COMPACT COMPOSITION OPERATORS
WITH SYMBOL A UNIVERSAL COVERING
MAP ONTO A MULTIPLY CONNECTED DOMAIN

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ABSTRACT. We generalise previous results of the author concerning the compactness of composition operators on the Hardy spaces $H^p$, $1 \leq p < \infty$, whose symbol is a universal covering map from the unit disk in the complex plane to general finitely connected domains. We demonstrate that the angular derivative criterion for univalent symbols extends to this more general case. We further show that compactness in this setting is equivalent to compactness of the composition operator induced by a univalent mapping onto the interior of the outer boundary component of the multiply connected domain.

1. Introduction

Let $\mathbb{D} = \{z: |z| < 1\}$ be the unit disk in the complex plane and $H^p$ the classic Hardy space of holomorphic functions $f$ on $\mathbb{D}$ satisfying

$$\|f\|_p^p = \lim_{r \to 1} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < \infty.$$ 

If $\phi: \mathbb{D} \to \mathbb{D}$ is a holomorphic mapping, then the composition operator $C_\phi: f \mapsto f \circ \phi$ is well defined and maps $H^p$ boundedly into itself for any $0 < p < \infty$.

Compactness of $C_\phi$, in contrast, depends on $\phi$ in a more subtle and interesting way. It was shown in [6] that $C_\phi$ is compact on $H^p$, $1 \leq p < \infty$, if and only if

$$\lim_{|w| \to 1} \frac{N_\phi(w)}{\log 1/|w|} = 0,$$

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where $N_\phi$ is the Nevanlinna counting function

$$N_\phi(w) = \begin{cases} 
\sum \phi(z) = w \log \frac{1}{|z|}, & w \in \phi(\mathbb{D}), \\
0, & \text{otherwise}.
\end{cases}$$

If $\phi$ is a univalent mapping of $\mathbb{D}$ onto a simply connected domain $D$, then the result above implies that $C_\phi$ is compact if and only if $D$ has no finite angular derivative, or, equivalently,

$$\lim_{|z| \to 1} \frac{1 - |\phi(z)|}{1 - |z|} = \infty.$$ 

In the next section, we will define the angular derivative and other quantities pertinent to this work. In general, although the angular derivative criterion is sufficient for compactness, it is not necessary, see [8]. For an introduction to the background to these results, see [7] or [2].

In [5], the author showed that if $\phi$ is a universal covering map onto a multiply connected domain of the form described below then the angular derivative criterion is both necessary and sufficient for $C_\phi$ to be compact on $H^p$. In particular, let $D = D_0 \setminus \{p_1, p_2, \ldots, p_n\}$ where $D_0$ is a simply connected domain in $\mathbb{D}$ and $p_i, i = 1, \ldots, n$ are isolated points in the interior of $D_0$. It was shown that if $\phi$ is a universal covering map of $\mathbb{D}$ onto $D$ then $C_\phi$ is compact on $H^p$, $1 \leq p < \infty$, if and only if

$$\lim_{|z| \to 1} \frac{1 - |\phi(z)|}{1 - |z|} = \infty.$$ 

This result highlights the importance of the geometry of the domain $D$ in the characterisation of compactness of $C_\phi$. In fact, it was shown that if $\psi$ is a univalent mapping of $\mathbb{D}$ onto $D_0$, then $C_\psi$ is compact on $H^p$, $1 \leq p < \infty$, if and only if $C_\psi$ is.

The purpose of this paper is to extend these results to arbitrary domains of finite multiplicity.

Throughout this paper, $D$ will represent a finitely connected domain contained in $\mathbb{D}$ whose boundary consists of $n$ components that may be either points or continua. Let $\phi$ be the universal covering map of $\mathbb{D}$ onto $D$. As in [5], we wish to characterize the compactness of $C_\phi: H^p \to H^p$. Our first main result is the following.

