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DUALITY OF UNIFORM APPROXIMATION PROPERTY IN OPERATOR SPACES

YANQI QIU

ABSTRACT. The duality of uniform approximation property for Banach spaces is well known. In this note, we establish, under the assumption of local reflexivity, the duality of uniform approximation property in the category of operator spaces.

1. Introduction

In this note, we will assume that the reader is familiar with the definitions of operator spaces and various classical properties of operator spaces.

We say that an operator space E has the operator space uniform approximation property (in short OUAP), if there is a constant $K \geq 1$ and a function $k(n)$ such that, for any n-dimensional subspace M of E, there exists a finite rank operator $T \in CB(E)$, such that

$$\|T\|_{cb} \leq K, \qquad \operatorname{rank} T \leq k(n) \quad \text{and} \quad T|_M = \mathrm{id}_M.$$

In the above situation, to emphasize the constant K and the function $k(n)$, we will say that E has the $(K, k(n))$-OUAP.

The main purpose of this note is to show that OUAP passes to the dual under the an assumption of local reflexivity.

THEOREM 1. *If E (resp. E^*) has the $(K, k(n))$-OUAP, and E^* (resp. E) is a locally reflexive operator space, then E^* (resp. E) has the*

$$\left(\frac{1}{1-1/m}\left(\big[(1+\varepsilon)K\big]^{1+m}+\frac{1}{m}\right), m^{2/m}\big[(1+\varepsilon)K\big]^{2+2/m}k(n)^{1+1/m}\right)\text{-OUAP},$$

for all $\varepsilon > 0$ and all integers $m > 1$.

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For simplicity, the locally reflexive in this note always means locally reflexive with constant 1. However, after a suitable modification of constants, Theorem 1 still holds if we use locally reflexive with constant $\lambda > 1$.

It is not known whether we can drop the assumption on the local reflexivity in Theorem 1. Here, a related problem seems to be open, we formulate it as Open Problem 1.

2. The main result

The notion of operator ideal norm in the category of Banach spaces can be generalized in an obvious way to the category of operator spaces (cf. e.g., [ER00]). Throughout the paper, all operator ideal norms are the operator ideal norm in the category of operator space.

Given an operator ideal norm α, we say that an operator space E has α-OUAP, if in the definition of OUAP, the condition $\operatorname{rank} T \leq k(n)$ is replaced by the condition $\alpha(T) \leq k(n)$. As before, if we need to emphasize the constant K and the function $k(n)$, we will say that E has the $(K, k(n))$-α-OUAP.

Let E be an operator space and let Y be a Banach space. Recall that an operator $u : E \to Y$ is called $(2, oh)$-summing if there is a constant C such that for all finite sequences (x_i) in E, we have

$$\sum_i \big\|u(x_i)\big\|^2 \leq C^2 \bigg\|\sum_i x_i \otimes \bar{x}_i\bigg\|_{E \otimes_{\min} \overline{E}},$$

and we denote by $\pi_{2,oh}(u)$ the smallest such constant.

Given an operator ideal norm α, we define α^d the dual ideal norm by

$$\alpha^d(T) = \alpha\big(T^*\big).$$

The operator ideal norm α is said to be 1-injective, if for any operator $T : E \to F$ and any completely isometric inclusion $i : F \hookrightarrow G$, we have

$$\alpha(T) = \alpha(i \circ T).$$

For an operator $T : E \to F$ and any integer $i \geq 1$, define the ith complete approximation number $b_i(T)$ of T by

$$b_i(T) = \inf\big\{\|T - S\|_{cb} : S \in CB(E, F), \operatorname{rank} S < i\big\}.$$

Remark 2. If E is a homogeneous operator space, that is, for all $T : E \to E$, we have $\|T\|_{cb} = \|T\|$, then $b_i(T) = a_i(T)$, where $a_i(T)$ stands for the usual ith approximation number of T. In particular, since the Piser's operator Hilbert space OH is homogeneous, we have $b_i(T) = a_i(T)$ for any $T \in CB(OH) = B(OH)$.

Let us recall the notion of locally reflexivity for operator spaces (see [Pis03]). An operator space E is called locally reflexive, if for any finite-dimensional operator space L, the natural linear isomorphism

$$CB\big(L, E^{**}\big) \to CB(L,E)^{**}$$

is isometric.

The following lemma is an immediate generalization of Lemma 1 in the article [Mas91].

