Illinois Journal of Mathematics
Volume 59, Number 4, Winter 2015, Pages 979–997
S 0019-2082

SPECTRALLY UNSTABLE DOMAINS

GERARDO A. MENDOZA

ABSTRACT. Let H be a separable Hilbert space, $A_c : \mathcal{D}_c \subset H \to H$ a densely defined unbounded operator, bounded from below, let $\mathcal{D}_{\min}$ be the domain of the closure of A_c and $\mathcal{D}_{\max}$ that of the adjoint. Assume that $\mathcal{D}_{\max}$ with the graph norm is compactly contained in H and that $\mathcal{D}_{\min}$ has finite positive codimension in $\mathcal{D}_{\max}$. Then the set of domains of selfadjoint extensions of A_c has the structure of a finite-dimensional manifold $\mathfrak{SA}$ and the spectrum of each of its selfadjoint extensions is bounded from below. If ζ is strictly below the spectrum of A with a given domain $\mathcal{D}_0 \in \mathfrak{SA}$, then ζ is not in the spectrum of A with domain $\mathcal{D} \in \mathfrak{SA}$ near $\mathcal{D}_0$. But $\mathfrak{SA}$ contains elements $\mathcal{D}_0$ with the property that for every neighborhood U of $\mathcal{D}_0$ and every $\zeta \in \mathbb{R}$ there is $\mathcal{D} \in U$ such that $\operatorname{spec}(A_{\mathcal{D}}) \cap (-\infty, \zeta) \neq \emptyset$. We characterize these "spectrally unstable" domains as being those satisfying a nontrivial relation with the domain of the Friedrichs extension of A_c.

1. Introduction

Throughout the paper, H is a separable Hilbert space,

$$A_c : \mathcal{D}_c \subset H \to H \tag{1.1}$$

is a densely defined unbounded operator which is semibounded from below, and

$$A : \mathcal{D}_{\max} \subset H \to H$$

is the adjoint operator, automatically an extension of the symmetric operator (1.1).

Received April 8, 2016.
2010 *Mathematics Subject Classification.* Primary 47B25, 47A10. Secondary 47F05, 58J05, 35P05.

The space $\mathcal{D}_{\max}$ is a Hilbert space with the inner product

$$(u,v)_A = (Au, Av) + (u,v), \quad u,v \in \mathcal{D}_{\max}, \tag{1.2}$$

where the inner product on the right is that of H. It is further assumed that the inclusion $\mathcal{D}_{\max} \hookrightarrow H$ is compact and that $\mathcal{D}_{\min}$, the domain of the closure of (1.1) (the closure of $\mathcal{D}_c$ in $\mathcal{D}_{\max}$) has finite positive codimension in $\mathcal{D}_{\max}$.

With these assumptions, all closed extensions of (1.1) are Fredholm and the set of domains of extensions with index 0 can be parametrized by the elements of a compact manifold (a Grassmannian) in which the domains of the selfadjoint extensions form a real analytic compact submanifold $\mathfrak{SA}$. It is a fact that all these selfadjoint extensions have discrete spectrum bounded from below. (See Section 2 for details.) Write $A_{\mathcal{D}}$ for the operator with domain $\mathcal{D}$. The assertion that

every $\mathcal{D}_0 \in \mathfrak{SA}$ has a neighborhood U_0 for which there is $C_0 \in \mathbb{R}$ such that $\mathcal{D} \in U_0 \implies \operatorname{spec}(A_{\mathcal{D}}) \subset \{\lambda : \Re\lambda > C_0\}$

is false. Namely, if it were to hold, then $\mathfrak{SA}$, being compact, would admit a finite cover by open sets U_j such that the spectrum of $A_{\mathcal{D}}$ is bounded from below by the same constant in each set U_j. Hence, there would be an absolute lower bound for the spectra of all selfadjoint extensions, which is not true (see Lemma 2.10 below). So in fact there is $\mathcal{D}_0 \in \mathfrak{SA}$ such that

(1.3) for every neighborhood U of $\mathcal{D}_0$ and every $\zeta \in \mathbb{R}$ there is $\mathcal{D} \in U$ such that $\operatorname{spec}(A_{\mathcal{D}}) \cap (-\infty, \zeta) \neq \emptyset$.

Such domains will be called spectrally unstable. The main purpose of this paper is to establish the following characterization of these domains (proof in Section 7).

THEOREM 1.4. *Let $\mathcal{D}_F \in \mathfrak{SA}$ be the domain of the Friedrichs extension of* (1.1). *The element $\mathcal{D} \in \mathfrak{SA}$ is spectrally unstable if and only if*

$$(\mathcal{D} \cap \mathcal{D}_F)/\mathcal{D}_{\min} \neq 0.$$

The set of elements in $\mathfrak{SA}$ for which $(\mathcal{D} \cap \mathcal{D}_F)/\mathcal{D}_{\min} \neq 0$ is a real analytic subvariety of codimension 1.

Viewing the problem from the perspective of the von Neumann theory [8] (see [9, Theorem X.2]), let $\mathcal{K}_{\pm i} = \ker(A_{\mathcal{D}_{\max}} \mp i)$. With the assumptions of the first two paragraphs above, these subspaces of H have the same finite dimension. Let $\mathcal{D}_0 \in \mathfrak{SA}$. The spectrum of $\mathrm{U}_{\mathcal{D}_0} = (A_{\mathcal{D}_0} - i)(A_{\mathcal{D}_0} + i)^{-1}$, the Cayley transform of $A_{\mathcal{D}_0}$, consists of 1 and a discrete subset of the circle $S^1 \subset \mathbb{C}$. The part of the spectrum of $\mathrm{U}_{\mathcal{D}_0}$ in $\Im\lambda < 0$ accumulates at 1, and so the fact that arbitrarily small perturbations of $\mathcal{D}_0$ to $\mathcal{D} \in \mathfrak{SA}$ can lead to an apparently spontaneous generation of spectrum of $A_{\mathcal{D}}$ arbitrarily close to $-\infty$ is not surprising. What Theorem 1.4 does, is characterize those domains $\mathcal{D}_0$ for which arbitrarily small perturbations lead to spectrum of the Cayley transform spilling over from $\Im\lambda \leq 0$ to $\Im\lambda > 0$ across 1.

Note in passing that for no $\mathcal{D} \in \mathfrak{S}\mathfrak{A}$ can the part of the spectrum of $\mathrm{U}_{\mathcal{D}}$ on the semicircle in $\Im\lambda > 0$ accumulate at 1, since the spectrum of any $A_{\mathcal{D}}$ is bounded below by [1, Theorem 7, pg. 217], quoted here as Theorem 2.11.

The key technical results are a very simple "regularity" result, Proposition 4.1, and Theorem 6.9, a statement concerning recovering the essential part of the domain of the Friedrichs extension as a limit of spaces associated with $\ker(A_{\mathcal{D}_{\max}} - \lambda)$. To describe these more precisely, let $\mathcal{E}$ be the orthogonal complement of $\mathcal{D}_{\min}$ in $\mathcal{D}_{\max}$ and $\pi_{\max}$ the orthogonal projection on $\mathcal{E}$, all with the inner product (1.2). Domains of closed extensions of (1.1) correspond to the various subspaces $D \subset \mathcal{E}$ via $\mathcal{D} = D + \mathcal{D}_{\min}$, with selfadjoint extensions corresponding to the points of a submanifold $\mathfrak{S}\mathfrak{A}$ of the Grassmannian of subspaces of $\mathcal{E}$ of a certain dimension (so it is not $\mathcal{D}_F$ that belongs to $\mathfrak{S}\mathfrak{A}$ in Theorem 1.4, but a certain subspace $D_F \subset \mathcal{E}$). Let $\mathcal{K}_\lambda = \ker(A_{\mathcal{D}_{\max}} - \lambda)$ and $K_\lambda = \pi_{\max}\mathcal{K}_\lambda$. Then $\lambda \mapsto K_\lambda$ is a smooth curve in $\mathfrak{S}\mathfrak{A}$ if λ is sufficiently negative, and $\lim_{\lambda\to-\infty} K_\lambda = D_F$. This is a consequence of the following. For any domain $\mathcal{D} = D + \mathcal{D}_{\min}$ with $D \in \mathfrak{S}\mathfrak{A}$ and any $s \geq 0$ we define Hilbert spaces $H^s_{\mathcal{D}}$ using $A_{\mathcal{D}}$; these Sobolev-like spaces give $H^0_{\mathcal{D}} = H$ and $H^1_{\mathcal{D}} = \mathcal{D}$. For $u \in D^\perp$, the linear functional δ_u defined by $\mathcal{D} \ni v \mapsto (Av, u) - (v, Au) \in \mathbb{C}$ is an element of the dual space of $H^1_{\mathcal{D}}$, and may also be in $H^{-s}_{\mathcal{D}^\dagger}$ for $0 < s < 1$, the dual of $H^s_{\mathcal{D}}$. We show that $\delta_u \notin H^{-1/2}_{\mathcal{D}^\dagger}$ for $\mathcal{D}_F = D_F + \mathcal{D}_{\min}$ if $u \neq 0$.