**Theorem 1.1.** Suppose $D$ is a finitely connected domain in $\mathbb{D}$. Let $\phi$ be a holomorphic universal covering map of $\mathbb{D}$ onto $D$. Then $C_\phi$ is compact on $H^p$, $1 \leq p < \infty$, if and only if

$$\lim_{|z| \to 1} \frac{1 - |\phi(z)|}{1 - |z|} = \infty.$$ 

As in [5], we develop this further in the following result.
Theorem 1.2. Suppose \( D \) is a finitely connected domain in \( \mathbb{D} \) that can be obtained by removing finitely many components from the interior of a simply connected domain \( D_0 \). Let \( \phi \) be a holomorphic universal covering map of \( \mathbb{D} \) onto \( D \) and \( \psi \) a univalent mapping of \( \mathbb{D} \) onto \( D_0 \). Then \( C_\phi \) is compact on \( H^p \), \( 1 \leq p < \infty \), if and only if \( C_\psi \) is.

In order to prove these results, we will require ideas from Fuchsian groups and Riemann surfaces. We will provide an overview of these but the reader may find more details in [1] and [3]. In Section 2, we will cover many of the prerequisites required for the proofs of the results. Sections 3 and 4 are devoted to the proofs of the main results.

2. Preliminaries

We begin this section with a discussion of the angular derivative and then move on to the construction of the universal covering map of \( \mathbb{D} \) onto a multiply connected domain.

Consider a holomorphic mapping \( \phi : \mathbb{D} \to \mathbb{D} \). At a point \( \zeta \in \partial \mathbb{D} \), \( \phi \) is said to have a finite angular derivative if there is a \( \eta \in \partial \mathbb{D} \) such that the ratio

\[
\frac{\phi(z) - \eta}{z - \zeta}
\]

converges as \( z \to \zeta \) non-tangentially. The angular derivative, when it exists, will be denoted by \( \phi'(\zeta) \). The existence of the angular derivative at a point \( \zeta \) has a number of geometric consequences on the mapping properties of \( \phi \). For example, it is known that it implies that \( \phi \) is conformal at \( \zeta \). See also Julia’s Lemma [2, Lemma 2.41].

We will require the following important result.

Theorem A (Julia–Caratheodory theorem). Let \( \phi : \mathbb{D} \to \mathbb{D} \) be a holomorphic function and suppose \( \zeta \in \partial \mathbb{D} \). The following are equivalent.

1. \( \phi \) has finite angular derivative \( \phi'(\zeta) \) at \( \zeta \).
2. \( D(\zeta) = \liminf_{z \to \zeta} \frac{1 - |\phi(z)|}{1 - |z|} < \infty \).
3. Both \( \phi \) and \( \phi' \) have non-tangential limits at \( \zeta \), with \( \lim_{r \to 1} \phi(r\zeta) = \eta \in \partial \mathbb{D} \).

When any (all) of these criteria hold we have that \( D(\zeta) \) is the non-tangential limit

\[
\lim_{z \to \zeta} \frac{1 - |\phi(z)|}{1 - |z|}
\]

and \( \phi'(\zeta) = D(\zeta)\bar{\zeta}\eta \).

In particular it is a consequence of this theorem that if

\[
\lim_{|z| \to 1} \frac{1 - |\phi(z)|}{1 - |z|} = \infty
\]

then \( \phi \) cannot have finite angular derivative at any point of \( \partial \mathbb{D} \).
When $\phi$ is univalent the existence of an angular derivative at a point depends only on the geometry of the simply connected domain $\phi(\mathbb{D})$, or, more precisely, on the boundary of $\phi(\mathbb{D})$. A good account of these results is contained in [4, §V.5]. We mention only that since the existence of an angular derivative at a point implies it is conformal there. Any simply connected domain contained in a polygon in $\mathbb{D}$, for example, cannot have finite angular derivative.