LEMMA 3. *Let α be a 1-injective operator ideal norm. If E is locally reflexive and E^* has the $(K, k(n))$-α^d-OUAP, then E has the*

$$\big(K(1+\varepsilon), k(n)(1+\varepsilon)\big)\text{-}\alpha\text{-OUAP},$$

for all $\varepsilon > 0$.

Proof. Assume E^* has the $(K, k(n))$-α^d-OUAP. Let $M \subset E$ be an n-dimensional subspace. Fix $(e_1, \dots, e_n)$ an Auberbach basis of M, that is, for all scalars $\lambda_i \in \mathbb{C}$, we have

$$\max_i |\lambda_i| \le \Big\| \sum_i \lambda_i e_i \Big\| \le \sum_i |\lambda_i|.$$

With the dual basis, it is easy to see that

$$\max_i \|a_i\| \le \Big\| \sum_i a_i \otimes e_i \Big\|_{\min} \le \sum_i \|a_i\|$$

for all elements a_i in some operator space G. Fix $\varepsilon > 0$ and define

$$\mathscr{R} = \big\{ T \in CB(E) : \|T\|_{cb} \le K(1+\varepsilon)^{1/2}, \\ \alpha(T) \le k(n)(1+\varepsilon)^{1/2}, \operatorname{rank} T < \infty \big\}$$

and

$$\mathscr{C} = \big\{ (Te_i, \dots, Te_n) : T \in \mathscr{R} \big\} \subset \ell_\infty^n(E).$$

We claim first that $(e_1, \dots, e_n) \in \overline{\mathscr{C}}$, the norm closure of $\mathscr{C}$ in $\ell_\infty^n(E)$. Otherwise, since $\mathscr{C}$ is convex, and the dual space $\ell_\infty^n(E)$ is identified with $\ell_1^n(E^*)$, by Hahn–Banach separating theorem, there exist $\xi_1, \dots, \xi_n$ in E^*, such that

$$\operatorname{Re}\Big(\sum_i (\xi_i, Te_i) \Big) < \operatorname{Re}\Big(\sum_i (\xi_i, e_i) \Big), \quad \forall T \in \mathscr{R}.$$

Since E^* has the $(K, k(n))$-α^d-OUAP, we can find a finite rank operator $S \in CB(E^*)$, such that

$$\|S\|_{cb} \le K, \qquad \alpha^d(S) \le k(n) \quad \text{and} \quad S\xi_i = \xi_i \quad \text{for all } i = 1, \dots, n.$$

Since E is locally reflexive, the range of S^* is a finite dimensional subspace $R(S^*)$ of E^{**}, and we can find an operator $\varphi : R(S^*) \to E$, such that

$$\|\varphi\|_{cb} \le (1+\varepsilon)^{1/2}$$

and

$$\big(\varphi(x), \xi_i\big) = (x, \xi_i) \quad \text{for all } i = 1, \dots, n \text{ and } x \in R\big(S^*\big).$$

Let us denote by $\overline{S^*}$ when S^* is considered as an operator $E^{**} \to R(S^*)$. Since α is 1-injective,

$$\alpha\big(\overline{S^*}\big) = \alpha\big(S^*\big) = \alpha^d(S) \le k(n).$$

Let T_0 be the composition of the following applications:

$$T_0 : E \xrightarrow{\ i_E\ } E^{**} \xrightarrow{\ \overline{S^*}\ } R(S^*) \xrightarrow{\ \varphi\ } E,$$

where i_E is the canonical inclusion. We have

$$\|T_0\|_{cb} \le \|i_E\|_{cb}\big\|\overline{S^*}\big\|_{cb}\|\varphi\|_{cb} \le K(1+\varepsilon)^{1/2}$$

and

$$\alpha(T_0) \le \|i_E\|_{cb}\alpha\big(\overline{S^*}\big)\|\varphi\|_{cb} \le k(n)(1+\varepsilon)^{1/2},$$

consequently $T_0 \in \mathscr{R}$. Moreover

$$(\xi_i, T_0 e_i) = \big(\xi_i, \varphi\big(S^*(e_i)\big)\big) = \big(\xi_i, S^* e_i\big) = (S\xi_i, e_i) = (\xi_i, e_i),$$

and hence T_0 satisfies

$$\sum_i (\xi_i, T_0 e_i) = \sum_i (\xi_i, e_i),$$

we get a contradiction.

