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STABLE SYMMETRIC POLYNOMIALS AND THE SCHUR–AGLER CLASS

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ABSTRACT. We call a multivariable polynomial an *Agler denominator* if it is the denominator of a rational inner function in the Schur–Agler class, an important subclass of the bounded analytic functions on the polydisk. We give a necessary and sufficient condition for a multi-affine, symmetric, and stable polynomial to be an Agler denominator and prove several consequences. We also sharpen a result due to Kummert related to three variable, multi-affine, stable polynomials.

1. Introduction

We say a multivariable polynomial $p \in \mathbb{C}[z_1, \dots, z_n]$ is *stable* if p has no zeros on the closed polydisk $\overline{\mathbb{D}}^n = \overline{\mathbb{D}} \times \cdots \times \overline{\mathbb{D}}$. "Stable" can refer to many variations on this idea, but we will stick with this definition throughout. Stable polynomials in their various related incarnations appear in complex analysis, orthogonal polynomials (see [12]), combinatorics, and statistical mechanics (see [11] or see [13] for a survey related to these last two). In particular, the paper [11] focuses on the class of "Lee–Yang polynomials" which satisfy a "nonstrict" form of stability, but are nonetheless closely related to the polynomials we study here.

This article has two goals: (1) further develop properties and examples of the Schur–Agler class on the polydisk, and (2) unify and explore connections between the following two classical theorems related to one variable polynomials. (We postpone discussion of the Schur–Agler class until Definition 1.3.)

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THEOREM 1.1 (The Christoffel–Darboux formula). *Let $p \in \mathbb{C}[z]$ be a stable one variable polynomial of degree d and write*

$$\tilde{p}(z) = z^d \overline{p(1/\bar{z})}.$$

Then, there exist linearly independent polynomials $A_1, \dots, A_d \in \mathbb{C}[z]$ such that

$$\frac{|p(z)|^2 - |\tilde{p}(z)|^2}{1-|z|^2} = \sum_{j=1}^{d} \left|A_j(z)\right|^2.$$

See [12] for more information.

THEOREM 1.2 (Grace–Walsh–Szegő). *Let $p \in \mathbb{C}[z]$ be a stable one variable polynomial of degree d. Then, the multi-affine symmetrization* (*defined below*) *$p_S \in \mathbb{C}[z_1, \dots, z_d]$ of p is stable.*

See [13] for more information and references.

Let us define the multi-affine symmetrization. Set $[d] = \{1, 2, \dots, d\}$. By *multi-affine* we mean a polynomial which has degree at most one in each variable separately. For such polynomials, it is convenient to replace multi-index notation with a set theory notation. Namely, if $\alpha \subset [d]$, then

$$z^\alpha = \prod_{j \in \alpha} z_j, \quad z^{\varnothing} = 1.$$

Now, if $p(z) = \sum_{j=0}^{d} p_j z^j$, then the *multi-affine symmetrization* is given by

$$p_S(z_1, \dots, z_d) = \sum_{\alpha \subset [d]} \binom{d}{|\alpha|}^{-1} p_{|\alpha|} z^\alpha$$

with $|\alpha|$ denoting cardinality of $\alpha \subset [d]$. The multi-affine symmetrization of p is the unique multi-affine symmetric polynomial $p_S \in \mathbb{C}[z_1, \dots, z_d]$ with $p_S(z, z, \dots, z) = p(z)$. Notice symmetrization is performed at a specific degree.

The Grace–Walsh–Szegő theorem can be useful in reducing questions about multivariable stable polynomials to questions about multi-affine stable polynomials by symmetrizing a given multivariable stable polynomial in each variable separately. See [13], which is a survey related to the works [3] and [4].

It is not clear how to generalize the Christoffel–Darboux formula to multivariable polynomials. *Two* variable stable polynomials satisfy a Christoffel–Darboux-like formula. If $p \in \mathbb{C}[z_1, z_2]$ is stable and of multidegree (d_1, d_2) (meaning degree d_1 in z_1 and d_2 in z_2), then writing

$$\tilde{p}(z_1, z_2) = z_1^{d_1} z_2^{d_2} \overline{p(1/\bar{z}_1, 1/\bar{z}_2)}$$

we have for $z = (z_1, z_2)$

$$\left|p(z)\right|^2 - \left|\tilde{p}(z)\right|^2 = \left(1 - |z_1|^2\right) \mathrm{SOS}_1(z) + \left(1 - |z_2|^2\right) \mathrm{SOS}_2(z),$$

where the terms $\mathrm{SOS}_1(z), \mathrm{SOS}_2(z)$ are each a sum of squared moduli of polynomials. Explicitly, there exist polynomials $A_1, \dots, A_N \in \mathbb{C}[z]$, such that

$\mathrm{SOS}_1(z) = \sum_{j=1}^{N} |A_j(z)|^2$ and $\mathrm{SOS}_2(z)$ can be written in a similar way. See [5], [6], or [7] for a proof of this formula.

This formula does not generalize straightforwardly to three or more variables. We give a special name to those polynomials for which it does.

DEFINITION 1.3. We say a stable polynomial $p \in \mathbb{C}[z_1, \dots, z_n]$ of multidegree $(d_1, \dots, d_n)$ is an *Agler denominator* if the following Christoffel–Darboux type of formula holds:

$$|p(z)|^2 - |\tilde{p}(z)|^2 = \sum_{j=1}^{n} (1 - |z_i|^2)\, \mathrm{SOS}_j(z), \tag{1.1}$$

where each SOS_j is a sum of squared moduli of polynomials in $\mathbb{C}[z_1, \dots, z_n]$ and as usual $\tilde{p}(z_1, \dots, z_n) = z_1^{d_1} \cdots z_n^{d_n} \overline{p(1/\bar{z_1}, \dots, 1/\bar{z_n})}$.

Let us explain the terminology. Given a stable polynomial $p \in \mathbb{C}[z_1, \dots, z_n]$,

$$\phi(z) = \frac{\tilde{p}(z)}{p(z)}$$

is a rational inner function on the polydisk. Inner just means ϕ has modulus 1 almost everywhere on the n-torus $\mathbb{T}^n := (\partial\mathbb{D})^n$, and this holds in our case because $|p(z)| = |\tilde{p}(z)|$ for all $z \in \mathbb{T}^n$. By the maximum principle, ϕ is in the *Schur class*, the set of bounded analytic functions on the polydisk with supremum norm at most one.