Let $\mathcal{D} \subset \mathbb{D}$ be an arbitrary multiply connected domain. Since $\mathcal{D}$ is hyperbolic, there is a Riemann surface $\mathcal{R}_\mathcal{D} \cong \mathbb{D}/\Gamma$ conformally equivalent to $\mathcal{D}$, where $\Gamma$ is a torsion-free Fuchsian group. The universal covering map is then constructed as in the following diagram.

$$
\mathbb{D} \xrightarrow{\hat{\phi}_\mathcal{D}} \mathcal{R}_\mathcal{D} \xrightarrow{\phi} \mathcal{D}
$$

Here the mapping $\hat{\phi}_\mathcal{D}$ exists as a consequence of the uniformization theorem. The mapping $\phi$ is conformal and locally univalent. It follows from the construction that for any $w \in \mathcal{D}$ the pre-image under $\phi$ of $w$ is a $\Gamma$-orbit, $\Gamma(z) = \{ g(z) : g \in \Gamma \}$. We will use this to estimate $N_\phi(w)$ in terms of the action of $\Gamma$. As such we suppose $\mathcal{F}$ is a locally finite fundamental domain for the action of $\Gamma$ on $\mathbb{D}$. Then $\tilde{\mathcal{F}}/\Gamma$ is homeomorphic to $\mathbb{D}/\Gamma$ where $\tilde{\mathcal{F}}$ denotes the relative closure of $\mathcal{F}$ in $\mathbb{D}$, see [1, §9.2]. The Dirichlet fundamental polygon is one such example, it is defined for $w \in \mathbb{D}$ as

$$
D(w) = \bigcap_{g \in \Gamma, g \neq \text{id}} \{ z \in \mathbb{D} : d_\mathbb{D}(z, w) < d_\mathbb{D}(z, g(w)) \}.
$$

Here we denote by $d_\mathbb{D}(z_1, z_2)$ the hyperbolic distance in $\mathbb{D}$,

$$
d_\mathbb{D}(z_1, z_2) = \inf_\gamma \int_\gamma \frac{2}{1 - |z|^2} |dz|,
$$

where the infimum is taken over all smooth curves $\gamma$ joining $z_1$ to $z_2$. This metric has as its geodesics radii and arcs of circles orthogonal to $\partial \mathbb{D}$. In particular

$$
d_\mathbb{D}(0, z) = \log \frac{1 + |z|}{1 - |z|}.
$$

Since automorphisms of $\mathbb{D}$ are isometries of the hyperbolic metric the action of $\Gamma$ gives rise to a hyperbolic metric on $\mathbb{D}/\Gamma$. Automorphisms are characterized as parabolic or hyperbolic according to whether they have one or two fixed points on $\partial \mathbb{D}$. The limit set of $\Gamma$, denoted $\Lambda(\Gamma)$, is the set of all limit points of orbits of a point under the action of $\Gamma$, it is a proper subset of $\partial \mathbb{D}$.
The conjugacy classes of parabolic elements of $\Gamma$ correspond to punctures in the Riemann surface, [3, pp. 214–216]. Similarly, there is a correspondence between boundary loops of $D$ and conjugacy classes of hyperbolic elements, called \textit{boundary hyperbolic elements} in [1, p. 265].

As in [5], we rely on estimating the Nevanlinna counting function by the Poincare series for $\Gamma$:

$$\rho_\Gamma(z, w; s) = \sum_{g \in \Gamma} \exp(-sd_D(z, g(w))).$$

The Poincare series converges for $s > \dim \Lambda(\Gamma)$ and so, in particular, for $s = 1$, the exponent that we will require.

3. Proof of Theorem 1.1

We assume throughout that $\partial D \cap \partial \mathbb{D} \neq \emptyset$. Otherwise both Theorem 1.1 and 1.2 are true trivially.

