Illinois Journal of Mathematics
Volume 60, Numbers 3–4, Fall and Winter 2016, Pages 641–654
S 0019-2082

LOCAL-TO-GLOBAL RIGIDITY OF BRUHAT–TITS BUILDINGS

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ABSTRACT. A vertex-transitive graph X is called local-to-global rigid if there exists R such that every other graph whose balls of radius R are isometric to the balls of radius R in X is covered by X. Let $d \geq 4$. We show that the 1-skeleton of an affine Bruhat–Tits building of type $\widetilde{A}_{d-1}$ is local-to-global rigid if and only if the underlying field has characteristic 0. For example, the Bruhat–Tits building of $\mathrm{SL}(d, \mathbf{F}_p((t)))$ is not local-to-global rigid, while the Bruhat–Tits building of $\mathrm{SL}(d, \mathbf{Q}_p)$ is local-to-global rigid.

A vertex-transitive graph X is called local-to-global rigid (LG-rigid) if there exists R such that every other graph whose balls of radius R are isometric to the balls of radius R in X is covered by X. This notion was introduced by Benjamini and Georgakopoulos and investigated in [BE], [G], [ST15]. More generally, given a class $\mathfrak{G}$ of graphs, X is called LG-rigid among $\mathfrak{G}$ if there exists R such that any graph of $\mathfrak{G}$ which R-locally X is covered by X. It follows from these works that in many cases, Cayley graphs of finitely presented groups are LG-rigid: for instance all Cayley graphs of torsion-free lattices in simple Lie groups, or Cayley graphs of torsion-free groups of polynomial growth. We also proved ([ST15]) that every finitely presented group which is not a quotient of a Burnside group admits LG-rigid Cayley graphs. On the other hand, in [ST15], we constructed many examples of such graphs which are not LG-rigid: for example, a Cayley graph of $\mathrm{SL}_4(\mathbf{Z})$; better, we constructed a Cayley graph of $F_2 \times F_2 \times \mathbf{Z}/2\mathbf{Z}$ that is not LG-rigid among vertex-transitive graphs. In this article, we investigate LG-rigidity for 1-skeletons of Bruhat–Tits buildings.

Received December 9, 2015; received in final form February 22, 2017.
2010 *Mathematics Subject Classification.* 20E42, 20F65.

By *non-Archimedean local skew field*, we will mean a locally compact discrete valuation field (not necessarily commutative). We will reserve the word *non-Archimedean local field* for commutative fields. If K is a non-Archimedean local skew field and $d \geq 3$, we denote by $X_d(K)$ the Bruhat–Tits building of type $\widetilde{A}_{d-1}$ constructed from K. Our main result characterizes, for $d \geq 4$, the fields for which $X_d(K)$ is LG-rigid.

THEOREM 0.1. *Let K be a* non-Archimedean local skew field. *If K has positive characteristic and $d \geq 3$, then $X_d(K)$ is not LG-rigid among vextex-transitive graphs. By contrast, if K has characteristic* 0 *and $d \geq 4$, then $X_d(K)$ is LG-rigid.*

Let us discuss the proof of Theorem 0.1. There is a natural locally compact Hausdorff topology on the isomorphism classes of non-Archimedean local skew fields where two fields are R-close if their residue rings $\mathcal{O}/\pi^R\mathcal{O}$ are isomorphic. For example, $\mathbf{Q}_p[p^{1/R}]$ and $\mathbf{F}_p((t))$ are R-close. Indeed, the elements of their rings of integers have a unique representation as a formal series $\sum_{n\geq 0}^{\infty} a_n t^n$ with $a_n \in \{0, 1, \ldots, p-1\}$ and as a formal series $\sum_{n\geq 0} a_n (p^{1/R})^n$ with $a_n \in \{0, 1, \ldots, p-1\}$ respectively. These two rings have therefore "the same elements", but the operations are different (without carry for $\mathbf{F}_p((t))$ and with carry for $\mathbf{Q}_p[p^{1/R}]$). However when R becomes large, the difference becomes smaller and smaller as the carry is sent at distance R, and in particular $\mathbf{Q}_p[p^{1/R}]$ and $\mathbf{F}_p((t))$ are R-close.

Similarly there is a locally compact Hausdorff topology on the isometry classes of vertex-transitive locally finite graphs where two graphs are R-close if they have the same balls of radius R. The idea of the proof of Theorem 0.1 is simple and can be summarized as follows.

(i) For these topologies, the map $K \mapsto X_d(K)$ is a homeomorphism on its image.
(ii) A non-Archimedean local skew field is isolated if and only if it has characteristic 0.
(iii) If $d \geq 4$, $\{X_d(K), K$ non-Archimedean local skew field$\}$ is open in the set of large-scale simply connected graphs.

The first statement in Theorem 0.1 is immediate from the continuity of X_d in (i) and from (ii). The second statement follows from (i)–(iii) and our work [ST15].