Now since $(e_1, \ldots, e_n) \in \overline{\mathscr{C}}$, for any $\mu > 0$, we can find an application $T \in \mathscr{R}$, such that $\|Te_i - e_i\| \le \mu$. When μ is small enough, the application $T|_M : M \to T(M)$ is invertible and admits an inverse $V : T(M) \to M$. For any n-tuple (a_i) in the operator space $\mathscr{K} = \mathscr{K}(\ell_2)$, we have

$$\begin{aligned}
\Big\|\sum_i a_i \otimes T(e_i)\Big\|_{\min} &\ge \Big\|\sum_i a_i \otimes e_i\Big\|_{\min} - \Big\|\sum_i a_i \otimes \big(T(e_i) - e_i\big)\Big\|_{\min} \\
&\ge \Big\|\sum_i a_i \otimes e_i\Big\|_{\min} - \mu \sum_i \|a_i\| \\
&\ge \Big\|\sum_i a_i \otimes e_i\Big\|_{\min} - n\mu \sup_i \|a_i\| \\
&\ge (1 - n\mu)\Big\|\sum_i a_i \otimes e_i\Big\|_{\min},
\end{aligned}$$

which implies that the mapping

$$\mathrm{id}_{\mathscr{K}} \otimes V : \mathscr{K} \otimes_{\min} T(M) \to \mathscr{K} \otimes_{\min} M$$

has norm less than $\frac{1}{1-n\mu}$, hence

$$\|V\|_{cb} \le \frac{1}{1 - n\mu}.$$

Let P be a projection from E onto $T(M)$, such that $\|P\|_{cb} \le n$, for example, let us denote by $(x_1, \ldots, x_n)$ an Auerbach basis for $T(M)$, and $(x_1^*, \ldots, x_n^*)$ its dual basis in $T(M)^*$, we can norm preservingly extend x_i^*, so that x_i^* can be viewed as an element in E^*, then the projection P defined by

$$Pe = \sum_i x_i^*(e) x_i, \quad \text{for all } e \in E$$

has c.b. norm less than n. Consider the following diagram (the right half is not commutative):

$$\begin{array}{ccccc} E & \xrightarrow{T} & E = T(M) \oplus E_1 & \xrightarrow{Q} & E \\ \uparrow{\scriptstyle \text{inclusion}} & & \downarrow{\scriptstyle P} & & \uparrow{\scriptstyle \text{inclusion}} \\ M & \xrightarrow{T} & T(M) & \xrightarrow{V} & M, \end{array}$$

where $E_1 = \ker P$ and $T(M) \oplus E_1$ is an algebraic direct sum, Q is defined by

$$Q = 1 - P + VP.$$

Hence, we have $Q|_{T(M)} = V$ and $Q|_{E_1}$ is the inclusion of E_1 into E. Now let $F = QT$, then

$$F|_M = \mathrm{id}_M, \qquad \operatorname{rank} F \le \operatorname{rank} T < \infty.$$

Let $J : T(M) \to E$ be the inclusion map and let $VP - P$ be the composition of the following maps:

$$E \xrightarrow{P} T(M) \xrightarrow{V-J} E.$$

We have

$$\|Q\|_{cb} \le 1 + \|VP - P\|_{cb}.$$

Consider the map

$$\mathrm{id}_{\mathscr{K}} \otimes (V - J) : \mathscr{K} \otimes_{\min} T(M) \to \mathscr{K} \otimes_{\min} E.$$

We have

$$\begin{aligned} \Big\| \sum_i a_i \otimes (e_i - Te_i) \Big\|_{\min} &\le \mu \sum_i \|a_i\| \le n\mu \sup_i \|a_i\| \\ &\le n\mu \Big\| \sum_i a_i \otimes e_i \Big\|_{\min} \\ &\le n\mu \|V\|_{cb} \Big\| \sum_i a_i \otimes Te_i \Big\|_{\min} \\ &\le \frac{n\mu}{1 - n\mu} \Big\| \sum_i a_i \otimes Te_i \Big\|_{\min}, \end{aligned}$$

which implies that $\|V - J\|_{cb} \le \frac{n\mu}{1-n\mu}$. Hence

$$\|Q\|_{cb} \le 1 + \frac{n^2\mu}{1 - n\mu},$$

when μ is small enough, we have $\|Q\|_{cb} \le (1+\varepsilon)^{1/2}$, consequently we have

$$\|F\|_{cb} \le K(1+\varepsilon) \quad \text{and} \quad \alpha(F) \le k(n)(1+\varepsilon). \qquad \square$$

We now list some properties about $(2, oh)$-summing norm (see [Pis96, pp. 88–89] for details).

