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A NOTE ON NONEXISTENCE OF MULTIPLE BLACK HOLES IN STATIC VACUUM EINSTEIN SPACE–TIMES

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ABSTRACT. The purpose of this note is to study the static vacuum Einstein space–time with half harmonic Weyl tensor, that is, $\delta W^{+}=0$. We prove that there are no multiple black holes on a four-dimensional static vacuum Einstein space–time with half harmonic Weyl tensor.

1. Introduction

In the last few decades have been a steadily growing interest in the study of the static space–times. A fundamental question on this subject is related with the uniqueness of black hole as well as the nonexistence of multiple black holes in static space. In this context, in a celebrated article [14], Israel gave the first answer for the uniqueness of black hole. More precisely, he proved that a static, topologically spherical black hole is described by the Schwarzschild or the Reissner–Nördström solutions. Afterward, inspired by [9], [14], [18], Bunting and Masood-ul-Alam [6] studied such a problem in an asymptotically Euclidean static vacuum space–time. In general, many authors have investigated this problem and provided important contributions to the development of this theory, we refer the reader to [11], [12], [10], [13] and [20] for an overview of the progress on such a subject.

DEFINITION 1. A Riemannian manifold (M^n,g), $n\geq 3$, is said to be a static vacuum Einstein space–time if there exist a lapse function $f:M\to(0,+\infty)$ satisfying the static vacuum Einstein equation

$$\nabla^2 f = f Ric \quad \text{and} \quad \Delta f = 0. \tag{1.1}$$

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A straightforward computation ensures $R=0$, where R stands for the scalar curvature of g. Moreover, it is known that the only complete solution to the static vacuum equations (1.1) with $f>0$ everywhere is a flat metric, with $f=\text{constant}$ (cf. Theorem 3.2 in [1]).

In the sequel, given a *static metric*

$$\bar{g} = g - f^2\,dt^2 \tag{1.2}$$

on $\overline{M}^{n+1} = M^n \times_f \mathbb{R}$ (cf. [10], [17], [15], [16], [20]), it is well known that:

- $Ric_{\bar{g}}(X,Y) = Ric_g(X,Y) - \frac{1}{f}\nabla_g^2 f(X,Y)$,
- $Ric_{\bar{g}}(V,H) = -g(V,H)\frac{\Delta_g f}{f}$ and
- $Ric_{\bar{g}}(X,V) = 0$,

where ∇_g^2 and Δ_g are, respectively, the Hessian and the Laplacian operator for g. Moreover, X and Y are horizontal vector fields, while H and V are vertical vector fields (see [5], [19]). From this, $\overline{M}$ is Ricci-flat if and only if the lapse function f satisfies (1.1).

Here, we consider non-trivial solutions of the static vacuum Einstein equation (1.1), complete and connected up to the boundary ∂M of M. Moreover, we assume that the set $f^{-1}(0) = \partial M$ is compact, and that the metric g and the function f extends smoothly to ∂M. To do so, let us recall that the set $\partial M = f^{-1}(0)$ is called the *horizon*, which corresponds to domains surrounding a collection of black holes. We say that there are no multiple black holes in (M^n, g) when the horizon $\partial M = f^{-1}(0)$ is connected. For more details see, for instance, [1] and [13].

It is already known that four-dimensional Riemannian manifolds are very special. For instance, it is well known that the bundle of 2-forms on a 4-dimensional compact oriented Riemannian manifold can be invariantly decomposed as a direct sum (cf. [5], [8]). Moreover, on an oriented Riemannian manifold (M^4, g), the Weyl curvature tensor W is an endomorphism of the bundle of 2-forms $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ such that

$$W = W^+ \oplus W^-,$$

where $W^\pm : \Lambda^2_\pm \longrightarrow \Lambda^2_\pm$ are called of the self-dual and anti-self-dual parts of W. Half conformally flat metrics are also known as self-dual or anti-self-dual if $W^- = 0$ or $W^+ = 0$, respectively.