Elliptic semibounded cone operators on compact manifolds $\mathcal{M}$ with boundary acting on weighted L^2-spaces of sections of a Hermitian vector bundle $E \to \mathcal{M}$,

$$A : C^\infty_c(\mathring{\mathcal{M}}; E) \subset x^{-\nu} L^2_b(\mathcal{M}; E) \to x^{-\nu} L^2_b(\mathcal{M}; E),$$

have the properties stated in the first two paragraphs, see Lesch [7, Proposition 1.3.16 and its proof]. The fine structure of the domain of the Friedrichs extension for these differential operators was given in [4, Theorem 8.12]; the interested reader may consult these references for detailed information about such operators. The research leading to the papers [5], [6] was the motivation for looking into the instability issue. Friedrichs defined his extension in [3]. The nature of the domain in the abstract context was elucidated by Freudenthal in [2].

The author is grateful to T. Krainer for suggestions that improved the manuscript and for pointing out reference [1].

2. Domains, selfadjointness

All closed extensions of (1.1) considered here will have as domain a subspace of $\mathcal{D}_{\max}$ containing $\mathcal{D}_{\min}$. Thus, the domain of every closed extension of (1.1) is of the form

$$\mathcal{D} = D + \mathcal{D}_{\min}$$

with D a subspace of the orthogonal complement, $\mathcal{E}$, of $\mathcal{D}_{\min}$ in $\mathcal{D}_{\max}$ with respect to the inner product (1.2); $\mathcal{E}$ is finite-dimensional by hypothesis. In particular, the domain of the Friedrichs extension of (1.1) has the form $\mathcal{D}_F = D_F + \mathcal{D}_{\min}$ for some subspace $D_F \subset \mathcal{E}$.

The resolvent family of

$$A : \mathcal{D}_F \subset H \to H$$

consists of compact operators $B_F(\lambda) : H \to H$, since they are also continuous as operators $H \to \mathcal{D}_F$ and the inclusion $\mathcal{D}_F \hookrightarrow H$ is compact. It follows that A with domain $\mathcal{D}_{\min}$ or $\mathcal{D}_{\max}$ is Fredholm, and from this and the finiteness of $\dim \mathcal{E}$, that every closed extension of (1.1) is Fredholm (with compact resolvent when it exists). It is easily verified that the index of A with domain $\mathcal{D} = D + \mathcal{D}_{\min}$ is

$$\operatorname{ind} A_{\mathcal{D}} = \operatorname{ind} A_{\mathcal{D}_{\min}} + \dim D. \tag{2.1}$$

Since $A_{\mathcal{D}_{\min}} - \lambda I$ is injective for large negative λ, $\operatorname{ind} A_{\mathcal{D}_{\min}} \leq 0$. And since $A_{\mathcal{D}_{\max}} - \lambda I$ is surjective for such λ, $\operatorname{ind} A_{\mathcal{D}_{\max}} \geq 0$. From $\operatorname{ind} A_{\mathcal{D}_{\max}} = \operatorname{ind} A_{\mathcal{D}_{\min}} + \dim \mathcal{E}$ and $\operatorname{ind} A_{\mathcal{D}_{\max}} = -\operatorname{ind} A_{\mathcal{D}_{\min}}$ (because $A_{\mathcal{D}_{\max}}$ and $A_{\mathcal{D}_{\min}}$ are adjoints of each other) one derives that $\dim \mathcal{E} = 2d$ with $d = -\operatorname{ind} A_{\mathcal{D}_{\min}}$; this is a positive number since $\dim \mathcal{E} > 0$. One can then view the set of domains of selfadjoint extensions of (1.1) as

$$\mathfrak{SA} = \{D \subset \mathcal{E} : A \text{ with domain } D + \mathcal{D}_{\min} \text{ is selfadjoint}\},$$

a subset of $\operatorname{Gr}_d(\mathcal{E})$, the Grassmannian of d-dimensional subspaces of $\mathcal{E}$. As such, $\mathfrak{SA}$ is a compact real analytic submanifold of dimension d^2 (see Proposition 2.9).

Let

$$[\cdot,\cdot]_A : \mathcal{D}_{\max} \times \mathcal{D}_{\max} \to \mathbb{C}$$

denote the skew-Hermitian form

$$[u, v]_A = (Au, v) - (u, Av).$$

Then $[u, v]_A = 0$ if either u or v belongs to $\mathcal{D}_{\min}$, so

$$[u, v]_A = [\pi_{\max} u, \pi_{\max} v]_A,$$

where

$$\pi_{\max} : \mathcal{D}_{\max} \to \mathcal{D}_{\max}$$

is the orthogonal projection on $\mathcal{E}$. The restriction of the Green form $[\cdot, \cdot\cdot]_A$ to $\mathcal{E}$ is non-degenerate because the Hilbert space adjoint of A with domain $\mathcal{D}_{\max}$ is A with domain $\mathcal{D}_{\min}$.

The facts collected in the following lemma can be verified directly, or following the arguments in [6, Section 6].

Lemma 2.2. *We have*

$$\mathcal{E} = \big\{u \in \mathcal{D}_{\max} : Au \in \mathcal{D}_{\max} \text{ and } A^2 u = -u\big\}. \tag{2.3}$$

If $u \in \mathcal{E}$, then $Au \in \mathcal{E}$, and the map

$$A|_{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \tag{2.4}$$

is an isometry with inverse $-A|_{\mathcal{E}}$. If u, $v \in \mathcal{E}$, then

$$[u, Av]_A = (u, v)_A. \tag{2.5}$$

Consequently, for any subspace $D \subset \mathcal{E}$, the adjoint of

$$A : D + \mathcal{D}_{\min} \subset H \to H$$

is

$$A : A\big(D^\perp\big) + \mathcal{D}_{\min} \subset H \to H, \tag{2.6}$$

where $D^\perp$ is the orthogonal complement of D in $\mathcal{E}$. Consequently

$$D \in \mathfrak{SA} \iff A\big(D^\perp\big) = D \iff A(D) = D^\perp. \tag{2.7}$$

and in particular, $D \in \mathfrak{SA} \implies D^\perp \in \mathfrak{SA}$.

We discuss the claim about the adjoint. The combination of (2.3) and (2.4) gives $A^2|_{\mathcal{E}} = -I$, so (2.5) can also be written as

$$[u, v]_A = -(u, Av)_A.$$

Suppose $\mathcal{D} = D + \mathcal{D}_{\min}$ with $D \subset \mathcal{E}$. The domain of the adjoint of $A_{\mathcal{D}}$ is $\mathcal{D}^* = D^* + \mathcal{D}_{\min}$ for some subspace $D^* \subset \mathcal{E}$. Since $A_{\mathcal{D}_{\min}}$ is symmetric, the condition that $v \in D^*$ reduces to the statement that $[u, v]_A = 0$ for all $u \in D$, equivalently,

$$v \in D^* \iff (u, Av)_A = 0 \quad \text{for all } u \in D.$$

Thus, $v \in D^* \iff Av \in D^\perp$, and so $D^* = (AD)^\perp$. Also $D = (AD^*)^\perp$, so $D^\perp = AD^*$, and using $A^2 = -I$ again we get $D^* = A(D^\perp)$, which gives the assertion in (2.6).

If $D \in \mathrm{Gr}_d(\mathcal{E})$ and $T : D \to D^\perp$ is a linear map, then

$$\operatorname{graph} T = \{u + Tu : u \in D\} \subset \mathcal{E}$$

is again an element of $\mathrm{Gr}_d(\mathcal{E})$. The set U_D of all such elements is a neighborhood of D in $\mathrm{Gr}_d(\mathcal{E})$.

LEMMA 2.8. *Suppose $D \in \mathfrak{SA}$. Then*

$$U_D \cap \mathfrak{SA} = \{\operatorname{graph} T : \text{ the map } AT : D \to D \text{ is selfadjoint}\}.$$

Here selfadjoint means with respect to the A-inner product.

Since $A|_{\mathcal{E}}$ is unitary, if $T : D \to D^\perp$ is such that $AT : D \to D$ is selfadjoint, then also $TA : D^\perp \to D^\perp$ is selfadjoint.

Proof of Lemma 2.8. Let $D \in \mathfrak{S}\mathfrak{A}$, let $T : D \to D^\perp$ be a linear map. In view of (2.7), the condition that $\operatorname{graph} T \in \mathfrak{S}\mathfrak{A}$ is that

$$\big(u + Tu, A(v + Tv)\big)_A = 0 \quad \text{for all } u, v \in D.$$

For a general $T : D \to D^\perp$ and $u, v \in D$ we have

$$\big(u + Tu, A(v + Tv)\big)_A = (u, Av)_A + (u, ATv)_A + (Tu, Av)_A + (Tu, ATv)_A.$$

Since $D \in \mathfrak{S}\mathfrak{A}$ and $u, v \in D$, $(u, Av)_A = 0$, and since $Tu, Tv \in D^\perp$ and $D^\perp \in \mathfrak{S}\mathfrak{A}$, also $(Tu, ATv)_A = 0$. Further, since A is an isometry on $\mathcal{E}$ and $A^2 = -I$, $(Tu, Av)_A = -(ATu, v)$. Thus,

$$\big(u + Tu, A(v + Tv)\big)_A = (u, ATv)_A - (ATu, v)_A$$

so $\operatorname{graph} T \in \mathfrak{S}\mathfrak{A}$ iff $AT : D \to D$ is selfadjoint with respect to the A-inner product. □

Thus $\mathfrak{S}\mathfrak{A}$, as a subset of $\operatorname{Gr}_d(\mathcal{E})$, is structurally simple.

