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VISCOSITY SOLUTIONS, ENDS AND IDEAL BOUNDARIES

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ABSTRACT. On a smooth, non-compact, complete, boundaryless, connected Riemannian manifold (M,g), there are three kinds of objects that have been studied extensively:

• Viscosity solutions to the Hamilton–Jacobi equation determined by the Riemannian metric;

• Ends introduced by Freudenthal and more general other remainders from compactification theory;

• Various kinds of ideal boundaries introduced by Gromov.

In this paper, we will present some initial relationship among these three kinds of objects and some related topics are also considered.

0. Background, preliminaries and results

This paper is a continuance to the papers [11], [10]. Our aim is to understand the dynamics of minimal geodesics on a non-compact Riemannian manifold from the viewpoint of Aubry–Mather theory.

Let M be a smooth, non-compact, complete, boundaryless (in the usual sense of point set topology), connected Riemannian manifold with Riemanian metric g. Let d be the distance on M and $|\cdot|_g$ the norm on the tangent bundle TM and/or the contangent bundle T^*M induced by the Riemannian metric g. Let ∇ be the gradient determined by the Riemannian metric g. Throughout this paper, all geodesic segments are always parameterized to be unit-speed. By a ray [5], we mean a geodesic segment $\gamma:[0,+\infty)\to M$ such that $d(\gamma(t_1),\gamma(t_2))=|t_2-t_1|$ for any $t_1,t_2\geq 0$. Throughout this paper, $|\cdot|$

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means Euclidean norm. By definition, the Busemann function associated to a ray γ, is defined as

$$b_\gamma(x) := \lim_{t\to+\infty}\big[d\big(x,\gamma(t)\big) - t\big].$$

Clearly, b_γ is a Lipschitz function with Lipschitz constant 1, that is,

$$\big|b_\gamma(x) - b_\gamma(y)\big| \le d(x,y).$$

Moreover, in [11], the authors showed that Busemann functions are in fact locally semi-concave with linear modulus (for the definition of local semi-concavity of linear modulus, we refer to [7], [11] or Appendix A).

There are also other two kinds of functions, both introduced by Gromov, may be regarded as the generalizations of Busemann functions. They are defined as follows. Let x_n be a sequence of points in M such that $d(y,x_n)\to\infty$ for some fixed point y (hence for any other fixed point in M) and

$$h(x) := \lim\big[d(x,x_n) - d(y,x_n)\big]$$

exists in the compact-open topology. Such a limit function will be called horo-function. More generally, let K_n be a sequence of nonempty closed subsets in M such that $d(y,K_n)\to\infty$ for some fixed point y (hence, for any other fixed point in M) and

$$h(x) := \lim\big[d(x,K_n) - d(y,K_n)\big]$$

exists in the compact-open topology. Such a limit function will be called *dl* (distance-like)-function (here, the definition of *dl*-function is slightly different from the original definition of Gromov [24, page 202], but it will not cause any confusion. See Remark 2.3 for further details).

As it is explained explicitly in [36] (see also [4]), we could define at least three kinds of ideal boundaries [23], [24] (all of them are equipped with the quotient compact-open topologies):

$$M(\infty) := \{\text{Busemann functions}\}/\{\text{constant functions}\};$$
$$M(\partial) := \{\text{horo-functions}\}/\{\text{constant functions}\};$$
$$M(\natural) := \{dl\text{-functions}\}/\{\text{constant functions}\}.$$

Remark 0.1. In the terminology of topology, here ideal boundary (not the usual one in the sense of point set topology and for short, we just refer it as boundary in the rest of this paper) should be understood in this way: We could compactify M (here M is regarded as a non-compact, but locally compact, hemi-compact, Hausdorff, completely regular, ... ,[1] topological space) in various ways. A compactification of M means that there exist a compact

[1] In this paper, the non-compact complete Riemannian manifold M, regarded as a topological space, is at least Hausdoff, separable, locally compact, hemi-compact (thus σ-compact), non-pseudo-compact, completely regular, Lindelöf (thus realcompact), perfectly normal and Tychonoff. For much more topological properties of non-compact complete Riemannian manifolds, we refer to [20, Theorem 2], [21, Theorem 1.1].

Hausdorff [2] topological space, say $\bar{M}$, and a topological embedding i of M into $\bar{M}$ such that $i(M)$ is a dense subset of $\bar{M}$. Then we call $\bar{M}\setminus i(M)$ to be a boundary ($\bar{M}\setminus i(M)$ is also called to be a remainder by topologists working in the field of compactification theory) of M. If the embedding i we considered here is defined in the usual way by $i(x)=\frac{d(x,\cdot)}{\{\text{constant functions}\}}$, then only $M(\partial)$ deserves the terminology "boundary" and $M(\infty)$ is only a part of the boundary $M(\partial)$. By abuse of notations, here we insist on calling $M(\infty)$ and $M(\natural)$ to be boundaries. We will see that calling $M(\natural)$ a boundary is reasonable in the light of Theorem 2 once the embedding is chosen suitably.

By definitions, we have $M(\infty)\subseteq M(\partial)\subseteq M(\natural)$. It has been realized [2], [9], [17], [30] that the set $M(\infty)$ is a good analogue of the set of static classes of Aubry sets in Aubry–Mather theory for positively definite Lagrangian systems (for details of Aubry–Mather theory, we refer to [37], [38], [16]), from the viewpoint that either an element in $M(\infty)$ or a static class could determine a fundamental viscosity solution in the associated settings. In [11], the authors began to study the geometric property of the Riemannian metric from this viewpoint. More precisely, [11, Theorem 1, Corollary 7.2] showed that all *dl*-functions (including horo-functions and Busemann functuions) are viscosity solutions with respect to the Hamilton–Jacobi equation

$$(*)\qquad |\nabla u|_g = 1.$$

A natural inverse problem is the following.

Problem 0.2. Whether any viscosity solution to the Hamilton–Jacobi equation $(*)$ must be a *dl*-function?

In this paper, we will show that the answer to this problem is (almost) yes! Precisely, we have the following result.