If p is an Agler denominator, then equation (1.1) is equivalent to ϕ being a member of a subclass of the Schur class called the *Schur–Agler class*, which we abbreviate to *Agler class*. Such analytic functions f satisfy the following more universal bound:

$$\|f(T_1, \dots, T_n)\| \leq 1 \tag{1.2}$$

for all n-tuples $(T_1, \dots, T_n)$ of commuting strict contractions on a separable Hilbert space. For $n = 1, 2$ the Schur class and the Agler class coincide, but they differ for larger n. See [8] for more background, including a discussion of the relationship between (1.1) and (1.2). Due to (1.2), the Agler class is natural from an operator theory perspective, yet it remains poorly understood. Agler class functions admit a nice matricial representation (called a transfer function realization; see [8]) which also allows one to produce examples of Agler class functions, but it still remains a difficult problem to determine whether a given function is indeed in the Agler class. In light of all of this background, we state our motivating question.

QUESTION 1.4. Are multi-affine symmetric stable polynomials always Agler denominators?

A positive answer would mean a strengthened Grace–Walsh–Szegő theorem holds, while any conclusive answer would at least enrich the study of the Agler class. This paper represents partial progress on this question, which we now summarize.

Theorem 3.3 gives a necessary and sufficient condition for a multi-affine symmetric polynomial to be an Agler denominator in terms of a certain $2^{d-1} \times 2^{d-1}$ matrix being positive semi-definite (where d is the number of variables).

Our condition yields the following corollary.

THEOREM 1.5. *Let $p \in \mathbb{C}[z_1, \ldots, z_n]$ be a multi-affine symmetric polynomial with $p(0, \ldots, 0) \neq 0$. Then, there exists an $r > 0$ such that $p_r(z) := p(rz)$ is an Agler denominator.*

Every polynomial with $p(0) \neq 0$ has a radius of stability (the supremum of r such that p_r is stable). (Note this concept is called the *inner radius* in [11].) The above theorem says that if we add the hypotheses multi-affine and symmetric, such polynomials possess an "Agler radius" (the supremum of r such that p_r is an Agler denominator) which is necessarily less than or equal to its radius of stability.

While this theorem appears to be a modest contribution, we know of no other nontrivial, naturally defined families of Schur class functions which happen to be Agler class functions. ("Trivial" examples can be obtained by taking convex combinations of Schur functions which depend on only two variables. One can also construct examples by using the earlier alluded to matricial representation of Agler class functions.) Furthermore, our approach gives a method for constructing sums of squares decompositions explicitly—something also not generally well understood.

What can be said for low numbers of variables?

It turns out that all 3 variable multi-affine stable polynomials are Agler denominators whether symmetric or not. This was proved in [9]. (Two decades ago the Agler class was of interest in electrical engineering in the construction of "wave digital filters" in the papers [10] and [9]. See also [1].) We shall give a proof of this fact in the appendix, since while it does not follow the main thrust of this paper, it is nonetheless closely related and we are able to sharpen Kummert's result slightly in the following theorem.

THEOREM 1.6. *If $p \in \mathbb{C}[z_1, z_2, z_3]$ is multi-affine and stable, then there exist sums of squares terms such that*

$$|p|^2 - |\tilde{p}|^2 = \sum_{j=1}^{3} \big(1 - |z_j|^2\big) \,\mathrm{SOS}_j(z),$$

where SOS_3 is a sum of two squares, while SOS_1, SOS_2 are sums of four squares.

This is related to Theorem 2.1 below and the main theme of [8]. Theorem 2.1 suggests we might have to use a sum of four squares in each SOS term above, but we can reduce one term to only contain two squares.

In the case of four variables, our necessary and sufficient condition from Theorem 3.3 can be significantly simplified.

THEOREM 1.7. *If $p \in \mathbb{C}[z_1, z_2, z_3, z_4]$ is stable, multi-affine, and symmetric, then p is an Agler denominator if and only if*

$$8\big(|p_0|^2 - |p_4|^2\big) - \big(|p_1|^2 - |p_3|^2\big) \geq 2\big|p_2\bar{p_1} - \bar{p}_2 p_3 - 2(p_1\bar{p_0} - \bar{p}_3 p_4)\big|,$$

where $p(z) = \sum_{\alpha \subset [4]} \binom{4}{|\alpha|}^{-1} p_{|\alpha|} z^\alpha$.

We do not know if this condition holds automatically under the assumption of stability. One difficulty is that both sides of the inequality are zero for symmetrizations of degree four polynomials with all zeros on the circle. These would be the typical extremal examples on which to test the inequality, for if it failed for one of them, it would fail for a nearby stable polynomial.

We have so far been unable to find a symmetric, stable, multi-affine polynomial that is not an Agler denominator. In Section 5, we present a few additional examples to illustrate.

2. Preliminaries

Let us reproduce the formula Agler denominators must satisfy:

$$\big|p(z)\big|^2 - \big|\tilde{p}(z)\big|^2 = \sum_{j=1}^{n} \big(1 - |z_j|^2\big) \operatorname{SOS}_j(z). \tag{2.1}$$

To begin our study, we use the following result.

THEOREM 2.1 ([8]). *If $p \in \mathbb{C}[z_1, \ldots, z_n]$ is an Agler class denominator of multi-degree $d = (d_1, \ldots, d_n)$, then the $\operatorname{SOS}_j(z)$ term in (2.1) is a sum of squares of polynomials of degree at most*

$$\begin{cases} d_j - 1 \ \textit{in}\ z_j, \\ d_k \ \textit{in}\ z_k \qquad \textit{for}\ k \neq j. \end{cases}$$

In particular, SOS_j can be written as a sum of at most $d_j \prod_{k \neq j}(d_k + 1)$ polynomials (by dimensionality).

