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ON INDEPENDENT RIGID CLASSES IN $H^*(\mathrm{WU}_q)$

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ABSTRACT. We introduce a family of rigid, linearly independent classes in $H^*(\mathrm{WU}_q)$. The family is different from the one studied by Hurder in (*Invent. Math.* **66** (1982), 313–323), and some of the classes are decomposed into products of elements of $H^*(\mathrm{WU}_q)$. We will show the independence by examining a complexification of Baker's example in (*Comment. Math. Helv.* **53** (1978), 334–363).

Introduction

One of classical questions in the study of secondary characteristic classes of foliations is that to determine the kernel and the image of the characteristic homomorphism. For example, if there are linearly dependent classes, then the relations belong to the kernel. If transversely holomorphic foliations are studied, these questions concern the mappings $H^*(\mathrm{WU}_q) \to H^*(B\Gamma_q^{\mathbb{C}})$ or $H^*(\mathrm{W}_q^{\mathbb{C}}) \to H^*(\overline{B\Gamma_q^{\mathbb{C}}})$. Some of related results are as follows. It is shown by Baum and Bott that all classes in $H^{2q+1}(\mathrm{WU}_q)$ and $H^{2q+1}(\mathrm{W}_q^{\mathbb{C}})$ admit continuous deformations [3]. These classes are said to be variable classes, which are also studied by Hurder [6]. On the other hand, classes rigid under deformations are called rigid classes. Hurder studied such classes in $H^*(\mathrm{W}_q^{\mathbb{C}})$ and gave a family of linearly independent classes [7]. In this paper, we give a family of rigid classes in $H^*(\mathrm{WU}_q)$ which are linearly independent and are different from Hurder's ones as elements of $H^*(\mathrm{W}_q^{\mathbb{C}})$. Some of these classes are shown to be products of a variable class and a rigid class. We will show the independence by studying a complexification of Baker's example [2] which will

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be constructed by using $\mathrm{SL}(k+n;\mathbb{C})$. The example is classical if $k=n=1$. Also, if $k=1$, then the non-triviality of the Godbillon–Vey class is shown in [1]. The case where $k=2$ and $n=3$ is studied in [4] by means of more direct calculations of differential forms.

1. Preliminaries

Throughout the paper, the Lie algebra of a Lie group is denoted by the corresponding fraktur letter. For example, $\mathfrak{g}$ denotes the Lie algebra of G. If $\mathfrak{g}$ is a complex Lie algebra, then the Lie algebra $\mathfrak{g}$ viewed as a real Lie algebra is denoted by $\mathfrak{g}_{\mathbb{R}}$. The coefficients of cohomology groups are chosen in $\mathbb{C}$ unless otherwise mentioned.

First, we introduce secondary characteristic classes.

Definition 1.1. Let $\mathbb{C}[v_1,\dots,v_q]$ be the polynomial ring generated by $v_1,\dots,v_q$, where the degree of v_j is set to be $2j$. We denote by I_q the ideal generated by monomials of degree greater than $2q$, and set $\mathbb{C}_q[v_1,\dots,v_q]=\mathbb{C}[v_1,\dots,v_q]/I_q$. We also define $\mathbb{C}_q[\bar{v}_1,\dots,\bar{v}_q]$ by replacing v_j by $\bar{v}_j$. We set

$$\begin{aligned}\mathrm{WU}_q&=\bigwedge[\widetilde{u}_1,\dots,\widetilde{u}_q]\otimes\mathbb{C}_q[v_1,\dots,v_q]\otimes\mathbb{C}_q[\bar{v}_1,\dots,\bar{v}_q],\\ \mathrm{W}_q&=\bigwedge[u_1,\dots,u_q]\otimes\mathbb{C}_q[v_1,\dots,v_q],\\ \mathrm{W}_q^{\mathbb{C}}&=\bigwedge[u_1,\dots,u_q,\bar{u}_1,\dots,\bar{u}_q]\otimes\mathbb{C}_q[v_1,\dots,v_q]\otimes\mathbb{C}_q[\bar{v}_1,\dots,\bar{v}_q].\end{aligned}$$

These algebras are equipped with derivations such that $d\widetilde{u}_i=v_i-\bar{v}_i$, $du_i=v_i$, $d\bar{u}_i=\bar{v}_i$, and $dv_i=d\bar{v}_i=0$. The degree of $\widetilde{u}_i$, u_i and $\bar{u}_i$ are set to be $2i-1$. We define $\overline{\mathrm{W}_q}$ by replacing u_i and v_j by $\bar{u}_i$ and $\bar{v}_j$ in W_q. Then, $\mathrm{W}_q^{\mathbb{C}}=\mathrm{W}_q\wedge\overline{\mathrm{W}_q}$.