Theorem 1. *Up to a constant, a function f is a viscosity solution with respect to the Hamilton–Jacobi equation*

$$|\nabla u|_g = 1$$

if and only f is a dl-function. Thus,

$$M(\natural) = \{\textit{viscosity solutions}\}/\{\textit{constant functions}\}.$$

By equipping compatible (with respect to the manifold topology) proximities or totally bounded uniformities, one could obtain various kinds compactifications and thus get various kinds of boundaries (i.e., remainders, cf. [41], [32]) from the viewpoint of general topology. It is known that proximity structures and totally bounded uniform structures are in equivalence

[2] In this paper, we are only interested in the Hausdorff compactifications. Since M, as a complete Riemannian manifold, is completely regular, Hausdorhoff compactifications do always exist, for example, [8].

one to one [3]. One can also correspond the compactifications to the maximal ideals of the algebra of bounded real functions (e.g., [39, Chapter 2.2]). Among the compactifications, the following three are particularly important: Alexandroff compactification (i.e., one-point compcatification), Freudenthal end-compactification and Stone–Čech compactification. On the set of compactifications, a very natural partial order " $\leq$" is well defined (e.g., Appendix B). Among the set of compactifications, the Alexandroff compactification is the smallest compactification and the Stone–Čech compactificaton is the largest one with respect to the partial order " $\leq$". We say that a compactification C_1 [3] is smaller than a compactification C_2 if $C_1 \leq C_2$ and that C_1 is strictly smaller than C_2 if $C_1 \leq C_2$ but $C_2 \nleq C_1$. We say that C_1 is equivalent to C_2 ($C_1 \simeq C_2$) if $C_1 \leq C_2$ and $C_2 \leq C_1$. That C_1 is equivalent to C_2 would imply that C_1 is homemorphic to C_2 and the converse is not true in general. The set of equivalence classes of compactifications is a complete lattice in our case, since M is locally compact. Also note that since M is completely regular, the set of equivalence classes of compactifications corresponds to the set of closed separating[4] subalgebra (containing the constant function) of real-valued function, for example, [28, page 71]. For more information on compactification theory, we refer to [8], [22], [40], [28].

Recall that a compact, connected, locally connected, metric space is called to be a Peano space.

THEOREM 2. *$M(\natural)$ is a Peano space and consequently $M(\natural)$ is a boundary (i.e., remainder) of M. Moreover, any compactification with $M(\natural)$ as boundary (i.e., remainder) is strictly smaller than the Stone–Čech compactification.*

In this paper, we use # to denote the cardinality of a set. For the initial relations among the three kinds of (ideal) boundaries, we have the following theorem.

THEOREM 3. *If $\#(M(\infty)) = 1$, then $\#(M(\natural)) = 1$. Consequently, the Hamilton–Jacobi equation admits no C^1 solutions.*

For a noncompact topological space, following Freudenthal [18] we could define its (topological) ends. In our special setting of noncompact, complete Riemannian manifold, we could define them by rays [1, III 2]. An equivalence class of cofinal (here two rays γ_1 and γ_2 are called to be cofinal if for any compact subset K, there exists $t_K > 0$ such that $\gamma_1(t_1)$ and $\gamma_2(t_2)$ lie in the same connected component of $M \setminus K$ for all $t_1, t_2 \geq t_K$) rays is called an end of M. Let $\mathscr{E}(M)$ be the set of ends, equipped with the natural topology. It is known that $\mathscr{E}(M)$ is a totally disconnected Hausdorff space, and it is

[3] Here, and somewhere else, we do not specify the embedding i and identify $i(M)$ with M whenever we believe that no confusion would be caused.

[4] We say a subset $\mathcal{S}$ of $C(M,\mathbb{R})$ is separating if for any closed subset K of M, and any point $x \notin K$, there exists $f \in \mathcal{S}$ such that $f(x) \notin \overline{f(K)}$, the closure of $f(K)$.

exactly a kind of boundary (i.e., remainder) with respect to the Freudenhal end compactification in the sense of Remark 0.1. For more details on ends theory, we refer to [18], [29], [1], [25], [13].

REMARK 0.3. Although here we represent an end by a ray and ray is a notion depending strongly on the Riemannian metric, end is indeed a topological notion. Here we only need to notice that confinal condition does not depend on the Riemannian metric. Indeed, an end can also be represented by other purely topological objects. For the sake of completeness, we collect the original definition and a description of end due to Hopf in Appendix B. Different from $\mathscr{E}(M)$, the ideal boundaries $M(\infty)$, $M(\partial)$ and $M(\natural)$ depend on the Riemannian metric.

For a ray, it could represent either an element of the (metric) ideal boundary $M(\infty)$ or an element of the (topological) $\mathscr{E}(M)$, thus connect these two objects. To state results along this line, we first introduce some notations.

Given a ray γ, we could define its coray as following. A ray $\gamma' : [0,\infty) \to M$ is called to be a coray to γ if there exist a sequence $x_k \to \gamma'(0)$, a sequence $t_k \to \infty$ and a sequence of minimal geodesic segments $\gamma_k : [0, d(x_k, \gamma(t_k))] \to M$ connecting x_k and $\gamma(t_k)$ such that γ_k converge to γ' uniformly on any compact interval of $[0,\infty)$. Now we have the following theorem.

THEOREM 4. (1) [10, Theorem 7] *For any ray γ, if γ' is a coray to γ, then γ and γ' are cofinal.*

(2) *If $\#(\mathscr{E}(M)) \geq 3$, then for any Riemannian metric g on M, the associated Hamilton–Jacobi equation*

$$|\nabla u|_g = 1$$

admits no C^1 solutions.

The significance of results in the present paper is that it builds some interesting connections among these three kinds of objects: viscosity solutions [33], [7]; ideal boundaries $M(\infty)$, $M(\partial)$ and $M(\natural)$ [23], [24]; the set of (topological) ends $\mathscr{E}(M)$ [18] or more general remainders in compactification theory [8]. It looks at the first glimpse that they belong somehow to different fields. By the results here, to understand the global property of viscosity solutions of Hamilton–Jacobi equation $(*)$, it is necessary to study deeply the structures of $M(\infty)$, $M(\partial)$, $M(\natural)$, $\mathscr{E}(M)$ and the relations among them. Also it would be interesting if one could relate deeply the structure of these sets to the geometric properties of the Riemannian metric (for Hadamard manifolds, manifolds of negative curve or Riemannian manifolds which are convex at infinity, progress are fruitful (e.g., [4], [12], [43]); for general cases, very little is known) or the dynamics of the geodesic flow. The results in the present paper may be regarded as initial progress and we hope to come back to this issue in the future.

We will divide the proof of Theorem 1 into two parts and leave it to the following two sections. In Section 3, we provide some simple applications of Theorem 1. In Section 4, we will analysis the topological structure of $M(\natural)$ and prove Theorem 2. In Section 5, we will consider some consequences when $M(\infty)$ is a singleton and prove Theorem 3. In Section 6, we consider some initial relations among ends, ideal boundaries and viscosity solutions and thus prove Theorem 4. Finally, we provide two appendices. We collect some necessary definitions and properties from PDE in Appendix A and the ones from topology in Appendix B. We hope this paper is in an almost self-contained level.

1. *dl*-functions are viscosity solutions

In [11], that Busemann functions are viscosity solutions is proved in details, but for *dl*-functions (or horo-functions), the result is stated as a corollary [11, Corollary 7.2] without details. Although the proof is almost the same to the case of Busemann functions, we still give a relatively detailed sketch as follows, for the sake of completeness.

Assume that f is a *dl*-function, that is, there exists a sequence of nonempty closed subsets K_n, with $d(x_0, K_n) \to \infty$ for some fixed point $x_0 \in M$ such that

$$\big[d(\cdot, K_n) - d(x_0, K_n)\big] \to f$$

in the compact-open topology. Then, by the following steps, we could prove that f is a viscosity solution and thus is locally semi-concave with linear modulus.