Proposition 2.9 ([6] Proposition 6.3). *The set $\mathfrak{S}\mathfrak{A}$ is a smooth real-algebraic subvariety of $\operatorname{Gr}_d(\mathcal{E})$.*

The dimension of the vector space of selfadjoint operators $D \to D$ (a real vector space) is d^2, so $\mathfrak{S}\mathfrak{A}$ is a real submanifold of $\operatorname{Gr}_d(\mathcal{E})$ of dimension d^2.

Lemma 2.10 ([6] Proposition 6.4). *Every $\lambda \in \mathbb{R}$ appears as eigenvalue of some selfadjoint extension of A.*

Proof. Let $\lambda \in \mathbb{R}$. If $\ker(A_{\mathcal{D}_{\min}} - \lambda) \neq 0$, then $\lambda \in \operatorname{spec}(A_{D+\mathcal{D}_{\min}})$ for every $D \in \mathfrak{S}\mathfrak{A}$, so the lemma holds in this case. Suppose now that $A_{\mathcal{D}_{\min}} - \lambda$ is injective and let $\mathcal{K}_\lambda = \ker(A_{\mathcal{D}_{\max}} - \lambda)$. Then $\mathcal{K}_\lambda \cap \mathcal{D}_{\min} = 0$, so $K_\lambda = \pi_{\max}\mathcal{K}_\lambda$ has the same dimension as $\mathcal{K}_\lambda$. The injectivity of $A_{\mathcal{D}_{\min}} - \lambda$ implies the surjectivity of its adjoint, $A_{\mathcal{D}_{\max}} - \lambda$, so the index of the latter, namely d, is equal to the dimension of its kernel. So $K_\lambda \in \operatorname{Gr}_d(\mathcal{E})$. Let $\mathcal{D} = K_\lambda + \mathcal{D}_{\min}$. To verify that $K_\lambda \in \mathfrak{S}\mathfrak{A}$ let $u, v \in \mathcal{K}_\lambda$ and $u_0, v_0 \in \mathcal{D}_{\min}$ (note that $\mathcal{D} = \mathcal{K}_\lambda + \mathcal{D}_{\min}$). Then $[u + u_0, v + v_0]_A = [u, v]_A$ using that the Hilbert space adjoint of $A_{\mathcal{D}_{\min}}$ is $A_{\mathcal{D}_{\max}}$ and that $A_{\mathcal{D}_{\min}}$ is symmetric. So

$$[u + u_0, v + v_0]_A = (u, Av) - (Au, v) = (u, \lambda v) - (\lambda u, v) = 0$$

since $\lambda \in \mathbb{R}$. It follows that $A_{\mathcal{D}}$ is symmetric, and from this and $\operatorname{ind} A_{\mathcal{D}} = 0$, that A is selfadjoint. □

We end with the following fundamental fact.

Theorem 2.11. *Let m be a lower bound of A_c. Every selfadjoint extension of A_c is semibounded from below and the part of its spectrum in $(-\infty, m)$ is discrete with at most d eigenvalues counting multiplicity.*

This is [1, Theorem 7, pg. 217]. Indeed, in view of the semiboundedness of (1.1), all we need to verify is that the deficiency indices of A_c are finite and equal. Since A_c is semibounded from below, $A_{\mathcal{D}_{\min}} - \lambda$ is injective if $\Im\lambda \neq 0$ or $\lambda \in \mathbb{R}$ is sufficiently negative. For such λ, $\mathcal{K}_\lambda = \ker(A_{\mathcal{D}_{\max}} - \lambda)$ has constant dimension d, because of (2.1) and the definition of d as $-\operatorname{ind} A_{\mathcal{D}_{\min}}$. In particular, the spaces $\mathcal{K}_i$ and $\mathcal{K}_{-i}$ have the same dimension. But these spaces are the orthogonal complements in H of the ranges of $A_{\mathcal{D}_{\min}} + i$ and $A_{\mathcal{D}_{\min}} - i$. We note in passing that both $\mathcal{K}_i$ and $\mathcal{K}_{-i}$ are subspaces of $\mathcal{E}$, with $\mathcal{E} = \mathcal{K}_i \oplus \mathcal{K}_{-i}$. This is the decomposition of $\mathcal{E}$ into the eigenspaces of the almost complex structure of $\mathcal{E}$ determined by A.

3. $\mathcal{D}$-Sobolev spaces

Let $A : \mathcal{D} \subset H \to H$ be a selfadjoint extension of (1.1), let

$$\Pi_{\mathcal{D},\lambda} : H \to H$$

be the orthogonal projection on $\ker(A_\mathcal{D} - \lambda)$. Define, for arbitrary $s \geq 0$,

$$H^s_\mathcal{D} = \Big\{ u \in H : \sum_{\lambda \in \operatorname{spec}(A_\mathcal{D})} \big(1 + |\lambda|\big)^{2s} \|\Pi_{\mathcal{D},\lambda} u\|^2 < \infty \Big\}.$$

This is a Hilbert space with inner product

$$(u,v)_s = \sum_{\lambda \in \operatorname{spec}(A_\mathcal{D})} \big(1 + |\lambda|\big)^{2s} (\Pi_{\mathcal{D},\lambda} u, \Pi_{\mathcal{D},\lambda} v).$$

We will write $\|\cdot\|_s$ for the norm of $H^s_\mathcal{D}$. We shall not make explicit the dependence on $\mathcal{D}$ of the norm or the inner product, and omit s altogether when $s = 0$.

Clearly $H^{s'}_\mathcal{D}$ is densely and continuously contained in $H^s_\mathcal{D}$ if $s' > s \geq 0$.

LEMMA 3.1. *The spaces $H^1_\mathcal{D}$ and $\mathcal{D}$ are equal and the A-norm on $\mathcal{D}$ and the norm of $H^1_\mathcal{D}$ are equivalent. The space $\mathcal{D}_c$ is contained in $H^s_\mathcal{D}$ for every $0 \leq s \leq 1$, and its closure in $H^1_\mathcal{D}$ is $\mathcal{D}_{\min}$.*

In particular, $H^1_\mathcal{D} \neq \mathcal{D}_{\max}$ since $\mathcal{D} \neq \mathcal{D}_{\max}$. We will write $\dot{H}^s_\mathcal{D}$ for the closure of $\mathcal{D}_c$ in $H^s_\mathcal{D}$ $(0 \leq s \leq 1)$. Evidently $\dot{H}^1_\mathcal{D}$ is independent of $\mathcal{D}$ (despite the notation), but $\dot{H}^s_\mathcal{D}$ may depend on $\mathcal{D}$ if $s < 1$.

Proof of Lemma 3.1. Suppose $v \in H^1_\mathcal{D}$, let

$$v_n = \sum_{\lambda < n} \Pi_{\mathcal{D},\lambda}(v)$$

and note that

$$Av_n = \sum_{\lambda < n} \lambda \Pi_{\mathcal{D},\lambda}(v).$$

Since $v \in H$, $v_n \to v$ in H, but since in fact $v \in H^1_{\mathcal{D}}$, Av_n also converges in H. Since $A_{\mathcal{D}}$ is closed, $v \in \mathcal{D}$. Thus $H^1_{\mathcal{D}} \subset \mathcal{D}$. The opposite inclusion follows from an application of the Spectral theorem. An explicit calculation gives

$$\frac{1}{4}\|u\|_1^2 \le \|u\|_A^2 \le \|u\|_1^2, \quad u \in \mathcal{D}.$$

That the closure of $\mathcal{D}_c$ in $H^1_{\mathcal{D}}$ is $\mathcal{D}_{\min}$ follows from this and that $\mathcal{D}_c \subset H^s_{\mathcal{D}}$ for $0 \le s \le 1$ follows form $H^1_{\mathcal{D}} \subset H^s_{\mathcal{D}}$ for such s. □

Let $H^{-s}_{\mathcal{D}^\dagger}$ be the dual of $H^s_{\mathcal{D}}$ with the norm topology. Denote the pairing of $\psi \in H^{-s}_{\mathcal{D}^\dagger}$ and $u \in H^s_{\mathcal{D}}$ by $\langle \psi, u\rangle_s$. Define $h^\sharp_s : H^s_{\mathcal{D}} \to H^{-s}_{\mathcal{D}^\dagger}$ by setting

$$\left\langle h^\sharp_s v, u\right\rangle_s = (u, v)_s. \tag{3.2}$$

The Riesz representation theorem gives that the map $h^\sharp_s$ is surjective, so invertible since it is also injective, and an antilinear isometry. The inverse will be denoted $h^\flat_s$.