By abuse of notations, elements of $H^*(\mathrm{WU}_q)$, etc., are denoted by their representatives. Since $H^*(\mathrm{W}_q^{\mathbb{C}})$ is isomorphic to $H^*(\mathrm{W}_q)\otimes H^*(\overline{\mathrm{W}_q})$, there is a natural inclusion from $H^*(\mathrm{W}_q)$ to $H^*(\mathrm{W}_q^{\mathbb{C}})$. There is also a natural mapping from $H^*(\mathrm{WU}_q)$ to $H^*(\mathrm{W}_q^{\mathbb{C}})$ which maps $\widetilde{u}_i$ to $u_i-\bar{u}_i$, v_i to v_i and $\bar{v}_i$ to $\bar{v}_i$, respectively. Indeed, this mapping corresponds to the natural mapping from $\overline{B\Gamma_q^{\mathbb{C}}}$ to $B\Gamma_q^{\mathbb{C}}$, which is a part of the homotopy fibration $\overline{B\Gamma_q^{\mathbb{C}}}\to B\Gamma_q^{\mathbb{C}}\to \mathrm{BGL}(q;\mathbb{C})$ and also is the classifying map of the $\Gamma_q^{\mathbb{C}}$-structure of $\overline{B\Gamma_q^{\mathbb{C}}}$, namely, the map which forgets the triviality of the complex normal bundle.

The following result is classical.

Theorem 1.2. *Let $B\Gamma_q^{\mathbb{C}}$ be the classifying space of transversely holomorphic foliations of complex codimension q, and let $\overline{B\Gamma_q^{\mathbb{C}}}$ the classifying space of transversely holomorphic foliations of complex codimension q with trivialized complex normal bundles. Then, there are well-defined homomorphisms $H^*(\mathrm{WU}_q)\to H^*(B\Gamma_q^{\mathbb{C}})$ and $H^*(\mathrm{W}_q^{\mathbb{C}})\to H^*(\overline{B\Gamma_q^{\mathbb{C}}})$.*

The above homomorphisms are called the characteristic homomorphisms. Let M be a manifold and $\mathcal{F}$ a transversely holomorphic foliation of M, of complex codimension q. Then, the classifying map induces a homomorphism from $H^*(\mathrm{WU}_q)$ to $H^*(M)$. If the complex normal bundle of $\mathcal{F}$ is trivial, then by fixing the homotopy type of a trivialization, we obtain a homomorphism from $H^*(\mathrm{W}_q^{\mathbb{C}})$ to $H^*(M)$.

If smooth (real) foliations with trivialized normal bundles are considered, then we can choose $\mathbb{R}$ as coefficients. We denote by $\mathrm{W}_q^{\mathbb{R}}$ the algebra W_q but the coefficients are chosen in $\mathbb{R}$. In order to distinguish $\mathrm{W}_q^{\mathbb{R}}$ from W_q, we denote u_i by h_i and v_j by c_j when we consider elements of $\mathrm{W}_q^{\mathbb{R}}$. Then, the characteristic homomorphism is defined as a mapping from $H^*(\mathrm{W}_q^{\mathbb{R}})$ to $H^*(\overline{B\Gamma_q};\mathbb{R})$, where $\overline{B\Gamma_q}$ denotes the classifying space of foliations of real codimension q with trivialized normal bundles. Elements of $H^*(\mathrm{WU}_q)$, $H^*(\mathrm{W}_q^{\mathbb{C}})$, $H^*(\mathrm{W}_q)$ and $H^*(\mathrm{W}_q^{\mathbb{R}})$ or their image under the characteristic homomorphisms which involve $\widetilde{u}_i$, u_i, $\bar{u}_i$ or h_i are called secondary characteristic classes.

There is a natural mapping, say ρ, from WU_{q+1} to WU_q such that if $i \leq q$ then $\rho(\widetilde{u}_i)=\widetilde{u}_i$, $\rho(v_i)=v_i$, $\rho(\bar{v}_i)=\bar{v}_i$ and if $i=q+1$ then $\rho(\widetilde{u}_i)=0$, $\rho(v_i)=0$ and $\rho(\bar{v}_i)=0$, respectively. The mapping ρ induces a homomorphism $\rho_*: H^*(\mathrm{WU}_{q+1}) \to H^*(\mathrm{WU}_q)$. The classes of $H^*(\mathrm{WU}_q)$ in the image of ρ_* are called rigid classes. Indeed, it is known that rigid classes are rigid under deformations of foliations. Similar homomorphisms from W_{q+1} to W_q, from $\mathrm{W}_{q+1}^{\mathbb{C}}$ to $\mathrm{W}_q^{\mathbb{C}}$ and from $\mathrm{W}_{q+1}^{\mathbb{R}} \to \mathrm{W}_q^{\mathbb{R}}$ are also defined. The classes in the image are also called rigid classes and are rigid under deformations. On the other hand, a class $\omega \in H^*(\mathrm{WU}_q)$ is said to be variable if ω varies continuously with respect to a family of foliations.