For what follows, we recall that the tensor W^+ is harmonic if $\delta W^+ = 0$, where δ is the formal divergence defined for any $(0,4)$-tensor F by

$$\delta F(X_1, X_2, X_3) = \text{trace}_g\big\{(Y,Z) \mapsto \nabla_Y F(Z, X_1, X_2, X_3)\big\},$$

where g is the metric of M^4. It is worth to point out that in dimension 4 we have

$$|\delta W|^2 = \left|\delta W^+\right|^2 + \left|\delta W^-\right|^2.$$

From here it follows that the half harmonic Weyl tensor assumption (that is, $\delta W^{+}=0$) is weaker than the harmonic Weyl tensor condition (that is, $\delta W=0$). Moreover, it is well-known that compact oriented 4-dimensional manifolds with parallel Ricci tensor must have $\delta W^{+}=0$. This implies that every four-dimensional Einstein manifold has half harmonic Weyl tensor (cf. 16.65 in [5], see also Lemma 6.14 in [8]). But, the converse statement is not necessarily true. Therefore, according to [5] "Besse's book", the assumption $\delta W^{+}=0$ can be seen as a generalization of the Einstein condition. For a detailed overview on the half harmonic Weyl tensor condition see Chapter 16 (Section H) in [5]. From these comments, it is natural to ask which geometric implications has the assumption of the harmonicity of the tensor W^{+} on a four-dimensional static space–times.

Before proceeding, it is convenient to recall that a Riemannian manifold (M^n,g) has f-weakly harmonic curvature if the Ricci tensor Ric_g satisfies

$$d^D Ric_g(\nabla f,\cdot,\nabla f)=0$$

for a function $f:M\to\mathbb{R}$, where d^D is the first-order differential operator from the space of sections of symmetric 2-tensors $C^\infty(S^2M)$ into $C^\infty(\bigwedge^2 T^*M\otimes T^*M)$ defined by

$$d^D\omega(X,Y,Z)=\nabla_X\omega(Y,Z)-\nabla_Y\omega(X,Z).$$

With these notations, recently, Hwang, Chang and Yun [13], studied static vacuum Einstein space–time with f-weakly harmonic curvature. More precisely, they proved the following result.

Theorem 1 (Hwang–Chang–Yun, [13]). *Let (M^n,g,f) be a static vacuum Einstein space–time satisfying* (1.1) *with f-weakly harmonic curvature. Then there are no multiple black holes in M^n.*

In this article, we shall replace the assumption of f-weakly harmonic curvature in the Hwang–Chang–Yun result by the hypotheses that the tensor W^{+} is harmonic on M. More precisely, we have established the following result.

Theorem 2. *Let (M^4,g,f) be a static vacuum Einstein space–time satisfying* (1.1) *with half harmonic Weyl tensor* (*i.e.*, $\delta W^{+}=0$). *Then there are no multiple black holes in M^4.*

Obviously if we change the condition $\delta W^{+}=0$ by the condition $\delta W^{-}=0$ the conclusion of Theorem 2 is the same. Furthermore, one should be emphasized that there is no relationship between f-weakly harmonic curvature and the condition that manifold has harmonic tensor W^{+}.

2. Preliminaries

In this section, we shall present some preliminaries which will be useful for the establishment of the desired result. We start recalling that for a Riemannian manifold (M^n,g), $n\geq 3$, the Weyl tensor W is defined by the following decomposition formula

$$R_{ijkl} = W_{ijkl} + \frac{1}{n-2}(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}) - \frac{R}{(n-1)(n-2)}(g_{jl}g_{ik} - g_{il}g_{jk}), \tag{2.1}$$

where R_{ijkl} stands for the Riemannian curvature operator. Moreover, the Cotton tensor C is given according to

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(\nabla_i R g_{jk} - \nabla_j R g_{ik}). \tag{2.2}$$

These two tensors are related as follows

$$C_{ijk} = -\frac{(n-2)}{(n-3)}\nabla^l W_{ijkl}, \tag{2.3}$$

provided $n \geq 4$.