Step 1: By [35, Theorem 3.1, Proposition 3.4], for any nonempty closed subset K_n, $d(x, K_n) - d(x_0, K_n)$ is a viscosity solution on $M \setminus K_n$ with respect to the Hamilton–Jacobi equation $(*)$ on $M \setminus K_n$.

Step 2: Since $d(x_0, K_n) \to \infty$ and $[d(\cdot, K_n) - d(x_0, K_n)] \to f$ in the compact-open topology (in our case, the compact-open topology coincides with the topology of uniform convergence on compacta), by the stability of viscosity solutions [7, Theorem 5.2.5], f is a global viscosity solution to the Hamilton–Jacobi equation $(*)$ on the whole manifold M.

Step 3: Since Hamilton–Jacobi equations $(*)$ and

$$(**) \qquad |du|_g^2 = 1$$

admit the same set of viscosity solutions, we may also regard f as a viscosity solution to the Hamilton–Jacobi equation $(**)$. Since $(**)$ is induced by the locally uniformly convex Hamiltonian $H(x, p) := |p|_g^2$, f, as a viscosity solution of a locally uniformly convex Hamilton–Jacobi equation, must be locally semi-concave with linear modulus [7, Theorem 5.3.6].

2. Viscosity solutions are dl-functions up to a constant

Given a viscosity solution f to the Hamilton–Jacobi equation $(*)$, we will show that f is a dl-function for a suitable sequence of closed subsets K_n up to a constant. The crucial point is to choose a suitable sequence of nonempty closed subsets K_n.

For any $a \in \mathbb{R}$, let $H_a := \{x : f(x) = a\}$. Choose any fixed point $x_0 \in M$ and assume that $f(x_0) = a_0 \in \mathbb{R}$. Let $K_n := H_{-n}$, we will show that the sequence of closed subsets K_n will be the one we are looking for.

First, K_n is a closed subset for any $n \in \mathbb{Z}^+$, since f is a continuous (in fact, Lipschitz) function. Also note that K_n is non-empty for all suitably large n (This point will be clearer in the following paragraph.) In the following, we will show that

$$\big[d(\cdot, K_n) - d(x_0, K_n)\big] \longrightarrow f$$

in the compact-open topology.

Now we will prove that for any two real numbers $a_1 > a_2 \in \mathbb{R}$, for any point $x \in H_{a_1}$, $d(x, H_{a_2}) = a_1 - a_2$. Choosing a reachable (unit) vector $V \in T_xM$ (i.e., there exists a sequence $x_i \to x$ such that f is differentiable at x_i and $\nabla f(x_i) \to V$), there exists a unique minimal geodesic segment $\gamma : (-\infty, 0] \to M$ with $\gamma(0) = x$ and $\dot\gamma(0) = V$ and $f(\gamma(t)) - f(x) = t$, since for the Hamilton–Jacobi equation whose Hamiltonian is locally uniformly convex, viscosity solutions coincide with variational (minmax) solutions [44], [45]. Now it is clear that K_n is nonempty for any sufficiently large n. Thus, there exists a unique t_0 such that $f(\gamma(t_0)) = a_2$. It is easy to see that $t_0 = a_2 - a_1$ and

$$d\big(x, \gamma(t_0)\big) = \text{length}\big(\gamma \mid [t_0, 0]\big) = |t_0| = -t_0.$$

Since $\gamma(t_0) \in H_{a_2}$, we get $d(x, H_{a_2}) \le -t_0 = a_1 - a_2$. If $a_3 := d(x, H_{a_2}) < a_1 - a_2$, applying Fubuni's theorem, we can get an absolutely continuous curve (parameterized by arc-length) $\xi : [-a_4, 0] \to M$ with $\xi(-a_4) \in H_{a_2}$ and $\xi(0) = x \in H_{a_1}$ such that $0 \le a_4 - a_3 \le \frac{1}{2}(a_1 - a_2 - a_3)$ and f is differentiable almost everywhere along ξ with respect to the 1-dimensional Lebesgue measure. Note that $f \circ \xi : [-a_4, 0] \to \mathbb{R}$ is still a Lipschitz function, thus differentiable almost everywhere with respect to the 1-dimensional Lebesgue measure. So we get

$$\begin{aligned} a_1 - a_2 &= f\big(\xi(0)\big) - f\big(\xi(-a_4)\big) \\ &= \int_{-a_4}^{0} g(\nabla f, \dot\xi)\, dt \\ &\le \text{length}(\xi|_{[-a_4,0]}) \quad \big(\text{since } |\nabla f|_g \le 1\big) \\ &= a_4 \\ &\le a_3 + \frac{1}{2}(a_1 - a_2 - a_3) \\ &< a_1 - a_2. \end{aligned}$$

This contradiction proves $d(x, H_{a_2}) = a_1 - a_2$.

By the discussions above, for any compact subset S, there exists a constant $n_S > 0$ such that for any $n > n_S$,

$$-n < \min_{x\in S} f(x), \qquad -n < f(x_0).$$

Then for any $x \in S$ and any $n > n_S$,

$$d(x,K_n) = d(x,H_{-n}) = f(x) + n$$

and

$$d(x_0,K_n) = d(x_0,H_{-n}) = f(x_0) + n.$$

It means that

$$d(x,K_n) - d(x_0,K_n) = f(x) - f(x_0)$$

for any $x \in S$ and any $n > n_S$. This is to say,

$$\big[d(x,K_n) - d(x_0,K_n)\big] \longrightarrow f(x) - f(x_0)$$

in the compact-open topology. □

NOTE 2.1. As the referee kindly suggested, there is a more direct proof of Theorem 1 based on Hopf–Lax formula. He/She also provided the details of this new proof. The preference will depend on the personal interest and academic background. Readers from the field of dynamical systems maybe like the original one and from the field of PDE maybe like the new one. Since the method we used will also be enlightening for the understanding of Sections 5 and 6, we insist on the original proof and provide the alternative proof in this note. We thank the referee this valuable advice.

Let f be a viscosity solution of $(*)$. Given any $x_0 \in M$, we take $n \in \mathbb{N}$ such that $f(x_0) > -n$. Now we consider the metric ball K centered at x_0 with radius $f(x_0) + n$. Now applying Hopf–Lax formula (e.g., Appendix A),

$$w(x) := \min\big\{f(y) + d(y,x) \mid y \in \partial K\big\}$$

must coincide with f by the uniqueness principle of viscosity solutions to Dirichlet problem. Here, and in the following, $\partial\cdot$ denotes the topological boundary of a subset of M. Consequently, we have

$$f(x_0) = f(y_n) + d(y_n,x_0) = f(y_n) + f(x_0) + n$$

for some $y_n \in \partial K$. It means that $f(y_n) = -n$. So the sets

$$K_n := \big\{y \mid f(y) = -n\big\}$$

are nonempty for any $n > -f(x_0)$. Moreover, we claim that for any $y \in K_n$,

$$d(x_0,y) \geq f(x_0) + n.$$

Otherwise, $K_n \cap \operatorname{int} K \neq \emptyset$. Choose $x^* \in K_n \cap \operatorname{int} K$, then by Hopf–Lax formula again,

$$-n = f\big(x^*\big) = \min\big\{f(y) + d\big(y,x^*\big) \mid y \in \partial K\big\}.$$

It means that there exists $y^* \in \partial K$ such that $-n = f(y^*) + d(y^*, x^*)$. So $f(y^*) < -n$. Now we use Hopf–Lax formula one more time,

$$\begin{aligned} f(x_0) &= \min\{f(y) + d(y, x_0) \mid y \in \partial K\} \\ &\le f(y^*) + d(y^*, x_0) \\ &< -n + f(x_0) + n \\ &= f(x_0). \end{aligned}$$

The contradiction proves the claim.

