

## ON THE BEHAVIOR OF THE COVARIANCE MATRICES IN A MULTIVARIATE CENTRAL LIMIT THEOREM UNDER SOME MIXING CONDITIONS

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ABSTRACT. In a paper that appeared in 2010, C. Tone proved a multivariate central limit theorem for some strictly stationary random fields of random vectors satisfying certain mixing conditions. The “normalization” of a given “partial sum” (or “block sum”) involved matrix multiplication by a “standard  $-1/2$  power” of its covariance matrix (a symmetric, positive definite matrix), and the limiting multivariate normal distribution had the identity matrix as its covariance matrix. The mixing assumptions in Tone’s result implicitly imposed an upper bound on the ratios of the largest to the smallest eigenvalues in the covariance matrices of the partial sums. The purpose of this note is to show that in Tone’s result, for the entire collection of the covariance matrices of the partial sums, there is essentially no other restriction on the relative magnitudes of the eigenvalues or on the (orthogonal) directions of the corresponding eigenvectors. For simplicity, the example given in this note will involve just random sequences, not the broader context of random fields.

### 1. Introduction

A multivariate central limit theorem was proved by C. Tone [26] for some strictly stationary random fields of random vectors satisfying certain mixing conditions. As in a somewhat related result in [6] under different dependence assumptions, the “normalization” of a given “partial sum” (or “block sum”) involved matrix multiplication by a “standard  $-1/2$  power” of its covariance matrix (a symmetric, positive definite matrix), and the limiting multivariate normal distribution had the identity matrix as its covariance matrix. (More

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on that below.) The mixing assumptions in Tone's [26] result implicitly imposed an upper bound on the ratios of the largest to the smallest eigenvalues in the covariance matrices of the partial sums. The purpose of this note is to show that in Tone's result, for the entire collection of the covariance matrices of the partial sums, there is essentially no other restriction on the relative magnitudes of the eigenvalues or on the (orthogonal) directions of the corresponding eigenvectors. This will be elucidated with an example described in Theorem 1.4 below, after a special case of Tone's result is stated in Theorem 1.3. For simplicity, our attention in this note will be confined to just *sequences* (of random vectors), instead of the broader context of random fields.

First, Notations 1.1 and 1.2 will give some definitions and notations and will also briefly review some well known, standard, elementary mathematics that will be needed.

NOTATIONS 1.1. In what follows, the entries of matrices are real numbers. The transpose of any given matrix  $M$  will be denoted  $M^t$ .

Now suppose  $m$  is a positive integer. In some of the notations below, the dependence on this given positive integer  $m$  will be tacitly understood and not indicated explicitly.

(A) A given element  $x \in \mathbf{R}^m$  will be represented as a "column vector" (an  $m \times 1$  matrix):  $x := [x_1, x_2, \dots, x_m]^t$ . For such an  $x$ , denote the Euclidean norm as  $\|x\| := (x_1^2 + x_2^2 + \dots + x_m^2)^{1/2}$ . The origin in  $\mathbf{R}^m$  will be denoted  $\mathbf{0}_m := [0, 0, \dots, 0]^t$ .

(B) A symmetric  $m \times m$  matrix  $A$  is "positive semi-definite" if  $x^t A x \geq 0$  for all  $x \in \mathbf{R}^m$ , and  $A$  is "positive definite" if  $x^t A x > 0$  (strict inequality) for all  $x \in \mathbf{R}^m - \{\mathbf{0}_m\}$ .

(C) If  $A$  is a symmetric, positive definite (hence nonsingular)  $m \times m$  matrix and  $r$  is a real number, then  $A^r$  denotes the symmetric, positive definite "rth power" matrix of  $A$ .

(It is of course defined by  $A^r := U D^r U^t$  where (i)  $U$  is an  $(m \times m)$  orthogonal matrix and  $D$  a diagonal matrix such that  $A = U D U^t$  and (ii)  $D^r$  is the diagonal matrix in which, for each  $i \in \{1, \dots, m\}$ , the  $i$ th diagonal element is  $d_i^r$  where  $d_i$  (a positive number, an eigenvalue of  $A$ ) is the  $i$ th diagonal element of  $D$ . The matrix  $A^r$  will thereby be uniquely defined, even though in general the choice of matrices  $U$  and  $D$  in this procedure is not unique.)

(D) For any given symmetric, positive definite  $m \times m$  matrix  $A = (a_{ij}, 1 \leq i, j \leq m)$ , define the following two quantities:

$$(1.1) \quad \eta_{\min}(A) := \min_{x \in \mathbf{R}^m: \|x\|=1} x^t A x, \quad \text{and}$$

$$(1.2) \quad \eta_{\max}(A) := \max_{x \in \mathbf{R}^m: \|x\|=1} x^t A x.$$

In (1.1)–(1.2), the min and max are both achieved for elements  $x$  on the unit sphere, and they are equal respectively to the smallest and largest eigenvalues of  $A$ . Each entry  $a_{ij}$  of  $A$  satisfies  $|a_{ij}| \leq \eta_{\max}(A)$ .

(E) For any two positive numbers  $a$  and  $b$  such that  $a < b$ , let  $\Lambda_{(m,a,b)}$  denote the set of all symmetric, positive definite  $m \times m$  matrices  $A$  such that  $a \leq \eta_{\min}(A) \leq \eta_{\max}(A) \leq b$  (that is, the set of all such matrices whose eigenvalues are all between  $a$  and  $b$  inclusive).

(F) For each  $\varepsilon > 0$ , let  $\mathbf{B}_{\text{sym}}^{(m)}[\varepsilon]$  denote the set of all symmetric (not necessarily positive semi-definite)  $m \times m$  matrices  $B := (b_{ij}, 1 \leq i, j \leq m)$  such that  $|b_{ij}| \leq \varepsilon$  for all  $(i, j) \in \{1, \dots, m\}^2$ .

(G) If  $a, b$ , and  $\varepsilon$  are positive numbers such that  $m\varepsilon < a < b$ , and  $A \in \Lambda_{(m,a,b)}$  and  $B \in \mathbf{B}_{\text{sym}}^{(m)}[\varepsilon]$ , then  $A + B \in \Lambda_{(m,a-m\varepsilon,b+m\varepsilon)}$ . (The point is that for such a  $B$ , if  $x \in \mathbf{R}^m$  is such that  $\|x\| = 1$ , then  $|x^t B x| \leq m\varepsilon$  simply by persistent trivial applications of the Cauchy inequality  $|y^t z| \leq \|y\| \cdot \|z\|$  for  $y, z \in \mathbf{R}^m$ .)

NOTATIONS 1.2. Now suppose  $(\Omega, \mathcal{F}, P)$  is a probability space. Again suppose  $m$  is a positive integer.

(A) An “ $\mathbf{R}^m$ -valued random variable” is a random vector with  $m$  (random real) coordinates. Such random vectors  $V$  will be represented as “random column vectors” (i.e.,  $m \times 1$  random matrices):  $V := [V_1, V_2, \dots, V_m]^t$ .

In the case where  $E\|V\|^2 < \infty$  (that is,  $EV_i^2 < \infty$  for each  $i \in \{1, \dots, m\}$ —recall Notations 1.1(A)), the  $(m \times m)$  covariance matrix of  $V$  will be denoted  $\Sigma_V$ . If also  $EV = \mathbf{0}_m$  (that is,  $EV_i = 0$  for each  $i$ ), then one has the trivial representation  $\Sigma_V = EVV^t$ . The matrix  $\Sigma_V$  is of course (symmetric and) positive semi-definite. (In the mean  $\mathbf{0}_m$  case, recall that for any  $x \in \mathbf{R}^m$ ,  $x^t \Sigma_V x = E(x^t V)(x^t V)^t = E(x^t V)^2 \geq 0$ .)

(B) Suppose  $X := (X_k, k \in \mathbf{Z})$  is a strictly stationary sequence of  $\mathbf{R}^m$ -valued random variables. For each  $n \in \mathbf{N}$ , define the partial sum (again, a “random  $m \times 1$  column vector”)  $S_n = S(X, n) := X_1 + X_2 + \dots + X_n$ . (Here and below,  $\mathbf{N}$  denotes the set of all positive integers.)

Our work will involve the case where  $EX_0 = \mathbf{0}_m$  and  $E\|X_0\|^2 < \infty$ . For typographical convenience, the covariance matrix of  $X_0$  will be written  $\Sigma_{X(0)}$ , and for each  $n \in \mathbf{N}$ , the covariance matrix of the normalized partial sum  $n^{-1/2}S_n$  will be written (with perhaps slight abuse of notation) as  $\Sigma_{S(X,n)/\sqrt{n}}$  (it is of course equal to  $n^{-1}\Sigma_{S(X,n)}$ ).

(C) Next, let us turn to measures of dependence. For any two  $\sigma$ -fields  $\mathcal{A}$  and  $\mathcal{B} (\subset \mathcal{F})$ , define the following four measures of dependence: First, define

$$(1.3) \quad \alpha(\mathcal{A}, \mathcal{B}) := \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |P(A \cap B) - P(A)P(B)|.$$

Next, define the “maximal correlation coefficient” [10]

$$(1.4) \quad \rho(\mathcal{A}, \mathcal{B}) := \sup | \text{Corr}(g, h) |,$$

where the supremum is taken over all pairs of real-valued, square-integrable random variables  $g$  and  $h$  such that  $g$  is  $\mathcal{A}$ -measurable and  $h$  is  $\mathcal{B}$ -measurable. Finally, define

$$(1.5) \quad \beta(\mathcal{A}, \mathcal{B}) := \sup \frac{1}{2} \sum_{i=1}^I \sum_{j=1}^J |P(A_i \cap B_j) - P(A_i)P(B_j)|$$

as well as the “coefficient of information” (see, e.g., [21] or [13])

$$(1.6) \quad I(\mathcal{A}, \mathcal{B}) := \sup \sum_{i=1}^I \sum_{j=1}^J P(A_i \cap B_j) \log \left( \frac{P(A_i \cap B_j)}{P(A_i)P(B_j)} \right),$$

where in each of (1.5) and (1.6) the supremum is taken over all pairs of finite partitions  $\{A_1, A_2, \dots, A_I\}$  and  $\{B_1, B_2, \dots, B_J\}$  of  $\Omega$  such that  $A_i \in \mathcal{A}$  for each  $i$  and  $B_j \in \mathcal{B}$  for each  $j$ . (Here and below, “log” denotes the natural logarithm.) In (1.6), the summand is taken to be 0 if either  $P(A_i)$  or  $P(B_j)$  is 0. It is well known (see, e.g., [3, v1, Proposition 3.11 and Theorem 5.3(III)]) that for any two  $\sigma$ -fields  $\mathcal{A}$  and  $\mathcal{B}$ ,

$$(1.7) \quad 4\alpha(\mathcal{A}, \mathcal{B}) \leq \rho(\mathcal{A}, \mathcal{B}), \quad \text{and}$$

$$(1.8) \quad 2\alpha(\mathcal{A}, \mathcal{B}) \leq \beta(\mathcal{A}, \mathcal{B}) \leq \sqrt{I(\mathcal{A}, \mathcal{B})}.$$

(D) Now again suppose  $X := (X_k, k \in \mathbf{Z})$  is a strictly stationary sequence of  $\mathbf{R}^m$ -valued random variables. (No assumptions on moments.) For any integer  $j$ , define the  $\sigma$ -fields  $\mathcal{F}_{-\infty}^j := \sigma(X_k, k \leq j)$  and  $\mathcal{F}_j^\infty := \sigma(X_k, k \geq j)$ . (Here and below,  $\sigma(\dots)$  denotes the  $\sigma$ -field  $\subset \mathcal{F}$  generated by  $(\dots)$ .) For each positive integer  $n$ , define the following five dependence coefficients:

$$(1.9) \quad \alpha(n) = \alpha(X, n) := \alpha(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^\infty);$$

$$(1.10) \quad \rho(n) = \rho(X, n) := \rho(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^\infty);$$

$$(1.11) \quad \beta(n) = \beta(X, n) := \beta(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^\infty);$$

$$(1.12) \quad I(n) = I(X, n) := I(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^\infty); \quad \text{and}$$

$$(1.13) \quad \rho^*(n) = \rho^*(X, n) := \sup \rho(\sigma(X_k, k \in \Gamma), \sigma(X_k, k \in \Delta))$$

where the supremum in (1.13) is taken over all pairs of nonempty, disjoint subsets  $\Gamma$  and  $\Delta$  of  $\mathbf{Z}$  such that  $\text{dist}(\Gamma, \Delta) := \min_{g \in \Gamma, h \in \Delta} |g - h| \geq n$ . (The sets  $\Gamma$  and  $\Delta$  can be “interlaced,” i.e., with each one containing elements from ones in the other set.) Of course by strict stationarity,  $\alpha(n) = \alpha(\mathcal{F}_{-\infty}^j, \mathcal{F}_{j+n}^\infty)$  for any integer  $j$ ; and the analogous comment applies to (1.10), (1.11), and (1.12).