In what follows, M^4 will denote an oriented 4-dimensional manifold and g is a Riemannian metric on M^4. As it was previously pointed out 4-manifolds are fairly special. For instance, following the notations used in [8], given any local orthogonal frame $\{e_1, e_2, e_3, e_4\}$ on an open set of M^4 with dual basis $\{e^1, e^2, e^3, e^4\}$, there exists a unique bundle morphism $*$ called *Hodge star* (acting on bivectors), such that

$$*\big(e^1 \wedge e^2\big) = e^3 \wedge e^4.$$

This implies that $*$ is an involution, that is, $*^2 = Id$. In particular, this ensures that the bundle of 2-forms on a 4-dimensional oriented Riemannian manifold can be invariantly decomposed as a direct sum $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$. From this, it follows that the Weyl tensor W is an endomorphism of $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ such that

$$W = W^+ \oplus W^-. \tag{2.4}$$

Recalling that the Weyl tensor is trace-free on any pair of indices, we have

$$W^+_{pqrs} = \frac{1}{2}(W_{pqrs} + W_{pq\overline{rs}}), \tag{2.5}$$

where $(\overline{rs})$, for instance, stands for the dual of (rs), that is, $(rs\overline{rs}) = \sigma(1234)$ for some even permutation σ in the set $\{1, 2, 3, 4\}$ (cf. Equation 6.17, p. 466 in [8]). For instance, we have

$$W^+_{1234} = \frac{1}{2}(W_{1234} + W_{1212}).$$

For more details we refer to [4], [3], [5], [8].

The next result, which can be found in [13], will be useful in the proof of our main result.

LEMMA 1 ([13]). *Let (M^n, g, f) be a static vacuum Einstein space–time. If f is non-trivial, then the set* $\mathrm{Crit}(f) = \{p \in M^n; \nabla f(p) = 0\}$ *has zero n-dimensional measure.*

3. Proof of the main result

Our approach is inspired by ideas outlined in [4], [3] and [2]. To start with, we show a formula relating the Cotton tensor with the Weyl tensor on a static vacuum Einstein space–time.

LEMMA 2. *Let (M^n, g, f) be a static vacuum Einstein space–time. Then*:

$$fC_{ijk} = W_{ijks}\nabla^s f + \frac{(n-1)}{(n-2)}(R_{ik}\nabla_j f - R_{jk}\nabla_i f) - \frac{1}{(n-2)}\big(R_{is}\nabla^s f g_{jk} - R_{js}\nabla^s f g_{ik}\big).$$

Proof. First, taking the covariant derivative of (1.1), we have

$$\nabla_i f R_{jk} + f\nabla_i R_{jk} = \nabla_i\nabla_j\nabla_k f.$$

Then, from Ricci equation we get that

$$R_{jk}\nabla_i f - R_{ik}\nabla_j f + f(\nabla_i R_{jk} - \nabla_j R_{ik}) = R_{ijkl}\nabla^l f.$$

Since $R = 0$, from (2.2), we obtain

$$R_{jk}\nabla_i f - R_{ik}\nabla_j f + fC_{ijk} = R_{ijkl}\nabla^l f \tag{3.1}$$

and, from the Weyl tensor formula (2.1) we achieve

$$R_{ijkl}\nabla^l f = W_{ijkl}\nabla^l f + \frac{1}{n-2}\big(R_{ik}\nabla_j f - R_{jk}\nabla_i f + R_{jl}\nabla^l f g_{ik} - R_{il}\nabla^l f g_{jk}\big).$$

Combining the above equation with (3.1), we get the promised result. ☐

Next, following the notations employed in [4], [3], we define the tensor T_{ijk} as follows

$$T_{ijk} = \frac{(n-1)}{(n-2)}(R_{ik}\nabla_j f - R_{jk}\nabla_i f) - \frac{1}{(n-2)}\big(R_{is}\nabla^s f g_{jk} - R_{js}\nabla^s f g_{ik}\big). \tag{3.2}$$

Taking into account this definition, we deduce from Lemma 2 that

$$fC_{ijk} = W_{ijks}\nabla^s f + T_{ijk}. \tag{3.3}$$

An analogous proof for the next lemma can be found in [3]. Nonetheless, since its proof is non-trivial, for sake of completeness, we shall sketch it here.