So far we get $d(x_0, K_n) = f(x_0) + n$. We assume that f vanishes at some point $z \in M$, up to addition of a constant. By the same argument as above, we get $d(z, K_n) = n$ for any $n \ge 1$. So

$$f(x_0) = d(x_0, K_n) - d(z, K_n)$$

for any sufficiently large n. By the arbitrariness of x_0, we conclude the proof.

REMARK 2.2. By the proof of Theorem 1, we could obtain this fact: If f is a viscosity solution and there exists a point x_0 such that $f(x_0) = 0$, then f itself is a *dl*-function. In other words, it is not necessary to add a constant in this case.

REMARK 2.3. In [24, page 202], *dl*-function is defined in a slightly different form: a function f is a *dl*-function if

$$f(x) = t + d\big(x, f^{-1}(-\infty, t]\big)$$

for all $t \in \mathbb{R}$ and those $x \in M$ where $f(x) \ge t$. By the procedure of the proof of Theorem 1 in Section 2, this definition coincides with the one we used up to a constant, thus we could use either of them to define the ideal boundary $M(\natural)$. If we prefer the original one, Theorem 1 could be restated as: f is a viscosity solution of Hamilton–Jacobi equation $(*)$ if and only if f is a *dl*-function. Also, the definition of *dl*-function is also slightly different from the one in [36], where an element in $M(\natural)$ is called to be a *dl*-function.

Combining the contents of Section 1 and Section 2, Theorem 1 is proved.

3. Some applications

By Theorem 1, $M(\natural)$ could be redefined as

$$M(\natural) = \{\text{viscosity solutions}\}/\{\text{constant functions}\}.$$

Thus, $M(\natural)$ should inherit some properties from viscosity solutions. Here we collect some well-known ones, which will be useful later.

COROLLARY 3.1 ([36, Lemma 4.5]). *f_1, f_2 are two dl-functions, then* $\min\{f_1, f_2\}$ *is still a dl-function up to a constant.*

The proof in [36] is totally different from the one presented below.

Proof. If f_1, f_2 are two *dl*-functions, then they are viscosity solutions and locally semi-concave with linear modulus.

First, we will show that $\min\{f_1, f_2\}$ is still a viscosity solution to the Hamilton–Jacobi equation $(*)$ (or equivalently $(**)$). By [7, Proposition 1.1.3] it is easy to obtain that $\min\{f_1, f_2\}$ is still a locally semi-concave function with linear modulus. Thus, by [7, Proposition 5.3.1], we only need to show that Hamilton–Jacobi equation $(**)$ is satisfied at all points of differentiable points of $\min\{f_1, f_2\}$. Let $M_1 := \{x : f_1(x) \neq f_2(x)\}$ and $M_2 := \{x : f_1(x) = f_2(x)\}$. We denote the interior of M_2 by $\mathrm{int}(M_2)$. Let $U := M_1 \cup \mathrm{int}(M_2)$, then $\min\{f_1, f_2\}$ satisfies the Hamilton–Jacobi equation $(**)$ at its differentiable points in U. Note that U is an open and dense subset, by [7, Proposition 3.3.4(a)], together with the continuity of $|\cdot|_g$, $\min\{f_1, f_2\}$ satisfies the Hamilton–Jacobi equation $(**)$ at any differentiable point. So far, we know that $\min\{f_1, f_2\}$ is really a viscosity solution to the Hamilton–Jacobi equation $(**)$ (or equivalently $(*)$). By Theorem 1, this means that $\min\{f_1, f_2\}$ is a *dl*-function up to a constant. □

We could state Corollary 3.1 alternatively as the following.

Corollary 3.2. *If f_1, f_2 are two viscosity solutions, then $\min\{f_1, f_2\}$ is a viscosity solution too.*

Corollary 3.3. *$M(\natural)$ is compact with respect to the quotient compact-open topology.*

Proof. Fix a point $x_0 \in M$ and we represent elements of $M(\natural)$ by *dl*-functions f with $f(x_0) = 0$. In other words, we identify $M(\natural)$ with

$$\mathcal{F} := \{f : f \text{ are } dl\text{-functions with } f(x_0) = 0\}.$$ [5]

For any sequence $f_n \in \mathcal{F}$, since f_n is uniformly Lipschitz (with Lipschitz constant 1, thus equi-continuous) and uniformly bounded on any compact subset, together with the fact that M is hemi-compact [20, Theorem 2] with respect to the manifold topology (or equivalently the topology induced by the distance d), there exists a subsequence f_{n_i} such that f_{n_i} converge to a continuous function f in the compact-open topology, by Arzela–Ascoli theorem. Further by the stability of viscosity solutions [7, Theorem 5.2.5], f itself is a viscosity solution and $f(x_0) = 0$. Namely, $f \in \mathcal{F}$. So far we have proved that $\mathcal{F}$ is sequentially compact.

[5] $\mathcal{F}$, equipped with the the compact-open topology, and $M(\natural)$, equipped with the quotient compact-open topology, are homemorphic. In fact, the map $\mathcal{F} \ni f \mapsto f/\sim$ is the homemorphism. But, here we only use the continuity of this map.

By [36, Theorem 4.6], $\mathcal{F}$ is metrizable.[6] It is well known for a metric space, sequential compactness and compactness are equivalent (e.g., [31, page 84, Proposition 3]). Thus, $\mathcal{F}$ is compact with respect to the compact-open topology. So, $M(\natural)$, as the image of the continuous map $\mathcal{F} \ni f \mapsto f/\sim \in M(\natural)$, is compact with respect to the quotient compact-open topology. $\square$

Let the horo-function compactification

$$\bar{M}(\partial) := \text{closure}\, \frac{\{d(\cdot,x) : x \in M\}}{\{\text{constant functions}\}},$$

then it is easy to see that $M(\partial)$ is the topological boundary of the set $\bar{M}(\partial)$, here "closure" and "boundary" are considered under the quotient compact-open topology. It is known that $\bar{M}(\partial)$ is a compact, metrizable (particularly, Hausdorff) space. Hence, $M(\partial)$ is a closed subset of $\bar{M}(\partial)$. By this fact, we could get the following well-known result.