Let G be a Lie group. Let H and K be Lie subgroups of G such that $H \supset K$. We assume that G/H admits a complex structure invariant under the left G-action. We also assume that there are an Ad_K-invariant splitting of $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ and an Ad_K-invariant Hermitian metric on $\mathfrak{g}/\mathfrak{h}$. We assume in addition that there is a discrete subgroup Γ of G such that $M=M_K=\Gamma\backslash G/K$ is a closed manifold. We denote by $\mathcal{F}_K$ the transversely holomorphic foliation of M induced by the left cosets of H. Note that if $K \subset K' \subset H$ then there is a natural projection, say π, from M_K to $M_{K'}$ and that $\mathcal{F}_K=\pi^*\mathcal{F}_{K'}$.

If M_K and $\mathcal{F}_K$ are as above, then the characteristic homomorphism $\chi: H^*(\mathrm{WU}_q) \to H^*(M_K)$ is well-studied. We state some results without proofs (see [1] and references therein).

Theorem 1.3. *Suppose that K is connected. If we denote by $\varphi: H^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{k}) \to H^*(M_K)$ the natural mapping, then there is a natural mapping $\chi_K: H^*(\mathrm{WU}_q) \to H^*(\mathfrak{g}_{\mathbb{R}},\mathfrak{k})$ such that $\chi=\varphi\circ\chi_K$ holds.*

Theorem 1.4. *Let G be a complex semisimple group. Let K be a compact connected subgroup and H a complex, closed, connected subgroup of G. Let $\mathfrak{g}_0$ be a compact real form of $\mathfrak{g}$ and G_0 the compact real form of G with Lie* *algebra* $\mathfrak{g}_0$. *Then, there is an isomorphism between* $H^*(\mathfrak{g}_{\mathbb{R}},\mathfrak{k})$ *and* $H^*(G_0)\otimes H^*(G_0/K)$.

Theorem 1.4 is shown by constructing an isomorphism of Lie algebras $\kappa:\mathfrak{g}\oplus\mathfrak{g}\to\mathfrak{g}_{\mathbb{R}}\otimes\mathbb{C}$ and a diffeomorphism $\tau: G_0\times(G_0/K)\to(G_0\times G_0)/K$. We set $f_K=\tau^*\circ(\kappa^*)^{-1}: C^*(\mathfrak{g}_{\mathbb{R}},\mathfrak{k})\to\Omega^*(G_0\times(G_0/K))$, where $C^*(\mathfrak{g}_{\mathbb{R}},\mathfrak{k})$ denotes the set of cochains on $\mathfrak{g}_{\mathbb{R}}$ relative to $\mathfrak{k}$ with coefficients in $\mathbb{C}$, and $\Omega(G_0\times(G_0/K))$ denotes the set of $\mathbb{C}$-valued differential forms on $G_0\times(G_0/K)$, respectively. Then, f_K induces an isomorphism $f_{K*}: H^*(\mathfrak{g}_{\mathbb{R}},\mathfrak{k})\to H^*(\mathfrak{g}_0)\otimes H^*(\mathfrak{g}_0,\mathfrak{k})$.

Let θ be the Bott connection as in [1, Lemma 3.1.11] or the unitary connection as in [1, Lemma 3.1.13], which is a $\mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$-valued 1-form on $G\times\mathrm{GL}(\mathfrak{g}/\mathfrak{h})$. Let θ^c be the complex conjugate of θ with respect to $\mathfrak{g}_0$, namely, θ^c is a $\mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$-valued 1-form on $G\times\mathrm{GL}(\mathfrak{g}/\mathfrak{h})$ defined as follows. First, let $\theta|_{\mathfrak{g}_0\oplus\mathfrak{gl}(\mathfrak{g}/\mathfrak{h})}=\alpha+\sqrt{-1}\beta$ be the decomposition of $\theta|_{\mathfrak{g}_0\oplus\mathfrak{gl}(\mathfrak{g}/\mathfrak{h})}$ into real and imaginary parts, namely, the both α and β are $\mathbb{R}$-valued. Then, we set $\theta^c=\alpha-\sqrt{-1}\beta$. By [1, Lemma 3.2.6] and [1, Proposition 3.2.7], we have the following.

PROPOSITION 1.5. *Let* θ *and* θ^c *be as above. If we denote by* p_i *the projection from* $\mathfrak{g}\oplus\mathfrak{g}$ *to* $\mathfrak{g}$ *as the* i*-th factor, then* $f_K\theta=p_1^*\theta+p_2^*\theta$ *and* $f_K\bar{\theta}=p_2^*\theta^c$.