We prove in Corollary 2.2 that the map $K \mapsto X_d(K)$ is continuous (actually 1-Lipschitz for the natural distances). Since it is injective by a deep Theorem of Tits and clearly proper, (i) follows. We could not find a direct proof of (i); for example we could not decide whether X_d is isometric. The point (ii) is classical, at least as far as commutative fields are concerned [K47]; this topology on the set of isomorphism classes of local fields has indeed been extensively used in representation theory [D84], [K86]. The simplest illustration of (ii) is, as recalled above, that $\mathbf{F}_p((t))$ is the limit of $\mathbf{Q}_p[p^{1/R}]$ as $R \to \infty$. We recall this in Section 1. The meaning of (iii) is made precise in Corollary 2.7; it is proved as a consequence of other deep results of Tits which give a local characterization of the graphs $X_d(K)$ among graphs with a special kind of labelling of the vertices called a *geometry of type* $\widetilde{A}_{d-1}$ and from our Proposition 2.3 where we show that such a labelling can be recovered locally.

Remark 0.2. Working with skew fields is very natural in view of the classification of locally finite buildings of type $\widetilde{A}_{d-1}$ with $d \geq 4$. Even if we were only interested in commutative fields, we would have to work with skew fields in the proof of the second statement. Indeed, we do not know of a direct proof showing that $\{X_d(K), K \text{ commutative}\}$ is open (this is true *a posteriori* because the set of non-Archimedean local fields is open in the space of non-Archimedean local skew fields).

Let us state two consequences of Theorem 0.1. Using the quasi-isometry rigidity of Bruhat–Tits buildings [KL97], one easily deduces the following special case of [BP89, Theorem A] (see Section 3.2 for the details).

Proposition 0.3. *Let d, N be integers and F be a finite field. Then for all but finitely many non-Archimedean local skew fields with residue field F, the building $X_d(K)$ does not admit any discrete group of isometry Γ such that*

- *the cardinality of the vertex set of the quotient graph $X_d(K)/\Gamma$ is at most N;*
- *for every vertex x of $X_d(K)$, the stabilizer of x in Γ has cardinality at most N.*

In particular, for all but finitely many $R \in \mathbf{N}$, $X_d(\mathbf{Q}_p[p^{1/R}])$ is not a Cayley graph.

For $d \geq 3$, $X_d(\mathbf{F}_p((t)))$ turns out to have a group of isometries acting simply transitively on its vertex set [CMSZ93], [CS98]. Hence, $X_d(\mathbf{F}_p((t)))$ can be seen as a Cayley graph of this group. In particular, Theorem 0.1 yields new examples of Cayley graphs of finitely presented groups which are not LG-rigid. With a slight modification, this example can be modified to provide the first example of torsion-free groups admitting non LG-rigid Cayley graphs.

Theorem 0.4. *There exists a Cayley graph X of some torsion-free finitely presented group, and for each $R > 0$, a 2-simply connected vertex transitive graph Y_R which is R-locally X, but is not even quasi-isometric to X. In particular, X is not LG-rigid.*

1. Non-Archimedean local fields

DEFINITION 1.1. A *non-Archimedean local skew field* is a (not necessarily commutative) field which is locally compact for the topology associated with a discrete valuation.

If K is a non-Archimedean local skew field with a discrete valuation $v\colon K \to \mathbf{Z} \cup \{\infty\}$, we will always assume that the image of v is $\mathbf{Z} \cup \{\infty\}$, and we will denote its *ring of integers* $\mathcal{O} = \{x \in K, v(x) \geq 0\}$ (or $\mathcal{O}^K$ if we need to keep track of K) and $\mathfrak{m} = \{x \in K, v(x) \geq 1\}$ the unique prime ideal in $\mathcal{O}$. Denote by π a *uniformizer* of K, that is a generator of the $\mathcal{O}$-module $\mathfrak{m}$. For an integer $R \geq 1$, denote by $\mathcal{O}_R$ the ring $\mathcal{O}/\pi^R\mathcal{O}$.

A skew field K with a discrete valuation v is locally compact if and only if it is complete and the residue field $\mathcal{O}/\pi\mathcal{O}$ is finite.

Two non-Archimedean local skew fields *are* $\mathbf{R}$-*close* if their residue rings $\mathcal{O}_R$ are isomorphic.

THEOREM 1.2. *The distance*

$$d\big(K,K'\big) = \inf\big\{e^{-R},\ K \text{ and } K' \text{ are } R\text{-close}\big\}$$

defines a locally compact Hausdorff topology on the isomorphism classes (*as topological fields*) *of non-Archimedean local skew fields.*

For this topology, a set of non-Archimedean local skew fields is relatively compact if and only if the cardinality of the residue field is bounded on this set.

A field is isolated if and only if it has characteristic 0.

The topology is Hausdorff because a non-Archimedean local skew field is determined as a topological field by the sequence of its residue rings. For the rest of the proof of Theorem 1.2, we will need an explicit description of all non-Archimedean local skew fields. We refer to [W74, Chapter 1] for the statements for which we do not provide a precise reference. If p is a prime number and $q = p^f$ for some $f \geq 1$, we denote by $\mathbf{Q}_q$ the totally unramified extension of degree f of $\mathbf{Q}_p$.

