Macaulay ring R such that $\text{inj.dim}_R(M)$ is finite and therefore it equals to $\dim R$. Then it is elementary to prove that if $\mu^{\dim R}(\mathfrak{p}, M) > 0$ then $\mathfrak{p}$ is a maximal ideal in R, use [4, Proposition 3.1.13]. Although this fact is not true for R-module M that is not finitely generated. For example Let $\mathfrak{p}$ be a prime ideal of R and M be the injective envelope of $R/\mathfrak{p}$.

For polynomial ring $R = k[x_1, \dots, x_n]$ with field k of characteristic zero, Puthenpurakal proved if $\text{inj.dim}_R(H^i_{\mathrm{I}}(R)) = c$ and $\mu^c(\mathfrak{p}, H^i_{\mathrm{I}}(R)) > 0$ for prime ideal $\mathfrak{p}$ of R, then $\mathfrak{p}$ is a maximal ideal of R, see [10, Theorem 1.1]. In the following proposition, we generalize his theorem to the case that R is a regular local ring which contains a field.

Proposition 3.8. *Let R be a regular local ring of dimension n which contains a field k. Let M be an R-module such that* $\text{inj.dim}_R(M) = c$ *and* $\mu^c(\mathfrak{p}, M) \neq 0$ *for a prime ideal $\mathfrak{p}$ of R. Assume that one of the following holds*:

- (i) *k is a field of characteristic $p > 0$ and M be a $\mathscr{F}$-finite.*
- (ii) *$R = k[[x_1, \dots, x_n]]$ and characteristic of k is 0 and M is a holonomic module.*
- (iii) *k is a field of characteristic 0 and $M = H^j_I(R)_f$ where I is an ideal of R and $f \in R$.*

Then $\text{ht}_R(\mathfrak{p}) \geq n - 1$.

Proof. We first show that $H^i_{\mathfrak{p}}(M)_{\mathfrak{p}}$ is an injective R-module for all positive integer i. In case (i), $H^i_{\mathfrak{p}}(M)_{\mathfrak{p}}$ is zero or an $\mathscr{F}_{R_{\mathfrak{p}}}$-finite module of dimension 0, see 2.4(c), (f). Then by 2.3(d) and 2.4(d) $H^i_{\mathfrak{p}}(M)_{\mathfrak{p}} \cong E_R(R/\mathfrak{p})^s$ where s is a positive integer. In case (ii), we note that $H^i_{\mathfrak{p}}(M)$ is a holonomic $\mathscr{D}$-module, see Remark 2.2(a). Let $\hat{R_{\mathfrak{p}}}$ denote the completion of $R_{\mathfrak{p}}$ with respect to the maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$. It follows that $H^i_{\mathfrak{p}}(M)_{\mathfrak{p}}$ has a natural structure of $\mathscr{D}(\hat{R_{\mathfrak{p}}}, k')$-module where k' is a suitable coefficient field of $\hat{R_{\mathfrak{p}}}$, see the proof of [7, Theorem 2.4(b)]. So, by Remark 2.1(a), $H^i_{\mathfrak{p}}(M)_{\mathfrak{p}}$ is a direct sum of copies of $E_{\hat{R_{\mathfrak{p}}}}(\hat{R_{\mathfrak{p}}}/\mathfrak{p}\hat{R_{\mathfrak{p}}})$. But as an R-module $E_{\hat{R_{\mathfrak{p}}}}(\hat{R_{\mathfrak{p}}}/\mathfrak{p}\hat{R_{\mathfrak{p}}})$ is isomorphic to $E_R(R/\mathfrak{p})$, so $H^i_{\mathfrak{p}}(M)_{\mathfrak{p}}$ is an injective R-module. Also $H^i_{\mathfrak{p}}(M)_{\mathfrak{p}}$ is a direct sum of finite copies of $E_R(R/\mathfrak{p})$, see Remark 2.2(f). In case (iii), $(H^i_{\mathfrak{p}}(H^j_{\mathrm{I}}(R)))_{\mathfrak{p}} \cong E_R(R/\mathfrak{p})^s$ where s is a positive integer, see [7, Theorem 3.4(b), (d)]. Then