LEMMA 3. *Let (M^4, g, f) be a complete static vacuum Einstein space–time with harmonic (anti-)self dual Weyl tensor. Then ∇f is an eigenvector of the Ricci curvature Ric.*

Proof. Since the scalar curvature is zero, we know that (2.2) becomes

$$C_{klj} = \nabla_k R_{lj} - \nabla_l R_{kj}.$$

So, as an immediate consequence of (2.3), we have

$$4\delta W^+_{jkl} = C_{klj} + C_{\overline{kl}j}. \tag{3.4}$$

From Lemma (2) and Eq. (3.4), we get

$$\begin{aligned} 4f\delta W^+_{jkl} &= f\big[(\nabla_k R_{jl} - \nabla_l R_{jk}) + (\nabla_{\overline{k}} R_{j\overline{l}} - \nabla_{\overline{l}} R_{j\overline{k}})\big] \\ &= \big[W_{kljs}\nabla^s f + W_{\overline{kl}js}\nabla^s f + T_{lkj} + T_{\overline{lk}j}\big]. \end{aligned} \tag{3.5}$$

In the sequel, we shall use our assumption $\delta W^+ = 0$. In order to do so, we consider an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ diagonalizing Ric at a point q, such that $\nabla f(q) \neq 0$, with associated eigenvalues λ_k $(k = 1, \ldots, 4)$, respectively. It is important to highlight that the regular points of M^4, denoted by $\{p \in M^4 : \nabla f(p) \neq 0\}$, is dense in M^4. Otherwise, f must be constant in an open set of M^4; for more details, see, for instance [7]. Therefore, from (3.2) and (3.5) we have

$$\begin{cases} (\lambda_1 - \lambda_2)\nabla_1 f \nabla_2 f + (\lambda_3 - \lambda_4)\nabla_3 f \nabla_4 f = 0, \\ (\lambda_1 - \lambda_3)\nabla_1 f \nabla_3 f + (\lambda_4 - \lambda_2)\nabla_4 f \nabla_2 f = 0, \\ (\lambda_1 - \lambda_4)\nabla_1 f \nabla_4 f + (\lambda_2 - \lambda_3)\nabla_2 f \nabla_3 f = 0. \end{cases} \tag{3.6}$$

We now claim that ∇f, whenever nonzero, is an eigenvector for Ric. In fact, taking into account that $\nabla f(p) \neq 0$ we have that, at least, one of the $(\nabla_j f) \neq 0$, $1 \leq j \leq 4$. If this occurs for exactly one of them, then $\nabla f = (\nabla_j f)e_j$ for some j, which gives that $Ric(\nabla f) = \lambda_j \nabla f$. On the other hand, if we have $(\nabla_j f) \neq 0$ for two directions, without loss of generality we can suppose that $\nabla_1 f \neq 0$, $\nabla_2 f \neq 0$, $\nabla_3 f = 0$ and $\nabla_4 f = 0$. Then, from (3.6) we have $\lambda_1 = \lambda_2 = \lambda$. In such a case we have $\nabla f = (\nabla_1 f)e_1 + (\nabla_2 f)e_2$. From this, we infer

$$\begin{aligned} Ric(\nabla f) &= Ric\big((\nabla_1 f)e_1 + (\nabla_2 f)e_2\big) = (\nabla_1 f)Ric(e_1) + (\nabla_2 f)Ric(e_2) \\ &= (\nabla_1 f)\lambda_1 e_1 + (\nabla_2 f)\lambda_2 e_2 = \lambda \nabla f. \end{aligned}$$