Let K be a *commutative* non-Archimedean local field with valuation v. Its residue field is isomorphic to $\mathbf{F}_{p^f}$ for a prime number p called the *residue characteristic* of K and an integer $f \geq 1$ called the *absolute residue degree* of K. The value $e = v(p) \in \mathbf{N} \cup \infty$ is called the *absolute ramification index.* If K has characteristic 0 (*i.e.* $e < \infty$), then K is isomorphic to a totally ramified commutative extension of $\mathbf{Q}_{p^f}$ of degree e (hence K is isomorphic to $\mathbf{Q}_{p^f}[X]/(P)$, where P is an Eisenstein polynomial of degree e with coefficients in $\mathcal{O}^{\mathbf{Q}_q}$, and a possible choice of uniformizer is X). We will use that, for every prime number p and all integers $f, e \geq 1$, there are finitely many [R00, §3.1.6], and at least one (for example $\mathbf{Q}_{p^f}[p^{1/e}]$), commutative non-Archimedean local fields with residue characteristic p, absolute ramification index e and absolute residue degree f. If K has positive characteristic (*i.e.*, $e=\infty$), K is isomorphic to $\mathbf{F}_{p^f}((t))$.

LEMMA 1.3. *If K is a local field of residue field $\mathbf{F}_q$ and absolute ramification index e, then K and $\mathbf{F}_q((t))$ are e-close.*

Proof. Note that K is a totally ramified commutative extension of $\mathbf{Q}_q$ of degree e. Hence we can write $K=\mathbf{Q}_q[X]/(P)$ where $q=p^f$ and $P=X^e+a_{e-1}X^{e-1}+\cdots+a_0$ is an Eisenstein polynomial of degree e with coefficients in $\mathbf{Z}_q=\mathcal{O}^{\mathbf{Q}_q}$. Recall that P satisfies $a_i\in p\mathbf{Z}_q$ for all $0\le i\le k-1$, and $a_0=bp$ where b is invertible in $\mathbf{Z}_q$. Note that $\mathcal{O}^K_e=\mathbf{Z}_q[X]/(J)$ where $J=(X^e)+(P)$. It follows that in $\mathcal{O}^K_e$ we have

$$p\big(b+a_1X+\cdots+a_{e-1}X^{e-1}\big)=0,$$

from which we deduce that $p=0$ because, b being invertible in $\mathbf{Z}_q$, $b+a_1X+\cdots+a_{e-1}X^{e-1}$ is invertible in $\mathcal{O}^K_e$. On the other hand, modulo p, one has $P=X^e$. So finally, we deduce that

$$O^K_e\simeq\mathbf{Z}_q[X]/\big(\big(X^e\big)+(p)\big)\simeq\big(\mathbf{Z}_q[X]/(p)\big)/\big(X^e\big)\simeq\mathbf{F}_q[X]/\big(X^e\big)\simeq\mathcal{O}^{\mathbf{F}_q((t))}_e,$$

and we are done. □

Let us now move to general skew fields. If K is a non-Archimedean local skew field with center L, the residue field $\mathbf{k}$ of K is an extension of the residue field $\mathbf{l}$ of L; denote by d the degree of $\mathbf{k}/\mathbf{l}$ (d is called the *residue degree* of K/L). The Galois group $\mathrm{Gal}(\mathbf{k}/\mathbf{l})$ is cyclic of order d with generator the Frobenius automorphism. Moreover, if π is a uniformizer of K, then the conjugation $x\in K\mapsto\pi^{-1}x\pi$ belongs to $\mathrm{Gal}(K/L)$. Its image α in $\mathrm{Gal}(\mathbf{k}/\mathbf{l})$ does not depend on the choice of the uniformizer because $\mathbf{k}$ is commutative, and corresponds to the rth power of the Frobenius for some $r\in\mathbf{Z}/d\mathbf{Z}$. The element $\frac{r}{d}\in\mathbf{Q}/\mathbf{Z}$ is called the *Hasse invariant* of K. The next theorem states that r is a generator of $\mathbf{Z}/d\mathbf{Z}$, and that the triple (L,d,r) with $d\ge1$ and r a generator of $\mathbf{Z}/d\mathbf{Z}$ determines a unique skew field K. Observe that the case where K is commutative corresponds to $d=1$, in which case $r=0$ (which is a generator of the trivial group $\mathbf{Z}/\mathbf{Z}$).

THEOREM 1.4 ([H32],[W74, Chapter 1 (pp. 20–22), Chapter XII]). *Let K, L, d, r as above. Then r is a generator of $\mathbf{Z}/d\mathbf{Z}$. Conversely, for every non-Archimedean local field L, every integer $d\ge1$ and every generator r of $\mathbf{Z}/d\mathbf{Z}$, there is a unique non-Archimedean local skew field K with center L, residue degree d over L and Hasse invariant r/d. It has degree d^2 over L and can be described as follows. It contains a maximal commutative extension K_1 of L of degree d which is unramified. Moreover, L has a uniformizer π and K has a uniformizer x such that $x^d=\pi$, $(1,x,\dots,x^{d-1})$ forms a basis of K as a K_1-vector space, and for all $a\in K_1$, $x^{-1}ax=f^r(a)$, where f is the unique automorphism of K_1 inducing the Frobenius automorphism of* $\mathrm{Gal}(\mathbf{k}_1/\mathbf{l})$.