2. Independent classes of $H^*(\mathrm{WU}_q)$

We study the following example, which is a complexification of Baker's example [2, Example 1(a)] (see also [4] for the case where $n=3$ and $k=2$). If $n=k=1$, then it is an example of Roussarie [5].

EXAMPLE 2.1. Let $G=\mathrm{SL}(k+n;\mathbb{C})$, where $n>k>0$ or $n=k=1$, $H=\{(a_{ij})\in G\mid a_{ij}=0 \text{ if } i>k \text{ and } j\le k\}$, $K=\mathrm{S}(\mathrm{U}(k)\times\mathrm{U}(n))$ and $T=T^{k+n-1}$ the maximal torus in G realized as diagonal matrices. Then, the left cosets of H induce transversely holomorphic foliations of complex codimension kn on $\Gamma\backslash G/T$ and $\Gamma\backslash G/K$, where Γ is a discrete subgroup of G such that $\Gamma\backslash G/K$ is a closed manifold.

We have a complexification of [2, Theorem 5.3] as follows. If $I=\{i_1,\dots,i_l\}$ then we set $\widetilde{u}_I=\widetilde{u}_{i_1}\cdots\widetilde{u}_{i_l}$. We denote $\widetilde{u}_I$ also by $\widetilde{u}_{i_1,\dots,i_l}$. If $I=\varnothing$ then we set $\widetilde{u}_I=1$ and regard $i_1=+\infty$. We define h_I in a similar way.

THEOREM 2.2. *Let* $I=\{i_1,\dots,i_l\}$ *and suppose that* $k<i_1<\cdots<i_l\le n$. *The classes of the form* $\widetilde{u}_{1,\dots,k}\widetilde{u}_I v_1^{kn}\bar{v}_1^{kn}$ *are non-trivial and linearly independent in* $H^*(\Gamma\backslash\mathrm{SL}(k+n;\mathbb{C})/T)$. *These classes are rigid classes, namely, rigid under deformations of foliations if* $k=1$ *and* $i_1>2$.

Proof. We retain the above notations. We denote by E_{ij} the element of $\mathfrak{gl}(k+n;\mathbb{C})$ such that the (i,j)-entry is 1 and the others are 0. Let $\{\omega_{ij}\}$ be the dual basis of $\{E_{ij}\}$ for $\mathfrak{gl}(k+n;\mathbb{C})$. We define a splitting $\sigma:\mathfrak{g}/\mathfrak{h}\to\mathfrak{g}$ by setting $\sigma([E_{ij}]) = E_{ij}$, where $j \leq k < i$. If we set $\omega = {}^t(\omega_{k+1,1}, \dots, \omega_{k+n,1}, \dots, \omega_{k+1,k}, \dots, \omega_{k+n,k})$, then $\ker \omega = \mathfrak{h}$. We set

$$\theta_0 = \begin{pmatrix} \omega_{k+1,k+1} & \cdots & \omega_{k+1,k+n} \\ \vdots & & \vdots \\ \omega_{k+n,k+1} & \cdots & \omega_{k+n,k+n} \end{pmatrix}, \qquad I_{ij} = E_{ij} \otimes I_n = \left(\begin{array}{c|c|c} O_n & \cdots & O_n \\ \hline \vdots & I_n & \vdots \\ \hline O_n & \cdots & O_n \end{array}\right)$$

for $1 \leq i, j \leq k$, and $\theta = \overbrace{\theta_0 \oplus \cdots \oplus \theta_0}^{k\text{-times}} - \sum_{1 \leq i,j \leq k} \omega_{ij} I_{ij}$. Then, we have $d\omega = -\theta \wedge \omega$. As in [1, Example 3.3.6], $\theta^u = \frac{\theta - {}^t\bar{\theta}}{2}$ induces a unitary connection with respect to the Hermitian metric given by $([A],[B]) \mapsto \mathrm{tr}({}^tA\bar{B})$. Hence, we can compute complex secondary classes as follows. We denote by $v_i(\theta)$ the i-th Chern form calculated by using a connection θ. The Chern–Simons form calculated by the i-th Chern polynomial and connections θ_1, θ_2 is denoted by $\Delta_{c_i}(\theta_1, \theta_2)$. Let $G_0 = \mathrm{SU}(k+n)$ and f_K as above. Then, by Proposition 1.5, $f_K v_i(\theta) = v_i(p_1^*\theta + p_2^*\theta)$ and $f_K \bar{v}_i(\theta) = f_K v_i(\bar{\theta}) = v_i(p_2^*\theta^c)$. Similarly, $f_K \widetilde{u}_i(\theta, \theta^u) = \Delta_{c_i}(p_1^*\theta + p_2^*\theta, p_1^*\theta^u + p_2^*\theta^u) - \Delta_{c_i}(p_2^*\theta^c, p_2^*\theta^{uc})$. We now set $X_{ij} = E_{ij} - E_{ji}$, $Y_{ij} = \sqrt{-1}(E_{ij} + E_{ji})$ and $K_k = \sqrt{-1}(E_{11} - E_{kk})$. Let β_{ij}, γ_{ij} and α_k be their duals. If $i \neq j$, then $\omega_{ij} = \beta_{ij} + \sqrt{-1}\gamma_{ij}$. We have $(p_2^*\theta^c)_{kk} = \sqrt{-1}(-\alpha_1 + \alpha_k)$ and $(p_2^*\theta^c)_{ij} = p_2^*\beta_{ij} - \sqrt{-1}p_2^*\gamma_{ij}$, where $k > 1$ and $i \neq j$. On the other hand, we have

