

SEMIGROUP-THEORETICAL CHARACTERIZATIONS OF ARITHMETICAL INVARIANTS WITH APPLICATIONS TO NUMERICAL MONOIDS AND KRULL MONOIDS

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ABSTRACT. Arithmetical invariants—such as sets of lengths, catenary and tame degrees—describe the non-uniqueness of factorizations in atomic monoids. We study these arithmetical invariants by the monoid of relations and by presentations of the involved monoids. The abstract results will be applied to numerical monoids and to Krull monoids.

1. Introduction

Factorization theory describes the non-uniqueness of factorizations into irreducible elements of atomic monoids by arithmetical invariants, and it studies the relationship between these arithmetical invariants and algebraic invariants of the objects under consideration. Here, an atomic monoid means a commutative cancellative semigroup with unit element such that every non-unit may be written as a product of atoms (irreducible elements), and main examples are the multiplicative monoids consisting of the non-zero elements from a noetherian domain. In abstract semigroup theory, minimal relations and presentations are key tools to describe the algebraic structure of semigroups. Thus, there should be natural connections between the arithmetical invariants of factorization theory and the presentations of the semigroup. However, only first steps have been made so far to unveil these connections and to apply them successfully for further investigations. We mention two results in this direction (more can be found in the references). In [9], it was proved that the

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catenary degree of a monoid allows a description in terms of \mathcal{R} -equivalence classes (see Proposition 4.6 for details). In [36], semigroup-theoretical descriptions are used in the study of the arithmetic of non-principal orders in algebraic number fields.

The aim of the present paper is to explore further the connections between arithmetical invariants and semigroup-theoretical invariants, such as the monoid of relations and presentations. We discuss central invariants from factorization theory, such as the ω -invariants (Section 3), the catenary and monotone catenary degrees (Section 4), the tame degrees (Section 5), and finally Section 6 deals with unions of sets of lengths. We provide—in the abstract setting of atomic monoids—new characterizations or new upper bounds (as in Propositions 3.3, 5.2, Corollary 6.4), and reveal the influence of special presentations to the arithmetic (as in Theorem 5.6). Throughout, we apply the abstract results to concrete classes of monoids, mainly to numerical monoids and to Krull monoids (Corollaries 5.7 and 5.8, Theorem 6.6). Moreover, some of the results have relevance from the computational point of view, as they can provide explicit algorithms which partly have been implemented in GAP (see Example 3.6 or Remarks 5.9, and [14]). A more detailed discussion of the results will be given at the beginning of each section as soon as we have the required terminology at our disposal.

2. Preliminaries

We denote by \mathbb{N} the set of positive integers, and we put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For every $n \in \mathbb{N}$, we denote by C_n a cyclic group with n elements. For real numbers $a, b \in \mathbb{R}$, we set $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$. Let $L, L' \subset \mathbb{Z}$. We denote by $L + L' = \{a + b \mid a \in L, b \in L'\}$ their *sumset*. Two distinct elements $k, l \in L$ are called *adjacent* if $L \cap [\min\{k, l\}, \max\{k, l\}] = \{k, l\}$. A positive integer $d \in \mathbb{N}$ is called a *distance* of L if there exist adjacent elements $k, l \in L$ with $d = |k - l|$. We denote by $\Delta(L)$ the *set of distances* of L . If $\emptyset \neq L \subset \mathbb{N}$, we call

$$\rho(L) = \sup \left\{ \frac{m}{n} \mid m, n \in L \right\} = \frac{\sup L}{\min L} \in \mathbb{Q}_{\geq 1} \cup \{\infty\}$$

the *elasticity* of L , and we set $\rho(\{0\}) = 1$. By a *monoid*, we mean a commutative, cancellative semigroup with unit element. *Throughout this paper, let S be a monoid.*

We denote by $\mathcal{A}(S)$ the set of atoms (irreducible elements) of S , by S^\times the group of invertible elements, by $S_{\text{red}} = \{aS^\times \mid a \in S\}$ the associated reduced monoid of S , and by $\mathfrak{q}(S)$ a quotient group of S with $S \subset \mathfrak{q}(S)$. A submonoid $T \subset S$ is said to be *saturated* if $T = S \cap \mathfrak{q}(T)$ (equivalently, if $a, b \in T$ and a divides b in S , then a divides b in T). We say that S is *reduced* if $|S^\times| = 1$. If not stated otherwise, we will use multiplicative notation. Submonoids of $(\mathbb{Z}^s, +)$, in particular numerical monoids, will of course be written additively.