We will say that a non-Archimedean local skew field K has type (p,f,e,d,r) if its center L has residue characteristic p, absolute ramification index e and absolute residue degree f, and if the extension K/L has residue degree d and Hasse invariant r/d. From the preceding discussion we conclude that for every prime number p, all integers $f,d \geq 1$, every $e \in \mathbf{N} \cup \{\infty\}$ and every $r \in (\mathbf{Z}/d\mathbf{Z})^*$, the number of skew fields of type (p,f,e,d,r) is finite and nonzero. As we have seen, there exists a unique skew field of type (p,f,∞,d,r). If $q=p^f$, it can be concretely defined as the quotient of the $\mathbf{F}_q((t))$-algebra freely generated by $\mathbf{F}_{q^d}((t))$ and by an element x, by the ideal generated by the following relations: $x^d = t$, and $ax = xf(a)$, for all $a \in \mathbf{F}_{q^d}((t))$, where f is the automorphism of $\mathbf{F}_{q^d}((t))$ uniquely defined by $f(t)=t$, and $f(z)=z^{q^r}$ for all $z \in \mathbf{F}_{q^d}$.

Lemma 1.5. *Let K (resp. F) be respectively a skew field of type (p,f,e,d,r) (resp. the skew field of type (p,f,∞,d,r)). Then K and F are ed-close: that is, the residue rings $\mathcal{O}^K_{ed}$ and $\mathcal{O}^F_{ed}$ are isomorphic.*

Proof. With the notation of Theorem 1.4, K is isomorphic to the quotient $\widetilde{K}$ of the ring freely generated by K_1 and x, by the ideal generated by the relations $x^d=\pi$, and $ax = xf^r(a)$, for all $a \in K_1$, where f is the Fröbenius automorphism of K_1. Indeed, one clearly has a morphism of L-algebras from $\widetilde{K}$ to K that is the identity on K_1 and on x. The fact that this is an isomorphism follows by comparing the dimensions over L. Note that the Fröbenius automorphism of K_1 respects the valuation, and hence induces an automorphism $\widetilde{f}$ on $\mathcal{O}^{K_1}/(\pi^e)$. We deduce that the residue ring $\mathcal{O}^K_{ed} = \mathcal{O}^K/(x^{ed}) = \mathcal{O}^K/(\pi^e)$ is isomorphic to the quotient of the ring freely generated by $\mathcal{O}^{K_1}/(\pi^e)$ and x, by the ideal generated by the relations $x^d=\pi$, and $ax = x\widetilde{f}^r(a)$. The same description of course applies to $\mathcal{O}^F_{ed}$, and we conclude by Lemma 1.3 because K_1 has absolute residue degree fd and absolute ramification index e. ☐

We can now prove Theorem 1.2. To do so, we show that the balls of radius $\frac{1}{2}$ are compact (here $\frac{1}{2}$ could be any number in $[\frac{1}{e},1)$), and that their only accumulation points are the fields of positive characteristic. Since two fields are at distance less than $\frac{1}{2}$ if and only if they have the same residue field, it amounts to investigating, for every finite field $\mathbf{F}_q$, the set of skew fields having $\mathbf{F}_q$ as residue field. This sets contains exactly the skew fields of type (p,f,e,d,r) for $q=p^{fd}$ and $k \in \mathbf{N} \cup \{\infty\}$. This determines p and forces f, d to take only finitely many values. Therefore (since there are finitely many fields of each type) a sequence of such fields either has a stationary subsequence, or a subsequence of type (p,f,e_n,d,r) for a sequence $e_n \to \infty$, which converges to the field of type (p,f,∞,d,r) by Lemma 1.5. This shows that the set of skew fields with residue field $\mathbf{F}_q$ is compact, and that the skew fields with characteristic 0 are isolated. Conversely, every non-Archimedean local skew field K of characteristic $p>0$ is the field of type (p,f,∞,d,r) for some f,

d, r. As we discussed, there is a sequence of skew fields of type (p, f, n, d, r), and it converges as $n \to \infty$ to K by Lemma 1.5.

2. Buildings

2.1. Graphs. In this paper "a graph" means a connected, locally finite, simplicial graph without multiple edges and loops. It is called vertex-transitive if its isometry group acts transitively on the set of vertices. A graph Y is R-locally X if every ball of radius R around a vertex in Y is isometric to a ball of radius R around a vertex in X. This defines a locally compact Hausdorff topology on the isomorphism classes of transitive graphs, for example, for the distance

$$\inf\left\{e^{-R}, Y \text{ is } R\text{-locally } X\right\}.$$

A set of vertex-transitive graphs if relatively compact if and only if the degree is bounded on this set.

2.2. Large-scale simple connectedness. Following the terminology of [ST15], let us call a graph k-simply connected if the 2-complex obtained by gluing j-gones along loops of length j for all $j \leq k$, is simply connected. Moreover, a graph is large-scale simply connected if it is k-simply connected for some k. We make the following trivial observation: let X be a simplicial complex such that all triangles are filled, then it is simply connected if and only its 1-skeleton is 3-simply connected.