The sums of squares terms may not be unique in (2.1), so we emphasize that the above theorem holds for all possible choices of a sums of squares decomposition.

REMARK 2.2. It is worth explaining the last sentence, using notation we find convenient for the rest of the paper. We will typically write sums of squares terms using vector polynomials. So,

$$\operatorname{SOS}(z) = \sum_{j=1}^{N} \big|A_j(z)\big|^2,$$

where the $A_j \in \mathbb{C}[z_1,\dots,z_n]$ will be written as

$$\mathrm{SOS}(z) = |A(z)|^2,$$

where $A(z) \in \mathbb{C}^N[z_1,\dots,z_n]$ is the vector polynomial $A = [A_1,\dots,A_N]^t$. Now, if $V = \mathrm{span}\{A_j : j = 1,\dots,N\}$ has dimension m, we can always rewrite $\mathrm{SOS}(z)$ using the square of a $\mathbb{C}^m$ valued vector polynomial. Indeed, if $B_1,\dots,B_m$ is a basis of V then there is an $N \times m$ matrix X such that

$$XB(z) = A(z),$$

where $B = [B_1,\dots,B_m]^t$. Then,

$$\mathrm{SOS}(z) = \left|XB(z)\right|^2 = B(z)^* X^* X B(z)$$

but X^*X is a $m \times m$ positive semi-definite matrix and so can be factored as $X^*X = Y^*Y$ with Y a $m \times m$ matrix. Hence,

$$\mathrm{SOS}(z) = \left|YB(z)\right|^2,$$

a sum of m squares.

Using the above conventions, we can rewrite the Christoffel–Darboux formula (Theorem 1.1) as

$$\left|p(z)\right|^2 - \left|\tilde{p}(z)\right|^2 = \left(1 - |z|^2\right)\left|A(z)\right|^2, \tag{2.2}$$

where now $A(z) = \sum_j A_j z^j$ is a vector polynomial. If $p(z) = \sum_j p_j z^j$, then by matching coefficients of both sides we get

$$p_j \bar{p}_k - \bar{p}_{d-j} p_{d-k} = \langle A_j, A_k \rangle - \langle A_{j-1}, A_{k-1} \rangle. \tag{2.3}$$

Here $\langle v, w \rangle = w^* v$ is the standard inner product of complex euclidean space (of dimension taken from context).

It is also useful (later) to point out that $|A(z)|^2 = |\tilde{A}(z)|^2 := |z^{d-1}|^2 |A(1/\bar{z})|^2$ and therefore

$$\langle A_j, A_k \rangle = \langle A_{d-1-k}, A_{d-1-j} \rangle. \tag{2.4}$$

3. Symmetric multi-affine Agler denominators

Again refer to equation (2.1).

Proposition 3.1. *If $p \in \mathbb{C}[z_1,\dots,z_d]$ is a symmetric multi-affine Agler denominator, then*:

- *For any choice of the sums of squares terms, $\mathrm{SOS}_j(z)$ does not depend on z_j, and hence is a function of $\hat{z_j}$, the $d-1$-tuple of all variables except z_j.*
- *The sums of squares terms can be chosen in a canonical way. Namely, there is a $d-1$-variable vector polynomial $B \in \mathbb{C}^{2^{d-1}}[t_1,\dots,t_{d-1}]$, such that*

$$\mathrm{SOS}_j(z) = \left|B(\hat{z_j})\right|^2.$$

- *Furthermore, $|B(t_1,\dots,t_{d-1})|^2$ is symmetric in $t_1,\dots,t_{d-1}$, and*

- $|B(t_1,\dots,t_{d-1})|^2$ *is "$\mathbb{T}^{d-1}$-symmetric", meaning*

$$\big|B(t_1,\dots,t_{d-1})\big|^2 = |t_1\cdots t_{d-1}|^2 \big|B(1/\bar{t}_1,\dots,1/\bar{t}_{d-1})\big|^2.$$

We emphasize that there are two types of symmetry here: symmetry in terms of permuting the variables and symmetry in terms of reflection across the torus, which we refer to as $\mathbb{T}^d$-symmetry. Also, note that $B(t)$ itself is not typically symmetric.

Proof of Proposition 3.1. The first item follows from Theorem 2.1 since p has multidegree $(1,1,\dots,1)$. For example, the theorem says $\mathrm{SOS}_1(z)$ is a sum of squares of polynomials with multidegrees bounded by $(0,1,1,\dots,1)$.

The second item follows from taking a given sum of squares decomposition and averaging over all permutations of the variables.

Indeed, if S_d denotes the set of permutations of $[d]$, define for each $\sigma\in S_d$, $z\in\mathbb{C}^d$

$$\sigma(z) = (z_{\sigma^{-1}(1)}, z_{\sigma^{-1}(2)},\dots,z_{\sigma^{-1}(n)})$$

(this puts z_j into $z_{\sigma(j)}$'s slot).

By symmetry of p and $\tilde{p}$,

$$\begin{aligned}
\big|p(z)\big|^2 - \big|\tilde{p}(z)\big|^2 &= d!^{-1}\sum_{\sigma\in S_d}\sum_{j=1}^{d}\big(1-|z_{\sigma^{-1}(j)}|^2\big)\,\mathrm{SOS}_j\big(\sigma(z)\big)\\
&= d!^{-1}\sum_{\sigma\in S_d}\sum_{j=1}^{d}\big(1-|z_j|^2\big)\,\mathrm{SOS}_{\sigma(j)}\big(\sigma(z)\big)\\
&= \sum_{j=1}^{d}\big(1-|z_j|^2\big)d!^{-1}\sum_{\sigma\in S_d}\mathrm{SOS}_{\sigma(j)}\big(\sigma(z)\big).
\end{aligned}\tag{3.1}$$

Then, by Remark 2.2 we may write

$$\big|B(\hat{z}_1)\big|^2 = d!^{-1}\sum_{\sigma\in S_d}\mathrm{SOS}_{\sigma(1)}\big(\sigma(z)\big),$$

where $B\in\mathbb{C}^{2^{d-1}}[\hat{z}_1]$. This is legitimate because each term $\mathrm{SOS}_{\sigma(1)}(\sigma(z))$ does not depend on z_1 and because the polynomials in the sums of squares decomposition span a space of dimension at most 2^{d-1} (the space in question being the polynomials of degree at most $(0,1,1,\dots,1)$).