Corollary 3.4. *$M(\partial)$ is compact with respect to the quotient compact-open topology.*

4. Proof of Theorem 2

For the simplicity of notations, in this section we use $\sim$ (as in Footnote 6) to denote the equivalence relation where two continuous functions are equivalent if they differ a constant. Now we will prove that $M(\natural)$ is a Peano space and thus $M(\natural)$ could also be regarded as a boundary (i.e., remainder). We will divide the proof to the following steps.

- *Compactness.* The compactness of $M(\natural)$ is proved in Corollary 3.3.
- *Connectedness.* For any two viscosity solutions u, v, let

$$f_t := \min\{u+t, v\}.$$

[6] Since M is hemi-compact, $\mathcal{C}(M)$, the set of continuous functions on M with the compact-open topology, is metrizable (e.g., [41, 43G]). The metric ρ could be defined by

$$\rho(u,v) = \sum_{n=1}^{\infty} \rho_n(u,v),$$

where $K_n = \overline{B_n(x_0)}$, the closed metric ball centered at some fixed point x_0 with radius n; $\rho_n(u,v) = \min\{\frac{1}{2^n}, \sup_{x\in K_n} |u(x)-v(x)|\}$. Thus the set $\mathcal{V}$ of viscosity solutions to the Hamilton–Jacobi equation $(*)$, as a subspace of $\mathcal{C}(M)$, is metrizable. We consider $\mathbb{R}$ as an additive group, acting on $\mathcal{V}$ by $tu := t+u$. Then for each $t \in \mathbb{R}$, t is an isometry and moreover the orbits of the action are closed. By [6, Theorem 2.1], $M(\natural)$, as a quotient space where the equivalence relation (denoted by $\sim$) is induced by the $\mathbb{R}$-action, is metrizable by quotient metric $\rho_\sim$, here $\rho_\sim$ is defined by

$$\rho_\sim(u/\sim, v/\sim) = \inf_{t,s\in\mathbb{R}} \rho(u+t, v+s).$$

Be careful that in general case on a quotient space of a metric space only quotient pseudo-metric is well defined.

By Corollary 3.2, for each $t \in \mathbb{R}$, f_t is a viscosity solution. Thus, the map

$$t \to f_t$$

is a continuous map from $\mathbb{R}$ to $\mathcal{V}$, the set (equipped with the compact-open topology) of viscosity solutions to the Hamilton–Jacobi equation ($*$). It is also easy to see that $f_t/\!\sim \longrightarrow u/\!\sim$ as $t \longrightarrow -\infty$ and $f_t/\!\sim \longrightarrow v/\!\sim$ as $t \longrightarrow \infty$. This shows that $\mathcal{V}$, and thus $M(\natural)$, is connected.

• *Local connectedness.* By [41, 27.16 Theorem], which is included in Appendix B, we only need to show that $M(\natural)$ is connected im Kleinen at every point $u/\!\sim \in M(\natural)$. In other words, we have to show that for any $u/\!\sim \in M(\natural)$ and for any open neighborhood $\mathcal{N}$ of $u/\!\sim$ in $M(\natural)$, there exists an open neighborhood $\mathcal{U}$ of $u/\!\sim$ with $\mathcal{U} \subset \mathcal{N}$ such that for any $v_1/\!\sim, v_2/\!\sim \in \mathcal{U}$, $v_1/\!\sim$ and $v_2/\!\sim$ are in a connect subset $\mathcal{M}$ of $\mathcal{N}$.

First, we choose $\epsilon > 0$ small enough such that the metric [7] ball $B_\epsilon(u/\!\sim)$ is contained in $\mathcal{N}$ and take $\mathcal{U} = B_{\frac{\epsilon}{8}}(u/\!\sim)$.

Now for any $v_1/\!\sim, v_2/\!\sim \in \mathcal{U} = B_{\frac{\epsilon}{8}}(u/\!\sim)$, we construct the connected subset $\mathcal{M}$ containing $v_1/\!\sim$ and $v_2/\!\sim$ as following. We fix a viscosity solution u as a representation of the class $u/\!\sim$. For any $v_1/\!\sim, v_2/\!\sim \in B_{\frac{\epsilon}{8}}(u/\!\sim)$, we fix a representation of $v_1/\!\sim$ and $v_2/\!\sim$ respectively, still denoted by v_1 and v_2, such that $\rho(u, v_1) < \frac{\epsilon}{8}$ and $\rho(u, v_2) < \frac{\epsilon}{8}$. Such a representation does exist since the $\mathbb{R}$-action, which induces the quotient equivalence relation, is an isometry (see Footnote 6). Hence, $\rho(v_1, v_2) < \frac{\epsilon}{4}$. For such two elements v_1 and v_2, let

$$f_t(v_1, v_2) := \min\{v_1 + t, v_2\}.$$

It is easy to see that

$$\begin{aligned} \rho_\sim\big(f_t(v_1,v_2)/\!\sim, v_1/\!\sim\big) &\le \rho\big(f_t(v_1,v_2), v_1 + t\big) \\ &\le \rho(v_1, v_2) < \frac{\epsilon}{4} \end{aligned}$$

for any $t \le 0$ and

$$\begin{aligned} \rho_\sim\big(f_t(v_1,v_2)/\!\sim, v_2/\!\sim\big) &\le \rho\big(f_t(v_1,v_2), v_2\big) \\ &\le \rho(v_1, v_2) < \frac{\epsilon}{4} \end{aligned}$$

for any $t \ge 0$. Consequently, $\rho_\sim(f_t(v_1,v_2)/\!\sim, u/\!\sim) < \frac{\epsilon}{2}$ for any $t \in \mathbb{R}$. Now we define

$$\mathcal{M} := \mathrm{closure}\Big\{\bigcup_{t\in\mathbb{R}} f_t(v_1,v_2)/\!\sim\Big\},$$

here v_i $(i = 1, 2)$ is a representation of $v_i/\!\sim$ as explained at the beginning of this paragraph. Since $f_t(v_1,v_2)/\!\sim \to v_1/\!\sim$ as $t \to -\infty$ and $f_t(v_1,v_2)/\!\sim \to v_2/\!\sim$ as $t \to \infty$, we have both $v_1/\!\sim$ and $v_2/\!\sim$ are contained in $\mathcal{M}$. It is easy to see that $\mathcal{M}$ is connected (in fact, $t \mapsto f_t(v_1, v_2)$ is a continuous curve

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[7] Recall that $M(\natural)$ is metrizable by the quotient metric $\rho_\sim$ introduced in Footnote 6.

in $M(\natural)$) and is contained in $B_\epsilon(u/\sim) \subseteq \mathcal{N}$. By definition, $M(\natural)$ is connected im Kleinen at $u/\sim$. By the arbitrariness of $u/\sim$, $M(\natural)$ is locally connected.

• *Metrizability.* As we said in the proof of Corollary 3.3, the metrizability of $M(\natural)$ is proved in [36, Theorem 4.6]. See also Footnote 6.