2.3. Classical buildings of type $\widetilde{A}_{d-1}$. Let $d \geq 2$. Let us recall the description of the building of $\mathrm{PGL}(d, K)$ (the building $\widetilde{A}_{d-1}(K, v)$) associated to a non-Archimedean local skew field K with discrete valuation v, see [R85, Chapter 9] for details.

An $\mathcal{O}$-lattice in K^d is a finitely generated $\mathcal{O}$-submodule which generates K^d as a K-vector space. Such a module is free of rank d, *that is*, of the form $\mathcal{O}v_1 + \cdots + \mathcal{O}v_d$ for a basis $(v_1, \ldots, v_d)$ of K^d. By the invariance property $a\mathcal{O} = \mathcal{O}a = \pi^k\mathcal{O}$ for any $a \in K^*$ and $k \in \mathbf{Z}$ with $v(a) = k$, we see that if L is an $\mathcal{O}$-lattice and $a \in K^*$, aL is also a lattice, so that it makes sense to talk about lattices modulo homothety.

The building $\widetilde{A}_{d-1}(K, v)$ is a simplicial complex of dimension $d-1$. Its 1-squeleton, that we denote by $X_d(F)$ (or X for short if there is no ambiguity) is described as follows. The vertices of X are the $\mathcal{O}$-lattices in K^d modulo homothety. There is an edge between two different vertices x and y if there are representatives L_1 and L_2 of x and y such that $\pi L_1 \subset L_2 \subset L_1$. This is the vertex transitive graph $X_d(F)$ we are interested in.

2.4. Continuity of $K \mapsto X_d(K)$. A lattice modulo homothety x has a unique representative, denoted by $L(x)$, contained in $\mathcal{O}^d$ but not in $\mathfrak{m}^d$. There is an edge between two different vertices x and y if and only if $\pi L(x) \subset L(y) \subset L(x)$ or $\pi L(y) \subset L(x) \subset L(y)$.

The following Lemma expresses that the ball of radius R around $\mathcal{O}^d$ in X is entirely described in terms of the ring $\mathcal{O}_R$.

LEMMA 2.1. *A lattice modulo homothety x belongs to the ball of radius R around o if and only if $\pi^R \mathcal{O}^d \subset L(x)$.*

Moreover the map $\overline{L}\colon x \mapsto L(x) \mod \pi^R\mathcal{O}^d$ is a bijection between the ball of radius R around $\mathcal{O}^d$ in X and the $\mathcal{O}_R$-submodules of $(\mathcal{O}_R)^d$ not contained in $(\pi\mathcal{O}_R)^d$.

Lastly two different vertices x and y in the ball of radius R around $\mathcal{O}^d$ in X are adjacent if and only if $\pi\overline{L}(x) \subset \overline{L}(y) \subset \overline{L}(x)$ or $\pi\overline{L}(y) \subset \overline{L}(x) \subset \overline{L}(y)$.

Proof. It is immediate that $\pi^R\mathcal{O}^d \subset L(x)$ if $d(x,o) \leq R$. The converse follows by applying, for any lattice $L \subset \mathcal{O}$, the invariant factor decomposition over $\mathcal{O}$-modules ([T37]) to $\mathcal{O}^d/L$, which provides a basis $v_1, \dots, v_d$ for the $\mathcal{O}$-module $\mathcal{O}^d$ and integers $n_1 \leq \dots \leq n_d$ such that $\pi^{n_1}v_1, \dots, \pi^{n_d}v_d$ is a basis for L. For the lattice $L(x)$, we have $n_1 = 0$, and if $\pi^R\mathcal{O}^d \subset L(x)$, we have $n_d \leq R$. If x_k is the equivalence class of the lattice $\oplus_{1\leq i\leq d}\mathcal{O}\pi^{\min(k,n_i)}v_i$ then x_k and x_{k+1} are adjacent in X_d, $x_0 = o$ and $x_R = x$, which shows that $d(x,y) \leq R$.

Since $L \mapsto L \mod \pi^R\mathcal{O}^d$ is a bijection between the lattices L such that $\pi^R\mathcal{O}^d \subset L \subset \mathcal{O}^d$ and the $\mathcal{O}_R$-submodules of $\mathcal{O}_R^d$, the second statement is immediate from the first.

The last statement is easy. □

We immediately deduce that if two local skew fields K, K' are R-close, then the graphs $X_d(K)$ is R-locally $X_d(K')$.

COROLLARY 2.2. *The ball of radius R in $X_d(K)$ does only depend (up to isometry) on the ring $\mathcal{O}_R$.*

Proof. By Lemma 2.1, the ball only depends on the pair $\pi\mathcal{O}_R \subset \mathcal{O}_R$. But $\pi\mathcal{O}_R$ is determined by $\mathcal{O}_R$ as its unique maximal ideal. □

2.5. $\{X_d(K)\}$ is open. We start by recalling some material from [T81].