(i) For any operator $u : OH \to E$ we have

$$\pi_{2,oh}(u) = \pi_2(u),$$

where $\pi_2(u)$ is the 2-summing norm of the operator u.

(ii) Any operator $u : E \to OH$ which is $(2, oh)$-summing is necessarily completely bounded and we have

$$\|u\|_{cb} \le \pi_{2,oh}(u).$$

(iii) Let M be any n-dimensional operator space, then there is an isomorphism $u : M \to OH_n$, such that

$$\pi_{2,oh}(u) = n^{1/2}, \qquad \left\|u^{-1}\right\|_{cb} = 1.$$

Let E, F be two operator spaces. For any linear map $T : E \to F$, we define a number $\delta(T) \in [0, \infty]$ as:

$$\delta(T) = \inf\left\{\|v\|_{cb}\pi_{2,oh}(w)\right\},$$

where the infimum runs over all possible factorizations of T through some operator Hilbert space $OH(I)$ as following:

(1)
$$\begin{tikzcd} & OH(I) \arrow[dr, "v"] & \\ E \arrow[ur, "w"] \arrow[rr, "T"] & & F. \end{tikzcd}$$

Proposition 4. *δ is a 1-injective operator ideal norm.*

Proof. If $T : E \to F$ has a factorization $T = vw$ as in (1) with

$$\|v\|_{cb}\pi_{2,oh}(w) < \infty,$$

then

$$\|T\|_{cb} \le \|v\|_{cb}\|w\|_{cb} \le \|v\|_{cb}\pi_{2,oh}(w),$$

and by definition of $\delta(T)$, we have

$$\|T\|_{cb} \le \delta(T).$$

It is easy to verify that if

$$S : \; L \xrightarrow{\alpha} E \xrightarrow{T} F \xrightarrow{\beta} G,$$

then we have

$$\delta(S) = \delta(\beta T\alpha) \le \|\beta\|_{cb}\delta(T)\|\alpha\|_{cb}.$$

Assume that $i: F \to G$ is a completely isometry, such that we have

$$\begin{tikzcd} & OH(I) \arrow[dr, "v"] & \\ E \arrow[ur, "w"] \arrow[r, "T"] & F \arrow[r, hook, "i"] & G. \end{tikzcd}$$

Let $\overline{R(w)}$ be the closure of the range of w in $OH(I)$, then there is some index set J such that we have an identification

$$\overline{R(w)} = OH(J)$$

completely isometrically. Now we define

$$\tilde{w} : E \to \overline{R(w)} = OH(J)$$

given by

$$\tilde{w}(e) = w(e), \quad \text{for any } e \in E.$$

Since the range of the operator $v|_{OH(J)}$ is contained in F, we may denote by $\tilde{v} : OH(J) \to F$ the mapping given by

$$\tilde{v}(x) = v(x), \quad \text{for any } x \in OH(J).$$

Since $i \circ T = v \circ w$, and by definitions of $\tilde{v}, \tilde{w}$, we obtain $T = \tilde{v} \circ \tilde{w}$. It follows that

$$\delta(T) \le \|\tilde{v}\|_{cb} \pi_{2,oh}(\tilde{w}) \le \|v\|_{cb} \pi_{2,oh}(w),$$

and thus $\delta(T) \le \delta(i \circ T)$. The inverse inequality has already been shown, thus δ is 1-injective.

We show now that δ satisfies the triangle inequality. Let $T_1, T_2 : E \to F$ be two operators with $\delta(T_1), \delta(T_2)$ finite. For any $\varepsilon > 0$, we can factorize T_i as

$$T_i : \ E \xrightarrow{w_i} OH(I_i) \xrightarrow{v_i} F,$$

such that

$$\|v_i\|_{cb} = \pi_{2,oh}(w_i) \le \sqrt{\delta(T_i) + \varepsilon}, \quad \text{for } i = 1, 2,$$

where I_1 and I_2 two disjoint index sets. We imbed $OH(I_i)$ canonically into $OH(I_1 \cup I_2) = OH(I_1) \oplus OH(I_2)$, and denote the inclusions by