The given strictly stationary sequence  $X$  is said to satisfy

“strong mixing” [23] if  $\alpha(n) \rightarrow 0$  as  $n \rightarrow \infty$ ,

“ $\rho$ -mixing” [15] if  $\rho(n) \rightarrow 0$  as  $n \rightarrow \infty$ ,

“absolute regularity” [29] if  $\beta(n) \rightarrow 0$  as  $n \rightarrow \infty$ ,

“information regularity” [21], [29] if  $I(n) \rightarrow 0$  as  $n \rightarrow \infty$ , and  
 “ $\rho^*$ -mixing” [24], [25] if  $\rho^*(n) \rightarrow 0$  as  $n \rightarrow \infty$ .

(The mixing condition in [24] looked somewhat different from  $\rho^*$ -mixing, but turned out to be equivalent to it in the context in that paper; see [3, v1, Theorem 5.13].) By (1.7)–(1.8) and (1.9)–(1.13), the following implications hold:

- (i)  $\rho^*$ -mixing implies  $\rho$ -mixing,
- (ii)  $\rho$ -mixing implies strong mixing,
- (iii) information regularity implies absolute regularity, and
- (iv) absolute regularity implies strong mixing.

With the possible exception of information regularity, all of these conditions have played a major role in limit theory for weakly dependent random variables; see, for example, the books [1], [3], [9], [17], and [22]. Information regularity is sometimes a handy tool in the study of stationary Gaussian sequences; see, for example, [13, Chapter 4] or [3, v3, Chapter 27].

Peligrad [19, Corollary 2.3] proved a central limit theorem for strictly stationary sequences of real-valued, square-integrable random variables satisfying the dependence assumptions  $\rho^*(1) < 1$  and  $\alpha(n) \rightarrow 0$  as  $n \rightarrow \infty$ . That result was generalized to strictly stationary random fields of real-valued random variables by Perera [20, Proposition 3] (with the sums being taken over a broad class of sets of indices, not just “rectangular blocks”). It was generalized again in [3, v3, Corollary 29.33]—again to strictly stationary random fields of real-valued random variables—with another, less restrictive generalization (to random fields) of the dependence coefficient  $\rho^*(1)$  (but with the sums taken over just the usual “rectangular blocks” of indices). Later, for an arbitrary positive integer  $m$ , Tone [26, Theorem 1.1] generalized that latter result to strictly stationary random fields of  $\mathbf{R}^m$ -valued random variables. For simplicity, we shall state her result here for just the special case of random sequences:

**THEOREM 1.3** (Tone [26]; Peligrad [19] for  $m = 1$ ). *Suppose  $m$  is a positive integer. Suppose  $X := (X_k, k \in \mathbf{Z})$  is a strictly stationary sequence of  $\mathbf{R}^m$ -valued random variables such that  $EX_0 = \mathbf{0}_m$  and  $E\|X_0\|^2 < \infty$ , and the covariance matrix  $\Sigma_{X(0)}$  is positive definite (hence nonsingular). Suppose also that  $\rho^*(X, 1) < 1$  and that  $\alpha(X, n) \rightarrow 0$  as  $n \rightarrow \infty$ . Then the following two statements hold:*

- (1) *For each  $n \in \mathbf{N}$ , the covariance matrix  $\Sigma_{S(X,n)}$  is positive definite (hence, nonsingular).*
- (2) *One has that (see Notations 1.1(C) and 1.2(A))*

$$(1.14) \quad \Sigma_{S(X,n)}^{-1/2} S(X, n) \Rightarrow N(\mathbf{0}_m, I_m) \quad \text{as } n \rightarrow \infty.$$

Here in (1.14), the notation  $\Rightarrow$  means convergence in distribution on (the Borel  $\sigma$ -field of)  $\mathbf{R}^m$ , and the notation  $N(\mathbf{0}_m, I_m)$  refers to the multivariate normal distribution on  $\mathbf{R}^m$  whose mean vector is  $\mathbf{0}_m$  and whose covariance matrix is the  $m \times m$  identity matrix  $I_m$ . The left-hand side of (1.14) is an  $\mathbf{R}^m$ -valued random variable (“random  $m \times 1$  column vector”) resulting from the matrix multiplication indicated there.

Under different dependence assumptions, again in the more general context of strictly stationary random fields, Bulinskii and Kryzhanovskaya [6, Equation (1.13) and Theorem 2] reformulated a multivariate central limit theorem in [7] into the form (1.14), with the same use of the “standard  $-1/2$  power” of the covariance matrix  $\Sigma_{S(X,n)}$  as “normalization,” and then treated a related central limit theorem of the form (1.14) involving the use of the “standard  $-1/2$  power” of a *sample* covariance matrix  $\hat{\Sigma}_{S(X,n)}$  as “normalization.” (Those results will not be treated further here.)

Here is our main result (recall Notations 1.1(E)).

**THEOREM 1.4.** *Suppose  $m$  is a positive integer. Suppose  $a, b$  and  $\tau$  are positive real numbers such that  $a < b$ . Then there exists a strictly stationary Gaussian sequence  $X := (X_k, k \in \mathbf{Z})$  of  $\mathbf{R}^m$ -valued, mean- $\mathbf{0}_m$  random variables with the following properties:*

- (1)  $\rho^*(X, 1) < 1$ .
- (2)  $\max\{I(X, 1), \beta(X, 1), \alpha(X, 1), \rho(X, 1)\} \leq \tau$ .
- (3)  $\max\{I(X, n), \beta(X, n), \alpha(X, n), \rho(X, n)\} \rightarrow 0$  as  $n \rightarrow \infty$ .
- (4) *For every element ( $m \times m$  matrix)  $G \in \Lambda_{(m,a,b)}$ , there exists an infinite set  $Q \subset \mathbf{N}$  such that*

$$(1.15) \quad \Sigma_{S(X,n)/\sqrt{n}} \rightarrow G \quad \text{as } n \rightarrow \infty, n \in Q.$$

Statements (2) and (3) have some redundancy (see (1.7)–(1.8)), but that is harmless. Of course (1.15) means that for every  $(i, j) \in \{1, 2, \dots, m\}^2$ , the  $(i, j)$ -entry of the matrix  $\Sigma_{S(X,n)/\sqrt{n}}$  converges to the  $(i, j)$ -entry of the matrix  $G$  as  $n \rightarrow \infty, n \in Q$ . Also, the statement that  $X$  is a “Gaussian sequence” means of course that for any positive integer  $L$  and any distinct integers  $k(1), k(2), \dots, k(L)$ , the joint distribution of the random vectors  $X_{k(1)}, X_{k(2)}, \dots, X_{k(L)}$  is a (possibly degenerate) multivariate normal distribution on  $\mathbf{R}^{Lm}$ .

Theorem 1.4 will be proved in Section 3, after some preliminary work is done in Section 2. In the rest of Section 1 here, a few comments on this theorem will be given.

Under the assumptions of Theorem 1.3, Tone [26, Claim 3.1] showed that for the covariance matrices  $\Sigma_{S(X,n)}$ , the ratio of the largest to smallest eigenvalues is bounded, and that in fact there exists a pair of positive numbers  $a < b$  such that  $\Sigma_{S(X,n)/\sqrt{n}} \in \Lambda_{(m,a,b)}$  for all  $n \in \mathbf{N}$ . Thus in property (4) in

Theorem 1.4, the restriction to matrices in  $\Lambda_{(m,a,b)}$  (for some pair of positive numbers  $a < b$ ) is unavoidable.

In Theorem 1.4, property (3) cannot be extended to include  $\rho^*(X, n) \rightarrow 0$  as  $n \rightarrow \infty$ , for that (in conjunction with certain other properties in Theorem 1.4) would force the covariance matrices  $\Sigma_{S(X,n)/\sqrt{n}}$  to converge to a limiting matrix as  $n \rightarrow \infty$  (a fact implicitly contained in another, somewhat related result of Tone [27, Theorem 3.2]), contradicting property (4). Also, in Theorem 1.4, the larger the ratio  $b/a$  is, the closer  $\rho^*(X, 1)$  has to be to 1. That insight ultimately goes back (in light of basic results in [15]) to work of Moore [18] involving a closely related condition.

For random sequences and random fields respectively, classes of examples constructed in [3, v3, Theorem 26.8] and [4, Theorem 1.9] “separate” various different but related mixing assumptions used in [2], [3], [19], [20], [26], [27], [28] and other related works. In particular, the latter class of examples (in [4]) “separates” the two generalizations (to random fields) of the dependence coefficient  $\rho^*(1)$  (in [20], and in [3] and [26]) implicitly alluded to prior to Theorem 1.3.

In (1.15), regardless of whether or not the eigenvalues of  $G$  are simple, one can trivially consider a further subsequence in which the eigenvalues and  $m$  orthogonal unit eigenvectors of the matrices  $\Sigma_{S(X,n)/\sqrt{n}}$  all converge; by a simple calculation, their limits must be the eigenvalues and  $m$  orthogonal unit eigenvectors of  $G$ . As a consequence, in Theorem 1.3, for the covariance matrices  $\Sigma_{S(X,n)}$ , the relative magnitudes of the eigenvalues, and the respective (orthogonal) directions of their eigenvectors, can range essentially arbitrarily—within some upper bound (as noted above) on the ratio of the largest to smallest eigenvalues. In this respect, Theorem 1.4 helps to “separate” Theorem 1.3 from other, more conventional multivariate central limit theorems (such as the one in [27, Theorem 3.2] alluded to above) in which there is a “limiting covariance matrix.”

It was noted above that in the special case of real-valued random variables (i.e.  $m = 1$ ), Theorem 1.3 boils down to a central limit theorem of Peligrad [19]. The author [2] (see also [3, v3, Theorem 27.12]) gave a construction (a variant of ones in [11] and [5]) that showed that in that result of Peligrad, the growth of the variances need not be asymptotically linear, but can instead “wobble” between two different linear rates of growth. That construction was in spirit (though not fully in letter) a version of Theorem 1.4 for the case  $m = 1$  (real-valued random variables).

As was noted above, Theorem 1.3 is actually just a special case of a result of Tone [26, Theorem 1.1], which in its full generality involved random fields (of  $\mathbf{R}^m$ -valued random variables) indexed by  $\mathbf{Z}^d$  for an arbitrary positive integer  $d$ . By modifying the arguments below, one can prove a version of Theorem 1.4 for such random fields for arbitrary ( $m$  and)  $d$ . However, in the case  $d \geq 2$ , for such a construction, the information in Theorem 1.4 that pertains to the

dependence coefficients  $\beta(n)$  and  $I(n)$  unavoidably becomes false and has to be omitted; see [3, v3, Theorem 29.9].