$$\begin{aligned} v_1(\theta) &= \frac{\sqrt{-1}}{2\pi} d\left(k \sum_{j=k+1}^{k+n} \omega_{jj} - n \sum_{j=1}^{k} \omega_{jj}\right) = \frac{-(k+n)\sqrt{-1}}{2\pi} \sum_{j=1}^{k} d\omega_{jj} \\ &= \frac{(k+n)\sqrt{-1}}{2\pi} \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq n}} \omega_{i,j+n} \wedge \omega_{j+n,i}. \end{aligned}$$

As $\beta_{ji} = -\beta_{ij}$ and $\gamma_{ji} = \gamma_{ij}$, we have

$$f_K \bar{v}_1(\theta) = \frac{(k+n)}{\pi} \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq n}} \beta_{i,j+n} \wedge \gamma_{i,j+n}.$$

Hence,

$$f_K \bar{v}_1(\theta)^{kn} = \left(\frac{k+n}{\pi}\right)^{nk} \bigwedge_{\substack{1 \leq i \leq k \\ 1 \leq j \leq n}} (\beta_{i,j+n} \wedge \gamma_{i,j+n}).$$

The right-hand side is equal to $C \,\mathrm{vol}_N$, where C is a non-zero real number and vol_N denotes the volume form of $N = \mathrm{SU}(k+n)/\mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(n))$. On the other hand, $p_1^*\theta$ (resp. $p_1^*\theta^u$) is the complexification of the real Bott connection (resp. metric connection with respect to the metric obtained by restriction to the real part) for the foliations modeled on $\mathrm{SL}(k+n;\mathbb{R})$ with leaves $\{gH_{\mathbb{R}}\}_{g\in\mathrm{SL}(k+n;\mathbb{R})}$, where $H_{\mathbb{R}}=H\cap\mathrm{SL}(k+n;\mathbb{R})$. Let $\chi_{K\mathbb{R}}$ be the characteristic homomorphism from $H^*(\mathrm{W}_q^{\mathbb{R}})$ to $H^*(\mathfrak{sl}(k+n;\mathbb{R});\mathbb{R})$ obtained in the same way as in Theorem 1.3 (see [2] for details). We regard $\chi_{K\mathbb{R}}$ as a homomorphism from $H^*(\mathrm{W}_q^{\mathbb{R}}\otimes\mathbb{C})$ to $H^*(G_0)$ by complexification and denote it again by $\chi_{K\mathbb{R}}$. Then $f_{K*}\chi_K(\omega)=C\chi_{K\mathbb{R}}(\omega_{\mathbb{R}})\otimes\mathrm{vol}_N$, where $\omega=\widetilde{u}_{1,\dots,k}\widetilde{u}_{i_1,\dots,i_l}v_1^{kn}\bar{v}_1^{kn}$, $\omega_{\mathbb{R}}=h_{1,\dots,k}h_{i_1,\dots,i_l}c_1^{kn}\in\mathrm{W}_q^{\mathbb{R}}$ and C is the constant as above. If K is replaced by T, then there is a natural projection $\pi:\varGamma\backslash G/T\to\varGamma\backslash G/K$ and $\mathcal{F}_T=\pi^*\mathcal{F}_K$. If we denote by π^* the natural mapping from $H^*(\mathfrak{sl}(k+n;\mathbb{C})_{\mathbb{R}},\mathfrak{k})$ to $H^*(\mathfrak{sl}(k+n;\mathbb{C})_{\mathbb{R}},\mathfrak{t})$ by abuse of notations, then $\chi_T(\omega)=\pi^*(\chi_K(\omega))$ by the naturality. If we denote by π' the projection from G_0/T to G_0/K, then $f_{T*}\circ\chi_T\circ\pi^*=(\mathrm{id}\otimes\pi'^*)\circ f_{K*}\circ\chi_K$. It is well-known that π'^* is injective on the subalgebra generated by the Chern classes so that $\chi_T(\omega)$ is also non-trivial. Therefore, the first part follows from Baker's theorem [2, Theorem 5.3]. Finally, the cochain $\widetilde{u}_{1,\dots,k}\widetilde{u}_I(v_1^{kn+1}\bar{v}_1^{kn-1}+v_1^{kn}\bar{v}_1^{kn}+v_1^{kn-1}\bar{v}_1^{kn+1})$ is mapped to the cocycle $\widetilde{u}_{1,\dots,k}\widetilde{u}_Iv_1^{kn}\bar{v}_1^{kn}$ under $\rho\colon\mathrm{WU}_{q+1}\to\mathrm{WU}_q$, and is closed if $k=1$ and $i_1>2$. Hence, $\widetilde{u}_{1,\dots,k}\widetilde{u}_Iv_1^{kn}\bar{v}_1^{kn}$ represents a rigid class if $k=1$ and $i_1>2$. □