$$J_i : OH(I_i) \to OH(I_1 \cup I_2).$$

Let P_i denote the orthogonal projection from $OH(I_1 \cup I_2)$ onto $OH(I_i)$ respectively. Then

$$T_1 + T_2 = v_1 w_1 + v_2 w_2 = AB,$$

where $B : E \to OH(I_1 \cup I_2)$ is defined by

$$B(x) = J_1 w_1(x) + J_2 w_2(x)$$

and $A : OH(I_1 \cup I_2) \to F$ is defined by

$$A(y) = v_1 J_1^{-1} P_1(y) + v_2 J_2^{-1} P_2(y).$$

For all finite sequences (x_i) in E, we have

$$\begin{aligned}\sum \big\|B(x_i)\big\|^2 &= \sum \big\|J_1 w_1(x_i) + J_2 w_2(x_i)\big\|^2\\ &= \sum \big\|w_1(x_i)\big\|^2 + \big\|w_2(x_i)\big\|^2\\ &\leq \big(\pi_{2,oh}(w_1)^2 + \pi_{2,oh}(w_2)^2\big)\Big\|\sum x_i \otimes \overline{x_i}\Big\|_{E\otimes_{\min}\overline{E}}\\ &\leq \big(\delta(T_1) + \delta(T_2) + 2\varepsilon\big)\Big\|\sum_i x_i \otimes \overline{x_i}\Big\|_{E\otimes_{\min}\overline{E}}.\end{aligned}$$

So we have

$$\pi_{2,oh}(B) \leq \sqrt{\delta(T_1) + \delta(T_2) + 2\varepsilon}.$$

For the c.b. norm of A, we will use the following description from [Pis96, Prop. 1.4]: assume that $(T_{i_1})_{i_1\in I_1}$ and $(T_{i_2})_{i_2\in I_2}$ are normalised orthogonal bases for $OH(I_1)$ and $OH(I_2)$ respectively, then

$$\begin{aligned}\|A\|_{cb}^2 = \sup\Big\{&\Big\|\sum_{i_1\in J_1} A(T_{i_1})\otimes\overline{A(T_{i_1})} + \sum_{i_2\in J_2} A(T_{i_2})\otimes\overline{A(T_{i_2})}\Big\|_{F\otimes_{\min}\overline{F}}:\\ &J_1\subset I_1, |J_1|<\infty; J_2\subset I_2, |J_2|<\infty\Big\}\\ \leq{}& \sup_{J_1\subset I_1,|J_1|<\infty}\Big\|\sum_{i_1\in J_1} v_1(T_{i_1})\otimes\overline{v_1(T_{i_1})}\Big\|_{F\otimes_{\min}\overline{F}}\\ &+\sup_{J_2\subset I_2,|J_2|<\infty}\Big\|\sum_{i_2\in J_2} v_2(T_{i_2})\otimes\overline{v_2(T_{i_2})}\Big\|_{F\otimes_{\min}\overline{F}}\\ ={}& \|v_1\|_{cb}^2 + \|v_2\|_{cb}^2 \leq \delta(T_1) + \delta(T_2) + 2\varepsilon.\end{aligned}$$

By the definition of δ, we have

$$\delta(T_1 + T_2) \leq \delta(T_1) + \delta(T_2) + 2\varepsilon$$

for any ε, hence we get

$$\delta(T_1 + T_2) \leq \delta(T_1) + \delta(T_2),$$

as desired. $\square$

Proposition 5. *For any finite rank operator $T: E\to F$, we have*

$$\delta(T) \leq \|T\|_{cb}\sqrt{\operatorname{rank} T}.$$

Proof. We can factorize T as following

$$E \xrightarrow{T} R(T) \xrightarrow{\mathrm{id}_{R(T)}} R(T) \hookrightarrow F.$$

The property (iii) of the $(2, oh)$-summing norm gives that

$$\delta(\mathrm{id}_{R(T)}) \leq \sqrt{\operatorname{rank} T}.$$

So we have

$$\delta(T) \le \|T\|_{cb}\sqrt{\operatorname{rank} T}. \qquad \square$$

REMARK 6. If E has the $(K, k(n))$-OUAP, then E has the

$$\big(K, Kk(n)^{1/2}\big)\text{-}\delta\text{-OUAP}$$

and also the $(K, Kk(n)^{1/2})$-δ^d-OUAP. The following lemma shows that in fact the OUAP and the δ-OUAP are equivalent.