As a simple corollary of Theorem 1.4 itself, one can derive a version of Theorem 1.4 in which the sequence  $X$  is not Gaussian. One can simply apply Theorem 1.4 itself with  $a$  replaced by some number  $a' \in (0, a)$ , then fix  $\varepsilon > 0$  such that  $a' + \varepsilon < a$ , and then replace  $X_k$  by  $X_k + [V_k^{(1)}, V_k^{(2)}, \dots, V_k^{(m)}]^t$  where  $(V_k^{(i)}, k \in \mathbf{Z}, i \in \{1, \dots, m\})$  is a family of independent, identically distributed real-valued random variables, this family being independent of the sequence  $X$ , with the  $V_k^{(i)}$ 's each taking the values  $\sqrt{\varepsilon}$  and  $-\sqrt{\varepsilon}$  with probability  $1/2$  each.

## 2. Preliminaries

This section will lay some groundwork for the proof, in Section 3, of Theorem 1.4.

The random sequence  $X$  described in Theorem 1.4 will be constructed (in Section 3) from a family of independent “building block” random sequences of a relatively simple structure. The following lemma will play a role in that process of “assembly.”

LEMMA 2.1. *Suppose  $(\Omega, \mathcal{F}, P)$  is a probability space,  $L$  is a positive integer, and  $\mathcal{A}_\ell$  and  $\mathcal{B}_\ell$ ,  $\ell \in \{1, 2, \dots, L\}$  are  $\sigma$ -fields ( $\subset \mathcal{F}$ ) such that the  $\sigma$ -fields  $\mathcal{A}_\ell \vee \mathcal{B}_\ell$ ,  $\ell \in \{1, \dots, L\}$  are independent. Then*

$$(2.1) \quad \rho\left(\bigvee_{\ell=1}^L \mathcal{A}_\ell, \bigvee_{\ell=1}^L \mathcal{B}_\ell\right) = \max_{1 \leq \ell \leq L} \rho(\mathcal{A}_\ell, \mathcal{B}_\ell), \quad \text{and}$$

$$(2.2) \quad I\left(\bigvee_{\ell=1}^L \mathcal{A}_\ell, \bigvee_{\ell=1}^L \mathcal{B}_\ell\right) = \sum_{\ell=1}^L I(\mathcal{A}_\ell, \mathcal{B}_\ell).$$

Proofs of these equalities can be found for example, in [3, v1, Theorems 6.1 and 6.2(VIII)]. Equation (2.1) is due to Csáki and Fischer [8, Theorem 6.2]. Equation (2.2) is a classic fact from information theory; see, for example, its role in Pinsker [21].

The “building blocks” for the construction (in Section 3) of the sequence  $X$  for Theorem 1.4 will be stationary Gaussian sequences of centered *real*-valued random variables. They will be identified (in Section 3) via a careful choice of their spectral densities. The rest of Section 2 here will lay some groundwork for that procedure.

NOTATIONS 2.2. With slight abuse of terminology, a real Borel function  $f$  on  $[-\pi, \pi]$  will be said to be “symmetric” if  $f(-\lambda) = f(\lambda)$  for a.e.  $\lambda \in [-\pi, \pi]$ .

(A) Suppose  $f$  is a real, nonnegative, Borel, symmetric, integrable function on  $[-\pi, \pi]$ . Suppose  $W := (W_k, k \in \mathbf{Z})$  is a strictly stationary sequence of real-valued, centered, square-integrable random variables. Then  $f$  is a “spectral



density function” for the sequence  $W$  if the following holds:

$$(2.3) \quad \forall k \in \mathbf{Z}, \quad EW_k W_0 = \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda) \frac{d\lambda}{2\pi}.$$

If  $W$  has a spectral density function, then it will be unique modulo sets of Lebesgue measure 0. The convention on spectral density used here is as in [3]; it differs by a factor of  $2\pi$  from a more standard convention used in other references.

(B) For each positive integer  $n$ , define the real, nonnegative, symmetric, continuous function (the Fejér kernel)  $F_n$  on  $[-\pi, \pi]$  as follows:

$$(2.4) \quad F_n(\lambda) := \begin{cases} (1/n) \cdot [\sin^2(n\lambda/2)]/[\sin^2(\lambda/2)] & \text{if } \lambda \in [-\pi, \pi] - \{0\}, \\ n & \text{if } \lambda = 0. \end{cases}$$

(C) It is well known that if  $W$  and  $f$  are as in (A) above, with  $f$  being the spectral density function of  $W$ , then for each positive integer  $n$ ,

$$(2.5) \quad E[(W_1 + W_2 + \dots + W_n)/\sqrt{n}]^2 = \int_{-\pi}^{\pi} F_n(\lambda) f(\lambda) \frac{d\lambda}{2\pi}.$$

See e.g. [3, v1, the Note after Lemma 8.18].

LEMMA 2.3. *Suppose  $W := (W_k, k \in \mathbf{Z})$  is a stationary real mean-zero Gaussian random sequence that has a spectral density  $f$  on  $[-\pi, \pi]$  that is bounded a.e. between two positive constants. Then  $\rho^*(W, 1) < 1$ .*

An elementary proof of this lemma can be found in [3, v1, Theorem 9.8(III)]. (It yields the inequality  $\rho^*(W, 1) \leq 1 - a/b$  where  $0 < a < b$  and  $a \leq f \leq b$  a.e. The sharper inequality  $\rho^*(W, 1) \leq (1 - a/b)/(1 + a/b)$  holds as a result of a more sophisticated argument of Moore [18] in a closely related context.)

The analysis that follows will now involve certain real, Borel, symmetric functions  $f$  on  $[-\pi, \pi]$  that can take (perhaps even exclusively) *negative* values—with the intent to use, for some such functions  $f$  later on, the positive function  $\lambda \mapsto e^{f(\lambda)}$  as the spectral density for a stationary Gaussian sequence.

NOTATIONS 2.4. (A) For any (not necessarily nonnegative) real, Borel, square-integrable, symmetric function  $f$  on  $[-\pi, \pi]$ , define the quantity

$$(2.6) \quad \Psi(f) := \sum_{k=1}^{\infty} k\psi_{f,k}^2,$$

where for each  $k \in \mathbf{N}$ ,

$$(2.7) \quad \psi_{f,k} := 2 \cdot \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda) \frac{d\lambda}{2\pi}.$$

Of course  $\sum_{k=1}^{\infty} \psi_{f,k}^2 < \infty$ ; and with  $\psi_{f,0} := (2\pi)^{-1} \int_{-\pi}^{\pi} f(\lambda) d\lambda$ , one has that  $\sum_{k=0}^{\infty} \psi_{f,k} \cos(k\lambda)$  converges in  $\mathcal{L}^2$  to  $f$  (and one can say more). However, the quantity  $\Psi(f)$  may be infinite.

(B) For any two real, Borel, square-integrable, symmetric functions  $f$  and  $g$  on  $[-\pi, \pi]$ , one has that  $\psi_{f+g,k} = \psi_{f,k} + \psi_{g,k}$  for each  $k$  (see (2.7)), and by (2.6) and Minkowski's inequality,  $[\Psi(f + g)]^{1/2} \leq [\Psi(f)]^{1/2} + [\Psi(g)]^{1/2}$  (where if necessary,  $\infty^{1/2} := \infty$ ).

(C) Suppose  $a$  and  $b$  are real numbers such that  $a < b$ . Suppose  $f, f_1, f_2, f_3, \dots$  is a sequence of real, Borel, symmetric functions on  $[-\pi, \pi]$  that are each bounded a.e. between  $a$  and  $b$ , and  $f_n \rightarrow f$  a.e. as  $n \rightarrow \infty$ . If  $\tau$  is a positive number and  $\Psi(f_n) \leq \tau$  for every  $n \in \mathbf{N}$ , then  $\Psi(f) \leq \tau$ .

(This formulation is unnecessarily restrictive, but will fit our applications later on. The point is that for each  $k$ ,  $\psi_{f(n),k}$  (where  $f(n)$  means  $f_n$ ) converges to  $\psi_{f,k}$  as  $n \rightarrow \infty$ , and hence for each positive integer  $L$ ,  $\sum_{k=1}^L k\psi_{f,k}^2 \leq \tau$ , and hence the same is true with  $L$  replaced by  $\infty$ .)

(D) If  $f$  is a real, Borel, square-integrable, symmetric function on  $[-\pi, \pi]$  such that  $\Psi(f) < \infty$ , then  $\int_{-\pi}^{\pi} e^{f(\lambda)} d\lambda < \infty$ . (This is a special case of a classic result of Lebedev and Milin [16]. For a detailed exposition of this, see, for example, [3, v3, Appendix, Theorem A2744(VII)].)

LEMMA 2.5. *For every  $\varepsilon > 0$ , there exists  $\delta = \delta(\varepsilon) > 0$  such that the following holds:*

*Suppose  $W := (W_k, k \in \mathbf{Z})$  is a stationary real mean-zero Gaussian random sequence with a spectral density function  $g$  of the form  $g(\lambda) = e^{f(\lambda)}$ ,  $\lambda \in [-\pi, \pi]$ , where  $f$  is a real, Borel, square-integrable, symmetric function on  $[-\pi, \pi]$  such that  $\Psi(f) \leq \delta$ ; then  $I(W, 1) \leq \varepsilon$ .*

This lemma is implicitly contained in arguments of Ibragimov, Rozanov, and Solev in [12], [14] (see also [13, Chapter 4]). A detailed, explicit proof of this lemma can be found in [3, v3, Theorem 27.11].

LEMMA 2.6. *Suppose  $\Upsilon_1$ , and  $\Upsilon_2$ , and  $\theta$  are real numbers such that*

$$(2.8) \quad \Upsilon_1 < \theta < \Upsilon_2.$$

*Suppose  $\delta > 0$  and  $\varepsilon > 0$ .*

*Suppose  $N$  is a positive integer.*

*Suppose  $f$  is a real, continuous, symmetric function on  $[-\pi, \pi]$  such that*

$$(2.9) \quad \Upsilon_1 < f(\lambda) < \Upsilon_2 \quad \text{for all } \lambda \in [-\pi, \pi] \quad \text{and}$$

$$(2.10) \quad \Psi(f) < \delta.$$

*Then one has that there exists a real, continuous, symmetric function  $h = h_{(f, \Upsilon(1), \Upsilon(2), \theta, \delta, \varepsilon, N)}$  on  $[-\pi, \pi]$  (where the notations  $\Upsilon(1)$  and  $\Upsilon(2)$  mean  $\Upsilon_1$  and  $\Upsilon_2$ ) with the following five properties:*

$$(2.11) \quad \text{For every } \lambda \in [-\pi, \pi], \quad \Upsilon_1 < h(\lambda) < \Upsilon_2;$$

$$(2.12) \quad \Psi(h) < \delta;$$

$$(2.13) \quad |h(0) - \theta| < \varepsilon;$$

$$(2.14) \quad \int_{-\pi}^{\pi} |h(\lambda) - f(\lambda)| d\lambda < \varepsilon; \quad \text{and}$$

$$(2.15) \quad \text{for every } n \in \{1, 2, \dots, N\},$$

$$\left| \int_{-\pi}^{\pi} F_n(\lambda) \cdot e^{h(\lambda)} d\lambda - \int_{-\pi}^{\pi} F_n(\lambda) \cdot e^{f(\lambda)} d\lambda \right| < \varepsilon.$$

*Proof.* Refer to (2.8) and (2.9). We shall first carry out the proof of Lemma 2.6 under the following extra assumption:

$$(2.16) \quad \theta > f(0).$$

Since  $f$  is (by assumption) continuous on the closed interval  $[-\pi, \pi]$ , it follows (see (2.8), (2.9), and (2.16)) that there exists a number  $c_0$  (henceforth fixed) with the following three properties:

$$(2.17) \quad 0 < c_0 < \min\{1, \theta - f(0)\};$$

$$(2.18) \quad \Upsilon_1 < f(\lambda) - c_0 < f(\lambda) + c_0 < \Upsilon_2 \quad \text{for all } \lambda \in [-\pi, \pi]; \quad \text{and}$$

$$(2.19) \quad |f(\lambda) - f(0)| < \Upsilon_2 - \theta \quad \text{for all } \lambda \in [-c_0, c_0].$$