We can also show the following by an essentially the same argument as above, using [8, Theorem 5.37] (see also [2, Theorem 5.9]) instead of [2, Theorem 5.3].

Theorem 2.3. *Let $k=1$ in Example 2.1 and $I=\{i_1,\dots,i_l\}$, and let $q(v_1,\dots,v_n)$ and $r(\bar{v}_1,\dots,\bar{v}_n)$ be monomials of degree $2i$ and $2n$, respectively. Then, we have the following.*

(1) *We fix $r(\bar{v}_1,\dots,\bar{v}_n)$ and denote it by $\bar{r}$. Then, the classes of the form $\widetilde{u}_{n-i+1}\widetilde{u}_Iq(v_1,\dots,v_n)\bar{r}$, $n-i+1<i_1<\cdots<i_l\le n$, are linearly independent in $H^*(\varGamma\backslash\mathrm{SL}(n+1;\mathbb{C})/T)$. These classes are multiplies of the classes $\widetilde{u}_1\widetilde{u}_Iv_1^n\bar{v}_1^n$ in $H^*(\varGamma\backslash\mathrm{SL}(n+1;\mathbb{C})/T)$.*
(2) *The classes of the form $\widetilde{u}_Iq(v_1,\dots,v_n)r(\bar{v}_1,\dots,\bar{v}_n)$, where $n-i+1<i_1<\cdots<i_l\le n$, are trivial in $H^*(\varGamma\backslash\mathrm{SL}(n+1;\mathbb{C})/T)$.*

Note that the classes in (2) are rigid ones. Some of the classes in (1) are also rigid. Suppose that $q(v_1,\dots,v_n)$ is of the form $v_jq'(v_1,\dots,v_n)$, where $n-i+1+j<i_1<\cdots<i_l$. Suppose also that $\bar{r}$ is of the form $\bar{v}_kr'(\bar{v}_1,\dots,\bar{v}_n)$, where $k+1<i_1$. In the both cases, we regard $i_1=+\infty$ and the conditions are satisfied if $I=\varnothing$. If we set $\omega=\widetilde{u}_j\widetilde{u}_Iq'(v_1,\dots,v_n)\bar{v}_{n-i+1}\bar{r}+\widetilde{u}_{n-i+1}\widetilde{u}_Iq(v_1,\dots,v_n)\bar{r}+\widetilde{u}_k\widetilde{u}_Iv_{n-i+1}q(v_1,\dots,v_n)r'(\bar{v}_1,\dots,\bar{v}_n)$, then ω determines an element of $H^*(\mathrm{WU}_{n+1})$ which is mapped to $\widetilde{u}_{n-i+1}\widetilde{u}_Iq(v_1,\dots,v_n)\bar{r}$. The condition seems artificial but classes of the form $\widetilde{u}_1\widetilde{u}_Iv_1^n\bar{v}_1^n$ satisfy it if $i_1>2$. We also remark that $\sqrt{-1}\widetilde{u}_1v_1^n\bar{v}_1^n$ represents the Godbillon–Vey class.