LEMMA 7. *If E has $(K, k(n))$-δ-OUAP, then E has*

$$\left(\frac{1}{1-1/m}\big(1/m + K^{m+1}\big), m^{2/m}k(n)^{2+2/m}\right)\text{-OUAP},$$

for all integers $m > 1$.

REMARK 8. For simplification, here we replace the inequality $\delta(T) \le k(n)$ in the definition of $(K, k(n))$-δ-OUAP by the strict inequality $\delta(T) < k(n)$, which of course is not an essential change.

Proof of Lemma 7. Assume E has $(K, k(n))$-δ-OUAP. Fix an integer $m > 1$ and an n-dimensional subspace M of E. Then we can find a finite rank operator $T : E \to E$, such that

$$T|_M = \mathrm{id}_M, \qquad \|T\|_{cb} \le K \quad \text{and} \quad \delta(T) < k(n).$$

By the definition of $\delta(T)$, we can factorize T as:

$$\begin{tikzcd} & OH \arrow[dr, "A"] & \\ E \arrow[ur, "B"] \arrow[rr, "T"] & & E \end{tikzcd}$$

such that $\pi_{2,oh}(B) < k(n)$ and $\|A\|_{cb} \le 1$. Since

$$T^{m+1} = (AB)^{m+1} = A(BA)^m B,$$

and BA is an operator $OH \to OH$, we have

$$\begin{aligned} b_i\big(T^{m+1}\big) &\le \|A\|_{cb}\|B\|_{cb} b_i\big((BA)^m\big) \\ &= \|A\|_{cb}\|B\|_{cb} a_i\big((BA)^m\big) \\ &\le \pi_{2,oh}(B) a_i\big((BA)^m\big). \end{aligned}$$

The sequence $(b_i(T))_{i\ge1}$ is nonincreasing, so we have:

$$\begin{aligned} \sup_i i^{m/2} b_i\big(T^{m+1}\big) &\le \left(\sum_i b_i\big(T^{m+1}\big)^{2/m}\right)^{m/2} \\ &\le \pi_{2,oh}(B)\left(\sum_i a_i\big((BA)^m\big)^{2/m}\right)^{m/2} \end{aligned}$$

$$
\begin{aligned}
&= \pi_{2,oh}(B)\big\|(BA)^m\big\|_{S_{2/m}} \\
&\le \pi_{2,oh}(B)\|BA\|_{S_2}^m \\
&= \pi_{2,oh}(B)\pi_2(BA)^m \\
&= \pi_{2,oh}(B)\pi_{2,oh}(BA)^m \\
&\le \pi_{2,oh}(B)^{m+1} \\
&< k(n)^{m+1},
\end{aligned}
$$

where we have used the following facts:

- Horn's inequality (cf. e.g., [Gar07, Cor. 15.8.1]) for the operator BA, that is, $\|(BA)^m\|_{S_{2/m}} \le \|BA\|_{S_2}^m$;
- the 2-summing norm and the Hilbert–Schmidt norm coincide for operators between two Hilbert spaces: $\pi_2(BA) = \|BA\|_{S_2}$;
- the $(2,oh)$-summing norm and the 2-summing norm for operators from a Piser's operator Hilbert space OH to some other Banach space coincide: $\pi_2(BA) = \pi_{2,oh}(BA)$.

Let i_0 be the smallest integer strictly greater than $m^{2/m}k(n)^{2+2/m}$, then $i_0^{m/2} > mk(n)^{m+1}$, so we have $b_{i_0}(T^{m+1}) < 1/m$. By the definition of $b_i(T)$, there exists $S : E \to E$, such that

$$\operatorname{rank} S < i_0 \quad \text{and} \quad \big\|T^{m+1} - S\big\|_{cb} < 1/m.$$

This implies that

$$\operatorname{rank} S \le m^{2/m}k(n)^{2+2/m}$$

and that $\mathrm{id}_E - T^{m+1} + S$ is invertible with an inverse V, whose c.b. norm satisfies

$$\|V\|_{cb} < \frac{1}{1-1/m}.$$

Consequently, if we define

$$T_0 = VS : E \to E,$$

then

$$T_0|_M = \mathrm{id}_M \quad \text{and} \quad \operatorname{rank} T_0 \le \operatorname{rank} S \le m^{2/m}k(n)^{2+2/m}.$$

For the c.b. norm of T_0, we have

$$
\begin{aligned}
\|T_0\|_{cb} &\le \frac{1}{1-1/m}\big(\big\|S - T^{m+1}\big\|_{cb} + \big\|T^{m+1}\big\|_{cb}\big) \\
&\le \frac{1}{1-1/m}\big(1/m + K^{m+1}\big),
\end{aligned}
$$

and this is exactly what we want. □

We will use the following proposition (cf. [GH01]).