For each  $c \in (0, c_0]$ , define the positive numbers  $a_{c,k}, k \in \mathbf{N}$  as follows:

$$(2.20) \quad a_{c,k} := \begin{cases} (c^2/\pi) \cdot (1/k) & \text{if } k = 1 \text{ or } 2, \\ (c^2/\pi) \cdot (1/k) \cdot 1/(\log k) & \text{if } k \geq 3. \end{cases}$$

Then for each  $c \in (0, c_0]$ , one has by (2.17) and (2.20) that

$$(2.21) \quad \theta - f(0) > a_{c,1} > a_{c,2} > a_{c,3} > \dots \downarrow 0$$

and that  $\sum_{k=1}^{\infty} a_{c,k} = \infty$ . Accordingly, for each  $c \in (0, c_0]$ , let  $M(c)$  denote the greatest positive integer such that (see the first inequality in (2.21))

$$(2.22) \quad \sum_{k=1}^{M(c)} a_{c,k} \leq \theta - f(0).$$

For each  $c \in (0, c_0]$ , define the real, continuous, symmetric function  $g_c$  on  $[-\pi, \pi]$  as follows: For  $\lambda \in [-\pi, \pi]$ ,

$$(2.23) \quad g_c(\lambda) := \sum_{k=1}^{M(c)} a_{c,k} \cos(k\lambda).$$

Now suppose  $c$  is an arbitrary fixed number such that  $c \in (0, c_0]$ . From (2.23), (2.20), the monotonicity in (2.21), and a standard fact for trigonometric series with nonnegative, monotonically decreasing coefficients (see [3, v3, Appendix, Lemma A2712]—take the real parts there—or [30, p. 3, Theorem 2.2]), one has that for any  $\lambda \in [c, \pi]$ ,

$$|g_c(\lambda)| \leq (\pi/\lambda) \cdot a_{c,1} = (\pi/\lambda) \cdot (c^2/\pi) \leq c.$$

Next, suppose for just a moment that  $\lambda \in (0, c]$ . Then  $0 < \lambda \leq c \leq c_0 < 1$  by (2.17). Let  $I$  denote the positive integer such that  $I < 1/\lambda \leq I + 1$ . Then for all  $k \in \{1, 2, \dots, I\}$ , one has that  $k\lambda < 1$  and hence  $\cos(k\lambda) > 0$ . If  $M(c) \leq I$ , then it follows from (2.23) and (2.20) that  $g_c(\lambda) > 0$ . If instead  $M(c) > I$ , then one has  $\sum_{k=1}^I a_{c,k} \cos(k\lambda) > 0$  and (since  $1 \leq \lambda \cdot (I + 1)$ ) again by (2.20), (2.17), and the monotonicity in (2.21) (again see [3, v3, Lemma A2712] or [30, p. 3])

$$\left| \sum_{k=I+1}^{M(c)} a_{c,k} \cos(k\lambda) \right| \leq (\pi/\lambda) \cdot a_{c,I+1} \leq (\pi/\lambda) \cdot (c^2/\pi) \cdot (1/(I + 1)) \leq c^2 < c,$$

and hence  $g_c(\lambda) \geq -c$  by (2.23). Putting all these pieces together (see also (2.22) and (2.23) again), one now has that

$$(2.24) \quad |g_c(\lambda)| \leq c \quad \text{for all } \lambda \in [c, \pi]; \quad \text{and}$$

$$(2.25) \quad -c \leq g_c(\lambda) \leq \sum_{k=1}^{M(c)} a_{c,k} \leq \theta - f(0) \quad \text{for all } \lambda \in [0, c].$$

(Equation (2.25) was shown above for  $\lambda \in (0, c]$ ; it extends to  $\lambda = 0$  by continuity of the function  $g_c$ .) By (2.18), (2.24), and (2.18) again (keeping in mind our ongoing assumption  $c \in (0, c_0]$ ), one has that for all  $\lambda \in [c, \pi]$ ,

$$\Upsilon_1 < f(\lambda) - c \leq f(\lambda) + g_c(\lambda) \leq f(\lambda) + c < \Upsilon_2.$$

By (2.18), (2.25), and (2.19), for all  $\lambda \in [0, c]$ ,

$$\Upsilon_1 < f(\lambda) - c \leq f(\lambda) + g_c(\lambda) \leq f(\lambda) + \theta - f(0) < \Upsilon_2 - \theta + \theta = \Upsilon_2.$$

Hence by symmetry, one now has that

$$(2.26) \quad \Upsilon_1 < f(\lambda) + g_c(\lambda) < \Upsilon_2 \quad \text{for all } \lambda \in [-\pi, \pi].$$

Equations (2.24), (2.25), and (2.26) were shown for any arbitrary  $c \in (0, c_0]$ . Our plan now is to let the function  $h$  be defined by

$$(2.27) \quad h := f + g_c$$

for some sufficiently small  $c \in (0, c_0]$ . To start off, note that under (2.27) for any given  $c \in (0, c_0]$ , (2.11) holds by (2.26).

Next, by (2.20), for each  $c \in (0, c_0]$ ,

$$\sum_{k=1}^{\infty} k \cdot a_{c,k}^2 = (c^4/\pi^2) \cdot \left[ 1 + (1/2) + \sum_{k=3}^{\infty} 1/[k(\log k)^2] \right] < \infty;$$

and in fact the middle term converges to 0 as  $c \rightarrow 0+$ . Hence by (2.23) and (2.6)–(2.7),  $\Psi(g_c) \rightarrow 0$  as  $c \rightarrow 0+$ . Hence by (2.10) and Notations 2.4(B),  $[\Psi(f + g_c)]^{1/2} < \delta^{1/2}$  for all  $c \in (0, c_0]$  sufficiently small. Thus under (2.27), Equation (2.12) holds for all  $c \in (0, c_0]$  sufficiently small.

Next, for each  $c \in (0, c_0]$ , by the definition of the positive integer  $M(c)$  (see the entire sentence containing (2.22)), followed by (2.20), one has that

$$0 \leq [\theta - f(0)] - \sum_{k=1}^{M(c)} a_{c,k} < \sum_{k=1}^{M(c)+1} a_{c,k} - \sum_{k=1}^{M(c)} a_{c,k} = a_{c,M(c)+1} \leq c^2/\pi.$$

That is, by (2.23),  $0 \leq [\theta - f(0)] - g_c(0) < c^2/\pi$ , that is,  $0 \leq \theta - [f(0) + g_c(0)] < c^2/\pi$ . Hence under (2.27), Equation (2.13) holds for all  $c \in (0, c_0]$  sufficiently small.

Next, by (2.24) and symmetry, for every  $\lambda \in [-\pi, \pi] - \{0\}$ ,  $g_c(\lambda) \rightarrow 0$  as  $c \rightarrow 0+$ . Hence by (2.9), (2.26), and dominated convergence, (2.14) holds (under (2.27)) for all  $c \in (0, c_0]$  sufficiently small. Also, since each Fejér kernel (see (2.4)) is bounded, and by (2.9) and (2.26) the functions  $\exp(f(\lambda))$  and  $\exp(f(\lambda) + g_c(\lambda))$  (for  $c \in (0, c_0]$ ) are uniformly bounded (between  $\exp \Upsilon_1$  and  $\exp \Upsilon_2$ ), one has by dominated convergence that (under (2.27)) Equation (2.15) holds for all  $c \in (0, c_0]$  sufficiently small. Thus under (2.27), Equations (2.11)–(2.15) hold for all  $c \in (0, c_0]$  sufficiently small. Thus, Lemma 2.6 holds under the extra assumption (2.16).

It will be useful to note that, again under the extra assumption (2.16), one can expand the statement of Lemma 2.6 to include the following variant of (2.15):

(2.28) For every  $n \in \{1, 2, \dots, N\}$ ,

$$\left| \int_{-\pi}^{\pi} F_n(\lambda) \cdot e^{-h(\lambda)} d\lambda - \int_{-\pi}^{\pi} F_n(\lambda) \cdot e^{-f(\lambda)} d\lambda \right| < \varepsilon.$$

To accomplish this, one shows that under (2.27), Equation (2.28) holds for all  $c \in (0, c_0]$  sufficiently small. The argument is essentially the same as the corresponding one for (2.15) in the preceding paragraph.

Now let us briefly take care of the cases where (2.16) does not hold. Refer to (2.8) and (2.9) again. If  $\theta = f(0)$ , then let  $h := f$  and we are done. Finally, if  $\theta < f(0)$ , then by replacing  $\Upsilon_1$ ,  $\Upsilon_2$ ,  $\theta$ , and  $f$  by  $-\Upsilon_2$ ,  $-\Upsilon_1$ ,  $-\theta$ , and  $-f$  (note that  $\Psi(-f) = \Psi(f)$  by (2.6)–(2.7)), one trivially converts to the case where (2.16) holds. (The resulting function, say  $\tilde{h}$ , is then multiplied by  $-1$  to produce the final function  $h$ . In order for (2.15) to result at the end of this “trivial conversion argument,” it was vital to derive the “extra” fact (2.28) at the end of the argument under (2.16) above.) That completes the proof of Lemma 2.6. □

### 3. Proof of Theorem 1.4

The proof will be written out here for the case  $m \geq 2$ . (The argument for the case  $m = 1$  is similar but less complicated.) The proof will be divided into several “steps.” (One of those “steps” will be a “lemma.”)

STEP 3.1. Refer to the statement of Theorem 1.4. Decreasing  $\tau$  and/or  $a$  and/or increasing  $b$  if necessary, we assume without loss of generality that

$$(3.1) \quad 0 < a < 1 < b \quad \text{and}$$

$$(3.2) \quad 0 < \tau < 1.$$

Let us identify the set of all (real)  $m \times m$  matrices with  $\mathbf{R}^{m^2}$  (with each entry in the matrix identified with a coordinate in  $\mathbf{R}^{m^2}$ ). The set  $\mathbf{R}^{m^2}$  is separable. Hence, every nonempty subset of  $\mathbf{R}^{m^2}$  is separable (an elementary fact—see for example, [3, v3, Appendix, Lemma A3101]). Accordingly, let  $\tilde{\Lambda}$  be a countable dense subset of  $\Lambda_{(m,a,b)}$ . Let  $G_1, G_2, G_3, \dots$  be a sequence of elements of  $\tilde{\Lambda}$  such that (for convenience) each element of  $\tilde{\Lambda}$  is listed infinitely many times in that sequence.

In order to prove Theorem 1.4, it suffices to construct a strictly stationary, mean- $\mathbf{0}_m$  Gaussian sequence  $X := (X_k, k \in \mathbf{Z})$  of  $\mathbf{R}^m$ -valued random variables such that properties (1), (2), and (3) in Theorem 1.4 hold as well as the following property: (4') There exists a strictly increasing sequence  $(N_1, N_2, N_3, \dots)$  of positive integers, and a positive number  $\Theta$ , such that (recall Notations 1.1(F)) for all  $n \in \mathbf{N}$  sufficiently large,

$$(3.3) \quad \Sigma_{S(X, N(n)) / \sqrt{N(n)}} - G_n \in \mathbf{B}_{\text{sym}}^{(m)} [2^{-n} \Theta].$$

(Here and throughout the rest of this note, when the notation  $N_n$  appears in a subscript, it will be written  $N(n)$  for typographical convenience.) It will then follow trivially that each member  $G \in \tilde{\Lambda}$  would be the limit of a subsequence of the matrices  $\Sigma_{S(X, N(n)) / \sqrt{N(n)}}$  (for the integers  $n$  such that  $G_n = G$ ); and property (4) in Theorem 1.4 would then follow as an easy consequence.

We shall return to the matrices  $G_n$  in Step 3.5 below.