The space $H^{-s}_{\mathcal{D}^\dagger}$ is again a Hilbert space with inner product

$$(\psi, \eta)_{-s} = \left(h^\flat_s \eta, h^\flat_s \psi\right)_s, \quad \psi, \eta \in H^{-s}_{\mathcal{D}^\dagger}.$$

The Hilbert space norm of an element of $H^{-s}_{\mathcal{D}^\dagger}$ is equal its norm as linear functional $H^s_{\mathcal{D}} \to \mathbb{C}$.

Suppose $0 \le s \le 1$, let $\dot H^{-s}_{\mathcal{D}^\dagger}$ be the dual of $\dot H^s_{\mathcal{D}}$. The inclusion map

$$\iota_s : \dot H^1_{\mathcal{D}} \to H^s_{\mathcal{D}}$$

gives the dual map

$$\iota^\dagger_s : H^{-s}_{\mathcal{D}^\dagger} \to \dot H^{-1}_{\mathcal{D}^\dagger}.$$

We are interested in the elements of the kernel of these maps.

The kernel of $\iota^\dagger_s$, the annihilator in $H^{-s}_{\mathcal{D}^\dagger}$ of the closure of $\dot H^1_{\mathcal{D}}$ in $H^s_{\mathcal{D}}$, is isomorphic via $h^\flat_s$ to the orthogonal complement of $\dot H^s_{\mathcal{D}}$ in $H^s_{\mathcal{D}}$, so $\dim \ker \iota^\dagger_s = \dim H^s_{\mathcal{D}}/\dot H^s_{\mathcal{D}}$. In particular, $\dim \ker \iota^\dagger_1 = d$, since by Lemma 3.1, $\dot H^1_{\mathcal{D}} = \mathcal{D}_{\min}$ and $H^1_{\mathcal{D}} = D + \mathcal{D}_{\min}$.

Suppose $0 \le s < s' \le 1$, and let $j_{s,s'} : H^{s'}_{\mathcal{D}} \hookrightarrow H^s_{\mathcal{D}}$ be the inclusion map. Then $\iota_s = j_{s,s'} \circ \iota_{s'}$, so $\iota^\dagger_s = \iota^\dagger_{s'} \circ j^\dagger_{s,s'}$. Since $j_{s,s'}$ has dense image, $j^\dagger_{s,s'}$ is injective. Consequently $u \in \ker \iota^\dagger_s$ if and only if $\iota^\dagger_{s'}(j^\dagger_{s,s'}(u)) = 0$ and we deduce that $j^\dagger_{s,s'}$ restricts to an injective map $\ker \iota^\dagger_s \to \ker \iota^\dagger_{s'}$. Identifying $H^{-s}_{\mathcal{D}^\dagger}$ with its image in $H^{-s'}_{\mathcal{D}^\dagger}$ by $j^\dagger_{s,s'}$ this means

$$\ker \iota^\dagger_s = H^{-s}_{\mathcal{D}^\dagger} \cap \ker \iota^\dagger_{s'}, \quad 0 \le s < s'. \tag{3.3}$$

All that is left is to determine $\ker \iota^\dagger_1$.

PROPOSITION 3.4. *The kernel of* $\iota_1^\dagger$ *consists of all maps* $\delta_u : H^1_{\mathcal{D}} \to \mathbb{C}$ *of the form*

$$H^1_{\mathcal{D}} \ni \psi \mapsto \langle \delta_u, \psi\rangle = [\psi, u]_A \in \mathbb{C} \tag{3.5}$$

for some $u \in D^\perp$. *Here, as before,* $D^\perp$ *is the orthogonal complement of* D *in* $\mathcal{E}$.

Proof. Let $u \in D^\perp$. The functional δ_u is clearly linear. Its continuity as a map $\delta_u : H^1_{\mathcal{D}} \to \mathbb{C}$ is an immediate consequence of the Cauchy–Schwarz inequality, the definition of the A-norm and the equivalence of the latter and that of $H^1_{\mathcal{D}}$. If $\psi \in \dot H^1_{\mathcal{D}}$, then $[\psi, u]_A = 0$ because $\dot H^1_{\mathcal{D}} = \mathcal{D}_{\min}$ and $D^\perp \subset \mathcal{D}_{\max}$, so $\delta_u \in \ker \iota_1^\dagger$. If $\delta_u = 0$, then $(A\psi, u) - (\psi, Au) = 0$ for all $\psi \in \mathcal{D}$, since $D^\perp + \mathcal{D}_{\min}$ is the domain of the adjoint of $A_{\mathcal{D}}$. So u belongs to the domain of the adjoint of $A_{\mathcal{D}}$. But since $A_{\mathcal{D}}$ is selfadjoint, we must have $u \in \mathcal{D}$, so $u = 0$. So the map

$$D^\perp \ni u \mapsto \delta_u \in H^{-1}_{\mathcal{D}^\dagger}$$

is an antilinear isomorphism into $\ker \iota_1^\dagger$. The surjectivity follows from the equality of the dimensions of $D^\perp$ and $H^1_{\mathcal{D}}/\dot H^1_{\mathcal{D}} \approx D$. ☐

4. Estimates

For $D \in \mathfrak{S}\mathfrak{A}$ we let $\mathcal{P}_{D^\perp}$ be the collection of functionals (3.5):

$$\mathcal{P}_{D^\perp} = \{\delta_u : u \in D^\perp\}.$$

Because of (3.3), elements of $\mathcal{P}_{D^\perp}$ may have better regularity (the number $-s$) than $H^{-1}_{\mathcal{D}^\dagger}$, but of course no element δ_u with $u \neq 0$ belongs to $H^0_{\mathcal{D}^\dagger}$. The following proposition gives an upper bound for the regularity of elements in $\ker \iota_1^\dagger$ in the case where $\mathcal{D}$ is the domain of the Friedrichs extension of A.

PROPOSITION 4.1. *Let* $\mathcal{D}_F = D_F + \mathcal{D}_{\min}$ *be the domain of the Friedrichs extension of* (1.1). *Then* $\mathcal{P}_{\mathcal{D}_F^\perp} \cap H^{-1/2}_{\mathcal{D}_F^\dagger} = 0$.

Proof. We show that $\dot H^{1/2}_{\mathcal{D}_F} = H^{1/2}_{\mathcal{D}_F}$ (so also $\dot H^s_{\mathcal{D}_F} = H^s_{\mathcal{D}_F}$ if $0 \le s \le 1/2$ because of (3.3)), an equality we obtain directly by following the construction of the Friedrichs extension of A. Let

$$\mathfrak{Q}(u, v) = (Au, v) + c(u, v), \quad u, v \in \dot H^1_{\mathcal{D}}$$

with a large enough constant c. The norms on $\dot H^1_{\mathcal{D}_F}$ induced by $\mathfrak{Q}$ and that of $H^{1/2}_{\mathcal{D}_F}$ are equivalent, so the $\mathfrak{Q}$-completion of $\dot H^1_{\mathcal{D}}$ can be identified with $\dot H^{1/2}_{\mathcal{D}_F}$. Let

$$B : H \to \dot H^{1/2}_{\mathcal{D}_F}$$

be the operator such that

$$\mathfrak{Q}(Bu, v) = (u, v) \quad \text{for all } u \in H, v \in \dot H^{1/2}_{\mathcal{D}_F}.$$

Then B is injective and its image is the domain of the Friedrichs extension of $A+cI$, which is the same as that of A. That is, $\mathcal{D}_F\subset \dot H^{1/2}_{\mathcal{D}_F}$, which is to say that $H^1_{\mathcal{D}_F}\subset \dot H^{1/2}_{\mathcal{D}_F}$. Since $H^1_{\mathcal{D}_F}$ is dense in $H^{1/2}_{\mathcal{D}_F}$, $\dot H^{1/2}_{\mathcal{D}_F}$ is a dense subspace of $H^{1/2}_{\mathcal{D}_F}$. Thus, $\dot H^{1/2}_{\mathcal{D}_F}=H^{1/2}_{\mathcal{D}_F}$. We note that this equality is standard. □

Returning to the case of an arbitrary domain $\mathcal{D}$ on which A is selfadjoint, let $\{\lambda_k\}_{k=1}^\infty$ be the sequence of eigenvalues of $A_{\mathcal{D}}$ repeated according to multiplicity and in increasing order, and let $\{\psi_k\}\subset\mathcal{D}$ be an orthonormal basis of H corresponding to these eigenvalues.