For a set P , we denote by $\mathcal{F}(P)$ the *free (Abelian) monoid* with basis P . Then every $a \in \mathcal{F}(P)$ has a unique representation in the form

$$a = \prod_{p \in P} p^{v_p(a)} \quad \text{with } v_p(a) \in \mathbb{N}_0 \quad \text{and } v_p(a) = 0 \quad \text{for almost all } p \in P.$$

We call $|a| = \sum_{p \in P} v_p(a)$ the *length* of a and $\text{supp}(a) = \{p \in P \mid v_p(a) > 0\} \subset P$ the *support* of a .

The free (Abelian) monoid $Z(S) = \mathcal{F}(\mathcal{A}(S_{\text{red}}))$ is called the *factorization monoid* of S , the unique homomorphism

$$\pi : Z(S) \rightarrow S_{\text{red}} \quad \text{satisfying } \pi(u) = u \text{ for each } u \in \mathcal{A}(S_{\text{red}})$$

is called the *factorization homomorphism* of S and

$$\sim_S = \{(x, y) \in Z(S) \times Z(S) \mid \pi(x) = \pi(y)\}$$

the *monoid of relations* of S . Clearly, we have $Z(S) \cong (\mathbb{N}_0^{\mathcal{A}(S_{\text{red}})}, +)$, and if S is written additively, then $Z(S)$ and \sim_S will be written additively too.

Let $\sigma \subset \sim_S$ be a subset. Then $\sigma^{-1} = \{(x, y) \in \sim_S \mid (y, x) \in \sigma\}$, and σ is called a *presentation* of S if the congruence generated by σ equals \sim_S (equivalently, if $(x, y) \in Z(S) \times Z(S)$, then $(x, y) \in \sim_S$ if and only if there exist $z_0, \dots, z_k \in Z(S)$ such that $x = z_0$, $z_k = y$, and, for all $i \in [1, k]$, $(z_{i-1}, z_i) = (x_{i-1}w_i, x_iw_i)$ with $w_i \in Z(S)$ and $(x_{i-1}, x_i) \in \sigma \cup \sigma^{-1}$). A presentation σ is said to be:

- *minimal* if no proper subset of σ generates \sim_S (see [38, Chapter 9] for characterizations of minimal presentations in our setting).
- *generic* if σ is minimal and for all $(x, y) \in \sigma$ we have $\text{supp}(xy) = \mathcal{A}(S_{\text{red}})$.

If S has a generic presentation, then S_{red} is finitely generated and has no primes.

For a subset $S' \subset S$, we set $Z(S') = \{z \in Z(S) \mid \pi(z) \in S'\}$. Let $Z \subset Z(S)$ be a subset. We say that an element $x \in Z$ is *minimal* in Z if for all elements $y \in Z$ with $y \mid x$ it follows that $x = y$. We denote by $\text{Min}(Z)$ the *set of minimal elements* in Z . Let $x \in Z$. Since the number of elements $y \in Z$ with $y \mid x$ is finite, there exists an $x^* \in \text{Min}(Z)$ with $x^* \mid x$.

For $a \in S$, the set

$$Z(a) = Z(\{a\}) \subset Z(S) \quad \text{is the set of factorizations of } a \quad \text{and}$$

$$L(a) = \{|z| \mid z \in Z(a)\} \subset \mathbb{N}_0 \quad \text{is the set of lengths of } a.$$

By definition, we have $Z(a) = \{1\}$ and $L(a) = \{0\}$ for all $a \in S^\times$. The monoid S is called *atomic* if $Z(a) \neq \emptyset$ for all $a \in S$ (equivalently, every non-unit can be written as a product of atoms), and it is called *factorial* if $|Z(a)| = 1$ for all $a \in S$. If S is reduced and atomic, then the set of atoms $\mathcal{A}(S)$ is the uniquely determined minimal generating set of S [26, Proposition 1.1.7]. We denote by

$\mathcal{L}(S) = \{\mathbf{L}(a) \mid a \in S\}$ the *system of sets of lengths* of S , and by

$$\Delta(S) = \bigcup_{L \in \mathcal{L}(S)} \Delta(L) \subset \mathbb{N}$$

the *set of distances* of S .

For $z, z' \in Z(S)$, we can write

$$z = u_1 \cdots u_l v_1 \cdots v_m \quad \text{and} \quad z' = u_1 \cdots u_l w_1 \cdots w_n,$$

where $l, m, n \in \mathbb{N}_0$ and $u_1, \dots, u_l, v_1, \dots, v_m, w_1, \dots, w_n \in \mathcal{A}(S_{\text{red}})$ are such that

$$\{v_1, \dots, v_m\} \cap \{w_1, \dots, w_n\} = \emptyset.$$

Then $\text{gcd}(z, z') = u_1 \cdots u_l$, and we call

$$\mathbf{d}(z, z') = \max\{m, n\} = \max\{|z \text{gcd}(z, z')^{-1}|, |z' \text{gcd}(z, z')^{-1}|\} \in \mathbb{N}_0$$

the *distance* between z and z' . For subsets $X, Y \subset Z(S)$, we set

$$\mathbf{d}(X, Y) = \min\{\mathbf{d}(x, y) \mid x \in X, y \in Y\} \in \mathbb{N}_0,$$

and thus $\mathbf{d}(X, Y) = 0$ if and only if $(X \cap Y \neq \emptyset \text{ or } X = \emptyset \text{ or } Y = \emptyset)$.