$$\begin{aligned} H^i_{\mathfrak{p}}(M)_{\mathfrak{p}} &= H^i_{\mathfrak{p}}\big(H^j_I(R)_f\big)_{\mathfrak{p}} \cong \big(H^i_{\mathfrak{p}}\big(H^j_I(R)\big)_f\big)_{\mathfrak{p}} \\ &\cong \big(H^i_{\mathfrak{p}}\big(H^j_{\mathrm{I}}(R)\big)\big)_{\mathfrak{p}} \otimes_R R_f \cong E_R(R/\mathfrak{p})^s \otimes_R R_f. \end{aligned}$$

Hence, $(H^i_{\mathfrak{p}}(M))_{\mathfrak{p}} \cong E_R(R/\mathfrak{p})^t$ where t is a positive integer. So $(H^i_{\mathfrak{p}}(M))_{\mathfrak{p}}$ is an injective R-module.

Therefore, in three cases, we have $\mu^0(\mathfrak{p}, H^c_{\mathfrak{p}}(M)) = \mu^c(\mathfrak{p}, M) > 0$, see [7, Lemma 1.4]. Note that by the above discussion $H^c_{\mathfrak{p}}(M)_{\mathfrak{p}} \cong E_R(R/\mathfrak{p})^s$ where $s > 0$ is an integer.

Suppose on the contrary $\operatorname{ht}_R(\mathfrak{p}) \le n-2$. Note that $\operatorname{Ass}_R(H^c_{\mathfrak{p}}(M))$ is finite, see Remarks 2.4(e), 2.2(f) and [7, Theorem 3.4(c)]. Let $\operatorname{Ass}_R(H^c_{\mathfrak{p}}(M)) = \{\mathfrak{p}, \mathfrak{q}_1, \dots, \mathfrak{q}_m\}$. Look at the exact sequence:

$$0 \to \Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}\big(H^c_{\mathfrak{p}}(M)\big) \to H^c_{\mathfrak{p}}(M) \to H^c_{\mathfrak{p}}(M)/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}\big(H^c_{\mathfrak{p}}(M)\big) \to 0.$$

Since $\mathfrak{p} \subsetneqq \mathfrak{q}_i$, we have $\mathfrak{p} \notin \operatorname{Ass}_R \Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H^c_{\mathfrak{p}}(M))$. Keep in mind that

$$\operatorname{Ass}_R H^c_{\mathfrak{p}}(M) = \operatorname{Ass}_R \Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}\big(H^c_{\mathfrak{p}}(M)\big) \cup \operatorname{Ass}_R H^c_{\mathfrak{p}}(M)/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}\big(H^c_{\mathfrak{p}}(M)\big).$$

It follows that $\operatorname{Ass}_R H^c_{\mathfrak{p}}(M)/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H^c_{\mathfrak{p}}(M)) = \{\mathfrak{p}\}$.

Let $g \in R \setminus \mathfrak{p}$. Then the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H^c_{\mathfrak{p}}(M)) \arrow[r] \arrow[d] & H^c_{\mathfrak{p}}(M) \arrow[r] \arrow[d] & H^c_{\mathfrak{p}}(M)/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H^c_{\mathfrak{p}}(M)) \arrow[r] \arrow[d, "\eta"] & 0 \\
0 \arrow[r] & (\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H^c_{\mathfrak{p}}(M)))_g \arrow[r] & H^c_{\mathfrak{p}}(M)_g \arrow[r] & (H^c_{\mathfrak{p}}(M)/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H^c_{\mathfrak{p}}(M)))_g \arrow[r] & 0.
\end{tikzcd}$$

Recall that $\operatorname{inj.dim}_R M = c$. Thus, there is an exact sequence

$$H^c_{(\mathfrak{p},g)}(M) \to H^c_{\mathfrak{p}}(M) \to H^c_{\mathfrak{p}}(M)_g \to H^{c+1}_{(\mathfrak{p},g)}(M) = 0.$$