The ψ_k are also a complete A-orthogonal system for $\mathcal{D}$. Therefore, an element $u\in\mathcal{D}_{\max}$ belongs to $D^\perp$ if and only if $(u,\psi_k)_A=0$ for all k:

$$u\in D^\perp \quad\iff\quad \lambda_k(Au,\psi_k)+(u,\psi_k)=0 \quad\text{for all } k.$$

Let $u\in D^\perp$. The relations

$$\begin{cases}\lambda_k(u,\psi_k)-(Au,\psi_k)=\overline{\langle\delta_u,\psi_k\rangle},\\ (u,\psi_k)+\lambda_k(Au,\psi_k)=0,\end{cases}$$

where the first identity comes from the definition of δ_u and the second is the orthogonality condition just mentioned, give

$$(u,\psi_k)=\lambda_k\frac{\overline{\langle\delta_u,\psi_k\rangle}}{1+\lambda_k^2},\qquad (Au,\psi_k)=-\frac{\overline{\langle\delta_u,\psi_k\rangle}}{1+\lambda_k^2}. \tag{4.2}$$

We will now express the elements of $\mathcal{P}_{D^\perp}$ as a Fourier series related to the orthonormal basis $\{\psi_k\}$. Recalling the maps $h^\sharp_s:H^s_{\mathcal{D}}\to H^{-s}_{\mathcal{D}^\dagger}$ defined in (3.2), let $\psi^0_k=h^\sharp_0\psi_k$. Since the inclusion map $j_s:H^s_{\mathcal{D}}\hookrightarrow H^0_{\mathcal{D}}$ has dense image, the dual map

$$j_s^\dagger:H^0_{\mathcal{D}^\dagger}\to H^{-s}_{\mathcal{D}^\dagger}$$

is injective with dense image. So ψ^0_k can be regarded as an element of $H^{-s}_{\mathcal{D}^\dagger}$ for any $s\ge 0$. From the definition of the inner product, we get $(\psi^0_k,\psi^0_\ell)_0=\delta_{k\ell}$. For $w\in H^s_{\mathcal{D}}$, we have

$$\big\langle j_s^\dagger\psi^0_k,w\big\rangle_s=\big\langle\psi^0_k,j_sw\big\rangle_0=(w,\psi_k)=\frac{(w,\psi_k)_s}{(1+|\lambda_k|)^{2s}}=\frac{\langle h^\sharp_s\psi_k,w\rangle_s}{(1+|\lambda_k|)^{2s}}$$

so, using the inverse $h^\flat_s$ of $h^\sharp_s$,

$$h^\flat_s\big(j_s^\dagger\psi^0_k\big)=\big(1+|\lambda_k|\big)^{-2s}\psi_k.$$

In particular,

$$\big\|j_s^\dagger\psi^0_k\big\|^2_{-s}=\big(j_s^\dagger\psi^0_k,j_s^\dagger\psi^0_\ell\big)_{-s}=\big(1+|\lambda_k|\big)^{-2s}\delta_{k\ell}.$$

If $v\in H^{-s}_{\mathcal{D}^\dagger}$, then

$$\big(v,j_s^\dagger\psi^0_k\big)_{-s}=\big(h^\flat_s\big(j_s^\dagger\psi^0_k\big),h^\flat_sv\big)_s=\big\langle v,h^\flat_s\big(j_s^\dagger\psi^0_k\big)\big\rangle_s=\frac{\langle v,\psi_k\rangle_s}{(1+|\lambda_k|)^{2s}}.$$

Thus the Fourier series representation of v is

$$v = \sum_k \langle v, \psi_k\rangle_s j_s^\dagger \psi_k^0.$$

The norm of an element $v = \sum_k v_k j_s^\dagger \psi_k^0 \in H^{-s}_{\mathcal{D}^\dagger}$ is given by

$$\|v\|_{-s}^2 = \sum_k \big(1+|\lambda_k|\big)^{-2s} |v_k|^2.$$

Suppose now $u \in D^\perp$ and $\delta_u \in H^{-s}_{\mathcal{D}^\dagger}$. Then

$$\langle \delta_u, \psi_k\rangle_s = \big(1+|\lambda_k|\big)^{2s} \big(\delta_u, j_m^\dagger \psi_k^0\big)_{-s},$$

hence

$$\|\delta_u\|_{-s}^2 = \sum \frac{|\langle \delta_u, \psi_k\rangle_s|^2}{(1+|\lambda_k|)^{2s}}. \tag{4.3}$$

Note that $\langle \delta_u, \psi_k\rangle_s$ is just $\langle \delta_u, \psi_k\rangle$ since $\psi_k \in H^s_{\mathcal{D}}$ for any $0 \le s \le 1$.

5. The bundle of kernels

The background spectrum of A, denoted bg-spec(A) is the set

$$\{\lambda \in \mathbb{C} : A_{\mathcal{D}_{\min}} - \lambda \text{ is not injective or } A_{\mathcal{D}_{\max}} - \lambda \text{ is not surjective}\},$$

see [6]. Its complement is denoted bg-res(A). The background spectrum is of interest in that it is a subset of the spectrum of every extension of A.

In the present case, since A is semibounded and admits an extension with compact resolvent, the set bg-spec(A) is (if not empty) a discrete subset of the real line with only $+\infty$ as a possible point of accumulation, equal to

$$\text{bg-spec}(A) = \{\lambda \in \mathbb{C} : A_{\mathcal{D}_{\min}} - \lambda \text{ is not injective}\}.$$

Indeed, if $\lambda \in \mathbb{R}$ then $\ker(A_{\mathcal{D}_{\min}} - \lambda) = \operatorname{rg}(A_{\mathcal{D}_{\max}} - \lambda)^\perp$.

For $\lambda \in$ bg-res(A) define

$$\mathcal{K}_\lambda = \ker(A_{\mathcal{D}_{\max}} - \lambda).$$

Since $A_{\min} - \lambda$ is injective if $\lambda \in$ bg-res(A), formula (2.1) with $\mathcal{D} = \mathcal{D}_{\max}$ gives $\dim \mathcal{K}_\lambda = d$. For these λ, $\mathcal{K}_\lambda \cap \mathcal{D}_{\min} = 0$. It follows that $K_\lambda = \pi_{\max}\mathcal{K}_\lambda$ also has dimension d for each $\lambda \in$ bg-res(A). (These spaces are the fibers of a holomorphic vector bundle over bg-res(A) that extends across bg-spec(A) as a holomorphic vector bundle. The latter fact, not obvious, will not be proved here as it is not needed.)

The following lemma makes explicit the relevancy of these spaces.

LEMMA 5.1. *Let $D \in \mathrm{Gr}_d(\mathcal{E})$. The spectrum of A with domain $\mathcal{D} = D + \mathcal{D}_{\min}$ is*

$$\big\{\lambda \in \text{bg-res}(A) : K_\lambda \cap D \neq 0\big\} \cup \text{bg-spec}(A).$$

Indeed, if $\lambda \in \operatorname{spec}(A_{\mathcal{D}})$ and $\lambda \in \text{bg-res}(A)$, then $\ker(A_{\mathcal{D}} - \lambda) = \mathcal{D} \cap \mathcal{K}_\lambda \neq 0$, and $u \in \ker(A_{\mathcal{D}} - \lambda)$ if and only if $\pi_{\max} u \in K_\lambda$ and $\pi_{\max} u \in D$.

Because of the property expressed in the lemma it is of interest to have a formula for the spaces $\mathcal{K}_\lambda$ when $\lambda \notin \text{bg-spec}(A)$. We get one such formula with the aid of the resolvent of an arbitrary selfadjoint extension $A_{\mathcal{D}}$ of (1.1).

Let then $D \in \mathfrak{S}\mathfrak{A}$, write $\pi_{D^\perp}$, $\pi_D : \mathcal{D}_{\max} \to \mathcal{D}_{\max}$ for the A-orthogonal projections on $D^\perp$ and D, respectively, and let $\pi_{\mathcal{D}} : \mathcal{D}_{\max} \to \mathcal{D}_{\max}$ be the orthogonal projection on $\mathcal{D}$ (so $\pi_{\mathcal{D}} = I - \pi_{D^\perp}$). Let $B_{\mathcal{D}}(\lambda)$ be the resolvent of $A_{\mathcal{D}}$. Suppose $\lambda \in \operatorname{res}(A_{\mathcal{D}})$ and $\phi \in \mathcal{K}_\lambda$. The identity

$$\phi = \pi_{D^\perp}\phi + \pi_{\mathcal{D}}\phi$$

gives

$$0 = (A - \lambda)\pi_{D^\perp}\phi + (A - \lambda)\pi_{\mathcal{D}}\phi.$$

Applying $B_{\mathcal{D}}(\lambda)$ get

$$\pi_{\mathcal{D}}\phi = -B_{\mathcal{D}}(\lambda)(A - \lambda)\pi_{D^\perp}\phi$$

since $\pi_{\mathcal{D}}\phi \in \mathcal{D}$. Thus,

$$\phi = \pi_{D^\perp}\phi - B_{\mathcal{D}}(\lambda)(A - \lambda)\pi_{D^\perp}\phi.$$

Conversely, it is easily verified that if $u \in D^\perp$, then

$$\phi_u(\lambda) = u - B_{\mathcal{D}}(\lambda)(A - \lambda)u$$

is an element of $\mathcal{K}_\lambda$ for each $\lambda \in \operatorname{res}(A_{\mathcal{D}})$. Evidently, the map $D^\perp \ni u \mapsto \phi_u(\lambda) \in \mathcal{K}_\lambda$ is bijective and depends holomorphically on $\lambda \notin \operatorname{spec}(A_{\mathcal{D}})$.