STEP 3.2. Refer again to (3.1). In what follows, for convenience, our attention will be “expanded” from  $\Lambda_{(m,a,b)}$  to  $\Lambda_{(m,a/2,2b)}$ .

Define the positive number

$$(3.4) \quad \gamma := a / (20m^2).$$

Define the (“lattice”) set

$$(3.5) \quad \mathbf{L} := \{ \dots, -3\gamma, -2\gamma, -\gamma, 0, \gamma, 2\gamma, 3\gamma, \dots \}$$

(that is, the set of all real numbers of the form  $k\gamma$ ,  $k \in \mathbf{Z}$ ). Let  $\Lambda_{\mathbf{L}}$  denote the set of all  $m \times m$  matrices  $H := (h_{ij}, 1 \leq i, j \leq m) \in \Lambda_{(m,a/2,2b)}$  such that  $h_{ij} \in \mathbf{L}$  for every  $(i, j) \in \{1, \dots, m\}^2$ . By Notations 1.1(D), (E) (see the second sentence after (1.2)), the set  $\Lambda_{(m,a/2,2b)}$  is bounded (as represented as a subset of  $\mathbf{R}^{m^2}$ ). It follows that  $\Lambda_{\mathbf{L}}$  is a finite set. Of course the set  $\Lambda_{\mathbf{L}}$  is nonempty.

(For example,  $cI_m \in \Lambda_{\mathbf{L}}$  where  $c$  is an element of  $\mathbf{L}$  such that  $a/2 < c < a$ —such a  $c$  exists by (3.4).) Define the positive integer

$$(3.6) \quad L := \text{card } \Lambda_{\mathbf{L}}.$$

Let the elements of  $\Lambda_{\mathbf{L}}$  be denoted as  $Q_1^{(1)}, Q_2^{(1)}, \dots, Q_L^{(1)}$ , with the representation

$$(3.7) \quad Q_\ell^{(1)} := (q_{\ell ij}^{(1)}, 1 \leq i, j \leq m)$$

for  $\ell \in \{1, 2, \dots, L\}$ . These matrices  $Q_\ell^{(1)}$  are of course symmetric and positive definite (since they belong to  $\Lambda_{(m, a/2, 2b)}$ ).

STEP 3.3. Two other classes of matrices will be needed. (These matrices will be symmetric but not positive definite.)

For each  $u \in \{1, \dots, m\}$ , let  $Q_u^{(2)} := (q_{uij}^{(2)}, 1 \leq i, j \leq m)$  denote the (symmetric)  $m \times m$  matrix defined by

$$(3.8) \quad q_{uij}^{(2)} := \begin{cases} 1 & \text{if } (i, j) = (u, u), \\ 0 & \text{for all other } (i, j). \end{cases}$$

Now recall the assumption  $m \geq 2$  made in the first sentence of Section 3. Let  $\mathbf{T}$  denote the set of all ordered pairs  $(u, v) \in \{1, \dots, m\}^2$  such that  $u < v$ . For each ordered pair  $(u, v) \in \mathbf{T}$ , let  $Q_{uv}^{(3)} := (q_{uvij}^{(3)}, 1 \leq i, j \leq m)$  denote the (symmetric)  $m \times m$  matrix defined by

$$(3.9) \quad q_{uvij}^{(3)} := \begin{cases} 1 & \text{if } (i, j) \in \{(u, u), (u, v), (v, u), (v, v)\}, \\ 0 & \text{for all other } (i, j). \end{cases}$$

Now to set the stage for the next lemma (and for some other calculations below), note that trivially by (3.1) and (3.4),  $\gamma/(2bL) < \gamma < 10m\gamma < 1$ .

LEMMA 3.4. *For every matrix  $G \in \Lambda_{(m, a, b)}$ , there exists an array*

$$(3.10) \quad c = c(G) := \{c_\ell^{(1)}, \ell \in \{1, 2, \dots, L\}; c_u^{(2)}, u \in \{1, 2, \dots, m\}; c_{uv}^{(3)}, (u, v) \in \mathbf{T}\}$$

*of positive numbers such that the following statements hold:*

$$(3.11) \quad \forall \ell \in \{1, \dots, L\}, \quad \gamma/(2bL) \leq c_\ell^{(1)} \leq 1;$$

$$(3.12) \quad \forall u \in \{1, \dots, m\}, \quad 2m\gamma \leq c_u^{(2)} \leq 10m\gamma;$$

$$(3.13) \quad \forall (u, v) \in \mathbf{T}, \quad 2\gamma \leq c_{uv}^{(3)} \leq 5\gamma;$$

and

$$(3.14) \quad G = \sum_{\ell=1}^L c_\ell^{(1)} Q_\ell^{(1)} + \sum_{u=1}^m c_u^{(2)} Q_u^{(2)} + \sum_{(u,v) \in \mathbf{T}} c_{uv}^{(3)} Q_{uv}^{(3)}.$$

*Proof.* Represent the matrix  $G$  by

$$(3.15) \quad G := (g_{ij}, 1 \leq i, j \leq m).$$

Of course by the hypothesis and Notations 1.1(E),  $G$  is symmetric. For each  $(i, j) \in \{1, \dots, m\}^2$ , let  $\kappa_{ij}$  denote the integer such that (see (3.4))

$$(3.16) \quad \kappa_{ij}\gamma \leq g_{ij} < (\kappa_{ij} + 1)\gamma.$$

Then  $\kappa_{ij} = \kappa_{ji}$ . Define the (symmetric)  $m \times m$  matrix  $H := (h_{ij}, 1 \leq i, j \leq m)$  as follows:

$$(3.17) \quad \forall i \in \{1, \dots, m\}, \quad h_{ii} := (\kappa_{ii} - 8m)\gamma; \quad \text{and}$$

$$(3.18) \quad \forall (i, j) \in \mathbf{T}, \quad h_{ij} = h_{ji} := (\kappa_{ij} - 3)\gamma.$$

Now for each  $(i, j) \in \{1, \dots, m\}^2$ ,

$$(3.19) \quad |g_{ij} - h_{ij}| \leq |g_{ij} - \kappa_{ij}\gamma| + |\kappa_{ij}\gamma - h_{ij}|.$$

In the right-hand side of (3.19), the first term is bounded above by  $\gamma$  (by (3.16)), and the second term is either  $8m\gamma$  (if  $i = j$ ) or  $3\gamma$  (if  $i \neq j$ ), by (3.17)–(3.18). Hence,  $G - H \in \mathbf{B}_{\text{sym}}^{(m)}[9m\gamma]$ . Recall from (3.4) and (3.1) that  $9m^2\gamma < a/2 < b$ . Since (by hypothesis)  $G \in \Lambda_{(m,a,b)}$ , it now follows from Notations 1.1(G) that  $H \in \Lambda_{(m,a/2,2b)}$ . Hence, by (3.17)–(3.18) and the sentence after (3.5),  $H \in \Lambda_{\mathbf{L}}$ . Accordingly (see the sentence after (3.6)), let  $\ell'$  denote the element of  $\{1, \dots, L\}$  such that

$$(3.20) \quad Q_{\ell'}^{(1)} = H.$$

Define the array  $c = c(G)$  in (3.10) (in a slightly unconventional order) as follows: First,

$$(3.21) \quad c_{\ell'}^{(1)} := 1 \quad \text{and} \quad \forall \ell \in \{1, \dots, L\} - \{\ell'\}, \quad c_{\ell}^{(1)} := \gamma/(2bL).$$

Next, for convenience, referring to (3.7), define the  $m \times m$  symmetric matrix  $S := (s_{ij}, 1 \leq i, j \leq m)$  as follows:

$$(3.22) \quad \forall (i, j) \in \{1, \dots, m\}^2, \quad s_{ij} := \sum_{\ell \in \{1, \dots, L\} - \{\ell'\}} c_{\ell}^{(1)} q_{\ell ij}^{(1)}.$$

(By (3.21),  $S = [\gamma/(2bL)] \sum_{\ell \in \{1, \dots, L\} - \{\ell'\}} Q_{\ell}^{(1)}$ ; however, the form (3.22) will be a little more natural for the calculations that follow.) Next, use  $S$  to continue the definition of the array in (3.10) as follows:

$$(3.23) \quad \forall (u, v) \in \mathbf{T}, \quad c_{uv}^{(3)} := [g_{uv} - h_{uv}] - s_{uv}.$$

Finally, use (3.23) itself to complete the definition of the array in (3.10) as follows:

$$(3.24) \quad \forall u \in \{1, \dots, m\}, \quad c_u^{(2)} := [g_{uu} - h_{uu}] - s_{uu} - \sum_{(i,j) \in \mathbf{T}: u \in \{i,j\}} c_{ij}^{(3)}.$$



Now recall from the entire last paragraph of Step 3.2 that  $Q_\ell \in \Lambda_{(m,a/2,2b)}$  for every  $\ell \in \{1, \dots, L\}$ . It follows from (3.7) and Notations 1.1(D), (E) (see the second sentence after (1.2)) that for each  $\ell \in \{1, \dots, L\}$  and each  $(i, j) \in \{1, \dots, m\}^2$ ,  $|q_{\ell ij}^{(1)}| \leq 2b$ . Hence by (3.21) and (3.22),

$$(3.25) \quad \forall (i, j) \in \{1, \dots, m\}^2, \quad |s_{ij}| \leq \gamma,$$

that is,  $S \in \mathbf{B}_{\text{sym}}^{(m)}[\gamma]$ .

Now we shall verify Equations (3.11)–(3.14) (though not quite in that order).

First, (3.11) holds by (3.21) and the sentence after (3.9).

Next, for each  $(u, v) \in \mathbf{T}$ , by (3.23) and (3.18),

$$(3.26) \quad \begin{aligned} c_{uv}^{(3)} &= [g_{uv} - \kappa_{uv}\gamma] + [\kappa_{uv}\gamma - h_{uv}] - s_{uv} \\ &= [g_{uv} - \kappa_{uv}\gamma] + 3\gamma - s_{uv}. \end{aligned}$$

By (3.16) and (3.25), the far right-hand side of (3.26) is bounded below by  $0 + 3\gamma - \gamma$  and bounded above by  $\gamma + 3\gamma + \gamma$ . Hence, (3.13) holds.

Next, let us verify (3.12). For any given  $u \in \{1, \dots, m\}$ , the following holds: The set  $\{(i, j) \in \mathbf{T} : u \in \{i, j\}\}$  has exactly  $m - 1$  elements  $((1, u), \dots, (u - 1, u)$  and  $(u, u + 1), \dots, (u, m))$ , and hence by (3.13) (just proved above),

$$(3.27) \quad 2(m - 1)\gamma \leq \sum_{(i,j) \in \mathbf{T}: u \in \{i,j\}} c_{ij}^{(3)} \leq 5(m - 1)\gamma.$$

Now by (3.24),

$$(3.28) \quad c_u^{(2)} = [g_{uu} - \kappa_{uu}\gamma] + [\kappa_{uu}\gamma - h_{uu}] - s_{uu} - \sum_{(i,j) \in \mathbf{T}: u \in \{i,j\}} c_{ij}^{(3)}.$$

By (3.16), (3.17), (3.25), and (3.27), the right side of (3.28) is bounded below by  $0 + 8m\gamma - \gamma - 5m\gamma$  and bounded above by  $\gamma + 8m\gamma + \gamma - 0$ . Hence, (3.12) holds.

