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UNIQUENESS THEOREM FOR NON-ARCHIMEDEAN ANALYTIC CURVES INTERSECTING HYPERPLANES WITHOUT COUNTING MULTIPLICITIES

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ABSTRACT. In this paper, we prove uniqueness theorems for analytic curves from $\mathbf{F}$ to $\mathbb{P}^n(\mathbf{F})$ sharing hyperplanes in general position without counting multiplicities, where $\mathbf{F}$ is a complete algebraically closed non-Archimedean field of arbitrary characteristic.

1. Introduction

Let $\mathbf{F}$ be an algebraically closed field complete with respect to a non-Archimedean absolute value $|\cdot|$.

In [1], Adams and Straus proved the following uniqueness theorem.

THEOREM A. *Let f and g be two nonconstant meromorphic functions on $\mathbf{F}$, where $\mathbf{F}$ has characteristic zero. Let a_1, a_2, a_3 and a_4 be four distinct values. Assume that $f^{-1}(a_i) = g^{-1}(a_i)$ for $i = 1, 2, 3, 4$. Then $f \equiv g$.*

Obviously, Theorem A is an analog of Nevanlinna's five-value theorem in the complex case (see [4]). Furthermore, they gave the example

$$f(z) = \frac{z}{z^2 - z + 1} \quad \text{and} \quad g(z) = \frac{z^2}{z^2 - z + 1}$$

to show that Theorem A is optimal since $f^{-1}(0) = g^{-1}(0)$, $f^{-1}(1) = g^{-1}(1)$, and $f^{-1}(\infty) = g^{-1}(\infty)$.

In 2001, Ru [5] extended Theorem A to non-Archimedean analytic curves in projective space.

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A non-Archimedean analytic curve f is a map $f=[f_0:\cdots:f_n]:\mathbf{F}\to\mathbb{P}^n(\mathbf{F})$, where $f_0,\dots,f_n$ are entire functions on $\mathbf{F}$ without common zeros. $(f_0,\dots,f_n)$ is called a reduced representation of f.

A non-Archimedean analytic curve $f:\mathbf{F}\to\mathbb{P}^n(\mathbf{F})$ is said to be linearly non-degenerate (over $\mathbf{F}$) if $f(\mathbf{F})$ is not contained in any proper linear subspace of $\mathbb{P}^n(\mathbf{F})$.

Hyperplanes $H_1,\dots,H_q$ in $\mathbb{P}^n(\mathbf{F})$ are said to be in general position if any $n+1$ of them are linearly independent.

Ru showed the following theorem.

THEOREM B ([5, Theorem 2.2]). *Let $f,g:\mathbf{F}\to\mathbb{P}^n(\mathbf{F})$ be two linearly non-degenerate analytic curves, where $\mathbf{F}$ has characteristic zero. Let $H_1,\dots,H_{3n+1}$ be hyperplanes in $\mathbb{P}^n(\mathbf{F})$ located in general position. Assume that $f^{-1}(H_j)=g^{-1}(H_j)$ for $1\le j\le 3n+1$ and $f^{-1}(H_i)\cap f^{-1}(H_j)=\emptyset$ for $i\neq j$. If $f(z)=g(z)$ for every point $z\in\bigcup_{j=1}^{3n+1}f^{-1}(H_j)$, then $f\equiv g$.*

In this paper, we will improve and generalize Theorem B as follows.

THEOREM 1. *Let $f,g:\mathbf{F}\to\mathbb{P}^n(\mathbf{F})$ be two linearly non-degenerate analytic curves, where $\mathbf{F}$ has characteristic zero. Let $H_1,\dots,H_{2n+2}$ be hyperplanes in $\mathbb{P}^n(\mathbf{F})$ located in general position. Assume that $f^{-1}(H_j)=g^{-1}(H_j)$ for $1\le j\le 2n+2$ and $f^{-1}(H_i)\cap f^{-1}(H_j)=\emptyset$ for $i\neq j$. If $f(z)=g(z)$ for every point $z\in\bigcup_{j=1}^{2n+2}f^{-1}(H_j)$, then $f\equiv g$.*

REMARK 1. (a) When $n=1$, Theorem 1 reduces to Theorem A.

(b) Our key technique is Lemma 5, which gives a new estimate for the divisor of $(f,H_i)(g,H_j)-(f,H_j)(g,H_i)\not\equiv 0$. This method does not work for $f_1\wedge\cdots\wedge f_\lambda$, where $f_1,\dots,f_\lambda$ are linearly non-degenerate analytic curves. Hence, we cannot improve Theorem 2.1 in [5].

Now, we consider that $\mathbf{F}$ has positive characteristic.