Using the orthonormal basis $\{\psi_k\}$ consisting of eigenfunctions of $A_{\mathcal{D}}$, the formula

$$B_{\mathcal{D}}(\lambda)f = \sum_k \frac{(f, \psi_k)}{\lambda_k - \lambda}\psi_k$$

and the formulas (4.2) give

$$\phi_u(\lambda) = u + \sum_k \frac{(1 + \lambda\lambda_k)\overline{\langle \delta_u, \psi_k \rangle}}{(1 + \lambda_k^2)(\lambda_k - \lambda)}\psi_k, \quad \lambda \notin \operatorname{spec}(A_{\mathcal{D}});$$

the series converges absolutely and uniformly in $H^1_{\mathcal{D}}$ on compact subsets of $\operatorname{res}(A_{\mathcal{D}})$. Alternatively, again using (4.2) in the expansion of u in terms of the ψ_k, we have

$$\phi_u(\lambda) = \sum_k \frac{\overline{\langle \delta_u, \psi_k \rangle}}{\lambda_k - \lambda}\psi_k, \quad \lambda \notin \operatorname{spec}(A_{\mathcal{D}}). \tag{5.2}$$

This series converges in $H^0_{\mathcal{D}}$ since

$$\sum_k \frac{|\langle \delta_u, \psi_k \rangle|^2}{(1 + |\lambda_k|)^2}$$

converges (because $\delta_u \in H^{-1}_{\mathcal{D}^\dagger}$).

6. Negativity and regularity

We continue our discussion with the selfadjoint operator $A_{\mathcal{D}}$ of the previous section; so $\mathcal{D} = D + \mathcal{D}_{\min}$ with $D \in \mathfrak{S}\mathfrak{A}$. Let $S : D^\perp \to D^\perp$ be selfadjoint with respect to the A-inner product, let $T = AS : D^\perp \to D$, and let

$$\operatorname{graph} T = \{u + Tu : u \in D^\perp\},$$

which by Lemma 2.8 is an element of $\mathfrak{S}\mathfrak{A}$. Let

$$\mathcal{D}_T = \operatorname{graph} T + \mathcal{D}_{\min}.$$

By Lemma 5.1, $\lambda \in \text{bg-res}(A)$ belongs to $\operatorname{spec}(A_{\mathcal{D}_T})$ if and only if $\operatorname{graph} T \cap K_\lambda \neq 0$. In particular, $\lambda \in \operatorname{res}(A_{\mathcal{D}})$ belongs to $\operatorname{spec}(A_{\mathcal{D}_T})$ if and only if there is $u \in D^\perp$, $u \neq 0$, such that

$$u - \pi_{\max} B_{\mathcal{D}}(\lambda)(A - \lambda)u = u + Tu,$$

that is, if and only if $-\pi_{\max} B_{\mathcal{D}}(\lambda)(A - \lambda)u = ASu$. Setting

$$F_{\mathcal{D}}(\lambda) = -A\pi_{\max} B_{\mathcal{D}}(\lambda)(A - \lambda)|_{D^\perp},$$

an operator $D^\perp \to D^\perp$ we thus have

$$\lambda \in \operatorname{spec}(A_{\mathcal{D}_T}) \cap \operatorname{res}(A_{\mathcal{D}}) \quad \Longleftrightarrow \quad F_{\mathcal{D}}(\lambda) + S \text{ has nontrivial kernel.} \tag{6.1}$$

Lemma 6.2. *The map $F_{\mathcal{D}}(\lambda)$ satisfies*

$$F_{\mathcal{D}}(\lambda)^* = F_{\mathcal{D}}(\overline{\lambda}), \quad \lambda \in \operatorname{res}(A_{\mathcal{D}}). \tag{6.3}$$

In addition, for any $\lambda \in \operatorname{res}(A_{\mathcal{D}})$,

$$\big(F_{\mathcal{D}}(\lambda)u, u'\big)_A = \sum_{k=0}^{\infty} \frac{\overline{\langle \delta_u, \psi_k\rangle} \langle \delta_{u'}, \psi_k\rangle}{1 + \lambda_k^2} \frac{1 + \lambda\lambda_k}{\lambda_k - \lambda}, \quad u, u' \in D^\perp. \tag{6.4}$$

Proof. Let u, $u' \in D^\perp$. Then

$$\begin{aligned} \big(F_{\mathcal{D}}(\lambda)u, u'\big)_A &= \big(-A\pi_{\max} B_{\mathcal{D}}(\lambda)(A - \lambda)u, u'\big)_A \\ &= \big(\pi_{\max} B_{\mathcal{D}}(\lambda)(A - \lambda)u, Au'\big)_A \\ &= \big(B_{\mathcal{D}}(\lambda)(A - \lambda)u, Au'\big)_A, \end{aligned} \tag{6.5}$$

where the first equality is the definition of $F_{\mathcal{D}}(\lambda)$, the second because $A|_{\mathcal{E}}$ is an isometry, and the third because $\mathcal{E} \perp \mathcal{D}_{\min}$ in the A-inner product. Using the definition of the A inner product in the last term, we thus have

$$\begin{aligned} \big(F_{\mathcal{D}}(\lambda)u, u'\big)_A &= \big(AB_{\mathcal{D}}(\lambda)(A - \lambda)u, -u'\big) + \big(B_{\mathcal{D}}(\lambda)(A - \lambda)u, Au'\big) \\ &= \big((A - \lambda)u + \lambda B_{\mathcal{D}}(\lambda)(A - \lambda)u, -u'\big) + \big(B_{\mathcal{D}}(\lambda)(A - \lambda)u, Au'\big) \\ &= -\big((A - \lambda)u, u'\big) + \big(B_{\mathcal{D}}(\lambda)(A - \lambda)u, (A - \overline{\lambda})u'\big). \end{aligned}$$

Likewise,

$$\big(u, F_{\mathcal{D}}(\overline{\lambda})u'\big)_A = -\big(u, (A - \overline{\lambda})u'\big) + \big((A - \lambda)u, B_{\mathcal{D}}(\overline{\lambda})(A - \overline{\lambda})u'\big).$$

Then (6.3) follows from noting that $((A-\lambda)u,u') = (u,(A-\overline{\lambda})u')$ because $D^\perp + \mathcal{D}_{\min}$ is a selfadjoint domain and $B_\mathcal{D}(\lambda)^* = B_\mathcal{D}(\overline{\lambda})$. This proves the first assertion of the lemma.

For the second, we have

$$\begin{aligned}\big(F_\mathcal{D}(\lambda)u,u'\big)_A &= \big(B_\mathcal{D}(\lambda)(A-\lambda)u,Au'\big)_A = -\big(u - B_\mathcal{D}(\lambda)(A-\lambda)u,Au'\big)_A\\ &= -\big(\phi_u(\lambda),Au'\big)_A = \lambda\big(\phi_u(\lambda),u'\big) - \big(\phi_u(\lambda),Au'\big)\end{aligned}$$

using (6.5). Using (5.2) and (4.2), we get

$$\lambda\big(\phi_u(\lambda),u'\big) = \sum_{k=0}^\infty \frac{\lambda\lambda_k \overline{\langle\delta_u,\psi_k\rangle}\langle\delta_{u'},\psi_k\rangle}{(1+\lambda_k^2)(\lambda_k-\lambda)}$$

and

$$-\big(\phi_u(\lambda),Au'\big) = \sum_{k=0}^\infty \frac{\overline{\langle\delta_u,\psi_k\rangle}\langle\delta_{u'},\psi_k\rangle}{(1+\lambda_k^2)(\lambda_k-\lambda)}.$$

The combination of these formulas gives (6.4). □

The following proposition is the key result.

PROPOSITION 6.6. *Let* $\mathcal{D} = D + \mathcal{D}_{\min}$ *with* $D \in \mathfrak{S}\mathfrak{A}$, *let*

$$D_0^\perp = \big\{u \in D^\perp : \delta_u \in H_{\mathcal{D}^\dagger}^{-1/2}\big\},$$

let $D_1^\perp \subset D^\perp$ *be complementary to* $D_0^\perp$ *in* $D^\perp$, *and let* $\pi_{D_1^\perp} : D^\perp \to D^\perp$ *be the orthogonal projection on* $D_1^\perp$. *Then for every selfadjoint operator* $S : D^\perp \to D^\perp$ *there is* $\zeta < 0$ *such that* $\pi_{D_1^\perp}(F_\mathcal{D}(\lambda)+S)|_{D_1^\perp}$ *is negative if* $\lambda < \zeta$.