Let $\tau\in S_d$. Observe that upon writing $\widehat{\tau(z)_1} = (z_{\tau^{-1}(2)},\dots,z_{\tau^{-1}(d)})$ (i.e. $\tau(z)$ with the first entry deleted) we have

$$\begin{aligned}
\big|B\big(\widehat{\tau(z)_1}\big)\big|^2 &= d!^{-1}\sum_{\sigma\in S_d}\mathrm{SOS}_{\sigma(1)}\big(\sigma\big(\tau(z)\big)\\
&= d!^{-1}\sum_{\sigma\in S_d}\mathrm{SOS}_{\sigma\tau^{-1}(1)}\big(\sigma(z)\big)
\end{aligned}$$

which is the sums of squares term in front of $(1-|z_j|^2)$ for $j=\tau^{-1}(1)$ as in (3.1). This also proves $|B(\hat{z_1})|^2$ is symmetric by considering all τ with $\tau(1)=1$.

If necessary we can modify $|B|^2$ to be $\mathbb{T}^{d-1}$-symmetric, by "reflecting" our sums of squares formula; i.e. given

$$\big|p(z)\big|^2-\big|\tilde{p}(z)\big|^2=\sum_{j=1}^{d}\big(1-|z_j|^2\big)\big|B(\hat{z_j})\big|^2$$

we will replace $(z_1,\dots,z_d)$ with $(1/\bar{z}_1,\dots,1/\bar{z}_d)$ and then multiply through by $|z_1z_2\cdots z_d|^2$ to get

$$\big|\tilde{p}(z)\big|^2-\big|p(z)\big|^2=\sum_{j=1}^{d}\big(|z_j|^2-1\big)\big|\tilde{B}(\hat{z_j})\big|^2,$$

where

$$\tilde{B}(t_1,\dots,t_{d-1})=t_1t_2\cdots t_{d-1}\overline{B(1/\bar{t_1},\dots,1/\bar{t}_{d-1})}.$$

Converting this to

$$\big|p(z)\big|^2-\big|\tilde{p}(z)\big|^2=\sum_{j=1}^{d}\big(1-|z_j|^2\big)\big|\tilde{B}(\hat{z_j})\big|^2$$

and then averaging with our original sums of squares formula yields

$$\big|p(z)\big|^2-\big|\tilde{p}(z)\big|^2=\sum_{j=1}^{d}\big(1-|z_j|^2\big)\frac{1}{2}\big(\big|B(\hat{z_j})\big|^2+\big|\tilde{B}(\hat{z_j})\big|^2\big).$$

We can then refactor $\frac{1}{2}(|B|^2+|\tilde{B}|^2)$ as a sum of at most 2^{d-1} squares to get sums of squares terms that are $\mathbb{T}^{d-1}$-symmetric. We show below that $|B|^2$ is truly canonical by showing that it can be solved for explicitly. □

Therefore, p is an Agler class denominator if and only if we can write

$$\big|p(z)\big|^2-\big|\tilde{p}(z)\big|^2=\sum_{j=1}^{d}\big(1-|z_j|^2\big)\big|B(\hat{z_j})\big|^2, \tag{3.2}$$

where $|B(t)|^2$ is symmetric and $\mathbb{T}^{d-1}$-symmetric in $t=(t_1,\dots,t_{d-1})$.

Let us examine what this implies in terms of coefficients. Write

$$B(t)=\sum_{\alpha\subset[d-1]}B_\alpha t^\alpha,\quad B_\alpha\in\mathbb{C}^{2^{d-1}}$$

then

$$\big|B(t)\big|^2=\sum_{\alpha,\beta}\langle B_\alpha,B_\beta\rangle t^\alpha\bar{t}^\beta.$$

Also, write

$$p(z_1, \dots, z_d) = \sum_{\alpha \subset [d]} \binom{d}{|\alpha|}^{-1} p_{|\alpha|} z^\alpha.$$

PROPOSITION 3.2.

(1) *Symmetry of* $|B(t)|^2$ *means each* $\langle B_\alpha, B_\beta \rangle$ *only depends on* $|\alpha|, |\beta|, |\alpha \cap \beta|$. *So, we may write*

$$B^i_{j,k} := \langle B_\alpha, B_\beta \rangle,$$

where $j = |\alpha|, k = |\beta|, i = |\alpha \cap \beta|$. *Notice that* i *has the following restriction*:

$$0 \le i \le j, k, d-1.$$

It is convenient to declare that for other configurations, including negative values of i, j, k, $B^i_{j,k} := 0$.

(2) $\mathbb{T}^{d-1}$*-symmetry means*

$$B^i_{j,k} = B^{d-1-j-k+i}_{d-1-k,d-1-j}. \tag{3.3}$$

(3) *Writing* $|\alpha| = j, |\beta| = k, |\alpha \cap \beta| = i$, *the term* $z^\alpha \bar{z}^\beta$ *appears with coefficient*

$$(d - j - k + i) B^i_{j,k} - i B^{i-1}_{j-1,k-1}$$

in the right-hand side of (3.2).

Proof. (1) This is straightforward.

(2) This follows from

$$\begin{aligned} \left|B(t)\right|^2 &= \left|\tilde{B}(t)\right|^2 \\ &= \sum_{\alpha,\beta} \langle B_\beta, B_\alpha \rangle t^{[d-1]-\alpha} \bar{t}^{[d-1]-\beta} \\ &= \sum_{\alpha,\beta} \langle B_{[d-1]-\beta}, B_{[d-1]-\alpha} \rangle t^\alpha \bar{t}^\beta. \end{aligned}$$

(3) Looking at the right hand side of (3.2), we pick up a copy of $B^i_{j,k}$ for every $r \in \alpha^c \cap \beta^c$, where we use α^c to denote the complement of $\alpha \subset [d]$ and note that $|\alpha^c \cap \beta^c| = d - j - k + i$. Finally, we pick up a copy of $-B^{i-1}_{j-1,k-1}$ for every $r \in \alpha \cap \beta$. □

Equating coefficients on both sides of (3.2), we get

$$\binom{d}{j}^{-1} \binom{d}{k}^{-1} (p_j \overline{p_k} - \overline{p_{d-j}} p_{d-k}) = (d - j - k + i) B^i_{j,k} - i B^{i-1}_{j-1,k-1} \tag{3.4}$$

which holds independently of i.