We continue to study Example 2.1 with $k=1$ and $n=q$. We assume that $1<i_1<\cdots<i_l\le q$. If $l>1$, then we also assume that $i_1+i_2>q+1$. We set

$$\begin{aligned}\xi_q &= \widetilde{u}_1\bigl(v_1^q+v_1^{q-1}\bar{v}_1+\cdots+\bar{v}_1^q\bigr),\\ \eta=\eta_{i_1,\dots,i_l,q} &= \widetilde{u}_{i_1,\dots,i_l}v_1^q+\sum_{\substack{1\le j\le q\\ 1\le m\le l}}(-1)^{m-1}\widetilde{u}_{1,i_1,\dots,\widehat{i_m},\dots,i_l}v_1^{q-j}\bar{v}_1^{j-1}\bar{v}_{i_m}.\end{aligned}$$

We have $\xi_q\eta=\widetilde{u}_{1,i_1,\dots,i_l}v_1^q\bar{v}_1^q$. The class ξ_q is the imaginary part of the Bott class and is a variable class. On the other hand, η is a rigid class if $i_1+i_2>q+2$. We do not know if η is a variable class if $i_1+i_2=q+2$. The cocycle η is closed as follows. We have

$$\begin{aligned}d\eta &= \sum_{m=1}^{l}(-1)^{m-1}\widetilde{u}_{i_1,\dots,\widehat{i_m},\dots,i_l}v_1^q(v_{i_m}-\bar{v}_{i_m})\\ &\quad+\sum_{m=1}^{l}(-1)^{m-1}\widetilde{u}_{i_1,\dots,\widehat{i_m},\dots,i_l}\bigl(v_1^q-\bar{v}_1^q\bigr)\bar{v}_{i_m}\\ &\quad\pm\sum_{\substack{1\le j\le q\\ m,n}}\widetilde{u}_{1,i_1,\dots,\widehat{i_n},\dots,\widehat{i_m},\dots,i_l}v_1^{q-j}\bar{v}_1^{j-1}\bar{v}_{i_m}(v_{i_n}-\bar{v}_{i_n})\\ &= -\sum_{m=1}^{l}(-1)^{m-1}\widetilde{u}_{i_1,\dots,\widehat{i_m},\dots,i_l}v_1^q\bar{v}_{i_m}+\sum_{m=1}^{l}(-1)^{m-1}\widetilde{u}_{i_1,\dots,\widehat{i_m},\dots,i_l}v_1^q\bar{v}_{i_m}\\ &\quad\pm\sum_{\substack{1\le j\le q\\ m,n}}\widetilde{u}_{1,i_1,\dots,\widehat{i_n},\dots,\widehat{i_m},\dots,i_l}v_1^{q-j}\bar{v}_1^{j-1}\bar{v}_{i_m}(v_{i_n}-\bar{v}_{i_n})\end{aligned}$$

in WU_{q+1} because $i_1>1$. If $l=1$, then the last term does not appear so that $d\eta=0$ in WU_{q+1}. If $l>1$, then we have $(q-j)+i_n+(j-1)+i_m\ge i_1+i_2+q$. Hence, η is closed in WU_{q+1} if $i_1+i_2>q+2$ and in WU_q if $i_1+i_2>q+1$.

By applying Theorem 2.2 with $k=1$ and $n=q$, we deduce the following.

Corollary 2.4. *The product of ξ_q with $\eta_{i_1,\dots,i_l,q}$, where $1<i_1<\cdots<i_l\le q$ and either $l=1$ or $l>1$ and $i_1+i_2>q+1$, are nontrivial in $H^*(\Gamma\backslash\mathrm{SL}(q+1;\mathbb{C})/T;\mathbb{C})$. The family $\{\xi_q,\eta_{i_1,\dots,i_l,q}\}$ consists of linearly independent classes. The class ξ_q is variable. The classes $\eta_{i_1,\dots,i_l,q}$ with $l=1$ or $i_1+i_2>q+2$ are rigid.*

Proof. If we take the product with ξ_q, then the family $\{\eta_{i_1,\dots,i_l,q}\}$, where $1<k\le q, 1<i_1<\cdots<i_l\le q,\ l>1$ and $i_1+i_2>q+1$, is mapped to a family which consists of linearly independent classes by Theorem 2.2. Hence, $\{\xi_q,\eta_{i_1,\dots,i_l,q}\}$ consists of linearly independent classes. □

Remark 2.5. Let $k=2$ and $n=3$ in Example 2.1, and ξ_6 and $\eta_{2,6}$ as above. Then $\xi_6\eta_{2,6}=\widetilde{u}_1\widetilde{u}_2v_1^6\bar{v}_1^6$, which is non-trivial by Theorem 2.2. Therefore, ξ_6 and $\eta_{2,6}$ are non-trivial. On the other hand, if we set $\mu=\widetilde{u}_1\widetilde{u}_2(v_1^6\bar{v}_1^5+v_1^5\bar{v}_1^6)$, then μ is closed and $\mu v_1 = \widetilde{u}_1\widetilde{u}_2 v_1^6\bar{v}_1^6$. Hence, μ and v_1 are also non-trivial. Actually v_1^6 is a generator of $H^{12}(\mathrm{SU}(5)/\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(3)))$. On the other hand, by examining the degree of generators of $H^*(\mathrm{SU}(5))\otimes H^*(\mathrm{SU}(5)/\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(3)))$, we see that the Godbillon–Vey class $\widetilde{u}_1 v_1^6\bar{v}_1^6$ is trivial. We have $\widetilde{u}_1 v_1^6\bar{v}_1^6 = \xi_6 v_1^6$. Hence, unlike the above cases, a product of non-trivial classes yields a trivial class.