By the result [34, Corollary (2.3)], any Peano space could be a boundary (i.e., remainder) of any locally compact, non-compact, metric space. Thus, $M(\natural)$ is a boundary (i.e., remainder) of M.

Since M is connected, locally compact but not pseudo-compact, the Stone–Čech compactifiaction $\beta(M)$ is connected but not locally connected [27, 2.5 Corollary], [40, 9.3.Theorem]. For the remainder $\beta(M) \setminus M$, the realcompactness of M implies that it is not connected im Kleinen at any point [42, Theorem 5]. Consequently, $\beta(M) \setminus M$ is not locally connected at any point. On the other hand, since $M(\natural)$ is locally connected, $M(\natural)$ is not homoemorphic to $\beta(M) \setminus M$, and thus any compactification with $M(\natural)$ as boundary (i.e., remainder) is not equivalent to the Stone–Čech compactification. Since Stone–Čech compactification is the largest compactification, any compactification with $M(\natural)$ as boundary is strictly smaller than the Stone–Čech compactification. □

Remark 4.1. One may ask which compactification can make $M(\natural)$ to be the boundary (i.e., remainder). One may express the compactification as the quotient of the Stone–Čech compactification, as in [34, Theorem 2.1]. Other more direct methods of compactification are not clear so far.

5. The case $M(\infty)$ is a singleton

In weak KAM theory [16], it is well known that if there is only one static class for some Aubry set, then the associated Hamilton–Jacobi equation has only one viscosity solution up to a constant. Here we also provide an analogous property (i.e., Theorem 3) in our setting. We restate the first statement of Theorem 3 as the following proposition.

Proposition 5.1. *If $M(\infty)$ is a singleton, then $M(\natural)$ is a singleton as well.*

Proof. For any fixed ray γ, we will prove that for any viscosity solution f, $f = b_\gamma$ up to a constant. Let D be the set on which both f and b_γ are differentiable. Clearly, D is of full measure with respect to the Lebesgue measure. Since f is a viscosity solution, for any $x \in D$, there exists a unique ray $\gamma_x : [0,\infty) \to M$ such that $\gamma_x(0) = x$, $-\dot\gamma_x(0) = \nabla f(x)$ and $f(\gamma_x(t_2)) - f(\gamma_x(t_1)) = t_1 - t_2$ for any $t_1, t_2 \in \mathbb{R}$. Since $M(\infty)$ consists of only one point, γ_x must be a coray to γ. Since b_γ is differentiable at x, γ_x is the only coray to γ. Thus, $\nabla f(x) = \nabla b_\gamma(x) = -\dot\gamma_x(0)$. Thus we get $\nabla(f - b_\gamma) = 0$ on D. Since D is of full measure and $f - b_\gamma$ is a Lipschitz function, applying Fubini's theorem we get $f - b_\gamma$ is a constant. So $M(\natural)$ is also a singleton. □

If the Hamilton–Jacobi equation $(*)$ admits a C^1 solution, say f, then f will be $C^{1,1}$ [15] and M will be foliated by the integral curves of ∇f, which are lines (recall that by definition a line $\gamma:\mathbb{R}\to M$ is a geodesic such that $d(\gamma(t_1),\gamma(t_2))=|t_1-t_2|$ for any $t_1,t_2\in\mathbb{R}$) [17], [26]. In fact, a more direct proof goes as follows. Otherwise, there is a (unit-speed) minimal geodesic $\xi:[t_3,t_4]\to M$ with $\xi(t_3)=\gamma(t_1)$ and $\xi(t_4)=\gamma(t_2)$ such that $t_4-t_3=d(\gamma(t_1),\gamma(t_2))<|t_2-t_1|$. But

$$\begin{aligned}|t_2-t_1| &= \big|f\big(\gamma(t_2)\big)-f\big(\gamma(t_1)\big)\big| \\ &= \big|f\big(\xi(t_4)\big)-f\big(\xi(t_3)\big)\big| \\ &= \left|\int_{t_3}^{t_4} g(\nabla f,\dot{\xi})\,dt\right| \\ &\le \int_{t_3}^{t_4} |\dot{\xi}|_g\,dt \\ &= t_4-t_3,\end{aligned}$$

and we get a contradiction.

It is easy to see that if there is a line, then $M(\infty)$ will contain at least two elements. It contradicts the assumption that $M(\infty)$ is a singleton. So far Theorem 3 is proved. □

To show the non-trivialness of Theorem 3, we provide an example for which $M(\infty)$ contains exactly one element.

EXAMPLE 5.2. In Euclidean space $\mathbb{R}^3$, we construct a surface of revolution as follows. First we choose a smooth function $f:[0,+\infty)\to[0,1]$ such that

- $f(0)=0$ and $f(x)>0$ for $x>0$.
- $f(x)=1$ for every $x\ge 1$.
- The graph of f and the y-axis are infinitely tangent at $(0,0)$.

Now we rotate the graph of f along the x-axis and get the surface of revolution S. Now we equip S the induced metric g from the Euclidean $\mathbb{R}^3$. For any ray $\gamma:[0,\infty)\to S$, there exists a $T>0$ such that $\gamma|_{[T,\infty)}$ is a half straight line parallel to the x-axis. So for any two rays, one is a coray to the other. It is easy to see that $M(\infty)$ is a singleton in this case.

6. Ends and ideal boundary

Since any two distinct ends can be connected by a line [1, III 2.3], we easily get

COROLLARY 6.1. *If $M(\infty)$ is a singleton, then $\mathscr{E}(M)$ must be a singleton.*

For corays, we have the following proposition.

PROPOSITION 6.2 ([10, Theorem 7]). *For any ray γ, all corays to γ are cofinal to γ.*

Theorem 4(1) is just a restatement of Proposition 6.2.

By Proposition 6.2, we easily get the following corollary.

Corollary 6.3. $\#(\mathscr{E}(M)) \le \#(M(\infty))$.

If $M = \mathbb{R}$, both $\mathscr{E}(M)$ and $M(\infty)$ always contain exactly two elements. For higher dimensional cases, $\mathbb{R}^n$ $(n \ge 2)$ is of one-end, that is, $\#(\mathscr{E}(M)) = 1$ ($\mathscr{E}(M)$ is a topological notion and independent of the Riemannian metric). We would like to pose

Problem 6.4. On $\mathbb{R}^n$ $(n \ge 2)$, characterize all Riemannian metrics such that the associated ideal boundary $M(\infty)$ is a singleton.

More generally, we pose the following problem.

Problem 6.5. On any non-compact, boundaryless, connected, paracompact manifold M, is there a complete Riemannian metric g such that the associated ideal boundary $M(\infty) = \mathscr{E}(M)$ (i.e., two rays γ and γ' are cofinal if and only if $b_\gamma - b_{\gamma'} = const.$)?

Now we restate Theorem 4(2) as the following proposition.