In order to utilise Shapiro’s compactness criterion, we must first establish that as $|w| \to 1$ in $D$ any $z \in \phi^{-1}(w)$ satisfies $|z| \to 1$. Consider a locally finite fundamental domain $F$ for the action of $\Gamma$ on $\mathbb{D}$. Then $F$ can be chosen to be a finite sided hyperbolic polygon with free sides $I_k, k = 1, 2, \ldots, N$. Each free side of $F$ lies in an interval of discontinuity, $\sigma_k$, for $\Gamma$ on $\partial \mathbb{D}$. The stabilizer

$$\{ g \in \Gamma : g(\sigma_k) = \sigma_k \}$$

is an infinite cyclic subgroup generated by a hyperbolic automorphism, say $h_k$.

Now the conjugacy class of each $h_k$, $Cl(h_k)$, maps $I_k$ to the pairwise disjoint sets

$$J_k = \bigcup_{\tilde{h} \in Cl(h_k)} \tilde{h}(I_k) = \bigcup_{\tilde{h} \in Cl(h_k)} \tilde{h}(\sigma_k).$$

Note that the unit circle then consists of points in the limit set $\Lambda(\Gamma)$ or in $J_k$ for some $k$.

Since $\mathbb{D}$ is finitely connected, there is a $0 < R < 1$ such that

$$A(R, 1) \cap \partial D = A(R, 1) \cap \partial D_0,$$

where $A(R, 1)$ denotes the annulus centered at the origin with inner radius $R$ and outer radius 1. Therefore as $|w| \to 1$, $w$ must converge to $\partial D_0$ which, considered as a boundary loop in $\mathbb{D}/\Gamma$, implies that $z \in \phi^{-1}(w)$ converges to $J_k$ for some $k$. Hence as $|w| \to 1, |z| \to 1$ as required. Without loss of generality, we will throughout assume $J_1$ corresponds to $\partial D_0$.

Proposition 3.1. With the notation above $C_\phi$ is compact on $H^p, 1 \leq p < \infty$, if and only if for each $\zeta \in J_1$

$$\lim_{z \to \zeta} \frac{\rho_\Gamma(0, z; 1)}{1 - |\phi(z)|} = 0.$$
Proof. We may write the Nevanlinna counting function as
\[ N_\phi(w) = \sum_{g \in \Gamma} \log \frac{1}{|g(z)|}, \]
where \( z \) is an arbitrary preimage of \( w \) under \( \phi \).

Now, since \( \Gamma \) is discontinuous in \( \mathbb{D} \), the set \( \{ g : |g(z)| \leq R \} \), for \( 1/2 < R < 1 \), is finite. Therefore, since
\[ \log \frac{1}{x} \leq 1 - x^2 \leq 2 \log \frac{1}{x}, \quad 1/2 < x < 1 \]
we have that
\[ N_\phi(w) \leq C \sum_{g \in \Gamma} \left( 1 - |g(z)|^2 \right) \]
\[ \leq C \sum_{g \in \Gamma} \frac{1 - |g(z)|}{1 + |g(z)|} \]
\[ = C \sum_{g \in \Gamma} \exp -d_\mathbb{D}(0, g(z)) \]
\[ = C \rho_\Gamma(0, z, 1), \]
where \( C \) denotes a constant not necessarily the same in each instance.

The opposite inequality
\[ N_\phi(w) \geq C \rho_\Gamma(0, z, 1) \]
follows similarly. The result now follows from the definition of \( J_1 \).

To complete the proof of the theorem, we will require the following result that was proved in \([5]\).

Lemma 3.2 ([5, Lemma 1]). If \( \Gamma \) uniformizes the domain \( \mathcal{D} \) then for \( z \in D(0) \) with \( z \) close enough to \( I_1 \), the free side of \( D(0) \) corresponding to \( \partial D_0 \),
\[ c_1 \exp -d_\mathbb{D}(0, z) \leq \rho_\Gamma(0, z, 1) \leq c_2 \exp -d_\mathbb{D}(0, z), \]
where \( c_1 \) and \( c_2 \) are constants depending only on \( \Gamma \).