For an integer $m \geq 2$, a generalized m-gon is a connected bipartite graph of diameter m and girth $2m$, in which every vertex has degree at least 2.

A Coxeter diagram over I is a function $M\colon I \times I \to \mathbf{N} \cup \{\infty\}$ such that for all $i, j \in I$, $M(i,i) = 1$ and $M(i,j) = M(j,i) \geq 2$ if $i \neq j$. A symmetry of a Coxeter diagram M is a permutation σ of I satisfying $M(i,j) = M(\sigma(i), \sigma(j))$ for all $i, j \in I$.

If I is a set, a geometry over I is a pair (X,τ) where X is a graph and $\tau\colon X\to I$ is a coloring of the vertices of X by labels in I satisfying that every pair of adjacent vertices have a different label. In a geometry, a complete subgraph is called a flag, and its type is the subset of I defined as its image by τ. The residue of a flag Z of type $J\subset I$ is the geometry over $I\setminus J$ given by $(Y,\tau|_Y)$ where Y is the set of vertices in $X\setminus Z$ adjacent to Z, with the same edges as in X. By convention the residue of the empty flag is (X,τ). We say that the geometry is simply connected if the simplicial complex obtained by filling all triangles in X is simply connected. This is therefore equivalent to saying that X is 3-simply connected.

If M is a Coxeter diagram over a set I, a geometry of type M is a geometry (X,τ) over I where for any subset $J\subset I$, the residue of any flag of type J is (1) nonempty if $|I\setminus J|\geq 1$, (2) connected if $|I\setminus J|\geq 2$, (3) a generalized $M(i,j)$-gon if $J=I\setminus\{i,j\}$ for some $i\neq j\in I$. We say that a graph admits a geometry of type M if there exists $\tau\colon X\to I$ such that (X,τ) is a geometry of type M.

The example important for us is the Coxeter diagram $\widetilde{A}_{d-1}$ over $\mathbf{Z}/d\mathbf{Z}$ given by where $\widetilde{A}_{d-1}(i,i)=1$, $\widetilde{A}_{d-1}(i,j)=3$ if $i-j\in\{-1,1\}$ and $\widetilde{A}_{d-1}(i,j)=2$ otherwise. Then $X_d(K)$ admits a geometry of type $\widetilde{A}_{d-1}$ (see [R85]). It is characterized by the following properties. The origin o is labelled by $\tau(o)=0$, and if x and y are two adjacent vertices with representatives L_1 and L_2 such that $\pi L_1\subset L_2\subset L_1$, then $L_1/\pi L_1$ is a vector space of dimension d over the finite field $\mathcal{O}/\mathfrak{m}$ and the dimension (modulo d) of the image of L_2 inside it is equal to $\tau(y)-\tau(x)$.

A particular case of a theorem of Tits [T81, Theorem 1.3] characterizes the buildings of type $\widetilde{A}_{d-1}$ as the simply connected geometries of type $\widetilde{A}_{d-1}$. This motivates the following result, which shows that for a 3-simply connected graph Y, admitting a geometry of type M is a local property.

Proposition 2.3. *Let M be a Coxeter diagram over a finite set I. Let X be a 3-simply connected graph. Assume that every ball of radius 3 in X is isomorphic to a ball of radius 3 in a graph admitting a geometry of type M. Then X admits a geometry of type M.*

Remark 2.4. More generally if X is k-simply connected, then X admits a geometry of type M if every ball of radius $\lfloor\frac{k+3}{2}\rfloor$ in X is isometric to a ball of the same radius in a graph admitting geometry of type M.

If X is a graph and $x\in X$, let $V(x)$ be the graph with vertex set $\{x'\in X, d(x,x')\leq 1\}$ and same edges as in X. A *germ of geometry of type M* at x is a coloring $\tau\colon V(x)\to I$ such that $(V(x),\tau)$ is a geometry over I and such that the conditions (1) (2) (3) hold for every flag in $V(x)$ containing x. Denote by $G(x)$ the set of all germs of geometry of type $\widetilde{A}_{d-1}$ at x. Observe that for a connected graph X and a map $\tau\colon X\to I$, (X,τ) is a geometry of type I if and only if the restriction of τ to $V(x)$ if a germ of geometry of type M for every $x \in X$. It is through this observation that we will construct a suitable labelling of a graph satisfying the local properties of Proposition 2.3.

We start by a lemma which implies that a germ of geometry of type M at x is characterized by its restriction to any flag of type I.

LEMMA 2.5. *Let X be a graph and $x \in X$. If $\tau \in G(x)$, $G(x)$ consists of all maps of the form $\sigma \circ \tau$ for a symmetry σ of M.*

Proof. It is clear that $\sigma \circ \tau \in G(x)$ if σ is a symmetry of M.