Proposition 6.6. *If $\#(\mathscr{E}(M)) \ge 3$, then for any Riemannian metric g on M, the associated Hamilton–Jacobi equation*

$$|\nabla u|_g = 1$$

admits no C^1 solutions.

Proof. Otherwise, there exists a Riemannian metric g such that the associated Hamilton–Jacobi equation

$$|\nabla u|_g = 1$$

admits a C^1 solution, say f. So $-f$ is also a C^1 solution to the Hamilton–Jacobi equation ($*$). So both f and $-f$ are locally semi-concave with linear modulus, and thus f is locally $C^{1,1}$ [7, Corollary 3.3.8]. So we obtain that ∇f is a locally Lipschitz vector field and consequently existence and uniqueness property of solutions of ODE holds. Clearly, the integral curves of ∇f form a locally Lipschitz foliation by lines. Now we fix any integral curve γ_0, and assume that γ_0^- representing $\mathcal{E}_-$ and γ_0^+ representing $\mathcal{E}_+$, here $\mathcal{E}_-, \mathcal{E}_+ \in M(\mathscr{E})$ may coincide. Here, and in the following, for a line $\gamma : \mathbb{R} \to M$, γ^+ and γ^- are rays defined respectively, by $\gamma^+(t) := \gamma(t)$ and $\gamma^-(t) := \gamma(-t)$ for all $t \ge 0$.

First, we prove for any other integral curve γ_1, we must have that γ_1^- represents $\mathcal{E}_-$ and γ_1^+ represents $\mathcal{E}_+$. Since M is connected, as a manifold, it is path-connected [19, Theorem 4]. So there exists a smooth curve $\xi : [0,1] \to M$ connecting $\gamma_0(0)$ and $\gamma_1(0)$, i.e. $\xi(0) = \gamma_0(0)$ and $\xi(1) = \gamma_1(0)$. We denote the flow generated by ∇f by ϕ_t. Now we will get two facts:

(1) For any $t \ge 0$, $\gamma_0^+(t)$ and $\gamma_1^+(t)$ are connected by the curve $\phi_t(\xi)$;

(2) Since f is a C^1 (in fact, $C^{1,1}$) solution to the Hamilton–Jacobi equation, for any compact subset K, there exists a $t_K > 0$ such that for $t > t_K$, $\phi_t(\xi) \cap K = \emptyset$.

Combining facts (1) and (2), we obtain that γ_0^+ and γ_1^+ represent the same end (in fact, γ_0^+ and γ_1^+ are strongly equivalent, which is a stronger condition introduced by Hopf than representing the same end, for definition and details, see [29], [25, 3.3]). Similarly, γ_0^- and γ_1^- represent the same end.

Up to now, we have proved that for any integral curves γ, γ^- represents $\mathcal{E}_-$ and γ^+ represents $\mathcal{E}_+$. Based on this fact, we could continue the proof as following.

Since $\#(\mathscr{E}(M)) \geq 3$, there exist a sufficiently large compact subset K, a ray ζ representing an end $\mathcal{E}$ different from $\mathcal{E}_+$ and $\mathcal{E}_-$, and a real number $T > 0$ such that $\zeta \mid [T, \infty)$ and $\gamma_0^+ \mid [T, \infty)$ lie in different connected components of $M \setminus K$, $\zeta \mid [T, \infty)$ and $\gamma_0^- \mid [T, \infty)$ lie in different connected components of $M \setminus K$. Denote the diameter of K by r ($r > 0$, since K cannot to be a single point set in our case). Choose S large enough such that $d(\zeta(T + S), K) > r$. Considering the integrable curve $\gamma' : \mathbb{R} \to M$ of ∇f with $\gamma'(0) = \zeta(T + S)$, since γ'^+ represents $\mathcal{E}_+$ and γ'^- represents $\mathcal{E}_-$, there exist two real numbers t^+ and t^- such that:

- $\gamma'(t^+) \in K$, $\gamma' \mid (t^+, \infty)$ and $\gamma_0^+ \mid (T, \infty)$ lie in the same connected component of $M \setminus K$.
- $\gamma'(t^-) \in K$, $\gamma' \mid (-\infty, t^-)$ and $\gamma_0^- \mid (T, \infty)$ lie in the same connected component of $M \setminus K$.

Since γ' is a line, $\gamma'(0) = \zeta(T + S)$ and $d(\zeta(T + S), K) > r$, we obtain

$$d\big(\gamma'(t^+), \gamma'(t^-)\big) = \big|t^+ - t^-\big| > 2r.$$

On the other hand, the fact that $\gamma'(t^+) \in K$ and $\gamma'(t^-) \in K$ will imply that $d(\gamma'(t^+), \gamma'(t^-)) \leq r$. This contradiction proves Proposition 6.6 and thus Theorem 4(2) is proved. ☐

Appendix A

In this appendix, we collect some fundamental definitions and properties from PDE.

Definition A.1. Given an open subset $\Omega \subset \mathbb{R}^n$, a continuous function

$$u : \Omega \to \mathbb{R}$$

is called locally semi-concave with linear modulus if, for any open convex subset $\Omega' \subset \Omega$ with compact support in Ω (i.e., $\Omega' \Subset \Omega$), there exists a constant C such that $u(x) - \frac{C}{2}|x|^2$ is a concave function in Ω' (here, $|\cdot|$ is the Euclidean norm).

We call the constant C (depending on the choice of Ω') to be the semi-concave constant.

For the functions defined on manifold M, we need the following definition.

DEFINITION A.2. A continuous function $u : M \to \mathbb{R}$ is called locally semi-concave with linear modulus if, for any $x \in M$, there exist an open neighborhood U and a smooth coordinate chart

$$\phi : U \to \mathbb{R}^n,$$

such that the function $u \circ \phi^{-1}$ is locally semi-concave on $\phi(U)$.

REMARK A.3. For two different charts ϕ_1, ϕ_2 both defined on U, $u \circ \phi_1^{-1}$ is locally semi-concave with linear modulus if and only $u \circ \phi_2^{-1}$ is locally semi-concave with linear modulus, although the semi-concave constants of $u \circ \phi_1^{-1}$ and of $u \circ \phi_2^{-1}$ may be different. So the definition is well posed.

REMARK A.4. For other (locally) semi-concave functions with more general modulus, see [7].

Let ∇ be the gradient determined by the Riemannian metric g. To state our main results, we need to introduce some more preliminary notions.

DEFINITION A.5. If $u : M \to \mathbb{R}$ is a locally Lipschitz function defined on Riemannian manifold (M, g), then a covector $V^* \in T_q^*M$ is called to be a subdifferential (resp. superdifferential) of u at $q \in M$, if there exist a neighborhood Ω of q and a C^1 function $\phi : \Omega \to \mathbb{R}$, $\phi(q) = u(q)$, $\phi(x) \le u(x)$ (resp. $\phi(x) \ge u(x)$) for every $x \in \Omega$ and $d\phi(q) = V^*$; similarly, a vector $V \in T_qM$ is called to be a subgradient (resp. supergradient) of u at $q \in M$, if there exist a neighborhood Ω of q and a C^1 function $\phi : \Omega \to \mathbb{R}$, $\phi(q) = u(q)$, $\phi(x) \le u(x)$ (resp. $\phi(x) \ge u(x)$) for every $x \in \Omega$ and $\nabla\phi(q) = V$.