Denote $\mathcal{E}$ the ring of entire functions on $\mathbf{F}$ and $\mathcal{M}$ the field of meromorphic functions on $\mathbf{F}$. If $\mathbf{F}$ has positive characteristic p and s is a positive integer, let $\mathcal{E}[p^s]=\{g^{p^s}|g\in\mathcal{E}\}$ and $\mathcal{M}[p^s]$ be the fraction field of $\mathcal{E}[p^s]$. Note that $\mathcal{M}[p^{s+1}]\subset\mathcal{M}[p^s]$ (see Proposition 3.4 in [2]).

If an analytic curve $f:\mathbf{F}\to\mathbb{P}^n(\mathbf{F})$ is linearly non-degenerate over $\mathbf{F}$, where $\mathbf{F}$ has positive characteristic p, then f is also linearly non-degenerate over $\mathcal{M}[p^s]$ for some integer $s\ge 1$ (see Lemma 5.2 in [2]). Hence, we can define the index of independence of f be the smallest integer s such that f linearly non-degenerate over $\mathbf{F}$ remains linearly non-degenerate over $\mathcal{M}[p^s]$.

We can generalize Theorem 1 to the case of positive characteristic.

THEOREM 2. *Let $\mathbf{F}$ have positive characteristic p, and $f,g:\mathbf{F}\to\mathbb{P}^n(\mathbf{F})$ be two analytic curves linearly non-degenerate over $\mathbf{F}$ with index of independence $\le s$. Let $H_1,\dots,H_{2p^{s-1}n+2}$ be $2p^{s-1}n+2$ hyperplanes in $\mathbb{P}^n(\mathbf{F})$ located in general position. Assume that $f^{-1}(H_j)=g^{-1}(H_j)$ for $1\le j\le 2p^{s-1}n+2$*

and $f^{-1}(H_i) \cap f^{-1}(H_j) = \emptyset$ *for* $i \neq j$. *If* $f(z) = g(z)$ *for every point* $z \in \bigcup_{j=1}^{2p^{s-1}n+2} f^{-1}(H_j)$, *then* $f \equiv g$.

There are several open questions related to the above results.

QUESTION 1. Is it true that the number of hyperplanes can be replaced by a smaller one?

QUESTION 2. The conditions "$f^{-1}(H_i) \cap f^{-1}(H_j) = \emptyset$ for $1 \leq i < j \leq q$" and "$f(z) = g(z)$ on $\bigcup_{j=1}^{q} f^{-1}(H_j)$" in the above theorems are not natural. Can one remove them?

2. Preliminaries

Let $\mathbf{F}$ be an algebraically closed field of characteristic $p \geq 0$, complete with respect to a non-Archimedean absolute value $|\cdot|$.

Recall that an infinite sum converges in a non-Archimedean norm if and only if its general term approaches zero. Thus, a function of the form

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbf{F}$$

is well defined whenever

$$\left|a_n z^n\right| \to 0 \quad \text{as } n \to \infty.$$

Functions of this type are called analytic functions of a non-Archimedean variable. If h is analytic on $\mathbf{F}$, then h is called an entire function on $\mathbf{F}$. Let

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbf{F}$$

be an analytic function on $|z| < R$. For $0 < r < R$, define

$$M_h(r) = \max_{|z|=r} |h(z)|.$$

We have the following lemma.

LEMMA 3. [1] *The following statements hold*:

(1) *We have* $M_h(r) = \max_{n \geq 0} |a_n| r^n$.
(2) *The maximum on the right of* (1) *is attained for a unique value of* n *except for a discrete sequence of values* $\{r_\nu\}$ *in the open interval* $(0, R)$.
(3) *If* $r \notin \{r_\nu\}$ *and* $|z| = r < R$, *then* $|h(z)| = M_h(r)$.
(4) *If* h *is a nonconstant entire function, then* $M_h(r) \to \infty$ *as* $r \to \infty$.
(5) *We have* $M_{fg}(r) = M_f(r) M_g(r)$ *for any analytic functions* f *and* g.

For a given entire function $h(z) = \sum_{n=0}^{\infty} a_n z^n$, define the kth Hasse derivative of h by

$$D^k h = \sum_{n=k}^{\infty} \binom{n}{k} a_n z^{n-k},$$

which is also analytic. Note that $D^0 h = h$ and $D^1 h = h'$. In characteristic zero, the Hasse derivative $D^k h$ is simply $h^{(k)}/k!$. Hasse derivatives are more useful than ordinary derivatives in positive characteristic and have similar properties (see [2]).

Lemma 4 (Logarithmic derivative lemma). *Let h be an entire function on $\mathbf{F}$. Then*

$$M_{\frac{D^k h}{h}}(r) \le \frac{1}{r^k} \quad (r > 0).$$

In particular, we have $M_{h^{(k)}/h}(r) \le \frac{1}{r^k}$ for characteristic zero.

For a nonzero entire function h on $\mathbf{F}$, we denote the divisor of h by ν_h. For $z_0 \in \mathbf{F}$, $\nu_h(z_0) := \mathrm{ord}_{z_0}(h)$.