Finally, (3.14) needs to be verified. First, for  $(i, j) \in \mathbf{T}$ , by (3.21), (3.22), (3.8), (3.9), (3.20) (with (3.7)), and (3.23),

$$(3.29) \quad \begin{aligned} &\sum_{\ell=1}^L c_\ell^{(1)} q_{\ell ij}^{(1)} + \sum_{u=1}^m c_u^{(2)} q_{uij}^{(2)} + \sum_{(u,v) \in \mathbf{T}} c_{uv}^{(3)} q_{uvij}^{(3)} \\ &= 1 \cdot q_{\ell' ij}^{(1)} + s_{ij} + 0 + c_{ij}^{(3)} \cdot 1 \\ &= h_{ij} + s_{ij} + c_{ij}^{(3)} = g_{ij}. \end{aligned}$$

Next, recall that the matrices  $G$ ,  $Q_\ell^{(1)}$ ,  $Q_u^{(2)}$ , and  $Q_{uv}^{(3)}$  (and  $S$ ) are symmetric. As a trivial consequence, for  $(i, j) \in \mathbf{T}$ , the far left and far right sides of

(3.29) remain equal if the indices  $i$  and  $j$  are switched. Finally, for each  $i \in \{1, \dots, m\}$ , by (3.21) (again with (3.20)), (3.22), (3.8), (3.9), and (3.24),

$$\begin{aligned} & \sum_{\ell=1}^L c_{\ell}^{(1)} q_{\ell ii}^{(1)} + \sum_{u=1}^m c_u^{(2)} q_{u ii}^{(2)} + \sum_{(u,v) \in \mathbf{T}} c_{uv}^{(3)} q_{uv ii}^{(3)} \\ &= 1 \cdot h_{ii} + s_{ii} + c_i^{(2)} \cdot 1 + \sum_{(u,v) \in \mathbf{T}: i \in \{u,v\}} c_{uv}^{(3)} \cdot 1 = g_{ii}. \end{aligned}$$

From all of these observations, (3.14) holds. That completes the proof of Lemma 3.4. □

STEP 3.5. This step will involve, after some preliminary work, repeated applications of Lemma 2.6. The notation  $h_{(f, \Upsilon(1), \Upsilon(2), \theta, \delta, \varepsilon, N)}$  in Lemma 2.6 (see the sentence after (2.10)) will be used repeatedly, and for typographical convenience it will be written below as  $h(f, \Upsilon_1, \Upsilon_2, \theta, \delta, \varepsilon, N)$ .

For the use of that notation, define (see (3.1), (3.2), (3.4), and (3.6)) the real numbers

$$(3.30) \quad \begin{aligned} \Upsilon_1 &:= \log(\gamma/(3bL)); & \Upsilon_2 &:= \log 2; & \text{and} \\ \delta &:= \delta(\tau^2/[2m(L+1+m)]), \end{aligned}$$

where in the last equality we are using the notation in Lemma 2.5. By (3.1), (3.4), and (3.30),  $\Upsilon_1 < 0 < \Upsilon_2$ . Referring to (2.6)–(2.7), we shall say that a given real, continuous, symmetric function  $f$  on  $[-\pi, \pi]$  satisfies “Condition **C**” if (2.9) and (2.10) hold for the given values in (3.30).

Next, refer to Notations 2.2(B), involving the Fejér kernels. Of course by Fejér’s Theorem, if  $f$  is a (say) real, continuous, symmetric function on  $[-\pi, \pi]$ , then  $(2\pi)^{-1} \int_{-\pi}^{\pi} F_n(\lambda) \cdot f(\lambda) d\lambda$  converges to  $f(0)$  as  $n \rightarrow \infty$ . For a given real, continuous, symmetric function  $f$  on  $[-\pi, \pi]$  and a given  $\varepsilon > 0$ , let  $\mathcal{N}(f, \varepsilon)$  be a positive integer such that

$$(3.31) \quad \forall n \geq \mathcal{N}(f, \varepsilon), \quad \left| f(0) - \int_{-\pi}^{\pi} F_n(\lambda) f(\lambda) \frac{d\lambda}{2\pi} \right| \leq \varepsilon.$$

Next, refer to the sequence  $G_1, G_2, G_3, \dots$  of matrices in  $\Lambda_{(m,a,b)}$  (in fact in  $\tilde{\Lambda}$ ) from the second paragraph of Step 3.1. Applying Lemma 3.4 and using the notations there, define for each positive integer  $n$  the array

$$(3.32) \quad \begin{aligned} \mathbf{c}_n &= c(G_n) \\ &:= \{c_{\ell,n}^{(1)}, \ell \in \{1, 2, \dots, L\}; c_{u,n}^{(2)}, u \in \{1, 2, \dots, m\}; c_{u,v,n}^{(3)}, (u,v) \in \mathbf{T}\} \end{aligned}$$

of positive numbers (satisfying (3.11)–(3.14) with  $G = G_n$ ). By (3.11), (3.12), and (3.13), together with (3.30) and the sentence after (3.9), one has that for each positive integer  $n$  and each number  $c$  in the array  $\mathbf{c}_n$ ,  $\Upsilon_1 < \log c \leq 0 < \Upsilon_2$ .

Now we shall define a sequence of positive integers  $(N_0, N_1, N_2, \dots)$ ; and we shall define, for each positive integer  $n$ , a collection

$$(3.33) \quad \mathcal{C}_n := \{f_{\ell,n}^{(1)}, \ell \in \{1, 2, \dots, L\}; f_{u,n}^{(2)}, u \in \{1, 2, \dots, m\}; f_{u,v,n}^{(3)}, (u, v) \in \mathbf{T}\}$$

of real, continuous, symmetric functions on  $[-\pi, \pi]$  that each satisfy Condition **C** (see the sentence after (3.30)). Notice that for a given positive integer  $n$ , there will be only finitely many functions in this array (3.33)—in fact  $L + m + m(m - 1)/2$  of them. The definition will be recursive in  $n$ , with  $N_{n-1}$  and  $\mathcal{C}_n$  being defined together for  $n = 1, 2, 3, \dots$ . It proceeds as follows:

To start off, define the positive integer  $N_0 := 1$ , and let each of the functions in the collection  $\mathcal{C}_1$  in (3.33) be the trivial constant function with range  $\{0\}$ . Of course a constant function  $f$  on  $[-\pi, \pi]$  satisfies  $\Psi(f) = 0$ . Since  $\Upsilon_1 < 0 < \Upsilon_2$  (as was noted above), it now follows that the (constant) functions in (3.33) (for  $n = 1$ ) satisfy Condition **C**.

Now suppose  $n \geq 1$  is an integer, and the positive integer  $N_{n-1}$  and the real, continuous, symmetric functions in  $\mathcal{C}_n$  in (3.33) have already been defined, and that those functions all satisfy Condition **C**. Define the positive integer

$$(3.34) \quad N_n := N_{n-1} + \max \mathcal{N}(e^f, 2^{-n}),$$

where this maximum is taken over all functions  $f$  in the collection  $\mathcal{C}_n$  in (3.33) for the given  $n$ . (Of course for each such  $f$ , the notation  $e^f$  simply refers to the real, continuous, symmetric function  $\lambda \mapsto e^{f(\lambda)}$  on  $[-\pi, \pi]$ .) Now referring to (3.30), (3.32), and the sentence after (3.32), and applying Lemma 2.6, define the functions in the collection  $\mathcal{C}_{n+1}$  as follows: First, for each  $\ell \in \{1, \dots, L\}$ , define the function  $f_{\ell,n+1}^{(1)}$  by

$$(3.35) \quad f_{\ell,n+1}^{(1)} := h(f_{\ell,n}^{(1)}, \Upsilon_1, \Upsilon_2, \log c_{\ell,n+1}^{(1)}, \delta, 2^{-n}, N_n).$$

Next, for each  $u \in \{1, \dots, m\}$ , define the function  $f_{u,n+1}^{(2)}$  by

$$(3.36) \quad f_{u,n+1}^{(2)} := h(f_{u,n}^{(2)}, \Upsilon_1, \Upsilon_2, \log c_{u,n+1}^{(2)}, \delta, 2^{-n}, N_n).$$

Finally, for each  $(u, v) \in \mathbf{T}$ , define the function  $f_{u,v,n+1}^{(3)}$  by

$$(3.37) \quad f_{u,v,n+1}^{(3)} := h(f_{u,v,n}^{(3)}, \Upsilon_1, \Upsilon_2, \log c_{u,v,n+1}^{(3)}, \delta, 2^{-n}, N_n).$$

That completes the definition of the collection  $\mathcal{C}_{n+1}$ . Note that from (2.11)–(2.12) in Lemma 2.6, each of the functions in this collection  $\mathcal{C}_{n+1}$  satisfies Condition **C**.

That completes the recursive definition of the positive integers  $N_0, N_1, N_2, \dots$  and the collections  $\mathcal{C}_n, n \in \mathbf{N}$ . From (3.34) and the definition of  $N_0$ , one has that

$$(3.38) \quad 1 = N_0 < N_1 < N_2 < \dots$$

STEP 3.6. The next task is to establish a collection

$$(3.39) \quad \mathcal{C} := \{f_\ell^{(1)}, \ell \in \{1, 2, \dots, L\}; f_u^{(2)}, u \in \{1, 2, \dots, m\}; f_{u,v}^{(3)}, (u, v) \in \mathbf{T}\}$$

of “limit functions” on  $[-\pi, \pi]$  from the collections  $\mathcal{C}_n$ .

First, suppose  $\ell \in \{1, \dots, L\}$ . For each positive integer  $n$ , from (3.35) and Equation (2.14) in Lemma 2.6, one has that  $\int_{-\pi}^\pi |f_{\ell,n+1}^{(1)}(\lambda) - f_{\ell,n}^{(1)}(\lambda)| d\lambda < 2^{-n}$ . Hence,  $\int_{-\pi}^\pi \sum_{n=1}^\infty |f_{\ell,n+1}^{(1)}(\lambda) - f_{\ell,n}^{(1)}(\lambda)| d\lambda < \infty$ . Hence  $\sum_{n=1}^\infty |f_{\ell,n+1}^{(1)}(\lambda) - f_{\ell,n}^{(1)}(\lambda)| < \infty$  for a.e.  $\lambda \in [-\pi, \pi]$ . Define the function  $f_\ell^{(1)}$  a.e. on  $[-\pi, \pi]$  as follows:

$$(3.40) \quad f_\ell^{(1)}(\lambda) := \lim_{n \rightarrow \infty} f_{\ell,n}^{(1)}(\lambda).$$

The right-hand side of (3.40) will be defined in  $\mathbf{R}$  for a.e.  $\lambda \in [-\pi, \pi]$ . On the null-set of values  $\lambda$  for which that limit does not exist in  $\mathbf{R}$ , the quantity  $f_\ell^{(1)}(\lambda)$  is left undefined here.

Next, for each  $u \in \{1, \dots, m\}$ , going through the same procedure, but using (3.36) instead of (3.35), define the function  $f_u^{(2)}$  a.e. on  $[-\pi, \pi]$  by

$$(3.41) \quad f_u^{(2)}(\lambda) := \lim_{n \rightarrow \infty} f_{u,n}^{(2)}(\lambda).$$

Finally, for each  $(u, v) \in \mathbf{T}$ , again going through the same procedure, this time using (3.37), define the function  $f_{u,v}^{(3)}$  a.e. on  $[-\pi, \pi]$  by

$$(3.42) \quad f_{u,v}^{(3)}(\lambda) := \lim_{n \rightarrow \infty} f_{u,v,n}^{(3)}(\lambda).$$

That completes the definition of the collection  $\mathcal{C}$  in (3.39). Since each of the functions in each of the collections  $\mathcal{C}_n$  is real and symmetric and satisfies Condition **C**, it follows from (3.40)–(3.42) that each of the functions in the collection  $\mathcal{C}$  is a.e. real and symmetric, with its range being bounded a.e. within the closed interval  $[\Upsilon_1, \Upsilon_2]$ .

STEP 3.7. Next, some calculations involving Fejér kernels will be given. Later on, they will play a key role in obtaining bounds on the covariance matrices for partial sums of sequences of random vectors ( $\mathbf{R}^m$ -valued random variables).