Proof. Suppose that the conclusion is false. Then there is a selfadjoint operator $S : D^\perp \to D^\perp$ and a sequence $\{\zeta_\ell\}_{\ell=1}^\infty$ decreasing to $-\infty$ such that $\pi_{D_1^\perp}(F_\mathcal{D}(\zeta_\ell)+S)|_{D_1^\perp}$ has a nonnegative eigenvalue for each ℓ. Let $u_\ell \in D_1^\perp$ be an eigenvector of $F_\mathcal{D}(\zeta_\ell)+S$ for such an eigenvalue, with $\|u_\ell\|_A = 1$. Thus

$$\big(F_\mathcal{D}(\zeta_\ell)u_\ell,u_\ell\big)_A + (Su_\ell,u_\ell)_A \geq 0$$

for all ℓ. Passing to a subsequence, we may assume that $\{u_\ell\}_{\ell=1}^\infty$ converges to some $u \in D_1^\perp$. Using (6.4), we have

$$(Su_\ell,u_\ell)_A \geq -\big(F_\mathcal{D}(\zeta_\ell)u_\ell,u_\ell\big)_A = -\sum_{k=0}^\infty \frac{|\langle\delta_{u_\ell},\psi_k\rangle|^2}{1+\lambda_k^2}\frac{1+\zeta_\ell\lambda_k}{\lambda_k-\zeta_\ell}$$

for every ℓ. If $k_0 = \min\{k : \lambda_k > 0\}$ and $k \geq k_0$, then

$$\frac{1+\zeta_\ell\lambda_k}{\lambda_k-\zeta_\ell} < 0$$

if $\zeta_\ell < -1/\lambda_{k_0}$, so bearing in mind that the λ_k increase monotonically with k,

$$\sum_{k=k_0}^\infty -\frac{|\langle\delta_{u_\ell},\psi_k\rangle|^2}{1+\lambda_k^2}\frac{1+\zeta_\ell\lambda_k}{\lambda_k-\zeta_\ell}$$

is a series of non-negative terms if $\ell > \ell_0$ so that $\zeta_\ell < -1/\lambda_{k_0}$ for such ℓ. Hence

$$(Su_\ell, u_\ell)_A \geq -\sum_{k=0}^{N} \frac{|\langle \delta_{u_\ell}, \psi_k\rangle|^2}{1+\lambda_k^2}\frac{1+\zeta_\ell\lambda_k}{\lambda_k-\zeta_\ell}$$

for every $N \geq k_0$ and all $\ell > \ell_0$. Taking the limit as $\ell \to \infty$ gives

$$(Su, u)_A \geq \sum_{k=0}^{N} \lambda_k \frac{|\langle \delta_u, \psi_k\rangle|^2}{1+\lambda_k^2}$$

for every N, so

$$\lim_{N\to\infty} \sum_{k=0}^{N} \lambda_k \frac{|\langle \delta_u, \psi_k\rangle|^2}{1+\lambda_k^2} \leq (Su, u)_A.$$

Since only finitely many λ_k can be negative, the estimate implies that

$$\sum_{k=0}^{\infty} |\lambda_k| \frac{|\langle \delta_u, \psi_k\rangle|^2}{1+\lambda_k^2}$$

converges. This in turn implies that the norm of δ_u as an element of $H^{-1/2}_{\mathcal{D}^\dagger}$ is finite, see (4.3). So $u \in D_0^\perp$, a contradiction since $\|u\|_A = 1$ and $u \in D_1^\perp \cap D_0^\perp$. □

In particular, if $\mathcal{P}_{D^\perp} \cap H^{-1/2}_{\mathcal{D}^\dagger} = 0$, then for every $c > 0$ there is $\zeta < 0$ such that $F_{\mathcal{D}}(\lambda) + cI$ is negative if $\lambda < \zeta$. In particular, we have the following corollary.

Corollary 6.7. *If $\mathcal{P}_{D^\perp} \cap H^{-1/2}_{\mathcal{D}^\dagger} = 0$, then $F_{\mathcal{D}}(\lambda)$ is invertible for every sufficiently negative λ, and $\|F_{\mathcal{D}}(\lambda)^{-1}\|_{\mathscr{L}(D^\perp)} \to 0$ as $\lambda \to -\infty$.*

The definition of $F_{\mathcal{D}}(\lambda)$ gives

$$K_\lambda = \{u - AF_{\mathcal{D}}(\lambda)u : u \in D^\perp\}.$$

Since $F_{\mathcal{D}}(\lambda)$ is invertible for every sufficiently negative λ, also

$$K_\lambda = \{v + F_{\mathcal{D}}(\lambda)^{-1}Av : v \in D\}. \tag{6.8}$$

Thus, if $\mathcal{P}_{D^\perp} \cap H^{-1/2}_{\mathcal{D}^\dagger} = 0$, Corollary 6.7 and (6.8) give that $K_\lambda \to D$ as $\lambda \to -\infty$. Applied to $D = D_F$ and bearing in mind Proposition 4.1 and that the Friedrichs extension of A is bounded below, we get:

Theorem 6.9. *Consider the curve*

$$\mathbb{R}_- \ni \lambda \mapsto K_\lambda \in \mathrm{Gr}_d(\mathcal{E}).$$

Then $K_\lambda \to D_F$ as $\lambda \to -\infty$.

The limit $\lim_{\lambda\to-\infty} K_\lambda$ is of course unique. Since K_λ is independent of its representation, we have that if in (6.8) $K_\lambda \to D$ then $D = D_F$. Consequently, THEOREM 6.10. *The Friedrichs domain of* A *is the only selfadjoint domain such that* $\mathcal{P}_{D^\perp} \cap H^{-1/2}_{\mathcal{D}^\dagger} = 0$.

PROPOSITION 6.11. *Suppose* $\{D_\ell\}_{\ell=1}^\infty \subset \mathfrak{S}\mathfrak{A}$ *is a sequence converging to* D *and there is* $\{\zeta_\ell\} \subset \mathbb{R}$ *with* $\zeta_\ell \to -\infty$ *as* $\ell \to \infty$ *such that* $D_\ell \cap K_{\zeta_\ell} \neq 0$. *Then* $D \cap D_F \neq 0$.

Proof. For each ℓ pick $v_\ell \in D_\ell \cap K_{\zeta_\ell}$ with $\|v_\ell\|_A = 1$. Passing to a subsequence, assume that $v_\ell \to v$ as $\ell \to \infty$. Using $\mathcal{E} = D_F \oplus D_F^\perp$ gives for each ℓ, a unique $w_\ell \in D_F$ such that $v_\ell = w_\ell + F_{\mathcal{D}_F}(\zeta_\ell)^{-1} A w_\ell$. The continuity of projections gives that w_ℓ converges. Now Corollary 6.7 applied to the Friedrichs domain gives $F_{\mathcal{D}_F}(\zeta)^{-1} A w_\ell \to 0$ as $\zeta \to -\infty$. Thus, $w_\ell \to v$. Since $w_\ell \in D_F$, $v \in D_F$. Now, $D_\ell = \operatorname{graph} T_\ell$ for a unique $T_\ell : D \to D^\perp$; the statement that $D_\ell \to D$ means that $T_\ell \to 0$. Thus $w_\ell = v'_\ell + T_\ell v'_\ell$ for a unique $v'_\ell \in D$ and as before v'_ℓ converges, so w_ℓ converges to an element of D which must be v. Since $\|v\|_A = 1$, $D \cap D_F \neq 0$. □

7. Spectrally unstable domains

The following, a restatement of Theorem 1.4, is our main result.

THEOREM 7.1. *Let* $\mathcal{D}_F = D_F + \mathcal{D}_{\min}$ *be the domain of the Friedrichs extension of* A. *The element* $D \in \mathfrak{S}\mathfrak{A}$ *has the property* (1.3) *if and only if* $D \in \mathfrak{V}_{D_F}$.

We have written $\mathfrak{V}_{D_F} = \{D \in \mathfrak{S}\mathfrak{A} : D \cap D_F \neq 0\}$. This is a real-algebraic subvariety of $\mathfrak{S}\mathfrak{A}$ of codimension 1.

Proof of Theorem 7.1. If $D \in \mathfrak{S}\mathfrak{A}$, then either $\pi_{D_F^\perp}|_D : D \to D_F^\perp$ is injective, or not. In the first case, $D \in U_{D_F^\perp}$, and in the second, $D \in \mathfrak{V}_{D_F}$. Thus

$$\mathfrak{S}\mathfrak{A} = (\mathfrak{S}\mathfrak{A} \cap U_{D_F^\perp}) \cup \mathfrak{V}_{D_F}$$

as a disjoint union.

Proposition 6.11 gives that every element of $\mathfrak{S}\mathfrak{A} \cap U_{D_F^\perp}$ is spectrally stable, so we only need to show that every element of $\mathfrak{V}_{\mathcal{D}_F}$ is spectrally unstable.