Numerical monoids. By a *numerical monoid*, we mean a submonoid $S \subset (\mathbb{N}_0, +)$ such that the complement $\mathbb{N} \setminus S$ is finite. The theory of numerical monoids is presented in the recent monograph [39]. The connection to semi-group algebras and to one-dimensional local domains (in particular, to power series domains $K[[S]]$) is documented in the surveys [7], [5], [6] (all these are domains which have received a lot of attention in factorization theory). We shall make use of this in Section 5.

Let S be a numerical monoid. Then S is reduced and finitely generated. Suppose that $\mathcal{A}(S) = \{n_1, \dots, n_t\}$ with $t \in \mathbb{N}$ and $1 < n_1 < \dots < n_t$. Then we write $S = \langle n_1, \dots, n_t \rangle$, and since $\mathbb{N} \setminus S$ is finite, it follows that $\text{gcd}(n_1, \dots, n_t) = 1$. Writing factorizations of an element $a \in S$ we put the atoms in boldface in order to distinguish between the atoms and the scalars. Thus $z = k_1 \mathbf{n}_1 + \dots + k_t \mathbf{n}_t$, with $k_1, \dots, k_t \in \mathbb{N}_0$, is the factorization of the element $a = \sum_{i=1}^t k_i n_i \in S$ of length $|z| = k_1 + \dots + k_t$. We denote by $\text{Ap}(S, a) = \{s \in S \mid s - a \notin S\}$ the *Apéry set* of a in S (see [39]).

Krull monoids. The monoid S is called a *Krull monoid* if it satisfies one of the following equivalent properties [26, Theorem 2.4.8]:

- (a) S is v -noetherian and completely integrally closed,
- (b) S has a divisor theory,
- (c) S_{red} is a saturated submonoid of a free monoid.

The theory of Krull monoids is presented in the monographs [32], [26]. Let S be atomic. Clearly, $\sim_S \subset Z(S) \times Z(S)$ is saturated and hence \sim_S is a Krull monoid by Property (c) (more on that can be found in [37, Lemma 11]), and hence, in particular, it is atomic. Moreover, if S_{red} is finitely generated, then \sim_S is finitely generated as a saturated submonoid of a finitely generated monoid (see [26, Proposition 2.7.5]). An integral domain R is a Krull domain

if and only if its multiplicative monoid $R \setminus \{0\}$ is a Krull monoid, and thus Property (a) shows that a noetherian domain is Krull if and only if it is integrally closed.

Main portions of the arithmetic of a Krull monoid—in particular, all questions dealing with sets of lengths—can be studied in the associated monoid of zero-sum sequences over its class group. To describe this concept, let G be an additive Abelian group, $G_0 \subset G$ a subset and $\mathcal{F}(G_0)$ the free monoid with basis G_0 . According to the tradition of combinatorial number theory, the elements of $\mathcal{F}(G_0)$ are called *sequences* over G_0 . For a sequence

$$U = g_1 \cdot \dots \cdot g_l = \prod_{g \in G_0} g^{v_g(U)} \in \mathcal{F}(G_0),$$

we call $v_g(U)$ the *multiplicity* of g in U ,

$$|U| = l = \sum_{g \in G} v_g(U) \in \mathbb{N}_0 \quad \text{the length of } U,$$

and

$$\sigma(U) = \sum_{i=1}^l g_i \quad \text{the sum of } U.$$

The monoid

$$\mathcal{B}(G_0) = \{U \in \mathcal{F}(G_0) \mid \sigma(U) = 0\}$$

is called the *monoid of zero-sum sequences* over G_0 , and Property (c) shows that $\mathcal{B}(G_0)$ is a Krull monoid. We define the *Davenport constant* of G_0 by

$$D(G_0) = \sup\{|U| \mid U \in \mathcal{A}(\mathcal{B}(G_0))\} \in \mathbb{N}_0 \cup \{\infty\},$$

which is a classical constant in Combinatorial Number Theory (see [19], [22]).

