Illinois Journal of Mathematics
Volume 55, Number 4, Winter 2011, Pages 1533–1549
S 0019-2082

PURE SUBGROUPS OF COMPLETELY DECOMPOSABLE GROUPS AND A GROUP CLASS PROBLEM

DANIEL HERDEN AND LUTZ STRÜNGMANN

ABSTRACT. In the work of Herden and Strüngmann (In *Models, modules and Abelian groups* (2008) 169–186 de Gruyter), an embedding problem for torsion-free Abelian groups was considered. It was shown for a large class of such groups, including the class of all bounded extensions of completely decomposable groups, that any member of the class can be purely embedded into some completely decomposable group. Moreover, an algorithm was given that determines explicitly the pure embedding and the completely decomposable overgroup. We continue the approach from the work of Herden and Strüngmann (In *Models, modules and Abelian groups* (2008) 169–186 de Gruyter) improving the algorithm and extending the main theorem to a broader class of torsion-free Abelian groups including some Hawaiian groups from the article of Mader and Strüngmann (*J. Algebra* **229** (2000) 205–233) and thus complementing the main result from the article of Strüngmann (*Proc. Amer. Math. Soc.* **137** (2009) 3657–3668).

A byproduct and starting point for this generalization will be a discussion of the following group class problem: Which groups G have the property that for any cardinal κ any subgroup U of the direct sum $G^{(\kappa)}$ is the kernel of some endomorphism of $G^{(\kappa)}$?

1. Introduction

Recall that Butler [5] defined a torsion-free Abelian group G to be a *Butler group* if it is an epimorphic image of a completely decomposable group of finite rank. It turned out that this is equivalent to saying that G is a pure

Received July 6, 2010; received in final form January 13, 2011.

The collaboration was supported by the Project No. I-963-98.6/2007 of the German–Israeli Foundation for Scientific Research & Development and by the Project STR627/6-1 of the German Research Foundation DFG.

2010 *Mathematics Subject Classification.* 20K15, 20K20, 20K25, 15A36.

subgroup of a completely decomposable group of finite rank. The class of Butler groups of finite rank has been studied extensively and was generalized in several ways to the infinite rank case implying interesting structure and independence results (see [4], [7], [13]). However, only little was proved on the straightforward class of pure subgroups of completely decomposable groups of arbitrary rank. Obviously, any pure subgroup H of a completely decomposable group is *locally Butler* meaning that every finite rank pure subgroup of H is a finite rank Butler group. However, Arnold [1] (see also [3]) gave an example of a countable group that is locally Butler but cannot be embedded as pure subgroup into any completely decomposable group. In some cases it is easy to describe all pure subgroups of a completely decomposable group D, for instance if D is homogeneous completely decomposable. However, to characterize the pure subgroups of completely decomposable groups in general seems to be a hard task.

The authors attacked this problem in [9] and obtained a very constructive proof of such pure embeddings for groups from a large class $\mathcal{G}$ of torsion-free Abelian groups. In particular, the almost completely decomposable groups and the bounded completely decomposable groups are contained in $\mathcal{G}$. An algorithm was obtained in [9] that produces for a given group G from the above class $\mathcal{G}$ a completely decomposable group D and a pure embedding $G \hookrightarrow D$. Later on, this algorithm was implemented in Maple for the case of finite rank groups (see [15]).

In this paper, we continue the work from [9] and improve the algorithm showing that even more groups are pure subgroups of completely decomposable groups. This way we are able to answer some of the open problems from [9]. The approach uses tools from linear algebra associating to a given group G a matrix B that mirrors the relations defining G. The main theorem says that if G is in *strong normal form* and there exists a matrix A that is *B-good* (i.e., relating the kernel of A and the image of B), then G is a pure subgroup of a completely decomposable group. As a starting point for this enhanced result will serve the investigation of the following related group class problem in Section 4: Which (not necessarily torsion-free) groups G have the property that for any cardinal κ and any subgroup $U \subseteq G^{(\kappa)}$ there exists some endomorphism of $G^{(\kappa)}$ having kernel U? We will give a complete classification for the case of reduced groups G and provide further results concerning the nonreduced case. It remains the crucial question:

> *Do all pure subgroups of completely decomposable groups admit B-good matrices?*

Here we will investigate the class of Hawaiian groups introduced in [11]. These groups are known to be pure subgroups of completely decomposable groups by the main theorem from [14]. We will give a negative example of a Hawaiian group without a B-good matrix and classify those countable Hawaiian groups which admit a B-good matrix. Thus it remains an interesting but still open challenge to characterize the pure subgroups of completely decomposable groups and unifying our result with [14] might be a first step to achieve this goal.

Our notation is standard as can be found for instance in [6] and we write maps from the left.

2. Preliminaries

In this section, we recall some basic definitions and terminology which shall be used frequently below. Let Π be the set of primes. A group X is called *rational* if it is isomorphic to a subgroup of the group of rationals $\mathbb{Q}$, that is, X is a torsion-free group of rank 1. If $[X]$ is the class of all groups isomorphic to X we may define an order relation by setting $[X] \leq [X']$ if $\mathrm{Hom}(X, X') \neq 0$. The class $[X]$ is called the *type* of X and if $x \in G$ is an element of the torsion-free group G, then we denote by type ${}^G(x)$ the type of x in G which is defined as type ${}^G(x) = [\langle x\rangle_*^G]$. Here, $\langle x\rangle_*^G$ is the pure subgroup of G generated by x. It is well known that types can also be described by equivalence classes of infinite sequences $(n_p : p \in \Pi)$ of natural numbers and the symbol ∞. For instance, one can choose $(n_p : p \in \Pi)$ such that $X = \langle 1/p^{m_p} : m_p < n_p + 1\rangle$ with the convention that $\infty + 1 = \infty$. We say that two such sequences $(n_p : p \in \Pi)$ and $(n'_p : p \in \Pi)$ are equivalent if they differ in finitely many finite entries only, that is, if $\sum_{p\in\Pi} |n_p - n'_p|$ is finite. For further details on types, see [6]. Since there is no danger of confusion, we will assume that $\mathbb{Z} \subseteq X$ for every rational group X and shall use X synonymously for its induced type $[X]$ respectively, the corresponding equivalence class $[(n_p : p \in \Pi)]$.

A torsion-free group G is called *completely decomposable* if G is a direct sum of rational groups. A *Butler group* is a pure subgroup of a finite rank completely decomposable group or equivalently an epimorphic image of a completely decomposable group of finite rank. Examples of Butler groups are the so-called *almost completely decomposable groups* (acd-groups) which are finite extensions of completely decomposable groups of finite rank. More generally, a torsion-free group G is a B_0*-group* if every pure finite rank subgroup of G is a Butler group. Examples of B_0-groups are countable bcd-groups. Recall that a *bounded completely decomposable group* (bcd-group) is a bounded extension of a completely decomposable group of arbitrary rank. For further terminology and details on Butler groups, acd-groups and bcd-groups see [2], [6], [10] and [11].