Denote ν_h^M the divisor of h with truncated multiplicity by a positive integer M. That means, for $z_0 \in \mathbf{F}$, $\nu_h^M(z_0) := \min\{M, \nu_h(z_0)\}$.

We define $\nu^1_{h,=k}$ be the divisor of all zeros of h with multiplicity k, without counting multiplicity. Hence,

$$\nu^1_{h,=k}(z_0) = \begin{cases} 0, & \text{if } \nu_h(z_0) \ne k, \\ 1, & \text{if } \nu_h(z_0) = k, \end{cases}$$

for $z_0 \in \mathbf{F}$.

3. Proof of main results

Assume that $f = [f_0 : \cdots : f_n]$ and $g = [g_0 : \cdots : g_n]$ are linearly non-degenerate analytic curves. Let $H_1, \ldots, H_q$ be q $(\ge 2n)$ hyperplanes, located in general position. We denote $H_j = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n(\mathbf{F}) | a_{j0}x_0 + \cdots + a_{jn}x_n = 0\}$, $(f, H_j) = a_{j0}f_0 + \cdots + a_{jn}f_n$, and $(g, H_j) = a_{j0}g_0 + \cdots + a_{jn}g_n$, $1 \le j \le q$. Obviously, $(f, H_j) \not\equiv 0$ and $(g, H_j) \not\equiv 0$ for $1 \le j \le q$.

Proof of Theorem 1. Suppose that $f \not\equiv g$. By changing indices if necessary, we may assume that

$$\underbrace{\frac{(f,H_1)}{(g,H_1)} \equiv \frac{(f,H_2)}{(g,H_2)} \equiv \cdots \equiv \frac{(f,H_{k_1})}{(g,H_{k_1})}}_{\text{group 1}}$$
$$\not\equiv \underbrace{\frac{(f,H_{k_1+1})}{(g,H_{k_1+1})} \equiv \cdots \equiv \frac{(f,H_{k_2})}{(g,H_{k_2})}}_{\text{group 2}}$$
$$\not\equiv \cdots \not\equiv \underbrace{\frac{(f,H_{k_{t-1}+1})}{(g,H_{k_{t-1}+1})} \equiv \cdots \equiv \frac{(f,H_{k_t})}{(g,H_{k_t})}}_{\text{group } t},$$

where $k_t = q$.

Since $f \not\equiv g$, the number of elements of every group is at most n.

We define the map $\sigma : \{1,\dots,q\} \to \{1,\dots,q\}$ by

$$\sigma(i) = \begin{cases} i+n, & \text{if } i+n \le q, \\ i+n-q, & \text{if } i+n > q. \end{cases}$$

It is easy to see that σ is bijective and $|\sigma(i) - i| \ge n$ (note that $q \ge 2n$). Hence, $\frac{(f,H_i)}{(g,H_i)}$ and $\frac{(f,H_{\sigma(i)})}{(g,H_{\sigma(i)})}$ belong to distinct groups, so that $(f,H_i)(g,H_{\sigma(i)}) - (f,H_{\sigma(i)})(g,H_i) \not\equiv 0$.

We consider $(f,H_i)(g,H_{\sigma(i)}) - (f,H_{\sigma(i)})(g,H_i)$, $1 \le i \le q$.

LEMMA 5. *For each $i \in \{1,\dots,q\}$ and a positive integer N, we have*

$$\begin{aligned}(1)\qquad \sum_{j=1, j\ne i,\sigma(i)}^{q} & \nu^1_{(f,H_j)} + \nu^N_{(f,H_i)} + \nu^N_{(g,H_i)} - N\nu^1_{(g,H_i)} \\ & + \nu^N_{(f,H_{\sigma(i)})}(r) + \nu^N_{(g,H_{\sigma(i)})} - N\nu^1_{(g,H_{\sigma(i)})} \\ & \le \nu_{(f,H_i)(g,H_{\sigma(i)})-(f,H_{\sigma(i)})(g,H_i)}.\end{aligned}$$

Proof. For any $j \in \{1,\dots,q\}\setminus\{i,\sigma(i)\}$, since $f = g$ on $f^{-1}(H_j)$ $(=g^{-1}(H_j))$, we have that a zero of (f,H_j) is also a zero point of $(f,H_i)(g,H_{\sigma(i)}) - (f,H_{\sigma(i)})(g,H_i)$.