The point now is that all values of $B^i_{j,k}$ can be solved for explicitly in terms of the coefficients of p. This is clear since the restrictions on i (in the above proposition) force $d - j - k + i$ to be nonzero, in which case $B^i_{j,k}$

is expressed in terms of $B^{i-1}_{j-1,k-1}$ and coefficients of p. One can even write down a complicated formula. This gives a concrete necessary and sufficient condition for p to be an Agler class denominator.

THEOREM 3.3. *A stable multi-affine symmetric polynomial $p \in \mathbb{C}[z_1, \dots, z_d]$*

$$p(z) = \sum_{\alpha \subset [d]} \binom{d}{|\alpha|}^{-1} p_{|\alpha|} z^\alpha$$

is an Agler class denominator if and only if the numbers $B^i_{j,k}$ which can be solved from (3.4) have the property that the $2^{d-1} \times 2^{d-1}$ matrix (indexed by subsets of $[d-1]$)

$$\mathcal{B} := \left(B^{|\alpha \cap \beta|}_{|\alpha|,|\beta|}\right)_{\alpha,\beta \subset [d-1]}$$

is positive semi-definite.

Proof. The "only if" direction follows from the preceding discussion. The "if" direction essentially follows from reversing all of the arguments and observing that if the given matrix is positive semi-definite then

$$\sum_{\alpha,\beta \subset [d-1]} B^{|\alpha \cap \beta|}_{|\alpha|,|\beta|} z^\alpha \bar{z}^\beta$$

can be factored as $|B(z)|^2$. □

Theorem 1.5 follows from this.

Proof of Theorem 1.5. We are assuming p is a symmetric, multi-affine polynomial, and we may assume $p(0) = 1$. For each r, set $p_r(z) := p(rz)$ construct the matrix $\mathcal{B}(r)$ as above. This matrix depends continuously on r and is positive definite when $r = 0$. Therefore, the matrix stays positive definite for r in some interval containing 0. By the previous theorem, for such r, p_r is an Agler class denominator. □

REMARK 3.4. Let us explicitly give the matrix $\mathcal{B}(0)$ from the proof because even in this trivial case it is useful to see the sums of squares decomposition.

Our "polynomial" is $p(z) = 1$ which we view as a multi-affine polynomial of d variables. So, $\tilde{p}(z) = z_1 \cdots z_d$. Solving the recurrence we get

$$\begin{aligned} B^i_{j,k} &= 0 \quad \text{if } j,k,i \text{ are not all equal,} \\ B^j_{j,j} &= \frac{1}{d\binom{d-1}{j}}. \end{aligned}$$

Then, $\mathcal{B}(0)$ is diagonal and clearly positive definite, and we get

$$\left|B(z)\right|^2 = \sum_{\alpha \subset [d-1]} \frac{|z^\alpha|^2}{d\binom{d-1}{|\alpha|}}$$

and hence

$$1-|z_1\cdots z_d|^2=\sum_{j=1}^{d}\big(1-|z_j|^2\big)\sum_{\alpha\subset[d]\setminus\{j\}}\frac{|z^\alpha|^2}{d\binom{d-1}{|\alpha|}}.$$

It turns out to be useful to apply the Christoffel–Darboux formula to

$$p(z,z,\dots,z)=\sum_{j=0}^{d}p_jz^j$$

(recall that we have weighted our multi-affine polynomial's coefficients to make this formula hold) and combine this with Theorem 3.3. Combining formula (2.3) with (3.4), we get

$$\begin{aligned}(3.5)\qquad &\binom{d}{j}^{-1}\binom{d}{k}^{-1}\big(\langle A_j,A_k\rangle-\langle A_{j-1},A_{k-1}\rangle\big)\\ &=(d-j-k+i)B^i_{j,k}-iB^{i-1}_{j-1,k-1}.\end{aligned}$$

The nice thing about this is that $\mathcal{B}$ is now expressed in terms of the matrix $\langle A_j,A_k\rangle$, which we know to be positive semi-definite (in fact, *positive* when p is stable).

4. Degree 4 case

We investigate the degree 4 situation and prove Theorem 1.7. Let

$$p(z_1,z_2,z_3,z_4)=\sum_{\alpha\subset\{1,2,3,4\}}\binom{4}{|\alpha|}^{-1}p_{|\alpha|}z^\alpha$$

which we assume to be stable. Solving for $\mathcal{B}$ from Theorem 3.3 in terms of the matrix $A_{j,k}=\langle A_j,A_k\rangle$ as in (3.5), we get

$$\begin{aligned}
&B^0_{0,0}=\frac{1}{4}A_{0,0},\qquad B^1_{1,1}=\frac{1}{4^2}A_{0,0}+\frac{1}{3\cdot4^2}A_{1,1},\\
&B^0_{1,0}=\frac{1}{12}A_{1,0},\qquad B^0_{2,0}=\frac{1}{12}A_{2,0},\\
&B^0_{3,0}=\frac{1}{4}A_{3,0},\qquad B^0_{1,1}=\frac{1}{2\cdot4^2}(A_{1,1}-A_{0,0}),\\
&B^0_{2,1}=\frac{1}{6\cdot4}(A_{2,1}-A_{1,0}),\qquad B^1_{2,1}=\frac{1}{2\cdot6\cdot4}(A_{2,1}+A_{1,0}),\\
&B^1_{3,1}=\frac{1}{12}A_{2,0}.
\end{aligned}$$

The remaining values follow from the relation

$$B^i_{j,k}=B^{3-j-k+i}_{3-k,3-j}.$$

(It is also useful to recall equation (2.4).)