3. Extensions of completely decomposable groups by torsion groups

In this section, we consider extensions of completely decomposable groups by torsion groups as it was done in [9]. Adopting notation from [9] let $C = \bigoplus_{i\in\kappa} R_i e_i$ be a completely decomposable group of rank κ for some cardinal κ

and rational groups R_i $(i \in \kappa)$. Moreover, let $D = \mathbb{Q} \otimes C = \bigoplus_{i \in \kappa} \mathbb{Q} e_i$ denote the divisible hull of C.

Following notation in [9], we denote by $\mathcal{G}(C)$ the class of all (torsion) extensions of C inside D. This is to say

$$\mathcal{G}(C) := \{G : C \subseteq G \subseteq D\}.$$

Clearly, any $G \in \mathcal{G}(C)$ satisfies that G/C is torsion.

In [9], it turned out to be convenient for the reader to discuss a running example to explain the theory that is developed. Our example is motivated by [12] where the class of Hawaiian groups was considered. Solving an open problem from [11] it was shown by the second author that these groups are in fact pure subgroups of completely decomposable groups. Recall from [9] the following definition where the *support* $[d]$ of an element $d = \sum_{i \in \kappa} q_i e_i \in D$ is defined as $[d] = \{i \in \kappa : q_i \neq 0\}$.

Definition 3.1. We say that a group $G \in \mathcal{G}(C)$ is given in standard form if there are elements $d_j \in D$, primes $p_j \in \Pi$ (not necessarily distinct) and integers $m_j \in \mathbb{N}_0$ for $j \in \kappa$ such that

(i) $G = \langle C, d_j : j \in \kappa \rangle$ and

(ii) $d_j = \frac{1}{p_j^{m_j}} \sum_{i \in \kappa} d_{i,j} e_i$ with $d_{i,j} \in \mathbb{Z}$ for all $i, j \in \kappa$.

Moreover, G is in normal form if in addition

(iii) $h_{p_j}^{R_i}(1) = 0$ for all $i \in [d_j]$ and $j \in \kappa$

holds. Finally, G is in strong normal form if the more general condition

(iii′) $h_{p_j}^{R_i}(1) = 0$ for all $i, j \in \kappa$

holds.

Here $h_{p_j}^{R_i}(1)$ denotes as usual the p_j-height of 1 in R_i. Note that in Definition 3.1, for all $j \in \kappa$ the elements $d_{i,j}$ are almost all equal to zero since $d_j \in D$ and that the standard form of G is by no means unique. In [9], it was proven that essentially every group in $\mathcal{G}(C)$ can be given in standard form by adding a free summand. However, for a group to possess a normal form one has to impose more conditions. Following [9], we associate to each group G in $\mathcal{G}(C)$ which is given in standard form a matrix $B(G)$ with rational entries. $B(G)$ describes the relations d_j $(j \in \kappa)$ that define the quotient G/C.

Definition 3.2. Let $G = \langle C, d_j : j \in \kappa \rangle \in \mathcal{G}(C)$ be given in standard form. If $d_j = \frac{1}{p_j^{m_j}} \sum_{i \in \kappa} d_{i,j} e_i$ with $d_{i,j} \in \mathbb{Z}$ put $B(G) = (\frac{d_{i,j}}{p_j^{m_j}})_{i,j \in \kappa}$, the $\kappa \times \kappa$ matrix with $\frac{d_{i,j}}{p_j^{m_j}}$ as entry in the ith row and jth column. We call $B(G)$ the matrix associated to G.

Note that the matrix $B(G)$ from Definition 3.2 is column finite and hence an endomorphism of $D = \mathbb{Q}^{(\kappa)}$ mapping e_i to d_i, that is, $B(G)(e_i) = d_i \in D$. From this, it is clear that $B(G)$ acts on D as the usual matrix operation on a vector space. However, notice that $B(G)$ does not define a homomorphism from C to G since e_i is of type R_i but d_i is not necessarily of this type inside G.

Sufficient conditions for the existence of a normal form were developed in [9]. However, our running example shows that these conditions are not necessary.

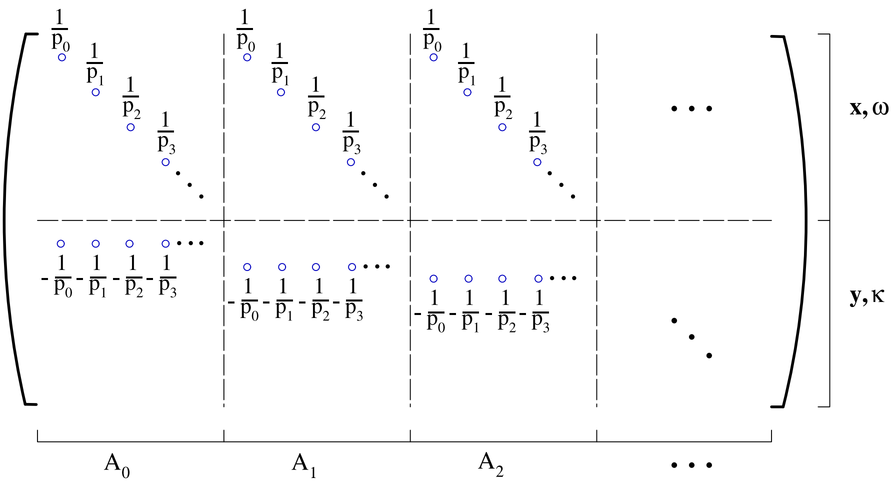
LEMMA 3.3 ([9]). *Let $G = \langle C, d_j : j \in \kappa \rangle \in \mathcal{G}(C)$ be given in standard form and $B(G)$ its associated matrix. If either*

(i) *$\{p_j : j \in \kappa\}$ is finite or*
(ii) *for every row $\bar{b}$ of $B(G)$ there is a natural number $n \in \mathbb{N}$ such that $n\bar{b} \in \mathbb{Z}^\kappa$ (e.g. $B(G)$ is row finite),*

then there is a completely decomposable group $C' = \bigoplus_{i\in\kappa} R_i' e_i' \cong C$ and a torsion-free group $G' \cong G$ such that $G' \in \mathcal{G}(C')$ is given in normal form. Moreover, the isomorphism $G \cong G'$ is induced by the isomorphism $C \cong C'$.

We now consider our example G^*.

EXAMPLE 3.4. Let $\aleph_0 \le \kappa$ be a cardinal, let $C := \bigoplus_{n\in\omega} \mathbb{Z}e_n \oplus \bigoplus_{\alpha\in\kappa} \mathbb{Z}e^\alpha$ and $\Pi = \{p_n : n \in \omega\}$ be an enumeration of the set of primes. Moreover, choose subsets $A_\alpha \subseteq \omega$ for $\alpha \in \kappa$. Then $G^* = \langle C, \frac{e_n - e^\alpha}{p_n} : n \in \omega, \alpha \in \kappa, n \in A_\alpha \rangle \in \mathcal{G}(C)$ is given in strong normal form. However, none of the conditions in Lemma 3.3 is satisfied. If $A_\alpha = \omega$ for all $\alpha \in \kappa$ the associated matrix $B = B(G^*)$ of G^* is



The matrix $B(G^*)$ for the case of general A_αs is obtained by omitting some of the columns.