To see the converse take $\tau' \in G(x)$. Let Z be a flag of type I containing x (such a flag exists by (1)). Then for every $z_1 \neq z_2 \in Z \setminus \{x\}$, by looking at the residue of $Z \setminus \{z_1, z_2\}$ we obtain from condition (3) that $M(\tau(z_1), \tau(z_2)) = M(\tau'(z_1), \tau'(z_2))$. This implies that there exists a unique symmetry σ_Z of M such that $\tau'(z) = \sigma_Z \circ \tau(z)$ for all $z \in Z$. We claim that $\sigma_Z = \sigma_{Z'}$ for every pair of flags Z and Z' of cardinality d containing x. The claim is proved by downwards induction on the cardinality of $Z \cap Z'$. The case when $|Z \cap Z'| = |I|$ or $|Z \cap Z'| = |I| - 1$ is obvious. Assuming that the claim is valid when $|Z \cap Z'| = k \leq |I| - 1$, let us prove the claim when $|Z \cap Z'| = k - 1$. Pick $z \in Z \setminus Z'$ and $z' \in Z' \setminus Z$. By (2) there is a path $z_0, \ldots, z_n$ contained in the residue of $Z \cap Z'$ such that $z_0 = z$ and $z_n = z'$. By (1) for each $i = 0, \ldots, n-1$ there is a flag Z_i of type I containing $(Z \cap Z') \cup \{z_i, z_{i+1}\}$, and by induction hypothesis applied to Z_i and Z_{i+1} we have $\sigma_{Z_i} = \sigma_{Z_{i+1}}$ for each $0 \leq i \leq n-1$. This proves that $\sigma_Z = \sigma_{Z'}$. Hence, σ_Z does not depend on Z, which proves the lemma. $\square$

From it we deduce the following lemma.

LEMMA 2.6. *Let X be a graph admitting a geometry of type M and x a vertex in X. For every germ τ of geometry of type M at x, there exists $\widetilde{\tau}\colon X \to I$ which extends τ and such that $(X, \widetilde{\tau})$ is a geometry of type M.*

Proof. Let $\tau_0\colon X \to I$ such that (X, τ_0) is a geometry of type I. By Lemma 2.5, there is a symmetry σ of the diagram M such that τ coincides with $\sigma \circ \tau_0$ on $V(x)$. Then $\widetilde{\tau} = \sigma \circ \tau_0$ extends τ and satisfies that $(X, \widetilde{\tau})$ is a geometry of type M. This shows the existence. $\square$

With these two lemmas, we can proceed to the proof of Proposition 2.3. The argument is the same as for the proof of [ST15, Theorem C]. Since every ball of radius 3 in X is isometric to a ball in a graph admitting a geometry of type M, $G(x)$ is nonempty for every $x \in X$, and it follows from Lemma 2.6 that if x, x' are two neighbors in X, then for every $\tau \in G(x)$, there is $\tau' \in G(x')$ which coincides with τ on $V(x) \cap V(x')$. It is unique because by Lemma 2.5, τ' is determined by its value on any complete subgraph with d vertices containing x' (and there is such a subgraph containing both x and x' by (1)). This defines a bijection that we denote $F_{x,x'}\colon G(x) \to G(x')$.

For every path $\gamma = (x_0, \dots, x_n)$ of adjacent vertices, we can define a bijection $F_\gamma\colon G(x_0) \to G(x_n)$ by composing the bijections $F_{x_i, x_{i+1}}$ along γ. We claim that F_γ only depends on the endpoints x_0 and x_n. Since X is 3-simply connected, we only have to check that F_γ is the identity of $A(x_0)$ if γ is a path of length $n \le 3$ with $x_0 = x_n$. This property clearly holds if X admits a geometry of type M, and hence also in X because $\bigcup_{k \le n} V(x_i)$ (and all its edges) is contained in the ball of radius 3 around x_0, which is isometric to a ball of radius 3 in a graph admitting a geometry of type M.

It remains to fix a vertex $x_0 \in X$ and $\tau_0 \in G(x_0)$. For every other vertex x, let $\tau_x \in G(x)$ be the common value of $F_\gamma(\tau_0)$ for all paths γ from x_0 to x. Define $\tau(x) = \tau_x(x)$. Since for adjacent edges x, x', $\tau_{x'} = F_{x,x'}(\tau_x)$ coincides with τ_x on $V(x) \cap V(x')$, we see that τ coincides with τ_x on $V(x)$. In particular, the restriction of τ to $V(x)$ belongs to $G(x)$ for all $x \in X$. This means that (X, τ) is a geometry of type $\widetilde{A}_{d-1}$. This concludes the proof of Proposition 2.3.

We deduce the following corollary.

Corollary 2.7. *Let $d \ge 4$. Let Y be a 3-simply connected graph which is 3-locally $X_d(K)$ for some non-Archimedean local skew field K. Then Y is isometric to $X_d(F)$ for a (unique up to isomorphism) non-Archimedean local skew field F.*

Proof. From the discussion before Proposition 2.3, Y is isometric to the 1-skeleton of a locally finite building of type $\widetilde{A}_{d-1}$ (and this holds whenever $d \ge 3$). By a theorem of Tits [T86, Corollaire 15], this forces Y to be isometric to $X_d(F)$ for some F if $d \ge 4$. Moreover, $X_d(F)$ determines the projective space $PG(d-1, F)$ up to collineation[1] [T74, Theorem 6.3], which determines F up to isomorphism by the fundamental theorem of projective geometry. □

3. Proof of main results

3.1. Proof of Theorem 0.1. The map $K \mapsto X_d(K)$ is continuous by Corollary 2.2. It is injective by the theorems of Tits recalled in the proof of Corollary 2.7. Let us show that it is proper, that is that if the cardinality of the residue field of K_n goes to ∞, then the degree in $X_d(K_n)$ also. But this holds because (Lemma 2.1) the degree in $X_d(K)$ is equal to the number of linear subspaces of dimension $\neq 0, d$ in[2] $(\mathcal{O}/\pi\mathcal{O})^d$. This implies (i): $K \mapsto X_d(K)$ is a homeomorphism on its image.