Suppose $D \in \mathfrak{V}_{D_F}$. We will show the existence of curves $\lambda \mapsto D_\lambda$ in $\mathfrak{S}\mathfrak{A}$ such that $D_\lambda \to D$ as $\lambda \to -\infty$ and $D_\lambda \cap K_\lambda \neq 0$. With such a curve we have that if U is a neighborhood of D and $\zeta < 0$, then there is $\zeta' < \zeta$ such that $D_\lambda \in U$ for every $\lambda < \zeta'$. Since $K_\lambda \cap D_\lambda \neq 0$, λ belongs to the spectrum of A with domain $\mathcal{D}_\lambda = D_\lambda + \mathcal{D}_{\min}$, which shows that D is spectrally unstable.

By Corollary 6.7 and Proposition 4.1, the operator $F_{\mathcal{D}_F}(\lambda) : D_F^\perp \to D_F^\perp$ is invertible for every sufficiently negative λ, so

$$K_\lambda = \big\{v + F_{\mathcal{D}_F}(\lambda)^{-1} A v : v \in D_F\big\},$$

see (6.8). Let V be a subspace of $D\cap D_F$, $V\neq 0$. As usual let π_D and $\pi_{D^\perp}$ be the orthogonal projections on D and $D^\perp$. If $v\in V$, then

$$\begin{aligned}v+F_{\mathcal{D}_F}(\lambda)^{-1}Av &= \pi_D\big(v+F_{\mathcal{D}_F}(\lambda)^{-1}Av\big)+\pi_{D^\perp}\big(v+F_{\mathcal{D}_F}(\lambda)^{-1}Av\big)\\ &= \big(v+\pi_D F_{\mathcal{D}_F}(\lambda)^{-1}Av\big)+\pi_{D^\perp}F_{\mathcal{D}_F}(\lambda)^{-1}Av.\end{aligned}$$

Let

$$V_\lambda=\big\{v+\pi_D F_{\mathcal{D}_F}(\lambda)^{-1}Av : v\in V\big\},$$

a subspace of D. Let W be the orthogonal complement of V in D. The mapping $D\to D$ given by

$$V\oplus W\ni(v\oplus w)\mapsto v+\pi_D F_{\mathcal{D}_F}(\lambda)^{-1}Av+w\in D$$

is invertible for every sufficiently negative λ because $\|F_{\mathcal{D}_F}(\lambda)^{-1}\|\to 0$ as $\lambda\to-\infty$. Its inverse tends to the identity as $\lambda\to-\infty$ and maps V_λ to V. Let $S_\lambda : V_\lambda\to V$ be the restriction to V_λ of this inverse and define $T_{\lambda,0}:V_\lambda\to D^\perp$ by

$$T_{\lambda,0}=\pi_{D^\perp}F_{\mathcal{D}_F}(\lambda)^{-1}AS_\lambda.$$

Then

$$\{v+T_{\lambda,0}v : v\in V_\lambda\}=\big\{v+F_{\mathcal{D}_F}(\lambda)^{-1}Av : v\in V\big\}\subset K_\lambda,$$

therefore

$$\big(v+T_{\lambda,0}v, A\big(v'+T_{\lambda,0}v'\big)\big)_A=0 \quad \text{for every } v,v'\in V_\lambda \tag{7.2}$$

(cf. the proof of Lemma 2.10). Let W_λ be the orthogonal complement of V_λ in D. We now look for $T_{\lambda,1}:W_\lambda\to D^\perp$ such that with $T_\lambda : D\to D^\perp$ defined as $T_{\lambda,0}$ on V_λ and as $T_{\lambda,1}$ on W_λ we have that $\operatorname{graph}T_\lambda\in\mathfrak{SA}$. Because of (2.7) this will be the case iff for arbitrary $v,v'\in V_\lambda$ and $w,w'\in W_\lambda$ the quantity

$$\big(v+w+T_{\lambda,0}v+T_{\lambda,1}w, A\big(v'+w'+T_{\lambda,0}v'+T_{\lambda,1}w'\big)\big)_A$$

vanishes. Using (7.2) first and then several times that D and $D^\perp$ are both in $\mathfrak{SA}$ (so we can take advantage of (2.7)) while keeping in mind that the ranges of $T_{\lambda,0}$ and $T_{\lambda,1}$ lie in $D^\perp$, the above expression is equivalent to

$$\begin{aligned}&\big(v,AT_{\lambda,1}w'\big)_A+\big(T_{\lambda,0}v,Aw'\big)_A+\big(w,AT_{\lambda,0}v'\big)_A+\big(T_{\lambda,1}w,Av'\big)_A\\ &\quad+\big(w,AT_{\lambda,1}w'\big)_A+\big(T_{\lambda,1}w,Aw'\big)_A.\end{aligned}$$

In order for this to vanish for all v,v',w,w' it is necessary and sufficient that

$$\big(v,AT_{\lambda,1}w'\big)_A+\big(T_{\lambda,0}v,Aw'\big)_A=0 \quad\text{and}\quad \big(w,AT_{\lambda,1}w'\big)_A+\big(T_{\lambda,1}w,Aw'\big)_A=0$$

for all $v\in V_\lambda$ and $w,w'\in W_\lambda$. Letting $T^*_{\lambda,0}:D\to V_\lambda$ be the adjoint of $T_{\lambda,0}$, the first condition is equivalent to the requirement that $AT_{\lambda,1}=-T^*_{\lambda,0}A$, that is,

$$T_{\lambda,1}=AT^*_{\lambda,0}A.$$

With this definition of $T_{\lambda,1}$ both $(w, AT_{\lambda,1}w')_A$ and $(T_{\lambda,1}w, Aw')_A$ vanish because $W_\lambda \perp V_\lambda$ and A is unitary. Thus $AT_\lambda : D \to D$ is selfadjoint, and since $T_\lambda \to 0$ as $\lambda \to -\infty$,

$$D_\lambda = \operatorname{graph} T_\lambda \in \mathfrak{SA}, \qquad K_\lambda \cap D_\lambda \neq 0 \quad \text{and} \quad D_\lambda \to D \quad \text{as } \lambda \to -\infty.$$

We have shown that $\mathfrak{V}_{D_F}$ consists of spectrally unstable domains. □

We end with an alternate argument to Proposition 6.11 that all elements of $\mathfrak{SA} \cap U_{D_F^\perp}$ are spectrally stable. Let $D_0 \in \mathfrak{SA} \cap U_{D_F^\perp}$ be arbitrary, let $T_0 : D_F^\perp \to D_F$ be such that $D_0 = \operatorname{graph} T_0$, let $S_0 = AT_0$, and let $M > \|S_0\|$. Then

$$U = \big\{\operatorname{graph} T : T \in \mathscr{L}\big(D_F^\perp, D_F\big), S = AT \text{ selfadjoint}, \|S\| < M\big\}$$

is a neighborhood of D_0 in $\mathfrak{SA}$. There is $\zeta < 0$ such that

$$\big(F_{\mathcal{D}_F}(\lambda)u, u\big)_A \leq -M\|u\|_A^2 \quad \forall u \in D_F^\perp, \lambda < \zeta.$$

Let $D \in U$, so $D = \operatorname{graph} T$ with $S = AT : D_F^\perp \to D_F^\perp$ selfadjoint and $\|S\| < M$. Then

$$\big(\big(F_{\mathcal{D}_F}(\lambda) - S\big)u, u\big)_A \leq \big(-M + \|S\|\big)\|u\|_A^2 \quad \forall u \in D_F^\perp, \lambda < \zeta$$

hence $\ker(F_{\mathcal{D}_F}(\lambda) - S) = 0$ if $\lambda < \zeta$. Therefore

$$\operatorname{spec}(A_{\mathcal{D}_T}) \subset [\zeta, \infty)$$

by (6.1).

References

[1] M. Birman and M. Z. Solomjak, *Spectral theory of selfadjoint operators in Hilbert space*, Mathematics and Its Applications (Soviet Series), Reidel, Dordrecht, 1987. MR 1192782

[2] H. Freudenthal, *Über die Friedrichssche Fortsetzung halbbeschränkter Hermitescher Operatoren*, Proc. Akad. Wet. Amsterdam **39** (1936), 832–833.

[3] K. O. Friedrichs, *Spektraltheorie halbbeschränkter Operatoren und Anwendung auf die Spektralzerlegung von Differentialoperatoren*, Math. Ann. **109** (1934), 465–487. MR 1512905

[4] J. B. Gil and G. A. Mendoza, *Adjoints of elliptic cone operators*, Amer. J. Math. **125** (2003), 357–408. MR 1963689

[5] J. B. Gil, T. Krainer and G. A. Mendoza, *Resolvents of elliptic cone operators*, J. Funct. Anal. **241** (2006), 1–55.

[6] J. B. Gil, T. Krainer and G. A. Mendoza, *Geometry and spectra of closed extensions of elliptic cone operators*, Canad. J. Math. **59** (2007), 742–794.

[7] M. Lesch, *Operators of Fuchs type, conical singularities, and asymptotic methods*, Teubner-Texte zur Math., vol. 136, Teubner, Leipzig, 1997.

[8] J. V. Neumann, *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren*, Math. Ann. **102** (1930), 49–131.

[9] M. Reed and B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press, New York, 1975.

Gerardo A. Mendoza, Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

E-mail address: gmendoza@temple.edu