4. A group class problem

In preparation for the pure embedding given in Section 5 we discuss the following basic group theoretic question:

(4.1) Which (not necessarily torsion-free) groups G have the property that for any cardinal κ and any subgroup $U \subseteq G^{(\kappa)}$ there is some $\varphi \in \mathrm{End}(G^{(\kappa)})$ with $U = \mathrm{Ker}\,\varphi$?

As every endomorphism $\varphi : G^{(\kappa)} \to G^{(\kappa)}$ with $\mathrm{Ker}\,\varphi = U$ is uniquely determined by some monomorphism $\varphi' : G^{(\kappa)}/U \to G^{(\kappa)}$ this question can be rephrased as:

(4.2) Which (not necessarily torsion-free) groups G have the property that for any cardinal κ and any subgroup $U \subseteq G^{(\kappa)}$ the group $G^{(\kappa)}/U$ embeds into $G^{(\kappa)}$?

The following lemma and theorem aim towards a partial answer of question (4.1).

Lemma 4.1. *Every torsion-free group fails property* (4.1).

Proof. Let G be a torsion-free group and set $U = \langle 2g \rangle \subseteq G$ for some element $0 \neq g \in G$. Then $0 \neq g + U \in G/U$ is an element of order 2 and the group G/U is not torsion-free. Thus, G/U does not embed into G contradicting (4.2). □

Theorem 4.2. *A reduced Abelian group satisfies property* (4.1) *if and only if it is a torsion group with bounded p-components.*

Proof. First, we show that every bounded p-group satisfies property (4.1):

Together with G also $G^{(\kappa)}$ and $G^{(\kappa)}/U$ are bounded p-groups and thus direct sums of cyclic p-groups. In particular, we can write

$$G^{(\kappa)} \cong \bigoplus_{i=1}^{n} \mathbb{Z}\big(p^i\big)^{(\kappa_i)} \quad \text{and} \quad G^{(\kappa)}/U \cong \bigoplus_{i=1}^{n} \mathbb{Z}\big(p^i\big)^{(\lambda_i)}$$

for some $n < \omega$ and suitable cardinals $0 \leq \kappa_i, \lambda_i$. Using that $\mathbb{Z}(p^i)$ embeds into $\mathbb{Z}(p^j)$ for $i \leq j$, the existence of a monomorphism $\varphi' : G^{(\kappa)}/U \to G^{(\kappa)}$ will be an easy consequence of

$$\sum_{i=m}^{n} \lambda_i \leq \sum_{i=m}^{n} \kappa_i \quad \text{for all } 1 \leq m \leq n. \tag{1}$$

Towards proving (1), we first observe that

$$r_p\big(p^{m-1}G^{(\kappa)}/p^m G^{(\kappa)}\big) = \sum_{i=m}^{n} \kappa_i \quad \text{and}$$

$$r_p\big(p^{m-1}\big(G^{(\kappa)}/U\big)/p^m\big(G^{(\kappa)}/U\big)\big) = \sum_{i=m}^{n} \lambda_i,$$

where r_p denotes the p-rank of the respective p-bounded groups. Now from

$$\begin{aligned} p^{m-1}\big(G^{(\kappa)}/U\big)/p^m\big(G^{(\kappa)}/U\big) &= \big(\big(p^{m-1}G^{(\kappa)} + U\big)/U\big)/\big(\big(p^m G^{(\kappa)} + U\big)/U\big) \\ &\cong \big(p^{m-1}G^{(\kappa)} + U\big)/\big(p^m G^{(\kappa)} + U\big) \end{aligned}$$

$$
\begin{aligned}
&= \big(p^{m-1}G^{(\kappa)} + \big(p^m G^{(\kappa)} + U\big)\big)/\big(p^m G^{(\kappa)} + U\big) \\
&\cong p^{m-1}G^{(\kappa)}/\big(p^{m-1}G^{(\kappa)} \cap \big(p^m G^{(\kappa)} + U\big)\big) \\
&\cong p^{m-1}G^{(\kappa)}/\big(p^m G^{(\kappa)} + \big(p^{m-1}G^{(\kappa)} \cap U\big)\big),
\end{aligned}
$$

and $p^m G^{(\kappa)} \subseteq p^m G^{(\kappa)} + (p^{m-1}G^{(\kappa)} \cap U)$ follows

$$
\begin{aligned}
\sum_{i=m}^{n} \lambda_i &= r_p\big(p^{m-1}\big(G^{(\kappa)}/U\big)/p^m\big(G^{(\kappa)}/U\big)\big) \\
&= r_p\big(p^{m-1}G^{(\kappa)}/\big(p^m G^{(\kappa)} + \big(p^{m-1}G^{(\kappa)} \cap U\big)\big)\big) \\
&\le r_p\big(p^{m-1}G^{(\kappa)}/p^m G^{(\kappa)}\big) = \sum_{i=m}^{n} \kappa_i
\end{aligned}
$$

for the p-ranks of the respective p-bounded groups. This verifies (1).

Now property (4.1) follows for every torsion group G whose p-components G_p are bounded:

As together with G also $G^{(\kappa)}$ and U are torsion groups we can write $G^{(\kappa)} = \bigoplus_{p\in\Pi} A_p$ and $U = \bigoplus_{p\in\Pi} B_p$ with bounded p-groups $B_p \subseteq A_p$ $(p \in \Pi)$. Thus

$$
G^{(\kappa)}/U \cong \bigoplus_{p\in\Pi} (A_p/B_p)
$$

holds and an embedding $\varphi' : G^{(\kappa)}/U \to G^{(\kappa)}$ is defined by setting $\varphi' = \bigoplus_{p\in\Pi} \varphi'_p$ for suitable embeddings $\varphi'_p : A_p/B_p \to A_p$.

It remains to be shown that every reduced Abelian group with property (4.1) has to be torsion with bounded p-components.

We start with the case of torsion groups: Thus, assume that G is torsion and satisfies property (4.1). Without loss of generality we may assume that G is p-torsion for some prime p. We choose a p-basic subgroup B of G. Hence, B is a direct sum of cyclic p-groups and the quotient G/B is divisible. If $G/B \neq 0$, property (4.1) implies that G/B embeds into G—a contradiction since G is reduced. Therefore, $G = B$ and it remains to prove that B is bounded. However, if B is unbounded, then the group $T = \bigoplus_{n\in\omega} \mathbb{Z}(p^n)$ is an epimorphic image of B. By [6], Exercise 17.14(a), p. 91, any countable p-group is an epimorphic image of T, hence of B and thus any countable p-group embeds into B by property (4.1). This includes countable divisible p-groups contradicting G being reduced.