Recall the $2^{4-1} \times 2^{4-1}$ matrix $\mathcal{B}$ is indexed by subsets of $[3] = \{1,2,3\}$. We will index according to the ordering:

$$\{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}.$$

It is convenient to break up $\mathcal{B}$ into blocks according to the size of subset and factor out a $\frac{1}{4}$:

$$\mathcal{B} = \frac{1}{4}\begin{bmatrix} S_{0,0} & S_{0,1} & S_{0,2} & S_{0,3} \\ S_{1,0} & S_{1,1} & S_{1,2} & S_{1,3} \\ S_{2,0} & S_{2,1} & S_{2,2} & S_{2,3} \\ S_{3,0} & S_{3,1} & S_{3,2} & S_{3,3}. \end{bmatrix}.$$

So, for example $S_{2,1}$ is a 3×3 matrix with rows indexed by $\{\{1,2\},\{2,3\},\{1,3\}\}$ and columns indexed by $\{\{1\},\{2\},\{3\}\}$.

Each block is now explicitly described.

$$\begin{aligned}
&S_{0,0} = S_{3,3} = A_{0,0}, \\
&S_{0,1} = S_{1,0}^{*} = S_{2,3}^{t} = \overline{S_{3,2}} = \tfrac{1}{3}A_{0,1}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \\
&S_{0,2} = S_{2,0}^{*} = S_{1,3}^{t} = \overline{S_{3,1}} = \tfrac{1}{3}A_{0,2}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \\
&S_{0,3} = S_{3,0}^{*} = A_{0,3}, \\
&S_{1,1} = \begin{bmatrix} \frac{1}{4}A_{0,0} + \frac{1}{12}A_{1,1} & \frac{1}{8}(A_{1,1} - A_{0,0}) & \frac{1}{8}(A_{1,1} - A_{0,0}) \\ \frac{1}{8}(A_{1,1} - A_{0,0}) & \frac{1}{4}A_{0,0} + \frac{1}{12}A_{1,1} & \frac{1}{8}(A_{1,1} - A_{0,0}) \\ \frac{1}{8}(A_{1,1} - A_{0,0}) & \frac{1}{8}(A_{1,1} - A_{0,0}) & \frac{1}{4}A_{0,0} + \frac{1}{12}A_{1,1} \end{bmatrix}, \\
&S_{1,2} = S_{2,1}^{*} = \begin{bmatrix} \frac{1}{12}(A_{1,2} + A_{0,1}) & \frac{1}{6}(A_{1,2} - A_{0,1}) & \frac{1}{12}(A_{1,2} + A_{0,1}) \\ \frac{1}{12}(A_{1,2} + A_{0,1}) & \frac{1}{12}(A_{1,2} + A_{0,1}) & \frac{1}{6}(A_{1,2} - A_{0,1}) \\ \frac{1}{6}(A_{1,2} - A_{0,1}) & \frac{1}{12}(A_{1,2} + A_{0,1}) & \frac{1}{12}(A_{1,2} + A_{0,1}) \end{bmatrix}, \\
&S_{2,2} = S_{1,1}.
\end{aligned}$$

(one must be careful in the last equality because the entries are indexed differently—$S_{1,1}$ is indexed by $\{\{1\},\{2\},\{3\}\}$ and $S_{2,2}$ is indexed by $\{\{1,2\},\{2,3\},\{1,3\}\}$).

This matrix, while complicated, has lots of symmetry, which we exploit by conjugating by the following circulant type matrix

$$R = 2\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \mu & \mu^2 \\ 1 & \mu^2 & \mu \end{bmatrix}$$

and $\mu = e^{i2\pi/3}$.

To compute $R\mathcal{B}R^*$ we observe that

$$CS_{1,0} = \begin{bmatrix} A_{1,0} \\ 0 \\ 0 \end{bmatrix},$$

$$CS_{1,1}C^* = \begin{bmatrix} A_{1,1} & 0 & 0 \\ 0 & \frac{1}{8}(9A_{0,0} - A_{1,1}) & 0 \\ 0 & 0 & \frac{1}{8}(9A_{0,0} - A_{1,1}) \end{bmatrix},$$

$$CS_{1,2}C^* = \begin{bmatrix} A_{1,2} & 0 & 0 \\ 0 & \frac{1}{4}\mu^2(A_{1,2} - 3A_{0,1}) & 0 \\ 0 & 0 & \frac{1}{4}\mu(A_{1,2} - 3A_{0,1}) \end{bmatrix}.$$

The matrix $\mathcal{B}$ is positive semi-definite if and only if $R\mathcal{B}R^*$ is, and after permuting index sets around $R\mathcal{B}R^*$ is positive semi-definite if and only if the following block matrix is

$$\begin{bmatrix} A & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & X^t \end{bmatrix},$$

where

$$X = \frac{1}{4}\begin{bmatrix} \frac{1}{2}(9A_{0,0} - A_{1,1}) & \mu(A_{2,1} - 3A_{1,0}) \\ \mu^2(A_{1,2} - 3A_{0,1}) & \frac{1}{2}(9A_{0,0} - A_{1,1}) \end{bmatrix}.$$

Since A is positive, we only need X positive semi-definite and this amounts to the following inequality

$$9A_{0,0} - A_{1,1} \geq 2|A_{2,1} - 3A_{1,0}|.$$

If we translate this into coefficients of p via (2.3) we get the inequality

$$8\big(|p_0|^2 - |p_4|^2\big) - \big(|p_1|^2 - |p_3|^2\big) \geq 2\big|p_2\bar{p}_1 - \bar{p}_2 p_3 - 2(p_1\bar{p}_0 - \bar{p}_3 p_4)\big|.$$

This proves Theorem 1.7.

5. Examples

We have been unable to locate a stable multi-affine symmetric polynomial which is not an Agler denominator. Let us present some of the simplest possible examples. Consider $q(z) = 1 - z$ which we can symmetrize at any degree we like:

$$p_3(z_1, z_2, z_3) \;=\; 1 - \frac{1}{3}\sum_{j=1}^{3} z_j,$$

$$p_4(z_1, \ldots, z_4) \;=\; 1 - \frac{1}{4}\sum_{j=1}^{4} z_j,$$

$$\ldots \text{etc.}$$

Note q is not "strictly" stable, but this is unimportant for what we are talking about—we really care about the existence of sums of squares decompositions as in the definition of Agler denominators and are not so worried about zeros on the boundary of the polydisk.