For each positive integer  $n$ , define the array

$$(3.43) \quad \mathbf{c}_n^* := \{c_{\ell,n}^{*(1)}, \ell \in \{1, 2, \dots, L\}; c_{u,n}^{*(2)}, u \in \{1, 2, \dots, m\}; c_{u,v,n}^{*(3)}, (u, v) \in \mathbf{T}\}$$

of positive numbers as follows: First, for each  $\ell \in \{1, \dots, L\}$ , referring to (3.38), (3.40), and (2.4), define the positive number

$$(3.44) \quad c_{\ell,n}^{*(1)} := \int_{-\pi}^\pi F_{N(n)}(\lambda) \cdot \exp(f_\ell^{(1)}(\lambda)) \frac{d\lambda}{2\pi}.$$

Next, for each  $u \in \{1, \dots, m\}$ , referring to (3.41), define the positive number

$$(3.45) \quad c_{u,n}^{*(2)} := \int_{-\pi}^{\pi} F_{N(n)}(\lambda) \cdot \exp(f_u^{(2)}(\lambda)) \frac{d\lambda}{2\pi}.$$

Finally, for each  $(u, v) \in \mathbf{T}$ , referring to (3.42), define the positive number

$$(3.46) \quad c_{u,v,n}^{*(3)} := \int_{-\pi}^{\pi} F_{N(n)}(\lambda) \cdot \exp(f_{u,v}^{(3)}(\lambda)) \frac{d\lambda}{2\pi}.$$

That completes the definition of the array  $\mathbf{c}_n^*$  in (3.43).

Our next task is to compare the arrays  $\mathbf{c}_n$  and  $\mathbf{c}_n^*$  in (3.32) and (3.43).

To start that process, suppose  $n \geq 2$ , and suppose  $\ell \in \{1, \dots, L\}$ . By (3.34),  $N_n > \mathcal{N}(\exp f_{\ell,n}^{(1)}, 2^{-n})$ ; and hence by (3.31),

$$(3.47) \quad \left| \exp(f_{\ell,n}^{(1)}(0)) - \int_{-\pi}^{\pi} F_{N(n)}(\lambda) \cdot \exp(f_{\ell,n}^{(1)}(\lambda)) \frac{d\lambda}{2\pi} \right| \leq 2^{-n}.$$

Also, for each integer  $p \geq n$ , one has that  $f_{\ell,p+1}^{(1)} := h(f_{\ell,p}^{(1)}, \Upsilon_1, \Upsilon_2, \log c_{\ell,p+1}^{(1)}, \delta, 2^{-p}, N_p)$  by (3.35), and since  $N_n \leq N_p$  by (3.38) one therefore has from Equation (2.15) in Lemma 2.6 that

$$(3.48) \quad \left| \int_{-\pi}^{\pi} F_{N(n)}(\lambda) \cdot \exp(f_{\ell,p+1}^{(1)}(\lambda)) \frac{d\lambda}{2\pi} - \int_{-\pi}^{\pi} F_{N(n)}(\lambda) \cdot \exp(f_{\ell,p}^{(1)}(\lambda)) \frac{d\lambda}{2\pi} \right| \leq 2^{-p}.$$

By (3.47) and (3.48), using a telescoping sum, one has that

$$(3.49) \quad \forall p \geq n + 1, \quad \left| \exp(f_{\ell,n}^{(1)}(0)) - \int_{-\pi}^{\pi} F_{N(n)}(\lambda) \cdot \exp(f_{\ell,p}^{(1)}(\lambda)) \frac{d\lambda}{2\pi} \right| \leq 2^{-n} + [2^{-n} + 2^{-(n+1)} + \dots + 2^{-(p-1)}] \leq 3 \cdot 2^{-n}.$$

Now recall that for each  $p \geq 1$ , the function  $f_{\ell,p}^{(1)}$  satisfies Condition **C** and is therefore bounded between  $\Upsilon_1$  and  $\Upsilon_2$ , and hence the function  $\exp f_{\ell,p}^{(1)}$  is bounded between  $\exp \Upsilon_1$  and  $\exp \Upsilon_2$ . Since any given Fejér Kernel is bounded, one now has by (3.44), (3.40), (3.49), and dominated convergence (taking the limit as  $p \rightarrow \infty$ ) that for our given fixed  $n$  and  $\ell$ ,

$$(3.50) \quad \begin{aligned} & \left| \exp(f_{\ell,n}^{(1)}(0)) - c_{\ell,n}^{*(1)} \right| \\ &= \left| \exp(f_{\ell,n}^{(1)}(0)) - \int_{-\pi}^{\pi} F_{N(n)}(\lambda) \cdot \exp(f_{\ell}^{(1)}(\lambda)) \frac{d\lambda}{2\pi} \right| \\ &\leq 3 \cdot 2^{-n}. \end{aligned}$$

Recall our supposition here that  $n \geq 2$ . From (3.35) (with  $n + 1$  replaced by  $n$ ) and Equation (2.13) in Lemma 2.6, one has that

$$(3.51) \quad |f_{\ell,n}^{(1)}(0) - \log c_{\ell,n}^{(1)}| < 2^{-(n-1)}.$$

Since  $f_{\ell,n}^{(1)}$  satisfies condition **C**, one trivially has (see (3.30) and the sentence after it) that  $f_{\ell,n}^{(1)}(0) < \log 2$ . From the sentence after (3.32), one also has that  $\log c_{\ell,n}^{(1)} \leq 0 < \log 2$ . Since  $de^x/dx = e^x \leq 2$  for  $x \leq \log 2$ , it now follows from (3.51) and trivial calculus that  $|\exp(f_{\ell,n}^{(1)}(0)) - c_{\ell,n}^{(1)}| \leq 4 \cdot 2^{-n}$ . Hence by (3.50),  $|c_{\ell,n}^{*(1)} - c_{\ell,n}^{(1)}| \leq 7 \cdot 2^{-n}$ .

Let us display for convenient reference what we have just verified:

$$(3.52) \quad \forall n \geq 2, \forall \ell \in \{1, \dots, L\}, \quad |c_{\ell,n}^{*(1)} - c_{\ell,n}^{(1)}| \leq 7 \cdot 2^{-n}.$$

With arguments exactly analogous to that of (3.52), using (3.41)–(3.42) and (3.45)–(3.46) in place of (3.40) and (3.44), one has that

$$(3.53) \quad \forall n \geq 2, \forall u \in \{1, \dots, m\}, \quad |c_{u,n}^{*(2)} - c_{u,n}^{(2)}| \leq 7 \cdot 2^{-n}$$

and that

$$(3.54) \quad \forall n \geq 2, \forall (u, v) \in \mathbf{T}, \quad |c_{u,v,n}^{*(3)} - c_{u,v,n}^{(3)}| \leq 7 \cdot 2^{-n}.$$

We shall return to (3.52)–(3.54) later on.

**STEP 3.8.** Our task in this step is to construct the random sequence  $X$  for Theorem 1.4. That will be done with a family of “building blocks” that are independent of each other, each one being a stationary real mean-zero Gaussian random sequence with a particular spectral density function. We shall use the well-known fact that any real, nonnegative, Borel, symmetric, integrable function on  $[-\pi, \pi]$  is the spectral density function of some stationary real mean-zero Gaussian random sequence.

Refer to (3.40), (3.41), and (3.42). For each  $\ell \in \{1, \dots, L\}$  and each  $p \in \{1, \dots, m\}$ , let  $X^{(1,\ell,p)} := (X_k^{(1,\ell,p)}, k \in \mathbf{Z})$  be a stationary real mean-zero Gaussian random sequence with spectral density function  $\exp f_\ell^{(1)}$  on  $[-\pi, \pi]$ . For each  $u \in \{1, \dots, m\}$ , let  $X^{(2,u)} := (X_k^{(2,u)}, k \in \mathbf{Z})$  be a stationary real mean-zero Gaussian random sequence with spectral density function  $\exp f_u^{(2)}$  on  $[-\pi, \pi]$ . For each  $(u, v) \in \mathbf{T}$ , let  $X^{(3,u,v)} := (X_k^{(3,u,v)}, k \in \mathbf{Z})$  be a stationary real mean-zero Gaussian random sequence with spectral density function  $\exp f_{u,v}^{(3)}$  on  $[-\pi, \pi]$ . Let these random sequences be constructed in such a way that they are all independent of each other.

Refer to Notations 1.1(C). For each  $\ell \in \{1, \dots, L\}$ , let  $(Q_\ell^{(1)})^{1/2}$  denote the symmetric positive definite  $m \times m$  “square root” matrix of  $Q_\ell^{(1)}$ . (Recall the sentence after (3.7).)

Define the sequence  $X := (X_k, k \in \mathbf{Z})$  of  $\mathbf{R}^m$ -valued random variables as follows: For each  $k \in \mathbf{Z}$ ,

$$(3.55) \quad X_k := \sum_{\ell=1}^L (Q_\ell^{(1)})^{1/2} [X_k^{(1,\ell,1)}, X_k^{(1,\ell,2)}, \dots, X_k^{(1,\ell,m)}]^t + \sum_{u=1}^m X_k^{(2,u)} \mathbf{e}_u + \sum_{(u,v) \in \mathbf{T}} X_k^{(3,u,v)} (\mathbf{e}_u + \mathbf{e}_v).$$

Here and below, for  $p \in \{1, \dots, m\}$ ,  $\mathbf{e}_p := [0, \dots, 0, 1, 0, \dots, 0]^t$  where the 1 is the  $p$ th coordinate. The first sum in the right-hand side of (3.55) involves matrix multiplication; the other two involve simple scalar multiplication. By elementary arguments,  $X$  is a strictly stationary, Gaussian sequence of  $\mathbf{R}^m$ -valued, mean- $\mathbf{0}_m$  random variables. Our task now is to verify properties (1)–(4) stipulated in Theorem 1.4. The “mixing properties” (1)–(3) will be verified in Step 3.9, and property (4) will be verified in Step 3.10.

STEP 3.9. In this step, the mixing properties (1), (2), and (3) stipulated in Theorem 1.4 will be verified (though not in that order).

For each positive integer  $n$ , by (3.55) and Lemma 2.1 (and the independence of the “building block” sequences in the second paragraph of Step 3.8),

$$(3.56) \quad \rho(X, n) \leq \max \left\{ \max_{1 \leq \ell \leq L, 1 \leq p \leq m} \rho(X^{(1,\ell,p)}, n), \max_{1 \leq u \leq m} \rho(X^{(2,u)}, n), \max_{(u,v) \in \mathbf{T}} \rho(X^{(3,u,v)}, n) \right\}$$

and

$$(3.57) \quad I(X, n) \leq \sum_{\ell=1}^L \sum_{p=1}^m I(X^{(1,\ell,p)}, n) + \sum_{u=1}^m I(X^{(2,u)}, n) + \sum_{(u,v) \in \mathbf{T}} I(X^{(3,u,v)}, n).$$

Next some calculations connected with information regularity are needed for the “building block” sequences.

Referring to the sentence containing (3.33), one has that for each  $\ell \in \{1, \dots, L\}$  and each positive integer  $n$ ,  $\Upsilon_1 < f_{\ell,n}^{(1)}(\lambda) < \Upsilon_2$  for all  $\lambda \in [-\pi, \pi]$ , and also  $\Psi(f_{\ell,n}^{(1)}) < \delta$ . Hence for each  $\ell \in \{1, \dots, L\}$ , by (3.40) and Notations 2.4(C),  $\Psi(f_\ell^{(1)}) \leq \delta$ . Hence for a given  $\ell \in \{1, \dots, L\}$  and a given  $p \in \{1, \dots, m\}$ , by the second paragraph in Step 3.8, one has from (3.30) and Lemma 2.5 that (i)  $I(X^{(1,\ell,p)}, 1) \leq \tau^2/[2m(L+1+m)]$ , and hence by [3, v3, Lemma 27.9(I)(II)], one also has that (ii)  $\rho(X^{(1,\ell,p)}, 1) \leq [2I(X^{(1,\ell,p)}, 1)]^{1/2} \leq \tau$ , and that (iii)  $I(X^{(1,\ell,p)}, n) \rightarrow 0$  and  $\rho(X^{(1,\ell,p)}, n) \rightarrow 0$  as  $n \rightarrow \infty$ . By exactly analogous arguments, using (3.41) and (3.42) in place of (3.40), one

obtains (i), (ii), and (iii) with  $X^{(2,u)}$  (for  $u \in \{1, \dots, m\}$ ) and with  $X^{(3,u,v)}$  (for  $(u, v) \in \mathbf{T}$ ) in place of  $X^{(1,\ell,p)}$ .