Since \( \exp -d(0, z) \sim (1 - |z|) \) it follows that \( C_\phi \) is compact if and only if
\[ \lim_{z \to \zeta} \frac{1 - |z|}{1 - |\phi(z)|} = 0 \]
for any \( \zeta \) in the free side of \( D(0) \) corresponding to \( \partial D_0 \). In the notation above we may call this free side \( I_1 \), then \( \sigma_1, J_1 \) and \( h_1 \) are implicitly defined.

To complete the proof of the theorem, we must consider, in turn, \( J_1, J_k \) \((k > 1)\) and \( \Lambda(\Gamma') \).

Consider first \( \zeta \in J_1 \). Then there exists a \( h \in \Gamma \) with \( h(\zeta) \in I_1 \). Now suppose without loss of generality that \( z \to \zeta \) inside \( D(h^{-1}(0)) = h^{-1}(D(0)) \). Then, with \( \zeta^* = h(\zeta) \) and \( z^* = h(z) \), we have that
\[ \exp -d_\mathbb{D}(0, z) \leq d_\mathbb{D}(0, h(z)), \]
and so
\[
\lim_{z \to \zeta} \frac{1 - |z|}{1 - |\phi(z)|} \leq C \lim_{z \to \zeta} \frac{\exp -d_D(0,z)}{1 - |\phi(z)|}
\]
\[
\leq C \lim_{z \to \zeta} \frac{\exp -d_D(0,h(z))}{1 - |\phi(z)|}
\]
\[
= C \lim_{z^* \to \zeta^*} \frac{\exp -d_D(0,z^*)}{1 - |\phi(z^*)|}
\]
\[
\leq C \lim_{z^* \to \zeta^*} \frac{\rho_\Gamma(0,z^*;1)}{1 - |\phi(z^*)|}.
\]

Conversely,
\[
\lim_{z \to \zeta} \frac{1 - |z|}{1 - |\phi(z)|} = \lim_{z \to \zeta} \frac{1 - |z|}{1 - |z^*|} \frac{1 - |z^*|}{1 - |\phi(z^*)|}
\]
\[
= \frac{1}{|h'(\zeta)|} \lim_{z^* \to \zeta^*} \frac{1 - |z^*|}{1 - |\phi(z^*)|}.
\]

Therefore,
\[
\lim_{z \to \zeta} \frac{1 - |z|}{1 - |\phi(z)|} = 0 \quad \text{if and only if} \quad \lim_{z^* \to \zeta^*} \frac{\rho_\Gamma(0,z^*;1)}{1 - |\phi(z^*)|} = 0
\]

for any $\zeta \in J_1$.

Now suppose that $\zeta \in J_k$, $k > 1$. By the comments at the beginning of this section, the sets $J_k$ correspond in the Riemann surface structure $\mathbb{D}/\Gamma$ to boundary loops corresponding to continua interior to $D_0$. In particular, as $z \to \zeta \in J_k$, we have that $\phi(z)$ is contained in a compact set interior to $\mathbb{D}$. It follows that at these points $\phi$ cannot have a finite angular derivative, by definition.

Finally, we consider the limit set $\zeta \in \Lambda(\Gamma)$. As above, these points necessarily have no finite angular derivative. This follows from the following result which may be found in [1, Theorem 10.2.5].

**Lemma 3.3.** $\Gamma$ is finitely generated if and only if each $\zeta \in \Lambda(\Gamma)$ is either
(1) a fixed point for a parabolic element of $\Gamma$; or
(2) a point of approximation—i.e. there is a sequence $g_n$, $n = 1, 2, \ldots$, of elements of $\Gamma$ such that $g_n(0) \to \zeta$ non-tangentially.

If $\zeta$ is a fixed point for a parabolic element, then this corresponds to a puncture in the Riemann surface structure (see [3, pp. 214–216]) and therefore to an isolated point in the boundary of $D$ interior to $\mathbb{D}$. Hence, as above, this implies that $\phi$ cannot have a finite angular derivative at $\zeta$.