Theorem A.1 implies p_3 is an Agler denominator, Theorem 1.7 implies p_4 is an Agler denominator, and Theorem 3.3 implies $p_5,\dots,p_{11}$ are Agler denominators after lengthy computations (which we necessarily performed with a computer since the computation for p_{11} involves checking whether a $2^{10}\times 2^{10}$ matrix is positive semi-definite).

So, for $d=3,\dots,11$, all of the following rational inner functions

$$\frac{d\prod_{j=1}^{d} z_j - \sum_{k=1}^{d}\prod_{j\neq k} z_j}{d-\sum_{j=1}^{d} z_j}$$

satisfy the von Neumann inequality (1.2).

Appendix: Three variable multi-affine stable polynomials

Here we give a proof of the following result due to Kummert and our sharpening (Theorem 1.6).

THEOREM A.1 ([9]). *If $p\in\mathbb{C}[z_1,z_2,z_3]$ is multi-affine and stable, then p is an Agler denominator.*

The proof we give is essentially Kummert's, although we have made it less computational and have removed the use of a classical theorem of Hilbert (viz. positive two variable degree 2 real polynomials are sums of three squares) to prove our sharpening.

LEMMA A.2. *Let $t(z_1,z_2)$ be a positive trig polynomial of degree one in each variable. Then, t is the sum of squared moduli of two polynomials.*

Proof. Write $t(z_1,z_2)=t_0(z_1)+t_1(z_1)z_2+\overline{t_1(z_1)z_2}$. Positivity implies $t_0(z_1)>2|t_1(z_1)|$ for all $z_1\in\mathbb{T}$ after minimizing over z_2. Then, the matrix

$$T(z_1)=\begin{bmatrix}\frac{1}{2}t_0(z_1) & t_1(z_1)\\ \overline{t_1(z_1)} & \frac{1}{2}t_0(z_1)\end{bmatrix}$$

is a positive matrix trig polynomial of degree one in z_1. By the matrix Fejér–Riesz theorem, it can be factored as $A(z_1)^*A(z_1)$ where $A(z_1)$ is a degree one 2×2 matrix polynomial. Then,

$$t(z_1,z_2)=\begin{bmatrix}1 & \bar{z}_2\end{bmatrix}T(z_1)\begin{bmatrix}1\\ z_2\end{bmatrix}=\left|A(z_1)\begin{bmatrix}1\\ z_2\end{bmatrix}\right|^2$$

which is a sum of two squares. $\square$

Proof of Theorems A.1 *and* 1.6. Write $p(z) = a(z_1, z_2) + b(z_1, z_2)z_3$. For $z_1, z_2 \in \mathbb{T}$, by direct computation

$$|p|^2 - |\tilde{p}|^2 = \left(1 - |z_3|^2\right)\left(\left|a(z_1, z_2)\right|^2 - \left|b(z_1, z_2)\right|^2\right). \tag{A.1}$$

Then, $|a(z_1, z_2)|^2 - |b(z_1, z_2)|^2$ is a nonnegative two variable trig polynomial of degree one in each variable. As p is stable, $|a|^2 - |b|^2$ is in fact strictly positive on $\mathbb{T}^2$, since a zero would imply $|p(z_1, z_2, \cdot)| = |\tilde{p}(z_1, z_2, \cdot)|$ and this would mean $z_3 \mapsto p(z_1, z_2, z_3)$ has a zero on $\mathbb{T}$.

By the lemma, we may write

$$\left|a(z_1, z_2)\right|^2 - \left|b(z_1, z_2)\right|^2 = \left|E(z_1, z_2)\right|^2 \quad \text{on } \mathbb{T}^2,$$

where E is a vector polynomial with values in $\mathbb{C}^2$.

We also remark that since p is stable, a is stable. By the maximum principle, we can then conclude that

$$\frac{\tilde{b}(z_1, z_2)}{a(z_1, z_2)}$$

is analytic and has modulus strictly less than one (since $|b| = |\tilde{b}|$ on $\mathbb{T}^2$ and since $|a| > |b|$ on $\mathbb{T}^2$). In particular, $a + \tilde{b}$ is stable.

We may polarize formula (A.1) and get for $z_1, z_2 \in \mathbb{T}$

$$\begin{aligned} &p(z_1, z_2, z_3)\overline{p(z_1, z_2, \zeta_3)} - \tilde{p}(z_1, z_2, z_3)\overline{\tilde{p}(z_1, z_2, \zeta_3)} \\ &\quad = (1 - z_3\bar{\zeta}_3)\left|E(z_1, z_2)\right|^2, \end{aligned} \tag{A.2}$$

which we rearrange into

$$\begin{aligned} &p(z_1, z_2, z_3)\overline{p(z_1, z_2, \zeta_3)} + z_3\bar{\zeta}_3\left|E(z_1, z_2)\right|^2 \\ &\quad = \tilde{p}(z_1, z_2, z_3)\overline{\tilde{p}(z_1, z_2, \zeta_3)} + \left|E(z_1, z_2)\right|^2. \end{aligned}$$

Then, for fixed $z_1, z_2 \in \mathbb{T}$ and for varying z_3, the map

$$\begin{bmatrix} p(z_1, z_2, z_3) \\ z_3 E(z_1, z_2) \end{bmatrix} \mapsto \begin{bmatrix} \tilde{p}(z_1, z_2, z_3) \\ E(z_1, z_2) \end{bmatrix} \tag{A.3}$$

gives a well-defined isometry $V(z_1, z_2)$ (which depends on z_1, z_2) from the span of the elements on the left to the span of the elements on the right (the span taken over the above vectors as z_3 varies). More concretely, by examining coefficients of z_3, we map

$$\begin{bmatrix} a(z_1, z_2) \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} \tilde{b}(z_1, z_2) \\ E(z_1, z_2) \end{bmatrix}, \qquad \begin{bmatrix} b(z_1, z_2) \\ E(z_1, z_2) \end{bmatrix} \mapsto \begin{bmatrix} \tilde{a}(z_1, z_2) \\ 0 \\ 0 \end{bmatrix}. \tag{A.4}$$

This is how the "lurking isometry argument" traditionally works, however $V(z_1, z_2)$ does not extend uniquely to define a unitary on $\mathbb{C}^3$ and we would like to extend $V(z_1, z_2)$ so that V is rational in z_1, z_2.