Next, the case $(\nabla_j f) \neq 0$ for three directions is analogous. Now, it remains to analyze the case $(\nabla_j f) \neq 0$ for $j = 1, 2, 3$ and 4. In this case we use again (3.6) to obtain

$$\begin{aligned} &(\lambda_1 - \lambda_2)^2(\nabla_1 f \nabla_2 f)^2 + (\lambda_3 - \lambda_4)^2(\nabla_3 f \nabla_4 f)^2 \\ &\quad + (\lambda_1 - \lambda_3)^2(\nabla_1 f \nabla_3 f)^2 + (\lambda_4 - \lambda_2)^2(\nabla_4 f \nabla_2 f)^2 \\ &\quad + (\lambda_1 - \lambda_4)^2(\nabla_1 f \nabla_4 f)^2 + (\lambda_2 - \lambda_3)^2(\nabla_2 f \nabla_3 f)^2 = 0. \end{aligned}$$

Therefore, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$. Of which follows that ∇f is an eigenvector for Ric. This finishes the proof of the lemma. □

3.1. Proof of Theorem 2.

Proof. Proceeding, for any point $p \in M$ where $\nabla f(p) \neq 0$, we consider a local coordinates systems $\{\theta^2, \theta^3, \theta^4\}$ on the level surface $\{x \in M : f(x) = f(p)\}$. In this case, for any neighbourhood of the level surface Σ where $|\nabla f| \neq 0$, we use the local coordinates system

$$\left(x^1, x^2, x^3, x^4\right) = \left(f, \theta^2, \theta^3, \theta^4\right)$$

adapted to level surfaces. Under this above notation, the metric g can be expressed as

$$ds^2 = \frac{1}{|\nabla f|^2}\, df^2 + g_{ab}(f,\theta)\, d\theta^a\, d\theta^b,$$

where $a, b \in \{2,3,4\}$. In what follows from Lemma 3, we will consider the normal vector field $e_1 = \frac{\nabla f}{|\nabla f|}$ to Σ_c and e_2, e_3, e_4 as an orthonormal frame on Σ_c such that $\{e_1, e_2, e_3, e_4\}$ orthogonalizes the Ricci tensor Ric.

With this notation in mind, since $Ric(\nabla f) = \lambda \nabla f$ and $\nabla_{e_a} f = g(\nabla f, e_a) = 0$ for $a = \{2,3,4\}$, we immediately deduce from (3.3) that

$$fC_{1a1} = W_{1a1s} \nabla^s f + T_{1a1} = 0.$$

In fact, since the Weyl tensor is skew-symmetric we have $W(\nabla f, \cdot, \nabla f, \nabla f) = 0$. Moreover, from (3.2) we get

$$T_{1a1} = \frac{3}{2}(R_{11}\nabla_a f - R_{a1}\nabla_1 f) - \frac{1}{2}\left(R_{1s}\nabla^s f g_{a1} - R_{as}\nabla^s f g_{11}\right) = 0.$$

This allows us to conclude that $fC_{1j1} = 0$ for $j \in \{1,2,3,4\}$ at a point p where $\nabla f(p) \neq 0$. Moreover, remember that $f > 0$ on M. Consequently, we deduce $C(\nabla f, \cdot \nabla f) = 0$ in $M \setminus \mathrm{Crit}(f)$. Therefore, from continuity of the Cotton tensor and Lemma 1 we conclude that, in fact, $C(\nabla f, \cdot, \nabla f)$ vanishes on M^4.

Finally, from the definition of the Cotton tensor (2.2) we arrive at

$$d^D Ric(\nabla f, \cdot, \nabla f) = 0,$$

and then we are in position to use Theorem 1 (see also Theorem 1 in [13]) in order to conclude that there are no multiples black holes in M^4. So, the proof is completed. □

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