The first part of Theorem 0.1 follows from Theorem 1.2 and (i).

Let us now prove the second half of Theorem 0.1. Let $d \ge 4$ and K be a non-Archimedean local skew field of characteristic 0. By Theorem 1.2 and

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[1] Both results of Tits can also be found in [R85, page 137].

[2] Precisely, if the residue field of K is $\mathbf{F}_q$, then the degree equals $\Pi_{i=1}^{d}(q^i - 1)/(q - 1) = \Pi_{i=1}^{d}(\sum_{v=0}^{i-1} q^v)$, which is a strictly increasing function of q.

(i), there exists $R > 0$ such that if K' is another non-Archimedean local skew field such that $X_d(K')$ is R-locally K, then K' is isomorphic to K, and in particular $X_d(K')$ is isometric to $X_d(K)$. By Corollary 2.7, this implies that if Y is a 3-simply connected graph which is $\max(3, R)$-locally $X_d(K)$, then it is isometric to $X_d(K)$. By [ST15, Proposition 1.5] $X_d(K)$ is LG-rigid.

3.2. Proof of Proposition 0.3. Assume, by contradiction, that there are infinitely many such fields K_n and discrete groups Γ_n. The assumptions imply that each group Γ_n admits a Cayley graph which is quasi-isometric to $X_d(K_n)$, with quasi-isometry constants independent from n. Using that $X_d(K_n)$ is 3-simply connected for all n, we obtain that each Γ_n has a presentation with $\leq f(N,F)$ generators and relations of length $\leq f(N,F)$. Since there are only finitely many groups with $\leq f(N,F)$ generators and relations of length $\leq f(N,F)$, we obtain that two of these groups coincide, and in particular there exists $n \neq m$ such that $X_d(K_n)$ and $X_d(K_m)$ are quasi-isometric. This contradicts [KL97].

3.3. Proof of Theorem 0.4. The group will be a cocompact lattice in $\mathrm{PGL}_d(\mathbf{F}_p((t)))$ for an arbitrary $d \geq 3$. We thank Pierre-Emmanuel Caprace for allowing us to include the argument.

Recall that in $X_d(K)$, the vertices are partitioned according to their *type* in $\mathbf{Z}/d\mathbf{Z}$ (when the building is identified with $\mathrm{PGL}_d(K)/\mathrm{PGL}_d(\mathcal{O})$, the type of $\gamma K^*\mathrm{PGL}_d(\mathcal{O})$ is $v(\det(\gamma)) + d\mathbf{Z}$). Consider the graph X, with vertex set the vertices of type 0 in $X_d(\mathbf{F}_p((t)))$, and an edge between two vertices at distance 2 in $X_d(\mathbf{F}_p((t)))$. It is a large-scale simply connected graph (because it is quasi-isometric to $X_d(\mathbf{F}_p((t)))$), and it is not LG-rigid (because if Y is a building R-close to $X_d(\mathbf{F}_p((t)))$, then its set of vertices of type 0 is $R/2$-close to X).

It remains to see that X is the Cayley graph of a torsion free lattice in $\mathrm{PGL}_d(\mathbf{F}_p((t)))$. By [CS98] there is a cocompact lattice Γ in $\mathrm{PGL}_d(\mathbf{F}_p((t)))$ which acts simply transitively on $X_d(\mathbf{F}_p((t)))$. We do not know whether Γ is torsion-free, but its subgroup $\Lambda = \{\gamma, v(\det\gamma) \in d\mathbf{Z}\}$ satisfies:

- Λ is an index d subgroup of Λ which acts simply transitively by isometries on X (and therefore X is a Cayley graph of Λ).
- Λ is torsion free. Indeed, if $g \in \Lambda$ has finite order, then the circumcenter of any of its orbits would be a point in the Bruhat–Tits building fixed by g. If C is the cell of minimal dimension which contains this fixed point, then g induces a permutation of the vertices of this cell. As g preserves the type, this permutation is the identity. In particular, g fixes one vertex of the building, hence g is the identity by [CS98].

Acknowledgments. We thank Sylvain Barré, Laurent Berger, Gaëtan Chenevier, Gabriel Dospinescu, Frédéric Haglund, Pierre Pansu, Mikael Pichot, Vincent Pilloni, Sandra Rozensztajn and the anonymous referee for useful discussions and comments. The argument leading to torsion-free examples in Theorem 0.4 was obtained in discussions together with Pierre-Emmanuel Caprace. We thank him for allowing us to include this argument.

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