Hence by (3.55) and (3.57) (and (3.2)),

$$I(X, 1) \leq [mL + m + m(m-1)/2] \cdot \tau^2 / [2m(L+1+m)] \leq \tau^2 \leq \tau;$$

and hence by (1.8),  $\alpha(X, 1) \leq \beta(X, 1) \leq \tau$ ; and also by (3.55) and (3.56),  $\rho(X, 1) \leq \tau$ . Also, by (3.55), (3.56), and (3.57),  $I(X, n) \rightarrow 0$  and  $\rho(X, n) \rightarrow 0$  as  $n \rightarrow \infty$ ; and hence also by (1.8),  $\alpha(X, n) \rightarrow 0$  and  $\beta(X, n) \rightarrow 0$  as  $n \rightarrow \infty$ . Thus properties (2) and (3) in Theorem 1.4 hold.

Next, recall from above that for a given  $\ell \in \{1, \dots, L\}$  and a given positive integer  $n$ , one has that  $\Upsilon_1 < f_{\ell,n}^{(1)}(\lambda) < \Upsilon_2$  for all  $\lambda \in [-\pi, \pi]$ . Hence by (3.40), for a given  $\ell \in \{1, \dots, L\}$ ,  $\Upsilon_1 \leq f_\ell^{(1)}(\lambda) \leq \Upsilon_2$  for a.e.  $\lambda \in [-\pi, \pi]$ . Hence by the second paragraph of Step 3.8, for a given  $\ell \in \{1, \dots, L\}$  and a given  $p \in \{1, \dots, m\}$ , the stationary Gaussian sequence  $X^{(1,\ell,p)}$  has a spectral density function that is bounded a.e. between the two positive constants  $\exp \Upsilon_1$  and  $\exp \Upsilon_2$ , and hence by Lemma 2.3 it satisfies  $\rho^*(X^{(1,\ell,p)}, 1) < 1$ . By exactly analogous arguments, using (3.41) and (3.42) in place of (3.40), one has that  $\rho^*(X^{(2,u)}, 1) < 1$  for  $u \in \{1, \dots, m\}$  and that  $\rho^*(X^{(3,u,v)}, 1) < 1$  for  $(u, v) \in \mathbf{T}$ . Now by (3.55) and Lemma 2.1, Equation (3.56) holds with each  $\rho$  replaced by  $\rho^*$ . It now follows that  $\rho^*(X, 1) < 1$ . Thus, property (1) in Theorem 1.4 holds.

STEP 3.10. In this final step, we shall verify property (4) in Theorem 1.4, by showing that for the sequence  $(N_1, N_2, N_3, \dots)$  of positive integers defined in Step 3.5 (see (3.38)), there exists a positive number  $\Theta$  such that (3.3) holds for all  $n \geq 2$ .

Refer again to the second paragraph of Step 3.8, where the sequences  $X^{(1,\ell,p)}$ ,  $X^{(2,u)}$ , and  $X^{(3,u,v)}$  are defined. One of course has that for each  $\ell \in \{1, \dots, L\}$ , each  $p \in \{1, \dots, m\}$ , and each  $n \in \mathbf{N}$ ,  $E[n^{-1/2}S(X^{(1,\ell,p)}, n)] = 0$ ; and the analogous comment applies with  $X^{(1,\ell,p)}$  replaced by  $X^{(2,u)}$  or  $X^{(3,u,v)}$ . (That should be kept in mind in the calculations that follow.) By (3.44) and Notations 2.2(B), (C) (and the second paragraph of Step 3.8), for each  $\ell \in \{1, \dots, L\}$ , each  $p \in \{1, \dots, m\}$  and each  $n \in \mathbf{N}$ ,

$$(3.58) \quad E[N_n^{-1/2}S(X^{(1,\ell,p)}, N_n)]^2 \\ = \int_{-\pi}^{\pi} F_{N(n)}(\lambda) \cdot \exp(f_\ell^{(1)}(\lambda)) \frac{d\lambda}{2\pi} = c_{\ell,n}^{*(1)}.$$

By similar arguments using (3.45) and (3.46) in place of (3.44), one has that for each  $u \in \{1, \dots, m\}$  and each  $n \in \mathbf{N}$ ,

$$(3.59) \quad E[N_n^{-1/2}S(X^{(2,u)}, N_n)]^2 \\ = \int_{-\pi}^{\pi} F_{N(n)}(\lambda) \cdot \exp(f_u^{(2)}(\lambda)) \frac{d\lambda}{2\pi} = c_{u,n}^{*(2)},$$



and that for each  $(u, v) \in \mathbf{T}$  and each  $n \in \mathbf{N}$ ,

$$(3.60) \quad \begin{aligned} E[N_n^{-1/2} S(X^{(3,u,v)}, N_n)]^2 \\ = \int_{-\pi}^{\pi} F_{N(n)}(\lambda) \cdot \exp(f_{u,v}^{(3)}(\lambda)) \frac{d\lambda}{2\pi} = c_{u,v,n}^{*(3)}. \end{aligned}$$

In what will now follow, we shall repeatedly use the fact that if  $V$  is an  $\mathbf{R}^m$ -valued random variable such that  $EV = \mathbf{0}_m$  and  $E\|V\|^2 < \infty$ , then the  $m \times m$  covariance matrix  $\Sigma_V$  can be written simply as  $\Sigma_V = EVV^t$ .

For each  $\ell \in \{1, \dots, L\}$  and each  $n \in \mathbf{N}$ , define the  $\mathbf{R}^m$ -valued random variable

$$Y_n^{(\ell)} := n^{-1/2} [S(X^{(1,\ell,1)}, n), S(X^{(1,\ell,2)}, n), \dots, S(X^{(1,\ell,m)}, n)]^t.$$

By (3.58) and the independence of the sequences  $X^{(1,\ell,p)}$ ,  $p \in \{1, \dots, m\}$  (again see the second paragraph of Step 3.8), one has that for each  $\ell \in \{1, \dots, L\}$  and each  $n \in \mathbf{N}$ , the  $\mathbf{R}^m$ -valued random variable  $Y_{N(n)}^{(\ell)}$  has mean vector  $\mathbf{0}_m$  and covariance matrix  $EY_{N(n)}^{(\ell)}(Y_{N(n)}^{(\ell)})^t = c_{\ell,n}^{*(1)} I_m$ . Hence for each  $\ell \in \{1, \dots, L\}$  and each  $n \in \mathbf{N}$  (recall that the matrix  $(Q_\ell^{(1)})^{1/2}$  is symmetric), the  $\mathbf{R}^m$ -valued random vector

$$N_n^{-1/2} \sum_{k=1}^{N(n)} (Q_\ell^{(1)})^{1/2} [X_k^{(1,\ell,1)}, X_k^{(1,\ell,2)}, \dots, X_k^{(1,\ell,m)}]^t = (Q_\ell^{(1)})^{1/2} Y_{N(n)}^{(\ell)}$$

has mean vector  $\mathbf{0}_m$  and covariance matrix

$$(3.61) \quad \begin{aligned} E(Q_\ell^{(1)})^{1/2} Y_{N(n)}^{(\ell)} (Y_{N(n)}^{(\ell)})^t ((Q_\ell^{(1)})^{1/2})^t \\ = (Q_\ell^{(1)})^{1/2} c_{\ell,n}^{*(1)} I_m (Q_\ell^{(1)})^{1/2} = c_{\ell,n}^{*(1)} Q_\ell^{(1)}. \end{aligned}$$

By (3.59) and the entire sentence containing (3.8), for each  $u \in \{1, \dots, m\}$ , the  $\mathbf{R}^m$ -valued random variable

$$N_n^{-1/2} \sum_{k=1}^{N(n)} X_k^{(2,u)} \mathbf{e}_u = N_n^{-1/2} [S(X^{(2,u)}, N_n)] \mathbf{e}_u$$

trivially has mean vector  $\mathbf{0}_m$  and covariance matrix

$$(3.62) \quad c_{u,n}^{*(2)} \mathbf{e}_u \mathbf{e}_u^t = c_{u,n}^{*(2)} Q_u^{(2)}.$$

Similarly, by (3.60) and the entire sentence containing (3.9), for each  $(u, v) \in \mathbf{T}$ , the  $\mathbf{R}^m$ -valued random variable

$$N_n^{-1/2} \sum_{k=1}^{N(n)} X_k^{(3,u,v)} (\mathbf{e}_u + \mathbf{e}_v) = N_n^{-1/2} [S(X^{(3,u,v)}, N_n)] (\mathbf{e}_u + \mathbf{e}_v)$$

has mean vector  $\mathbf{0}_m$  and covariance matrix

$$(3.63) \quad c_{u,v,n}^{*(3)} (\mathbf{e}_u + \mathbf{e}_v) (\mathbf{e}_u + \mathbf{e}_v)^t = c_{u,v,n}^{*(3)} Q_{u,v}^{(3)}.$$

Now we use the elementary equality  $\Sigma_{Y+Z+\dots+V} = \Sigma_Y + \Sigma_Z + \dots + \Sigma_V$  for an arbitrary finite collection  $Y, Z, \dots, V$  of independent  $\mathbf{R}^m$ -valued random variables whose coordinates have finite second moments. By (3.55) and the independence of the sequences in the second paragraph of Step 3.8, followed by the entire sentences containing (3.61), (3.62), and (3.63), one has that for each  $n \in \mathbf{N}$ , the  $\mathbf{R}^m$ -valued random variable  $N_n^{-1/2}S(X, N_n)$  has mean vector  $\mathbf{0}_m$  and covariance matrix

$$(3.64) \quad G_n^* := \sum_{\ell=1}^L c_{\ell,n}^{*(1)} Q_\ell^{(1)} + \sum_{u=1}^m c_{u,n}^{*(2)} Q_u^{(2)} + \sum_{(u,v) \in \mathbf{T}} c_{u,v,n}^{*(3)} Q_{u,v}^{(3)}.$$

Now from (3.32) and (3.14), for each  $n \in \mathbf{N}$ ,

$$(3.65) \quad G_n = \sum_{\ell=1}^L c_{\ell,n}^{(1)} Q_\ell^{(1)} + \sum_{u=1}^m c_{u,n}^{(2)} Q_u^{(2)} + \sum_{(u,v) \in \mathbf{T}} c_{u,v,n}^{(3)} Q_{u,v}^{(3)}.$$

Recall from the final paragraph of Step 3.2 that  $Q_\ell^{(1)} \in \Lambda_{(m,a/2,2b)}$  for each  $\ell \in \{1, \dots, L\}$ . By (3.7) and Notations 1.1(D), (E) (see the second sentence after (1.2)), one has that  $|q_{\ell ij}^{(1)}| \leq 2b$  for each  $\ell \in \{1, \dots, L\}$  and each  $(i, j) \in \{1, \dots, m\}^2$ . Taking that together with the entire sentences containing (3.8) and (3.9), and then using (3.52), (3.53), and (3.54), one obtains from (3.64) and (3.65) that for each  $n \geq 2$ ,

$$G_n^* - G_n \in \mathbf{B}_{\text{sym}}^{(m)} [((2b \cdot L) + (1 \cdot m) + (1 \cdot m(m-1)/2)) \cdot 7 \cdot 2^{-n}].$$

Referring again to the entire sentence containing (3.64), one has that for the positive number  $\Theta := 7 \cdot [2bL + m + m(m-1)/2]$ , Equation (3.3) holds for all  $n \geq 2$ . That completes the proof of property (4) in Theorem 1.4. The proof of Theorem 1.4 is complete.

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