Write $E = [E_1, E_2]^t$. Define $F = [-\tilde{E}_2, \tilde{E}_1]^t$. Then, $\langle F(z_1,z_2), E(z_1,z_2)\rangle = 0$ which means the vector

$$X(z_1,z_2) = \begin{bmatrix} 0 \\ F(z_1,z_2) \end{bmatrix}$$

is orthogonal to both the left and right sides of (A.3). So, to extend V to a rational unitary, it is only a matter of assigning

$$V(z_1,z_2)X(z_1,z_2) = \phi(z_1,z_2)X(z_1,z_2), \tag{A.5}$$

where ϕ is a unimodular function, in such a way that V is rational.

Kummert cleverly gives the matrix V explicitly.

Claim 1. Define

$$V = \frac{1}{a}\begin{bmatrix} \tilde{b} & \tilde{E}^t \\ E & \frac{E\tilde{E}^t - a(\tilde{a}+b)I}{a+\tilde{b}} \end{bmatrix}.$$

Then, V is holomorphic in $\mathbb{D}^2$ and unitary valued on $\mathbb{T}^2$, and V satisfies (A.3) for $(z_1,z_2) \in \mathbb{T}^2$ and hence for all $(z_1,z_2) \in \overline{\mathbb{D}}^2$ by analyticity.

First, V is holomorphic since a and $a + \tilde{b}$ are stable. Using this definition of V, the fact that V is unitary valued on $\mathbb{T}^2$ will follow from checking that (A.3) and (A.5) hold (i.e., $V(z_1,z_2)$ performs the mapping as indicated in (A.3) and (A.5)).

Indeed, it can be directly checked that the equivalent condition in (A.4) holds because of the relation

$$\begin{aligned}\tilde{E}(z_1,z_2)^t E(z_1,z_2) &= z_1 z_2 \big|E(z_1,z_2)\big|^2 \\ &= z_1 z_2\Big(\big|a(z_1,z_2)\big|^2 - \big|b(z_1,z_2)\big|^2\Big) = a\tilde{a} - b\tilde{b}.\end{aligned}$$

In addition, (A.5) holds because

$$V(z_1,z_2)X(z_1,z_2) = -\frac{\tilde{a}+b}{a+\tilde{b}}X(z_1,z_2)$$

since $\tilde{E}^t F = 0$, which is indeed a unimodular multiple of X. This proves the claim.

This means V is a two variable rational matrix valued inner function. It was proved in [10] (see also [2]) that such functions have transfer function representations. Namely, there exists a $(2+n_1+n_2)\times(2+n_1+n_2)$ block unitary

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix},$$

where B is a $2\times(n_1+n_2)$ matrix, C is a $(n_1+n_2)\times 2$, D is a $(n_1+n_2)\times(n_1+n_2)$ (all subdivided as indicated) such that $V(z_1,z_2) = A + Bd(z_1,z_2)(I -$

$Dd(z_1,z_2))^{-1}C$ where

$$d(z_1,z_2)=\begin{bmatrix} z_1I_1 & 0 \\ 0 & z_2I_2 \end{bmatrix}.$$

Here I_1, I_2 are the n_1, n_2-dimensional identity matrices, respectively.

Such a representation is equivalent to the formula

$$U\begin{bmatrix} I \\ z_1G_1(z_1,z_2) \\ z_2G_2(z_1,z_2) \end{bmatrix} = \begin{bmatrix} V(z_1,z_2) \\ G_1(z_1,z_2) \\ G_2(z_1,z_2) \end{bmatrix}, \tag{A.6}$$

where G_1, G_2 are some C^{n_1}, $\mathbb{C}^{n_2}$ valued functions (which can in fact be explicitly solved for).

Define

$$Y=\begin{bmatrix} p \\ z_3E \end{bmatrix} \quad \text{and} \quad H_j = G_jY \quad \text{for } j=1,2.$$

Then,

$$U\begin{bmatrix} I \\ z_1G_1 \\ z_2G_2 \end{bmatrix} Y = U\begin{bmatrix} Y \\ z_1G_1Y \\ z_2G_2Y \end{bmatrix} = U\begin{bmatrix} p \\ z_3E \\ z_1H_1 \\ z_2H_2 \end{bmatrix} = \begin{bmatrix} VY \\ H_1 \\ H_2 \end{bmatrix} = \begin{bmatrix} \tilde{p} \\ E \\ H_1 \\ H_2 \end{bmatrix},$$

where the equations follow in order by: algebra, definitions of Y, H_j, (A.6), and (A.3).

Since U is a unitary and since

$$U\begin{bmatrix} p \\ z_3E \\ z_1H_1 \\ z_2H_2 \end{bmatrix} = \begin{bmatrix} \tilde{p} \\ E \\ H_1 \\ H_2 \end{bmatrix}$$

we have

$$\begin{aligned} &|p|^2+|z_3|^2|E|^2+|z_1|^2|H_1|^2+|z_2|^2|H_2|^2 \\ &\quad =|\tilde{p}|^2+|E|^2+|H_1|^2+|H_2|^2 \end{aligned}$$

which can be rearranged to give

$$|p|^2-|\tilde{p}|^2=\sum_{j=1,2}\big(1-|z_j|^2\big)|H_j|^2+\big(1-|z_3|^2\big)|E|^2.$$

Even though we have not verified that H_1 and H_2 are polynomials, this is enough to prove p is an Agler denominator by [8]. In fact, Theorem 2.1 forces H_1, H_2 to be polynomials of multi-degree $(0,1,1)$, $(1,0,1)$ and the sums of squares $|H_1|^2, |H_2|^2$ can be rewritten as sums of four squares each (by dimensionality; see Remark 2.2). □

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