For any $z_0 \in f^{-1}(H_i)$ $(= g^{-1}(H_i))$, z_0 is a zero of $(f,H_i)(g,H_{\sigma(i)}) - (f,H_{\sigma(i)})(g,H_i)$ with

$$\nu_{(f,H_i)(g,H_{\sigma(i)})-(f,H_{\sigma(i)})(g,H_i)}(z_0) \ge \min\big\{\nu_{(f,H_i)}(z_0), \nu_{(g,H_i)}(z_0)\big\}.$$

Note that the set $f^{-1}(H_i)$ is the union of $\{z | \min\{\nu_{(f,H_i)}(z), \nu_{(g,H_i)}(z)\} = \nu_{(f,H_i)}(z)\} \cap f^{-1}(H_i)$ and $\{z | \min\{\nu_{(f,H_i)}(z), \nu_{(g,H_i)}(z)\} = \nu_{(g,H_i)}(z)\} \cap f^{-1}(H_i)$.

Case 1. If $z_0 \in \{z | \min\{\nu_{(f,H_i)}(z), \nu_{(g,H_i)}(z)\} = \nu_{(f,H_i)}(z)\}$, then

$$\min\big\{\nu_{(f,H_i)}(z_0), \nu_{(g,H_i)}(z_0)\big\} = \nu_{(f,H_i)}(z_0) \ge \min\big\{\nu_{(f,H_i)}(z_0), N\big\}.$$

Case 2. Consider $z_0 \in \{z | \min\{\nu_{(f,H_i)}(z), \nu_{(g,H_i)}(z)\} = \nu_{(g,H_i)}(z)\}$.

For $z_0 \in \{z | \min\{\nu_{(f,H_i)}(z), \nu_{(g,H_i)}(z)\} = \nu_{(g,H_i)}(z)\} \cap \{z | \nu_{(g,H_i)}(z) \ge N\}$, we have

$$\min\big\{\nu_{(f,H_i)}(z_0), \nu_{(g,H_i)}(z_0)\big\} = \nu_{(g,H_i)}(z_0) \ge N = \min\big\{\nu_{(f,H_i)}(z_0), N\big\}.$$

For $z_0 \in \{z | \min\{\nu_{(f,H_i)}(z), \nu_{(g,H_i)}(z)\} = \nu_{(g,H_i)}(z)\} \cap \{z | \nu_{(g,H_i)}(z) = k\}$, $k = 1,\dots,N-1$, we have

$$\begin{aligned}\min\big\{\nu_{(f,H_i)}(z_0), \nu_{(g,H_i)}(z_0)\big\} &= \nu_{(g,H_i)}(z_0) = k \\ &\ge \min\big\{\nu_{(f,H_i)}(z_0), N\big\} - (N-k)\nu^1_{(g,H_i)}(z_0).\end{aligned}$$

For any $z_0 \in f^{-1}(H_{\sigma(i)})$ $(= g^{-1}(H_{\sigma(i)}))$, z_0 is a zero of $(f,H_i)(g,H_{\sigma(i)}) - (f,H_{\sigma(i)})(g,H_i)$ with

$$\nu_{(f,H_i)(g,H_{\sigma(i)})-(f,H_{\sigma(i)})(g,H_i)}(z_0) \ge \min\big\{\nu_{(f,H_{\sigma(i)})}(z_0), \nu_{(g,H_{\sigma(i)})}(z_0)\big\}.$$

By the same argument, if

$$z_0 \in \big\{z \big| \min\big\{\nu_{(f,H_{\sigma(i)})}(z), \nu_{(g,H_{\sigma(i)})}(z)\big\} = \nu_{(f,H_{\sigma(i)})}(z)\big\},$$

then

$$\min\big\{\nu_{(f,H_{\sigma(i)})}(z_0), \nu_{(g,H_{\sigma(i)})}(z_0)\big\} = \nu_{(f,H_{\sigma(i)})}(z_0) \geq \min\big\{\nu_{(f,H_{\sigma(i)})}(z_0), N\big\}.$$

If $z_0 \in \{z| \min\{\nu_{(f,H_{\sigma(i)})}(z), \nu_{(g,H_{\sigma(i)})}(z)\} = \nu_{(g,H_{\sigma(i)})}(z)\} \cap \{z|\nu_{(g,H_{\sigma(i)})}(z) \geq N\}$, we have

$$\begin{aligned}&\min\big\{\nu_{(f,H_{\sigma(i)})}(z_0), \nu_{(g,H_{\sigma(i)})}(z_0)\big\}\\ &\quad = \nu_{(g,H_{\sigma(i)})}(z_0) \geq N = \min\big\{\nu_{(f,H_{\sigma(i)})}(z_0), N\big\}.\end{aligned}$$

If $z_0 \in \{z| \min\{\nu_{(f,H_{\sigma(i)})}(z), \nu_{(g,H_{\sigma(i)})}(z)\} = \nu_{(g,H_{\sigma(i)})}(z)\} \cap \{z|\nu_{(g,H_{\sigma(i)})}(z) = k\}$, $k = 1, \ldots, N-1$, we have

$$\begin{aligned}&\min\big\{\nu_{(f,H_{\sigma(i)})}(z_0), \nu_{(g,H_{\sigma(i)})}(z_0)\big\}\\ &\quad = \nu_{(g,H_{\sigma(i)})}(z_0) = k\\ &\quad \geq \min\big\{\nu_{(f,H_{\sigma(i)})}(z_0), N\big\} - (N-k)\nu^1_{(g,H_{\sigma(i)})}(z_0).\end{aligned}$$