Now for the case of general reduced Abelian groups: Assume that G is reduced and satisfies property (4.1). As shown the torsion subgroup (which satisfies property (4.1) as a fully invariant subgroup of G) has bounded p-components. Choose a prime p and let p^n denote the bound of the p-component of $t(G)$, hence $p^n(t(G))_p = 0$. If now x is a torsion-free element of G, then $x + \langle p^{n+1}x\rangle$ is an element of order p^{n+1} inside $G/\langle p^{n+1}x\rangle$ and with property

(4.1) also inside G, a contradiction. Thus, G itself is a torsion group with bounded p-components. $\square$

Finally, we have the following result which shows that groups with large divisible part always satisfy property (4.1). Let D_0 denote the torsion-free part of a divisible Abelian group D.

LEMMA 4.3. *Let $G = D \oplus R$ be an Abelian group with D divisible and R reduced. If for all $p \in \Pi \cup \{0\}$ the p-component D_p of D has size at least $|G|$, then G satisfies property* (4.1).

Proof. For any subgroup U of $G^{(\kappa)}$ the corresponding quotient $G^{(\kappa)}/U$ has size less than or equal to $|D_0^{(\kappa)}|$ by our assumptions. Moreover, any p-component $(G^{(\kappa)}/U)_p$ of $G^{(\kappa)}/U$ (for $p \in \Pi$) has size less than or equal to $|D_p^{(\kappa)}|$. Thus, $G^{(\kappa)}/U$ can be embedded into $D^{(\kappa)} \subseteq G^{(\kappa)}$. $\square$

5. The pure embedding

We now return to matrices. If α, β are ordinals and $R \subseteq \mathbb{Q}$, then let $\mathrm{Mat}_{(\alpha\times\beta)}(R)$ consist of all column finite matrices with α rows and β columns and with entries in R. Note that any $B \in \mathrm{Mat}_{(\alpha\times\beta)}(R)$ defines a homomorphism from $\mathbb{Q}^{(\beta)}$ to $\mathbb{Q}^{(\alpha)}$ in the obvious way.

We need to introduce a subclass of $\mathcal{G}(C)$ as follows. Let

$$\mathcal{G}_{bd}(C) = \left\{ G \in \mathcal{G}(C) \mid (G/C)_p \textit{ is bounded by some } p^{n_p^G} \textit{ for all } p \in \Pi \right\}.$$

If $G \in \mathcal{G}_{bd}(C)$, then we will assume that $n_p := n_p^G$ denotes the corresponding bound of $(G/C)_p$ and that n_p is chosen to be minimal with this property for every prime $p \in \Pi$. Note that $n_p = n_p^G$ depends on G but there will be no danger of confusion if we suppress the index G in the sequel. Moreover, for $G \in \mathcal{G}_{bd}(C)$ we define

$$R(G) := \left\langle \frac{1}{p^n} : p \in \Pi, n \leq n_p \right\rangle$$

and if $p \in \Pi$ we let

$$R_p(G) := \left\langle \frac{1}{q^n} : p \neq q \in \Pi, n \leq n_q \right\rangle.$$

Clearly, for $G \in \mathcal{G}_{bd}(C)$ in standard form we have the following properties

- without loss of generality, we can choose $m_j \leq n_p$ whenever $p_j = p$.
- $B(G) \in \mathrm{Mat}_{(\kappa\times\kappa)}(R(G))$.
- $p^{n_p} B(G) \in \mathrm{Mat}_{(\kappa\times\kappa)}(R_p(G))$.
- $p^{n_p} B(G)$ defines a homomorphism from $\mathbb{Z}^{(\kappa)}$ to $R_p(G)^{(\kappa)}$.

Different to [9], we now define B-good matrices as follows.

DEFINITION 5.1. Let $G \in \mathcal{G}_{bd}(C)$ be in standard form and $B = B(G) \in \mathrm{Mat}_{(\kappa\times\kappa)}(R(G))$ its associated matrix. A matrix

$$A = (a_{i,j})_{i\in\kappa, j\in\kappa} \in \mathrm{Mat}_{(\kappa\times\kappa)}(\mathbb{Z})$$

is B-good if the following condition is satisfied for all $p \in \Pi$:

$$x \in \mathrm{Im}\big(p^{n_p}B\big) + \big(p^{n_p}R_p(G)\big)^{(\kappa)} \quad\Longleftrightarrow\quad A(x) \in \big(p^{n_p}R_p(G)\big)^{(\kappa)}$$

if one regards $p^{n_p}B$ and A as homomorphisms from $\mathbb{Z}^{(\kappa)}$ to $R_p(G)^{(\kappa)}$, respectively, from $R_p(G)^{(\kappa)}$ to $R_p(G)^{(\kappa)}$.

We have a first easy lemma that shows the existence of B-good matrices locally.

LEMMA 5.2. *Let $G \in \mathcal{G}_{bd}(C)$ be in standard form and*

$$B = B(G) \in \mathrm{Mat}_{(\kappa\times\kappa)}\big(R(G)\big)$$

its associated matrix. For every $p \in \Pi$, there is a matrix $A_p = (a_{i,j})_{i\in\kappa, j\in\kappa} \in \mathrm{Mat}_{(\kappa\times\kappa)}(\mathbb{Z})$ such that

$$x \in \mathrm{Im}\big(p^{n_p}B\big) + \big(p^{n_p}R_p(G)\big)^{(\kappa)} \quad\Longleftrightarrow\quad A_p(x) \in \big(p^{n_p}R_p(G)\big)^{(\kappa)}.$$

Proof. We start with the observation that

$$\big(g + p^{n_p}R_p(G)\big) \cap \mathbb{Z} \neq \emptyset \quad \text{for all } g \in R_p(G). \tag{2}$$

We represent $g = \frac{s}{t} \in R_p(G)$ with $s, t \in \mathbb{Z}$ and $\gcd(t,p) = 1$. Choosing $a, b \in \mathbb{Z}$ with $at + bp^{n_p} = 1$ we have

$$g = \big(at + bp^{n_p}\big)g = as + bp^{n_p}g$$

and $as \in (g + p^{n_p}R_p(G)) \cap \mathbb{Z}$ proving (2).

Now Theorem 4.2 holds for $R_p(G)^{(\kappa)}/(p^{n_p}R_p(G))^{(\kappa)}$ as bounded p-group and there exists an endomorphism $\varphi \in \mathrm{End}(R_p(G)^{(\kappa)}/(p^{n_p}R_p(G))^{(\kappa)})$ with $\mathrm{Ker}\,\varphi = (\mathrm{Im}(p^{n_p}B) + (p^{n_p}R_p(G))^{(\kappa)})/(p^{n_p}R_p(G))^{(\kappa)}$. With (2), we can represent φ by a matrix $A_p = (a_{i,j})_{i\in\kappa, j\in\kappa} \in \mathrm{Mat}_{(\kappa\times\kappa)}(\mathbb{Z})$ and the claim of the lemma is immediate. ☐

Clearly, if the set of all primes p such that $(G/C)_p$ is nontrivial for $G \in \mathcal{G}_{bd}(C)$ is finite, then the above Lemma 5.2 together with the Chinese Remainder theorem implies the following corollary (see [9]). Note that by [8] (see also Lemma 3.3) every bcd-group can be assumed to be in strong normal form.