PROPOSITION 9. *For an operator space E, there are an infinite set I and a nontrivial ultrafilter $\mathscr{U}$ on I, a completely isometric embedding $j : E^{**} \to E^I/\mathscr{U}$, and $j(E^{**})$ is completely complemented in $E^I/\mathscr{U}$ (i.e., there is a completely contractive surjective projection $P : E^I/\mathscr{U} \to j(E^{**})$), such that we have the following commutative diagram*:

$$\begin{tikzcd} E \arrow[rr, "i"] \arrow[dr, "i_E"'] & & E^I/\mathscr{U} \\ & E^{**}, \arrow[ur, "j"'] & \end{tikzcd}$$

where i and i_E are canonical inclusions.

PROPOSITION 10. *The property for* (*operator spaces*) *of having the* $(K, k(n))$-OUAP *is stable under ultraproducts. In particular*, *if E has the* $(K, k(n))$-OUAP, *then so does E^{**}.*

Proof. Let $(E_i)_{i\in I}$ be a family of operator spaces having the $(K, k(n))$-OUAP, $\mathscr{U}$ an ultrafilter on I. We want to show that $\prod_{i\in I} E_i/\mathscr{U}$ has the $(K, k(n))$-OUAP. For any n-dimensional subspace

$$M \subset \prod_{i\in I} E_i/\mathscr{U},$$

choose an algebraic basis $x^1, \ldots, x^n$ of M, with $x^k = (x_i^k)_{\mathscr{U}}$. Let M_i be the linear span of x_i^k for $k = 1, \ldots, n$. Obviously, we have

$$M = \prod_{i\in I} M_i/\mathscr{U}.$$

Since each E_i has the $(K, k(n))$-OUAP, we can find $T_i : E_i \to E_i$ such that

$$\|T_i\|_{cb} \le K, \qquad \operatorname{rank} T_i \le k(n) \quad \text{and} \quad T_i|_{M_i} = \mathrm{id}_{M_i}.$$

Let

$$T = (T_i)_{\mathscr{U}} : \prod_{i\in I} E_i/\mathscr{U} \to \Pi_{i\in I} E_i/\mathscr{U},$$

then

$$\|T\|_{cb} \le \lim_{\mathscr{U}} \|T_i\|_{cb} \le K, \qquad \operatorname{rank} T \le k(n), \qquad T|_M = \mathrm{id}_M.$$

According to Proposition 9, since E^{**} is completely complemented in some ultrapower of E, it is easy to show E^{**} has the $(K, k(n))$-OUAP when E has it. $\square$

Proof of Theorem 1. Assume that E has the $(K, k(n))$-OUAP, then so does E^{**}. As in Remark 6, E^{**} has the $(K, Kk(n)^{1/2})$-δ^d-OUAP. If E^* is locally reflexive, and since δ is 1-injective, then we can apply Lemma 3 to show that E^* has

$$\big(K(1+\varepsilon), Kk(n)^{1/2}(1+\varepsilon)\big)\text{-}\delta\text{-OUAP},$$

for all $\varepsilon > 0$. Now by applying Lemma 7, we get the desired result. The case from E^* to E is more direct without the argument of ultraproducts. □

It seems to be interesting to ask whether we can drop the assumption on local reflexivity in Theorem 1. The following question seems to be open.

Open Problem 1. Does the OUAP property of E (resp. E^*) imply that E^* (resp. E) is locally reflexive?

The above open problem is related to the following result of Ozawa, see Section 4 of [Oza01].

Proposition 11 (Ozawa). *The CBAP property does not imply locally reflexivity.*

Remark 12. After writing this note, the author was told by Pisier that in fact the ideal norm δ defined here coincides with the completely 2-summing norm π_2° (cf. [Pis98], p. 62).

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Yanqi Qiu, Aix-Marseille Université, Centrale Marseille, CNRS, I2M, UMR7373, 39 Rue F. Juliot Curie 13453, Marseille, France

E-mail address: yqi.qiu@gmail.com