Hence, the natural map η is surjective. As $g \notin \mathfrak{p}$, we get that η is also injective. Thus, $H^c_{\mathfrak{p}}(M)/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H^c_{\mathfrak{p}}(M)) = (H^c_{\mathfrak{p}}(M)/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H^c_{\mathfrak{p}}(M)))_g$ for all $g \in R \setminus \mathfrak{p}$. It follows that $H^c_{\mathfrak{p}}(M)/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H^c_{\mathfrak{p}}(M)) = (H^c_{\mathfrak{p}}(M)/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H^c_{\mathfrak{p}}(M)))_{\mathfrak{p}}$.

Note that $(\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H^c_{\mathfrak{p}}(M)))_{\mathfrak{p}} = 0$. We deduce that

$$H^c_{\mathfrak{p}}(M)_{\mathfrak{p}} \cong \big(H^c_{\mathfrak{p}}(M)/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}\big(H^c_{\mathfrak{p}}(M)\big)\big)_{\mathfrak{p}} \cong H^c_{\mathfrak{p}}(M)/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}\big(H^c_{\mathfrak{p}}(M)\big).$$

Now we prove the proposition

(i) Clearly $H^c_{\mathfrak{p}}(M)/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H^c_{\mathfrak{p}}(M))$ is $\mathscr{F}$-finite. Putting this along with $(H^c_{\mathfrak{p}}(M))_{\mathfrak{p}} \cong E_R(R/\mathfrak{p})^s$, we conclude that $E_R(R/\mathfrak{p})$ is $\mathscr{F}$-finite. So we reach to a contradiction because by Lemma 3.6 $E_R(R/\mathfrak{p})$ cannot be $\mathscr{F}$-finite.

(ii) Exactly same (i): $H^c_{\mathfrak{p}}(M)/\Gamma_{\mathfrak{q}_1 \dots \mathfrak{q}_m}(H^c_{\mathfrak{p}}(M))$ is holonomic and it is contradicts with Lemma 3.1.

(iii) Let $\hat{R}$ be the completion of R with respect to maximal ideal $\mathfrak{m}$. Then

$$\big(H^c_{\mathfrak{p}\hat{R}}\big(H^j_{\mathrm{I}\hat{R}}(\hat{R})_f\big)\big)/\Gamma_{(\mathfrak{q}_1 \dots \mathfrak{q}_m)\hat{R}}\big(H^c_{\mathfrak{p}\hat{R}}\big(H^j_{\mathrm{I}\hat{R}}(\hat{R})_f\big)\big) \cong E_R(R/\mathfrak{p})^s \otimes_R \hat{R}.$$

But $E_R(R/\mathfrak{p})^s \otimes_R \hat{R}$ is not holonomic by Lemma 3.2. □

EXAMPLE 3.9. Let $R = k[[x,y,z]]$ be a power series ring over a field k and let I be the ideal $(xy, xz)R$ of R. Then $\dim_R H^i_{\mathrm{I}}(R) = 1$ and $\operatorname{inj.dim}_R(H^i_{\mathrm{I}}(R)) = 0$, see [5, Examples 2.9]. Thus, there exists $\mathfrak{p} \in \operatorname{Ass}_R(H^i_{\mathrm{I}}(R))$ such that $\operatorname{ht}_R(\mathfrak{p}) = 2$. It is well known that for all R-module M, $\mathfrak{q} \in \operatorname{Ass}_R(M)$ if and only if $\mu^0(\mathfrak{q}, M) > 0$. It follows that $\mu^0(\mathfrak{p}, H^i_{\mathrm{I}}(R)) > 0$. Thus, the lower bound for the prime ideal $\mathfrak{p}$ in the Proposition 3.8 is not strict.

4. Main theorem

In this section, we prove our main result about injective dimension of local cohomology.