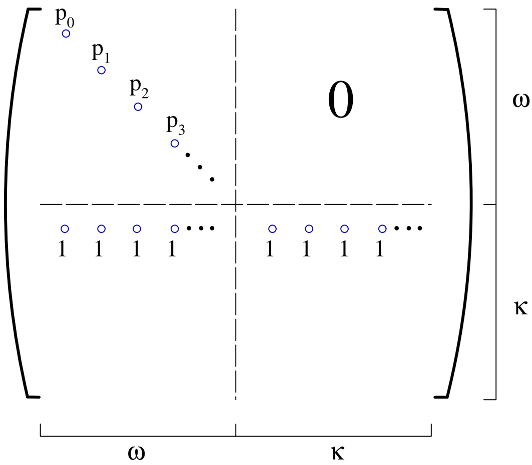
COROLLARY 5.3. *Let $G \in \mathcal{G}(C)$ be a bcd-group and*

$$B = B(G) \in \mathrm{Mat}_{(\kappa\times\kappa)}\big(R(G)\big)$$

be the associated matrix of its strong normal form. Then there exists a B-good matrix $A \in \mathrm{Mat}_{(\kappa\times\kappa)}(\mathbb{Z})$.

We return to our running example G^* from Example 3.4.

EXAMPLE 5.4. Let G^* be as in Example 3.4 with $A_\alpha = \omega$ for all $\alpha \in \kappa$. Then $G^* \in \mathcal{G}_{bd}$ has a B-good matrix of the following form:



where this matrix is of dimension $(\omega, \kappa) \times (\omega, \kappa)$.

Proof. First, note that for every $p \in \Pi$ we have $R_p(G^*) = \langle \frac{1}{q} : q \neq p \rangle$. We have to check Definition 5.1 for all $p \in \Pi$ but will only do this for $p = p_0$. The general case is then obtained by obvious modification. To start, let be

$$x = \begin{pmatrix} x_0 & x_1 & \cdots \mid x^0 & x^1 & \cdots & x^\alpha & \cdots \end{pmatrix}^T \in R_p\left(G^*\right)^{(\omega,\kappa)}$$

and assume $x \equiv p_0 By \bmod p_0 R_p(G^*)^{(\omega,\kappa)}$ for some vector

$$y = \begin{pmatrix} y_0^0 & y_1^0 & \cdots \mid y_0^1 & y_1^1 & \cdots \mid \cdots \mid y_0^\alpha & y_1^\alpha & \cdots \mid \cdots \end{pmatrix}^T \in \mathbb{Z}^{(\omega\cdot\kappa)}.$$

Hence, with matrix B from Example 3.4 we can write

$$x \equiv p_0 By = \left(\begin{array}{c} \sum_{\alpha\in\kappa} y_0^\alpha \\ \frac{p_0}{p_1} \sum_{\alpha\in\kappa} y_1^\alpha \\ \vdots \\ \hline -\sum_{i\in\omega} \frac{p_0}{p_i} y_i^0 \\ \vdots \\ -\sum_{i\in\omega} \frac{p_0}{p_i} y_i^\alpha \\ \vdots \end{array} \right) \equiv \left(\begin{array}{c} \sum_{\alpha\in\kappa} y_0^\alpha \\ 0 \\ \vdots \\ \hline -y_0^0 \\ \vdots \\ -y_0^\alpha \\ \vdots \end{array} \right) \bmod p_0 R_p\left(G^*\right)^{(\omega,\kappa)}$$

and any element in the image of $p_0 B$ is particularly of the form

$$\begin{pmatrix} \sum_{\alpha\in\kappa} y_0^\alpha & 0 & \cdots \mid -y_0^0 & \cdots & -y_0^\alpha & \cdots \end{pmatrix}^T \bmod p_0 R_p\left(G^*\right)^{(\omega,\kappa)}.$$

Clearly, in this case $Ax = A(p_0By) \equiv 0 \bmod p_0R_p(G^*)^{(\omega,\kappa)}$.

We now calculate Ax for any vector x as above and obtain

$$Ax = \begin{pmatrix} p_0x_0 & p_1x_1 & \cdots \mid \sum_{n\in\omega} x_n + \sum_{\alpha\in\kappa} x^\alpha & 0 & \cdots \end{pmatrix}^T.$$

Thus, if $Ax \in p_0R_p(G^*)^{(\omega,\kappa)}$, then $x_n \in p_0R_p(G^*)$ for all $n \geq 1$. Moreover, $x_0 \equiv -\sum_{\alpha\in\kappa} x^\alpha \bmod p_0R_p(G^*)$ and hence

$$x \equiv \begin{pmatrix} -\sum_{\alpha\in\kappa} x^\alpha & 0 & \cdots \mid x^0 & x^1 & \cdots & x^\alpha & \cdots \end{pmatrix}^T \bmod p_0R_p\big(G^*\big)^{(\omega,\kappa)}$$

which is in the image of p_0B. □

That in general Hawaiian groups fail to have a B-good matrix is shown by the following

Example 5.5. Let G^* be as in Example 3.4 with $A_0 = \omega \setminus \{0\}$ and $A_\alpha = \omega$ for all $0 < \alpha \in \kappa$. Then $G^* \in \mathcal{G}_{bd}$ has no B-good matrix.

Proof. We retain the notations from Examples 3.4 and 5.4 and assume that a B-good matrix A exists. Let

$$a = \begin{pmatrix} a_0 & a_1 & \cdots \mid a^0 & a^1 & \cdots & a^\alpha & \cdots \end{pmatrix} \in \mathbb{Z}^{(\omega,\kappa)}$$

denote a row vector of this matrix A. Then for $n \geq 1$, $\alpha \in \kappa$ setting $p = p_n$ we have $e^0 - e^\alpha = (e_n - e^\alpha) - (e_n - e^0) \in \operatorname{Im}(p_nB)$ and hence $a^0 - a^\alpha = a^T \cdot (e^0 - e^\alpha) \in \mathbb{Z} \cap p_nR_p(G^*) = p_n\mathbb{Z}$. Thus, $a^0 - a^\alpha = 0$ and $a^\alpha = a^\beta$ holds for all $\alpha, \beta \in \kappa$. But then $a^T \cdot (e_0 - e^0) = a_0 - a^0 = a_0 - a^1 = a^T \cdot (e_0 - e^1) \in p_0R_p(G^*)$ and $A(e_0 - e^0) \in p_0R_p(G^*)^{(\omega,\kappa)}$ follows though obviously $e_0 - e^0 \notin \operatorname{Im} p_0B + p_0R_p(G^*)^{(\omega,\kappa)}$, a contradiction. □

The last two examples can easily be generalized to characterize all those countable Hawaiian groups which admit a B-good matrix.