We denote by $D^-u(q)$ (resp. $D^+u(q)$) the set of subdifferentials (resp. superdifferentials) of u at q, and $\nabla^-u(q)$ (resp. $\nabla^+u(q)$) the set of subgradients (resp. supergradients) of u at q.

Now we can reformulate the definition of viscosity solution as follows.

DEFINITION A.6. A continuous function u is called a viscosity subsolution of the Hamilton–Jacobi equation $|\nabla u|_g = 1$ if for any $q \in M$,

$$|V|_g \le 1 \quad \text{for every } V \in \nabla^+u(q).$$

Similarly, a continuous function is called a viscosity supersolution of the Hamilton–Jacobi equation $|\nabla u|_g = 1$ if for any $q \in M$,

$$|V|_g \ge 1 \quad \text{for every } V \in \nabla^-u(q).$$

A continuous function is a viscosity solution if it is a viscosity subsolution and a viscosity supersolution simultaneously.

Once subdifferentials and superdifferentials, instead of subgradients and supergradients, are involved in, one can define the viscosity solution of the Hamilton–Jacobi equation $|du|_g^2 = 1$ similarly.

For a locally semi-concave function u, it is a viscosity solution of $|\nabla u|_g = 1$ if and only if it satisfies the equation at its differential points [7, Proposition 5.3.1]. For more precise details on viscosity solutions, we refer to [33], [7].

We present a version of Hopf–Lax formula, which is provided by the referee. This formula is only used in Note 2.1. It could be slightly modified from the results in [33, Chapter 5].

Theorem 5 (Hopf–Lax formula). *For any compact subset K of M, and any trace u defined on ∂K and 1-Lipschitz with respect to the distance d, the unique viscosity solution agreeing with u on ∂K is given by*

$$f(x) = \min\{u(y) + d(y,x) \mid y \in \partial K\}.$$

Appendix B

In this appendix, we collect some well-known definitions and properties from topology.

Definition B.1. A topology space X is pseudo-compact if every continuous function $f \in C(X, \mathbb{R})$ is bounded.

Definition B.2. A topology space X is hemi-compact if there is a sequence

$$K_1, \dots, K_n, \dots$$

of compact subsets of X such that if K is any compact subset of X, then $K \subset K_n$ for some n.

Definition B.3. A Hausdorff space X is called a Raum if X is connected, locally connected, locally compact and σ-compact.

Definition B.4. A topology space X is completely regular if whenever K is a closed subset of X and $x \notin K$, there is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(K) = 1$. A completely regular T_1-space is called a Tychonoff space.

Definition B.5. A topological space X is said to be realcompact if it can be embedded homemorphically as a subset of some (not necessarily finite) Cartesian power of the reals, with the product topology.

It is known that any Lindelöf space is realcompact [14, Theorem 3.11.12]. For a complete, connected, non-compact Riemannian manifold, it is a non-pseudo-compact, hemi-compact Raum.

Definition B.6. Let X be a non-compact Raum. We say that a decreasing sequence G_K of nonempty subsets of X represents an end if

(i) G_k if open.
(ii) G_k is connected.

(iii) ∂G_k is compact.
(iv) $\bigcap_{n=1}^{\infty} \bar{G_k} = \emptyset$.

Two such sequences (G_k) and (H_k) represent the same end if $G_l \cap H_l \neq \emptyset$ for all $l \in \mathbb{N}$. In this case, we say (G_k) and (H_k) are equivalent. We let the set $\mathscr{E}(X)$ of ends be the set of such equivalent classes.

For further understanding of ends, we recall a description of ends due to Hopf [29].

THEOREM 6. (i) *For every end of a Raum X, it is represented by a proper map $\gamma : [0,\infty) \to X$.*

(ii) *Two proper maps $\gamma_i : \{i\} \times [0,\infty) \to X$ $(i = 0,1)$ represent the same end if and only if they have a proper extension $f : L \to X$, where L is the "infinite ladder" $\{0,1\} \times [0,\infty) \cup [0,1] \times \{0,1,2,\ldots\}$.*

Let $C^*(X)$ be the collection of all bounded continuous real-valued functions on X, the range of each $f \in C^*(X)$ can be taken as a closed bounded interval I_f in $\mathbb{R}$. Once X is Tychonoff, the collection $C^*(X)$ separates points from closed sets in X and thus, the evaluation map $e : X \to \Pi\{I_f \mid f \in C^*(X)\}$, defined by

$$\big[e(x)\big]_f = f(x),$$

is a topological embedding of X in ΠI_f.

DEFINITION B.7. The Stone–Čech compactification of X is the closure βX of $e(X)$ in the product ΠI_f.

Now for two compactifications (C_1, i_1) and (C_2, i_2), we write $(C_1, i_1) \le (C_2, i_2)$ if there exists a continuous map $F : C_2 \to C_1$ such that $F \circ i_2 = i_1$. If $(C_1, i_1) \le (C_2, i_2)$ and $(C_2, i_2) \le (C_1, i_1)$ hold simultaneously, we say that they are topologically equivalent and denote this relation by $(C_1, i_1) \simeq (C_2, i_2)$. We say (C_1, i_1) is strictly smaller than (C_2, i_2), namely $(C_1, i_1) < (C_2, i_2)$, if $(C_1, i_1) \le (C_2, i_2)$ and $(C_2, i_2) \nleq (C_1, i_1)$. When no confusion is caused, we write $C_1 \le C_2$, $C_1 < C_2$, $C_1 \simeq C_2$ and $C_1 \nleq C_2$ respectively for short. It is known that Stone–Čech compactification is the largest compactification under the partial order $\le$.

DEFINITION B.8. A space X is said to be locally connected at a point $x \in X$ if for every open neighborhood N of x in X, there exists a connected open neighborhood N' of x such that $N' \subset N$. X is said to be locally connected if it is locally connected at each of its points.

DEFINITION B.9. A space X is said to be connected im kleinen at a point $x \in X$ if for every open neighborhood N of x in X, there exists an open neighborhood N' of x with $N' \subset N$ such that any pair of points in N' lie in some connected subset of N.

In general, a space X that is locally connected at $x \in X$ is connected im Kleinen at x and the converse is not true. But we have the following theorem.

THEOREM 7 ([41, 27.16 Theorem]). *If X is connected im kleinen at each point, then X is locally connected.*

DEFINITION B.10. A topology space X is called to be a Peano space if it is a compact, connected, locally connected metric space.

THEOREM 8 ([34, Corollary 2.3]). *Let X be any locally compact, non-compact, metric space and let Y be any Peano space. Then there exists a compactification (C, i) of X such that $C \setminus i(X)$ is homeomorphic to Y.*

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