Theorem 4.1. *Let* $(R, \mathfrak{m})$ *be a regular local ring which contains a field. Let* I *be an ideal of* R. *Suppose that one of the following two conditions* (i) *or* (ii) *holds*:

(i) *R is of prime characteristic $p > 0$ and M is an $\mathscr{F}$-finite module.*
(ii) *R is of characteristic 0 and $M = H^i_{\mathrm{I}}(R)_f$ for some $f \in R$.*

Then $\dim_R M - 1 \leq \operatorname{inj.dim}_R M$.

Proof. We prove the theorem by induction on $\dim(M)$. If $\dim(M) \leq 1$, we have nothing to prove. In case (i), assume that for every $\mathscr{F}$-finite module of the dimension less than n the theorem is true. In case (ii), assume that for every R module $\mathcal{N} = H^j_I(R)_g$ of dimension less than n the theorem is true such that $g \in R$.

Now suppose M be an R-module of dimension $n > 1$ which satisfies either (i) or (ii).

Let $\mathfrak{p}$ be a prime ideal of R such that $\dim_{R_\mathfrak{p}}(M)_\mathfrak{p} = n-1$. Then $M_\mathfrak{p}$ satisfies induction hypothesis. Hence $n-2 \leq \operatorname{inj.dim}_{R_\mathfrak{p}}(M)_\mathfrak{p}$. If $\operatorname{inj.dim}_{R_\mathfrak{p}}(M)_\mathfrak{p} = n-1$, we are done. So we assume $\operatorname{inj.dim}_{R_\mathfrak{p}}(M)_\mathfrak{p} = n-2$. We claim that there is a prime ideal $\mathfrak{q} \subsetneq \mathfrak{p}$ such that $\mu^{n-2}(\mathfrak{q}, M) \neq 0$. Suppose on the contrary there is not such prime ideal. Pick $g \in \mathfrak{p}$ such that $\dim_{R_\mathfrak{p}}((M)_g) = n-1$. Then $(M)_g$ satisfies the induction hypothesis, see Remark 2.4(b). But $\operatorname{inj.dim}_{R_\mathfrak{p}}(M)_g < n-2$ and this contradicts with the induction hypothesis.

So there is a prime ideal $\mathfrak{q} \subsetneq \mathfrak{p}$ such that $\mu^{n-2}(\mathfrak{q}, M) \neq 0$. In view of Proposition 3.8(i), (iii) we conclude that $n-1 \leq \operatorname{inj.dim} M$, as desired. $\square$

Remark 4.2. Note that in view of Example 3.9, the lower bound in the main theorem is not strict.

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References

[1] B. Bhatt, M. Blickle, G. Lyubeznik, A. Singh and W. Zhang, *Local cohomology modules of a smooth $\mathbb{Z}$-algebra have finitely many associated primes*, Invent. Math. **197** (2014), 509–519. MR 3251828

[2] J.-E. Björk, *Rings of differential operators*, North Holland, Amsterdam, 1979. MR 0549189

[3] M. Brodmann and R. Y. Sharp, *Local cohomology: An algebraic introduction with geometric application*, vol. 60, Cambridge University Press, Cambridge, 1998. MR 1613627

[4] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, vol. 39, Cambridge University Press, Cambridge, 1998. MR 1251956

[5] M. Hellus, *A note on the injective dimension of local cohomology modules*, Proc. Amer. Math. Soc. **136** (2008), 2313–2321. MR 2390497

[6] C. Huneke and R. Sharp, *Bass numbers of local cohomology modules*, Trans. Amer. Math. Soc. **339** (1993), 765–779. MR 1124167

[7] G. Lyubeznik, *Finiteness properties of local cohomology modules (an application of D-modules to commutative algebra)*, Invent. Math. **113** (1993), 41–55. MR 1223223

[8] G. Lyubeznik, *F-modules: Applications to local cohomolgy and D-modules in chracteristic $P>0$*, J. Reine Angew. Math. **491** (1997), 65–130. MR 1476089

[9] H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Math, vol. 8, 1986. MR 0879273

[10] T. J. Puthenpurakal, *On injective resolution of local cohomology modules*, Illinois J. Math. **58** (2014), 709–718. MR 3395959

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