Lemma 5.6. *Let G^* be as in Example 3.4 with $\kappa = \omega$. Then G^* admits a B-good matrix if and only if for all $\alpha \neq \beta < \omega$ the sets A_α and A_β are either almost disjoint or $A_\alpha = A_\beta = \omega$ holds.*

Proof. We again retain the notations from Examples 3.4 and 5.4.

Our condition on the sets A_α $(\alpha < \omega)$ is obviously necessary: If $|A_\alpha \cap A_\beta| = \omega$ with $n \notin A_\beta$ for some $\alpha \neq \beta < \omega$, then an argument similar to Example 5.5 contradicts the existence of a B-good matrix A as $A(e^\alpha - e^\beta) = 0$ but $e^\alpha - e^\beta \notin \operatorname{Im} p_nB + p_nR_{p_n}(G^*)^{(\omega,\omega)}$.

To show that our condition is also sufficient, we will describe how to construct a B-good matrix A' from a given family of sets A_α $(\alpha < \omega)$. We will start off from the matrix A given in Example 5.4 which is B-good in the special case of $A_\alpha = \omega$ for all $\alpha < \omega$. From Example 5.4, we know

$$\begin{aligned} &Ax \in p_nR_{p_n}\big(G^*\big)^{(\omega,\omega)} \\ &\quad\iff \quad x \in U_n := \big\langle R_{p_n}\big(G^*\big)\big(e_n - e^\alpha\big) \big| \alpha < \omega\big\rangle + p_nR_{p_n}\big(G^*\big)^{(\omega,\omega)} \end{aligned}$$

and we have

$$x \in \operatorname{Im}(p_n B) + p_n R_{p_n}\left(G^*\right)^{(\omega,\omega)}$$
$$\iff \quad x \in V_n := \left\langle R_{p_n}\left(G^*\right)\left(e_n - e^\alpha\right) | n \in A_\alpha \right\rangle + p_n R_{p_n}\left(G^*\right)^{(\omega,\omega)} \subseteq U_n.$$

Our goal is to realize for all $n < \omega$

$$A'x \in p_n R_{p_n}\left(G^*\right)^{(\omega,\omega)} \quad \iff \quad x \in V_n$$

by adding appropriate additional lines

$$a = \begin{pmatrix} a_0 & a_1 & \cdots \mid a^0 & a^1 & \cdots & a^\alpha & \cdots \end{pmatrix} \in \mathbb{Z}^{(\omega,\kappa)}$$

to the matrix A. For this let some $m < \omega$ and $y \in U_m \setminus V_m$ be given. Without loss of generality, we can assume a representation $y = \sum_{m \notin A_\alpha} z_\alpha(e_m - e^\alpha)$ with integers $z_\alpha \not\equiv 0 \pmod{p_m}$. We now can easily choose the coefficients $a_n, a^\alpha \in \mathbb{Z}$ of our new line vector a by induction over $\alpha < \omega$ such that the following conditions hold:

(1) If $\alpha, \beta < \omega$ with $A_\alpha = A_\beta$, then $a^\alpha = a^\beta$.
(2) If $\alpha, n < \omega$ with $n \in A_\alpha$, then $a^\alpha \equiv a_n \pmod{p_n}$.
(3) $\sum_{m \notin A_\alpha} z_\alpha(a_m - a^\alpha) \not\equiv 0 \pmod{p_m}$.

Both conditions (2) and (3) make crucial use of the Chinese Remainder theorem and the almost disjointness of the sets $A_\alpha \neq \omega$ $(\alpha < \omega)$.

Now, adding the line a to our starting matrix A will guarantee $A'y \notin p_m R_{p_m}(G^*)^{(\omega,\omega)}$ while at the same time preserving $A'x \in p_n R_{p_n}(G^*)^{(\omega,\omega)}$ for all $x \in V_n$ $(n < \omega)$. Thus, adding to A a suitable line for each single bad candidate $y \in U_m \setminus V_m$ will result in a B-good matrix A'. □

We now prove the main result of this section.

THEOREM 5.7. *Let $C = \bigoplus_{i \in \kappa} R_i e_i$ and $G \in \mathcal{G}_{bd}(C)$ be in strong normal form and $B = B(G) \in \operatorname{Mat}_{(\kappa \times \kappa)}(R(G))$ its associated matrix. Moreover, assume there exists a B-good matrix A. Then G is a pure subgroup of some completely decomposable group D.*

Proof. Let $G \in \mathcal{G}_{bd}(C)$ be given in strong normal form with $C = \bigoplus_{i \in \kappa} R_i e_i$. Since G is in strong normal form we have $h_p^{R_i}(1) = 0$ for all $i \in \kappa$ and $\frac{1}{p} \in R(G)$ and hence $R_i \cap R(G) = \mathbb{Z}$ for all $i \in \kappa$. Let $B = B(G)$ be the matrix associated to G and let $A = (a_{i,j})_{i \in \kappa, j \in \kappa}$ be B-good.

Define $A' = (a'_{i,j})_{i \in \kappa+\kappa, j \in \kappa} \in \operatorname{Mat}_{((\kappa+\kappa)\times\kappa)}(\mathbb{Z})$ as follows:

- $a'_{i,j} = a_{i,j}$ and $a'_{\kappa+i,j} = \delta(i,j)$ for $i, j \in \kappa$ where δ denotes the Kronecker function, that is, $\delta(i,j) = 1$ if $i = j$ and 0 otherwise.

For $i \in \kappa$, we now put

- $R'_i = \mathbb{Z} + \sum_{a_{i,j} \neq 0, j \in \kappa} R_j \subseteq \mathbb{Q}$,
- $R'_{\kappa+i} = R_i + R(G)$ and
- $C' = \bigoplus_{i \in \kappa+\kappa} R'_i e'_i$.