In the second case, if $\zeta = \lim_{n \to \infty} g_n(0)$, where $(g_n(0))_{n \in \mathbb{Z}}$ is a non-tangential sequence, then $\phi$ is constant and has absolute value less than 1 on $(g_n(0))_{n \in \mathbb{Z}}$. Therefore, by the Julia–Caratheodory theorem, $\phi$ cannot have finite angular derivative at $\zeta$. This completes the proof.
4. Proof of Theorem 1.2

In order to prove this result, we will consider the function
\[ \omega = \psi^{-1} \circ \phi. \]
Now \( \omega \) is a universal covering map of \( \mathbb{D} \) onto a multiply connected domain with the same configuration as \( D \) whose outer boundary is \( \partial \mathbb{D} \). We claim that at each point \( \zeta \in \partial \mathbb{D} \) for which
\[
\lim_{z \to \zeta} |\omega(z)| = 1
\]
we have that \( |\omega'(\zeta)| < \infty \).

Suppose that \( \Gamma_0 \) is the Fuchsian group uniformizing \( \omega(\mathbb{D}) \), so that \( \omega(\mathbb{D}) \cong \mathbb{D}/\Gamma_0 \).

Then, as in the previous proof, we let \( I \) be the free side of a locally finite fundamental polygon for \( \Gamma_0 \). Now \( I \subset \sigma \), an interval of discontinuity for \( \Gamma_0 \), and we let \( J \) be the image of \( I \) under the conjugacy class of the stabiliser of \( \sigma \).

Clearly, \( \zeta \in J \) if and only if \( |\omega(z)| \to 1 \) as \( z \to \zeta \). Fix one such \( \zeta \). Then we may find a neighborhood \( N \) of \( \zeta \) such that \( \omega \) is univalent on \( N \cap \mathbb{D} \) and continuous on \( N \cap \partial N \). The continuity of \( \omega \) on the boundary follows, since \( \omega(N \cap \mathbb{D}) \) is a Jordan domain for small enough \( N \). Therefore, \( \omega \) can be extended to be holomorphic in \( N \) by the reflection principle.

Indeed the same conclusion can be made by considering the Schottky double of \( \omega(\mathbb{D}) \), defined as
\[
\Omega(\Gamma_0)/\Gamma_0, \quad \Omega(\Gamma_0) = \mathbb{D} \cup \mathbb{D}^* \cup (\partial \mathbb{D} \setminus \Lambda(\Gamma_0)),
\]
where \( \mathbb{D}^* = \{z: |z| > 1\} \cup \{\infty\} \).

Since \( \omega \) can be extended to be holomorphic at \( \zeta \), it must have a derivative there.

An application of the Julia–Carathéodory theorem implies that the angular derivative coincides with the absolute value of the derivative of \( \omega \) at \( \zeta \), as required.

To complete the proof, first note that
\[ C_\phi = C_\psi C_\omega \]
so if \( C_\psi \) is compact then so is \( C_\phi \). Conversely, if \( C_\phi \) is compact, then for each point \( \zeta \in \partial \mathbb{D} \), \( |\phi'(\zeta)| = \infty \) by Theorem 1.1. Suppose then that \( \eta \) is a point at which \( |\psi'(\eta)| < \infty \). We may find \( \zeta \in \partial \mathbb{D} \) such that
\[
\omega(r\zeta) \to \eta \quad (r \to 1)
\]
and, furthermore, \( \zeta \in J \). However, we have shown that \( |\omega'(\zeta)| < \infty \), so that by the Julia–Carathéodory theorem
\[
\lim_{r \to 1} \phi'(r\zeta) = \lim_{r \to 1} \psi'(\omega(r\zeta)) \omega'(r\zeta)
= \lambda \psi'(\eta)|\omega'(\zeta)|
\]
for some $|\lambda| = 1$. It follows that $\phi$ has finite angular derivative at $\zeta$, contradicting the compactness of $C_\phi$. Therefore, $|\psi'(\eta)| = \infty$ for all $\eta \in \partial \mathbb{D}$ and $C_\psi$ is compact.

References


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