Let $q = 2q'-2$ with $q' > 1$. Let $\mathscr{R}$ be a subset of $H^*(\mathrm{W}_q)\subset H^*(\mathrm{W}_q^{\mathbb{C}})$ defined by $\mathscr{R} = \{u_{2,i_2,\ldots,i_s}v_2^{q'-1} \mid 2 < i_2 < \cdots < i_s \le q\} \cup \{u_{q',i_2,\ldots,i_s}v_{q'} \mid q' < i_2 < \cdots < i_s \le q\}$. The family $\mathscr{R}$ is a set of rigid secondary classes studied by Hurder in [7]. Indeed, $\mathscr{R}$ consists of linearly independent secondary classes. The families in Theorems 2.2 and 2.3 differ from $\mathscr{R}$. At present, we know only the following classes α_q and β_q as elements of WU_q which belongs to the subspace of $H^*(\mathrm{W}_q^{\mathbb{C}})$ spanned by $\mathscr{R}$ and $\overline{\mathscr{R}}$. Let $\alpha_{q'} = \widetilde{u}_2(v_2^{q'-1}+\cdots+\bar{v}_2^{q'-1})$ and $\beta_{q'} = \widetilde{u}_{q'}(v_{q'}+\bar{v}_{q'})$. Note that $\alpha_2=\beta_2$. These classes are rigid classes also in $H^*(\mathrm{WU}_q)$. We have the following.

Corollary 2.6. *α_2 is non-trivial in $H^7(\mathit{\Gamma}\backslash\mathrm{SL}(3;\mathbb{C})/T;\mathbb{C})$. If $q'>2$, then $\alpha_{q'}$ and $\beta_{q'}$ are linearly independent in $H^{4q'-1}(\mathit{\Gamma}\backslash\mathrm{SL}(q+1;\mathbb{C})/T;\mathbb{C})$.*

Proof. Let $\gamma = \widetilde{u}_1 v_2^{q'-1} + \widetilde{u}_2(v_2^{q'-2}+\cdots+\bar{v}_2^{q'-2})\bar{v}_1$. Then, γ is closed and $\gamma\alpha_{q'} = \widetilde{u}_1\widetilde{u}_2 v_2^{q'-1}\bar{v}_2^{q'-1}$. This class is non-trivial in $H^{4q+4}(\mathit{\Gamma}\backslash\mathrm{SL}(q+1;\mathbb{C})/T;\mathbb{C})$ by Theorem 2.3. On the other hand, we have $\alpha_{q'}\beta_{q'} = \widetilde{u}_2\widetilde{u}_{q'}(v_2^{q'-1}\bar{v}_{q'} + v_{q'}\bar{v}_2^{q'-1})$. Let $\delta = \widetilde{u}_1(v_1^{2q'-2}\bar{v}_1^{q'-2}+\cdots+v_1^{q'-2}\bar{v}_1^{2q'-2})$. If $q'>2$, then $\delta\alpha_{q'}=0$ because $(q'-2+2q'-2)+2(q'-1) = 5q'-6 > 2q = 4q'-4$. On the other hand, $\delta\beta_{q'} = \widetilde{u}_1\widetilde{u}_{q'}(v_1^{2q'-2}\bar{v}_1^{q'-2}\bar{v}_{q'} + v_1^{q'-2}v_{q'}\bar{v}_1^{2q'-2})$. By Theorem 2.3, there is a non-zero real number C such that $\widetilde{u}_1\widetilde{u}_{q'}v_1^{2q'-2}\bar{v}_1^{q'-2}\bar{v}_{q'} = \widetilde{u}_1\widetilde{u}_{q'}v_1^{q'-2}v_{q'}\bar{v}_1^{2q'-2} = Ch_1h_{q'}c_1^q\otimes c_1^q$. Hence, $\delta\beta_{q'}$ is non-trivial. $\square$

Remark 2.7. If $q'>3$, then $\alpha_{q'}\beta_{q'}v_{q'-3} = \widetilde{u}_2\widetilde{u}_{q'}v_{q'-3}v_{q'}\bar{v}_2^{q'-1}$, which is non-trivial in $H^*(\mathit{\Gamma}\backslash\mathrm{SL}(q+1;\mathbb{C})/T;\mathbb{C})$ by Theorem 2.3.

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