We will view A' as an embedding of C into C' and define a mapping

$$\varphi : C \to C', \quad e_j \mapsto \sum_{i\in\kappa} a_{i,j} e'_i + e'_{\kappa+j} = A(e_j) + e'_{\kappa+j} = A'(e_j).$$

By the choice of the R'_i $(i \in \kappa+\kappa)$ the mapping φ is well-defined and we have that $\kappa + i \in [\varphi(e_j)]$ if and only if $i = j$ for $i, j \in \kappa$. We claim that φ extends to $\varphi : G \to C'$ and is a pure embedding. Let $\tilde{\varphi}$ be the extension of φ to $D = \mathbb{Q} \otimes G = \mathbb{Q}^{(\kappa)}$. We have to prove first that $\tilde{\varphi}(d_k) \in C'$ for all $k \in \kappa$. Fix $k \in \kappa$, put $p = p_k$ and recall that $d_k = \frac{1}{p^{m_k}} \sum_{i\in\kappa} d_{i,k} e_i$. Then

$$p^{n_p} d_k \in \mathbb{Z}^{(\kappa)} \quad \text{and} \quad p^{n_p} d_k = p^{n_p} B(e_k) \in \operatorname{Im}\bigl(p^{n_p} B\bigr).$$

Note that $m_k \le n_p$ since $p = p_k$. Thus,

$$A\bigl(p^{n_p} d_k\bigr) \in \bigl(p^{n_p} R_p(G)\bigr)^{(\kappa)},$$

$A(p^{n_p} d_k) \in (p^{n_p} R_p(G) \cap \mathbb{Z})^{(\kappa)} = (p^{n_p}\mathbb{Z})^{(\kappa)}$ and

$$\tilde{\varphi}(d_k) = A(d_k) + \frac{1}{p^{m_k}} \sum_{i\in\kappa} d_{i,k} e'_{\kappa+i} \in \bigoplus_{i\in\kappa} \mathbb{Z} e'_i \oplus \bigoplus_{i\in\kappa} R(G) e'_{\kappa+i} \subseteq C'.$$

Therefore, φ extends to $\varphi : G \to C'$.

In order to see that φ is a monomorphism, assume that $\varphi(g) = 0$ for some $g \in G$ and represent g as $g = \sum_{i\in\kappa} r_i e_i$ with $r_i \in R_i + R(G)$. Then $\varphi(g) \restriction_{R'_{\kappa+i} e'_{\kappa+i}} = r_i e'_{\kappa+i}$ and hence $r_i = 0$ follows for all $i \in \kappa$. Therefore, $g = 0$ and φ is a monomorphism.

Finally, we have to show that $\operatorname{Im}(\varphi)$ is pure in C'. Let q be a prime and $g = \sum_{i\in\kappa} r_i e_i \in G$ with $r_i \in R_i + R(G)$. Assume that

$$\varphi(g) = q \sum_{i\in\kappa+\kappa} r'_i e'_i =: q g' \in C'$$

for some $r'_i \in R'_i$. We distinguish two cases.

Case 1: $(G/C)_q = \{0\}$.

In this case, we conclude $\frac{1}{q} \notin R(G)$, hence there is $t \in \mathbb{Z}$ such that $\gcd(t, q) = 1$ and $t r_i \in R_i$ for all $i \in \kappa$. Choose $s_1, s_2 \in \mathbb{Z}$ such that $s_1 q + s_2 t = 1$. Then

$$g = s_1 q g + s_2 t g \in qG + C.$$

In order to prove that g is divisible by q inside G we may therefore assume without loss of generality that $g = s_2 t g \in C$ and hence $r_i \in R_i$ for all $i \in \kappa$. We consider the $R'_{\kappa+j} e'_{\kappa+j}$ component of C' and deduce $q r'_{\kappa+j} = r_j \in R_j$ for all $j \in \kappa$. Since by assumption $\frac{1}{q} \notin R(G)$ it is easy to see that q divides r_j inside R_j for all $j \in \kappa$ and hence q divides g inside C. Thus, $\varphi(\frac{1}{q} g) = \sum_{i\in\kappa+\kappa} r'_i e'_i \in \operatorname{Im}(\varphi)$.

Case 2: $(G/C)_q \neq \{0\}$.

As in Case 1, we obtain that $qr'_{\kappa+j} = r_j \in R_j + R(G)$ for all $j \in \kappa$. Since G is in strong normal form we have $\frac{1}{q} \notin R_j$ for all $j \in \kappa$. Thus there is a least integer h such that

- $(h,q) = 1$,
- $q^{n_q-1}hr_i = q^{n_q}hr'_{\kappa+i} \in \mathbb{Z}$ by definition of $R'_{\kappa+i}$ for all $i \in \kappa$,
- $hr'_i \in \mathbb{Z}$ by definition of R'_i for all $i \in \kappa$.

We now look at the first κ components of g'. Let $g'' = \sum_{i\in\kappa} r'_i e'_i$. It follows that $q^{n_q}hg'' = A(q^{n_q-1}hg)$. Since $q^{n_q-1}hg \in \bigoplus_{i\in\kappa} \mathbb{Z}e_i$ and $hg'' \in \bigoplus_{i\in\kappa} \mathbb{Z}e'_i$ we deduce that $A(q^{n_q-1}hg) \in \bigoplus_{i\in\kappa} q^{n_q} R_q(G)e'_i$ and hence $q^{n_q-1}hg \in \mathrm{Im}(q^{n_q}B) + \bigoplus_{i\in\kappa} q^{n_q} R_q(G)e_i$. Therefore, $q^{n_q-1}hg = \sum_{k\in\kappa} n_k q^{n_q} d_k + q^{n_q}c$ for some integers $n_k \in \mathbb{Z}$ $(k \in \kappa)$ and $c \in \bigoplus_{i\in\kappa} R_q(G)e_i$. We deduce $hg = q\sum_{k\in\kappa} n_k d_k + qc$ and

$$hh'g = q\left(h' \sum_{k\in\kappa} n_k d_k + h'c\right) \in qG$$

for some integer h' with $(h',q) = 1$ and $h'c \in \bigoplus_{i\in\kappa} \mathbb{Z}e_i \subseteq C$. Again using Euclid and the fact that $\gcd(q,hh') = 1$, we obtain that g is divisible by q inside G and thus $g' = \varphi(\frac{1}{q}g) \in \mathrm{Im}(\varphi)$. □

Clearly the above result together with Corollary 5.3 implies the main result from [9].

Corollary 5.8. *Every bcd-group is a pure subgroup of some completely decomposable group.*

Moreover, together with Theorem 2.6 from [14] we have the following negative result

Corollary 5.9. *Let $C = \bigoplus_{i\in\kappa} R_i e_i$ and $G \in \mathcal{G}_{bd}(C)$ be in strong normal form and $B = B(G) \in \mathrm{Mat}_{(\kappa\times\kappa)}(R(G))$ its associated matrix. Then G being a pure subgroup of some completely decomposable group D does not imply the existence of a B-good matrix A.*

Proof. The group G^* from Example 5.5 has no B-good matrix while at the same time we can choose with $R_\alpha = \langle \frac{1}{p_n} : n \in A_\alpha \rangle$ the completely decomposable group

$$D = \bigoplus_{n\in\omega} \mathbb{Z}f_n \oplus \bigoplus_{\alpha\in\kappa} R_\alpha f^\alpha \oplus \mathbb{Z}g$$

and the pure embedding

$$\varphi : G^* \to D \quad \text{via } e_n \mapsto p_n f_n + g \text{ and } e^\alpha \mapsto f^\alpha + g \quad \text{(see [14]).}$$ □

Finally, we are able to answer Question 7.2 from [9] partially.