Note that $f^{-1}(H_i) \cap f^{-1}(H_j) = \emptyset$ for all $1 \leq i < j \leq q$. We have

$$\begin{aligned}\sum_{j=1, j\neq i, \sigma(i)}^{q} &\nu^1_{(f,H_j)} + \nu^N_{(f,H_i)} - (N-1)\nu^1_{(g,H_i),=1} - (N-2)\nu^1_{(g,H_i),=2}\\ &- \cdots - \nu^1_{(g,H_i),=N-1} + \nu^N_{(f,H_{\sigma(i)})} - (N-1)\nu^1_{(g,H_{\sigma(i)}),=1}\\ &- (N-2)\nu^1_{(g,H_{\sigma(i)}),=2} - \cdots - \nu^1_{(g,H_{\sigma(i)}),=N-1}\\ \leq\ &\nu_{(f,H_i)(g,H_{\sigma(i)})-(f,H_{\sigma(i)})(g,H_i)}.\end{aligned} \tag{2}$$

On the other hand, for each j, $1 \leq j \leq q$,

$$\begin{aligned}&(N-1)\nu^1_{(g,H_j),=1} + (N-2)\nu^1_{(g,H_j),=2} + \cdots + \nu^1_{(g,H_j),=N-1}\\ &\quad = N\nu^1_{(g,H_j)} - \nu^N_{(g,H_j)}.\end{aligned} \tag{3}$$

Combining (2) and (3), we have (1). □

Take summation of (1) over $1 \leq i \leq q$, we have

$$\begin{aligned}(q-2)&\sum_{j=1}^{q} \nu^1_{(f,H_j)} + \sum_{i=1}^{q}\big(\nu^N_{(f,H_i)} + \nu^N_{(g,H_i)}\big)\\ &+ \sum_{i=1}^{q}\big(\nu^N_{(f,H_{\sigma(i)})} + \nu^N_{(g,H_{\sigma(i)})}\big) - N\sum_{i=1}^{q}\big(\nu^1_{(g,H_i)} + \nu^1_{(g,H_{\sigma(i)})}\big)\\ &\leq \sum_{i=1}^{q} \nu_{(f,H_i)(g,H_{\sigma(i)})-(f,H_{\sigma(i)})(g,H_i)}.\end{aligned}$$

Since σ is bijective, this gives

$$
\begin{aligned}
&(q-2)\sum_{j=1}^{q}\nu^1_{(f,H_j)}+2\sum_{i=1}^{q}\big(\nu^N_{(f,H_i)}+\nu^N_{(g,H_i)}\big)-2N\sum_{i=1}^{q}\nu^1_{(g,H_i)}\\
&\quad\le\sum_{i=1}^{q}\nu_{(f,H_i)(g,H_{\sigma(i)})-(f,H_{\sigma(i)})(g,H_i)}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&(q-2)\sum_{j=1}^{q}\nu^1_{(g,H_j)}+2\sum_{i=1}^{q}\big(\nu^N_{(f,H_i)}+\nu^N_{(g,H_i)}\big)-2N\sum_{i=1}^{q}\nu^1_{(f,H_i)}\\
&\quad\le\sum_{i=1}^{q}\nu_{(f,H_i)(g,H_{\sigma(i)})-(f,H_{\sigma(i)})(g,H_i)}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\frac{(q-2N-2)}{2}\sum_{j=1}^{q}\big(\nu^1_{(f,H_j)}+\nu^1_{(g,H_j)}\big)+2\sum_{j=1}^{q}\big(\nu^N_{(f,H_j)}+\nu^N_{(g,H_j)}\big)\\
&\quad\le\sum_{i=1}^{q}\nu_{(f,H_i)(g,H_{\sigma(i)})-(f,H_{\sigma(i)})(g,H_i)}.
\end{aligned}
\tag{4}
$$

Take $N=n$ and $q=2n+2$, we have

$$
2\sum_{j=1}^{2n+2}\big(\nu^n_{(f,H_j)}+\nu^n_{(g,H_j)}\big)\le\sum_{i=1}^{2n+2}\nu_{(f,H_i)(g,H_{\sigma(i)})-(f,H_{\sigma(i)})(g,H_i)}.
\tag{5}
$$

Denote by $W(f_0,\dots,f_n)$ (or $W(g_0,\dots,g_n)$) the Wronskian of $f_0,\dots,f_n$ (or $g_0,\dots,g_n$). Since f and g are linearly non-degenerate, we have $W(f_0,\dots,f_n)\not\equiv 0$ and $W(g_0,\dots,g_n)\not\equiv 0$.