Lemma 5.10. *Let $C = \bigoplus_{i\in\omega} R_i e_i$ and $G = \langle C, \frac{e_0+e_i}{p_i} : i \geq 1 \rangle$ where $\{p_i : i \geq 1\}$ is a list of primes. Moreover, assume that G is in strong normal form,* *i.e.* $h_{p_i}^{R_j}(1) = 0$ *for all* i, j. *Then* G *is a pure subgroup of some completely decomposable group.*

Before we prove Lemma 5.10, we note that Lemma 7.1 from [9] shows that in some cases, namely when G is not in strong normal form, the above conclusion is indeed wrong. This shows that being a pure subgroup of some completely decomposable group does not only depend on the defining relations but also on the base group C in the definition of G.

Proof of Lemma 5.10. We only verify the case when $C = \bigoplus_{i\in\omega} \mathbb{Z}e_i$ and $\{p_i : i \geq 1\}$ lists the set of primes in increasing order. The general case is then obtained by obvious modification. In order to apply the Main Theorem 5.7, we need to see that G has a B-good matrix A. We first calculate the matrix B associated to G.

$$B = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \frac{1}{11} & \cdots \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{5} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{7} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now, one easily sees that the following matrix A is B-good.

$$A = \begin{pmatrix} -1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 5 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 7 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We exemplify this for $p = p_2 = 3$ (the general case is similar). In this case, $3B$ is

$$B = \begin{pmatrix} \frac{3}{2} & 1 & \frac{3}{5} & \frac{3}{7} & \frac{3}{11} & \cdots \\ \frac{3}{2} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{3}{5} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{3}{7} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus, an element $x = 3By \in \mathrm{Im}(3B)$ is of the form

$$x = 3By = \begin{pmatrix} y_1 \\ 0 \\ y_1 \\ 0 \\ \vdots \end{pmatrix} + \begin{pmatrix} \frac{3}{2}y_0 + \frac{3}{5}y_2 + \frac{3}{7}y_3 + \cdots \\ \frac{3}{2}y_0 \\ 0 \\ \frac{3}{5}y_2 \\ \vdots \end{pmatrix} \equiv \begin{pmatrix} y_1 \\ 0 \\ y_1 \\ 0 \\ \vdots \end{pmatrix} \bmod 3R_3(G)^{(\omega)}.$$

Thus,

$$Ax \equiv \begin{pmatrix} -y_1 + y_1 \\ 0 \\ 3y_1 \\ 0 \\ \vdots \end{pmatrix} \equiv 0 \bmod 3R_3(G)^{(\omega)}.$$

Conversely, if $Ax \in 3R_3(G)^{(\omega)}$, then

$$Ax = \begin{pmatrix} -x_0 + \sum_{i \geq 1} x_i \\ 2x_1 \\ 3x_2 \\ 5x_3 \\ \vdots \end{pmatrix} \in 3R_3(G)^{(\omega)}$$

which implies that $x_1, x_3, x_4, \dots \in 3R_3(G)$. Hence, also $-x_0 + x_2 \in 3R_3(G)$ and we conclude that

$$x = \begin{pmatrix} x_2 \\ 0 \\ x_2 \\ 0 \\ \vdots \end{pmatrix} + \begin{pmatrix} x_0 - x_2 \\ x_1 \\ 0 \\ x_3 \\ \vdots \end{pmatrix} \equiv \begin{pmatrix} x_2 \\ 0 \\ x_2 \\ 0 \\ \vdots \end{pmatrix} \bmod 3R_3(G)^{(\omega)}.$$

But the latter is certainly in the image of $3B$, so A is B-good.

Thus, the algorithm in the Main Theorem 5.7 applies and we obtain the following pure embedding:

$$\begin{aligned} \varphi : G \longrightarrow D = \bigoplus_{k \in \omega} \mathbb{Z} f_k \oplus \bigoplus_{k \in \omega} S g_k \\ e_0 \mapsto -f_0 + g_0 \\ e_j \mapsto f_0 + p_j f_j + g_j \quad \text{for } j \geq 1 \end{aligned}$$

with $S = R(G) = \langle \frac{1}{p} : p \in \Pi \rangle$. $\square$

References

[1] D. M. Arnold, *Notes on Butler groups and balanced extensions*, Boll. Unione Mat. Ital. A (6) **5** (1986), 175–184. MR 0850285

[2] D. M. Arnold, *Abelian groups and representations of finite partially ordered sets*, CMS Books in Mathematics, Springer-Verlag, New York, 2000. MR 1764257

[3] D. M. Arnold and K. M. Rangaswamy, *A note on countable Butler groups*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **10** (2007), 605–611. MR 2351532

[4] L. Bican and L. Salce, *Butler groups of infinite rank*, Abelian group theory, Lecture Notes in Math., vol. 1006, Springer-Verlag, New York, 1965, pp. 680–698. MR 0722617

[5] M. C. R. Butler, *A class of torsion-free abelian groups of finite rank*, Proc. Lond. Math. Soc. **15** (1965), 680–698. MR 0218446

[6] L. Fuchs, *Infinite Abelian groups*, vols. 1 & 2, Academic Press, New York, 1970 & 1973.

[7] L. Fuchs, *A survey on Butler groups of infinite rank*, Contemp. Math. **171** (1994), 121–139. MR 1293137

[8] M. Gryga, *Die Einbettung fast vollständig zerlegbarer Gruppen im globalen Fall*, Staatsexamensarbeit, University of Duisburg-Essen, 2009.

[9] D. Herden and L. Strüngmann, *Pure subgroups of completely decomposable groups—An algorithmic approach*, Models, modules and Abelian groups, de Gruyter, Berlin, 2008, pp. 169–186. MR 2513234

[10] A. Mader, *Almost completely decomposable groups*, Algebra, logic and applications, vol. 13, Gordon and Breach Science Publishers, Amsterdam, 2000. MR 1751515

[11] A. Mader and L. Strüngmann, *Bounded essential extensions of completely decomposable groups*, J. Algebra **229** (2000), 205–233. MR 1765779

[12] A. Mader and L. Strüngmann, *A class of finitely Butler groups and their endomorphism rings*, Hokkaido Math. J. **37** (2008), 399–425. MR 2415908

[13] S. Shelah and L. Strüngmann, *It is consistent with ZFC that B_1 groups are not B_2 groups*, Forum Math. **15** (2003), 507–524. MR 1978332

[14] L. Strüngmann, *On endomorphism rings of B_1-groups that are not B_2-groups when adding $\aleph_4$ reals to the ground model*, Proc. Amer. Math. Soc. **137** (2009), 3657–3668. MR 2529872

[15] C. Tschinke, *Die Einbettung fast vollständig zerlegbarer Gruppen im lokalen Fall – Ein Algorithmus und seine Realisierung in Maple*, Staatsexamensarbeit, University of Duisburg-Essen, 2009.

Daniel Herden, Fakultät für Mathematik, Universität Duisburg-Essen, Campus Essen, 45117 Essen, Germany

E-mail address: daniel.herden@uni-due.de

Lutz Strüngmann, Fakultät für Mathematik, Universität Duisburg-Essen, Campus Essen, 45117 Essen, Germany

E-mail address: lutz.struengmann@uni-due.de