LEMMA 6. *Let $H_1,\dots,H_{2n+2}$ be the hyperplanes in $\mathbb{P}^n(\mathbf{F})$, located in general position. Then*

$$
\sum_{j=1}^{2n+2}\nu_{(f,H_j)}-\nu_{W(f_0,\dots,f_n)}\le\sum_{j=1}^{2n+2}\nu^n_{(f,H_j)}.
\tag{6}
$$

Proof. Since $f^{-1}(H_i)\cap f^{-1}(H_j)=\emptyset$ for all $1\le i<j\le 2n+2$, each point $z\in\bigcup_{j=1}^{2n+2}f^{-1}(H_j)$ satisfies $z\in f^{-1}(H_{i_0})$ for some i_0 with $1\le i_0\le 2n+2$, and $z\notin f^{-1}(H_j)$ for $j\ne i_0$. Hence $(f,H_j)(z)\ne 0$ for $j\ne i_0$. Assume that (f,H_{i_0}) vanishes at z with vanishing order m. Without loss of generality, we assume that $a_{i_00}\ne 0$. Then, $W(f_0,f_1,\dots,f_n)=a_{i_00}^{-1}W((f,H_{i_0}),f_1,\dots,f_n)$ and $W(f_0,\dots,f_n)$ vanishes at z with vanishing order at least $m-n$. Hence, we have (6). ∎

By Lemma 6 and (5), we have

$$2\left(\sum_{j=1}^{2n+2}\nu_{(f,H_j)}-\nu_{W(f_0,\dots,f_n)}+\sum_{j=1}^{2n+2}\nu_{(g,H_j)}-\nu_{W(g_0,\dots,g_n)}\right)\le\sum_{i=1}^{2n+2}\nu_{(f,H_i)(g,H_{\sigma(i)})-(f,H_{\sigma(i)})(g,H_i)}. \tag{7}$$

Define

$$\Psi=\big(W(f_0,\dots,f_n)W(g_0,\dots,g_n)\big)^2\times\prod_{i=1}^{2n+2}\big((f,H_i)(g,H_{\sigma(i)})-(f,H_{\sigma(i)})(g,H_i)\big)\Big/\left(\prod_{j=1}^{2n+2}(f,H_j)(g,H_j)\right)^2.$$

By (7), Ψ is entire. Furthermore, $\Psi\not\equiv 0$.

By Lemma 3, there exists a sequence $z_k\in\mathbf{F}$ such that $r_k=|z_k|\to\infty$, $r_k\notin\{r_\nu\}$, and $(f,H_j)(z_k)\neq 0$ for $1\le j\le 2n+2$, where the set $\{r_\nu\}$ is a discrete set.

Assume that

$$\big|f_{i_k}(z_k)\big|=\max_{0\le i\le n}\big\{\big|f_i(z_k)\big|\big\}\quad\text{and}\quad\big|g_{j_k}(z_k)\big|=\max_{0\le j\le n}\big\{\big|g_j(z_k)\big|\big\}. \tag{8}$$

Now, for each fixed z_k, we suppose that

$$\big|(f,H_{\mu_1})(z_k)\big|\le\big|(f,H_{\mu_2})(z_k)\big|\le\cdots\le\big|(f,H_{\mu_{2n+2}})(z_k)\big|$$

and

$$\big|(g,H_{\nu_1})(z_k)\big|\le\big|(g,H_{\nu_2})(z_k)\big|\le\cdots\le\big|(g,H_{\nu_{2n+2}})(z_k)\big|.$$

Solving the system of linear equations

$$a_{\mu_l 0}f_0(z_k)+\cdots+a_{\mu_l n}f_n(z_k)=(f,H_{\mu_l})(z_k),\quad 1\le l\le n+1,$$

we have

$$\big|f_{i_k}(z_k)\big|\le C_1\max_{1\le l\le n+1}\big\{\big|(f,H_{\mu_l})(z_k)\big|\big\}=C_1\big|(f,H_{\mu_{n+1}})(z_k)\big|$$

for a constant C_1 dependent only on $H_1,\dots,H_{2n+2}$.

Similarly, we have

$$\big|g_{j_k}(z_k)\big|\le C_2\max_{1\le l\le n+1}\big\{\big|(g,H_{\nu_l})(z_k)\big|\big\}=C_2\big|(g,H_{\nu_{n+1}})(z_k)\big|$$

for $C_2>0$.

Hence, we obtain

$$\big|f_{i_k}(z_k)\big|\le B\big|(f,H_{\mu_{n+1}})(z_k)\big|\le B\big|(f,H_{\mu_{n+2}})(z_k)\big|\le\cdots\le B\big|(f,H_{\mu_{2n+2}})(z_k)\big|$$

and

$$\big|g_{j_k}(z_k)\big|\le B\big|(g,H_{\nu_{n+1}})(z_k)\big|\le B\big|(g,H_{\nu_{n+2}})(z_k)\big|\le\cdots\le B\big|(g,H_{\nu_{2n+2}})(z_k)\big|,$$

where $B>0$ is a constant independent of z_k.

Thus,

$$
\begin{aligned}
(9)\quad & \big|\Psi(z_k)\big| \\
&= \frac{|W(f_0,\dots,f_n)(z_k)|^2|W(g_0,\dots,g_n)(z_k)|^2}{(\prod_{j=1}^{2n+2}|(f,H_j)(z_k)||(g,H_j)(z_k)|)^2} \\
&\quad\times \prod_{i=1}^{2n+2}\big|\big((f,H_i)(g,H_{\sigma(i)})-(f,H_{\sigma(i)})(g,H_i)\big)(z_k)\big| \\
&\le \frac{B^{4n+4}|W(f_0,\dots,f_n)(z_k)|^2|W(g_0,\dots,g_n)(z_k)|^2}{(\prod_{l=1}^{n+1}|(f,H_{\mu_l})(z_k)||(g,H_{\nu_l})(z_k)|)^2|f_{i_k}(z_k)|^{2n+2}|g_{j_k}(z_k)|^{2n+2}} \\
&\quad\times \prod_{i=1}^{2n+2}\big|\big((f,H_i)(g,H_{\sigma(i)})-(f,H_{\sigma(i)})(g,H_i)\big)(z_k)\big|.
\end{aligned}
$$

By Lemma 4, for $1\le\alpha\le n$,

$$M_{\frac{(f,H_j)^{(\alpha)}}{(f,H_j)}}(r)\le\frac{1}{r^\alpha},$$

and hence

$$\left|\frac{(f,H_j)^{(\alpha)}}{(f,H_j)}(z_k)\right|\le\frac{1}{|z_k|^\alpha}. \tag{10}$$

By the properties of the Wronskian, we have

$$\frac{|W(f_0,\dots,f_n)(z_k)|}{\prod_{l=1}^{n+1}|(f,H_{\mu_l})(z_k)|}=\frac{C_3|W((f,H_{\mu_1}),\dots,(f,H_{\mu_{n+1}}))(z_k)|}{\prod_{l=1}^{n+1}|(f,H_{\mu_l})(z_k)|},$$

where $C_3>0$ is a constant.

By the properties of the non-Archimedean norm and (10), we have

$$
\begin{aligned}
(11)\quad & \frac{|W((f,H_{\mu_1}),\dots,(f,H_{\mu_{n+1}}))(z_k)|}{\prod_{l=1}^{n+1}|(f,H_{\mu_l})(z_k)|} \\
&\le \max_{\alpha_1+\cdots+\alpha_{n+1}=\frac{n(n+1)}{2}}\left|\frac{(f,H_{\mu_1})^{(\alpha_1)}}{(f,H_{\mu_1})}(z_k)\right|\cdots\left|\frac{(f,H_{\mu_{n+1}})^{(\alpha_{n+1})}}{(f,H_{\mu_{n+1}})}(z_k)\right| \\
&\le \frac{1}{|z_k|^{\frac{n(n+1)}{2}}}.
\end{aligned}
$$

Similarly, we have

$$\frac{|W((g,H_{\nu_1}),\dots,(g,H_{\nu_{n+1}}))(z_k)|}{\prod_{l=1}^{n+1}|(g,H_{\nu_l})(z_k)|}\le\frac{1}{|z_k|^{\frac{n(n+1)}{2}}}. \tag{12}$$

On the other hand, by (8) and the properties of the non-Archimedean norm, we also have

$$\prod_{i=1}^{2n+2}\big|\big((f,H_i)(g,H_{\sigma(i)})-(f,H_{\sigma(i)})(g,H_i)\big)(z_k)\big| \le C_4\big|f_{i_k}(z_k)\big|^{2n+2}\big|g_{j_k}(z_k)\big|^{2n+2} \tag{13}$$

for a constant C_4 independent of z_k.

Combining (9), (11), (12) and (13), we have

$$\big|\Psi(z_k)\big| \le \frac{B^{4n+4}\cdot C}{|z_k|^{2n(n+1)}}$$

for all k, where $C>0$ is a constant which depends only on the hyperplanes. Let $k\to\infty$, this implies that $\Psi\equiv 0$, which is a contradiction. So $f\equiv g$. This completes the proof of Theorem 1. □

Proof of Theorem 2. Suppose that $f\not\equiv g$. Repeating the argument in the proof of Theorem 1, we have

$$\frac{(q-2N-2)}{2}\sum_{j=1}^{q}\big(\nu^1_{(f,H_j)}+\nu^1_{(g,H_j)}\big)+2\sum_{j=1}^{q}\big(\nu^N_{(f,H_j)}+\nu^N_{(g,H_j)}\big) \le \sum_{i=1}^{q}\nu_{(f,H_i)(g,H_{\sigma(i)})-(f,H_{\sigma(i)})(g,H_i)}.$$

Take $N=p^{s-1}n$ and $q=2p^{s-1}n+2$, we have

$$2\sum_{j=1}^{2p^{s-1}n+2}\big(\nu^{p^{s-1}n}_{(f,H_j)}+\nu^{p^{s-1}n}_{(g,H_j)}\big) \le \sum_{i=1}^{2p^{s-1}n+2}\nu_{(f,H_i)(g,H_{\sigma(i)})-(f,H_{\sigma(i)})(g,H_i)}.$$

In the positive characteristic case, we should use the generalized Wronskian instead of the ordinary Wronskian.

Since $f=[f_0:\cdots:f_n]$ is linearly non-degenerate over $\mathcal{M}[p^s]$, by Theorem 3.5 in [2], there exist positive integers $\gamma_1,\dots,\gamma_n$ with $\gamma_i\le\gamma_{i-1}+p^{s-1}$ such that

$$\begin{vmatrix} f_0 & \cdots & f_n \\ D^{\gamma_1}f_0 & \cdots & D^{\gamma_1}f_n \\ D^{\gamma_2}f_0 & \cdots & D^{\gamma_2}f_n \\ \vdots & \vdots & \vdots \\ D^{\gamma_n}f_0 & \cdots & D^{\gamma_n}f_n \end{vmatrix} \not\equiv 0.$$

This determinant is called the generalized Wronskian of f. For more properties of the generalized Wronskian, we refer readers to [3].

Denote by $\tilde{W}(f_0,\dots,f_n)$ (or $\tilde{W}(g_0,\dots,g_n)$) the generalized Wronskian of f (or g), which is not identically zero.

Similar to (6), we have

$$\sum_{j=1}^{2p^{s-1}n+2} \nu_{(f,H_j)} - \nu_{\tilde{W}(f_0,\dots,f_n)} \leq \sum_{j=1}^{2p^{s-1}n+2} \nu^{p^{s-1}n}_{(f,H_j)}$$

and

$$\sum_{j=1}^{2p^{s-1}n+2} \nu_{(g,H_j)} - \nu_{\tilde{W}(g_0,\dots,g_n)} \leq \sum_{j=1}^{2p^{s-1}n+2} \nu^{p^{s-1}n}_{(g,H_j)}.$$

Hence,

$$2\left(\sum_{j=1}^{2p^{s-1}n+2} \nu_{(f,H_j)} - \nu_{\tilde{W}(f_0,\dots,f_n)} + \sum_{j=1}^{2p^{s-1}n+2} \nu_{(g,H_j)} - \nu_{\tilde{W}(g_0,\dots,g_n)}\right)$$
$$\leq \sum_{i=1}^{2p^{s-1}n+2} \nu_{(f,H_i)(g,H_{\sigma(i)})-(f,H_{\sigma(i)})(g,H_i)}.$$

We consider

$$\Psi = \frac{(\tilde{W}(f_0,\dots,f_n)\tilde{W}(g_0,\dots,g_n))^2}{(\prod_{j=1}^{2p^{s-1}n+2}(f,H_j)(g,H_j))^2} \times \prod_{i=1}^{2p^{s-1}n+2}\big((f,H_i)(g,H_{\sigma(i)}) - (f,H_{\sigma(i)})(g,H_i)\big),$$

which is a nonzero entire function.

By Lemma 3, we can take a sequence $z_k \in \mathbf{F}$ such that $r_k = |z_k| \to \infty$, $r_k \notin \{r_\nu\}$, and $(f,H_j)(z_k) \neq 0$ for $1 \leq j \leq 2p^{s-1}n+2$, where the set $\{r_\nu\}$ is a discrete set. Assume that

$$\big|f_{i_k}(z_k)\big| = \max_{0\leq i\leq n}\big\{\big|f_i(z_k)\big|\big\} \quad \text{and} \quad \big|g_{j_k}(z_k)\big| = \max_{0\leq j\leq n}\big\{\big|g_j(z_k)\big|\big\}.$$

Hence, we have $|f_{i_k}(z_k)| \to \infty$, $|g_{j_k}(z_k)| \to \infty$ as $k \to \infty$.

By the same argument as in the proof of Theorem 1, there exist positive constants B and C, dependent only on the hyperplanes, such that

$$\big|\Psi(z_k)\big| \leq \frac{B^{4(2p^{s-1}-1)n+4}\cdot C}{|z_k|^{2n(n+1)}|f_{i_k}(z_k)|^{2(p^{s-1}-1)n}|g_{j_k}(z_k)|^{2(p^{s-1}-1)n}}$$

for all k. This yields that $\Psi \equiv 0$, which is a contradiction. □

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