We will use that for a reduced finitely generated monoid S the following statements are equivalent [26, Theorem 2.7.14]:

- S is a Krull monoid,
- S is isomorphic to a monoid $\mathcal{B}(G_0)$ with $G_0 \subset G$ as above,
- S is isomorphic to a monoid of non-negative integer solutions of a system of linear Diophantine equations.

3. The ω -invariants

DEFINITION 3.1. Let S be atomic. For $b \in S$, let $\omega(S, b)$ denote the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

For all $n \in \mathbb{N}$ and $a_1, \dots, a_n \in S$, if $b \mid a_1 \cdot \dots \cdot a_n$, then there exists a subset $\Omega \subset [1, n]$ such that $|\Omega| \leq N$ and

$$b \mid \prod_{\nu \in \Omega} a_\nu.$$

Furthermore, we set

$$\omega(S) = \sup\{\omega(S, u) \mid u \in \mathcal{A}(S)\} \in \mathbb{N}_0 \cup \{\infty\}.$$

Let S be atomic. By definition, an element $b \in S$ is a prime element if and only if $\omega(S, b) = 1$, and S is factorial if and only if $\omega(S) = 1$. Thus these ω -invariants (together with the associated tame degrees, see in particular Equation 5.2) measure in particular how far away atoms are from primes (see [29], [28], [27], [30]). An algorithm to compute the $\omega(S, \cdot)$ values in numerical monoids was recently presented in [4]. Here we start by showing that a slight variant of the property in the definition of $\omega(S, \cdot)$ does not change its value. Also the forthcoming characterization of $\omega(S)$ will be easy to prove, but it is useful from a computational point of view. Although special cases have been handled in the literature before (Proposition 3.3 was done for numerical monoids in [4, 3.3.1], and Lemma 3.2 for domains in [3]), we provide short proofs because these technical results are the key ingredient in the proof of our main results (see Theorem 5.6).

LEMMA 3.2. *Let S be atomic and $b \in S$. Then $\omega(S, b)$ is the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:*

For all $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathcal{A}(S)$, if $b \mid a_1 \cdots a_n$, then there exists a subset $\Omega \subset [1, n]$ such that $|\Omega| \leq N$ and

$$b \mid \prod_{\nu \in \Omega} a_\nu.$$

Proof. Let $\omega'(S, b)$ denote the smallest integer $N \in \mathbb{N}_0 \cup \{\infty\}$ satisfying the property mentioned in the lemma. We show that $\omega(S, b) = \omega'(S, b)$. By definition, we have $\omega'(S, b) \leq \omega(S, b)$. (Note, if $b \in S^\times$, then $\omega(S, b) = \omega'(S, b) = 0$).

In order to show that $\omega(S, b) \leq \omega'(S, b)$, let $n \in \mathbb{N}$ and $a_1, \dots, a_n \in S$ with $b \mid a_1 \cdots a_n$. After renumbering if necessary there is an $m \in [0, n]$ such that $a_1, \dots, a_m \in S \setminus S^\times$ and $a_{m+1}, \dots, a_n \in S^\times$. Then $b \mid a_1 \cdots a_m$, and for every $i \in [1, m]$ we pick a factorization $a_i = u_{i,1} \cdots u_{i,k_i}$ with $k_i \in \mathbb{N}$ and $u_{i,1}, \dots, u_{i,k_i} \in \mathcal{A}(S)$. Then there is a subset $I \in [1, m]$ and, for every $i \in I$, a subset $\emptyset \neq \Lambda_i \subset [1, k_i]$ such that

$$|I| \leq \sum_{i \in I} |\Lambda_i| \leq \omega'(S, b) \quad \text{and} \quad b \mid \prod_{i \in I} \prod_{\nu \in \Lambda_i} u_{i,\nu}$$

which implies that $b \mid \prod_{i \in I} a_i$. □

PROPOSITION 3.3. *Let S be atomic.*

1. *For every $s \in S$ we have*

$$\omega(S, s) = \sup\{|x| \mid x \in \text{Min}(\mathbb{Z}(sS))\}.$$

2. $\omega(S) = \sup\{|x| \mid x \in \text{Min}(\mathbb{Z}(uS)) \text{ for some } u \in \mathcal{A}(S)\}.$

Proof. Obviously, it is sufficient to prove the first statement. Furthermore, we may assume that S is reduced.

Let $x = \prod_{u \in \mathcal{A}(S)} u^{m_u} \in \text{Min}(Z(sS))$. Since $x \in Z(sS)$, it follows that s divides $\prod_{u \in \mathcal{A}(S)} u^{m_u}$ (in S), and since x is minimal in $Z(sS)$, s does not divide a proper subproduct. Therefore, we get that $\omega(S, s) \geq \sum_{u \in \mathcal{A}(S)} m_u = |x|$.

Conversely, let $(m_u)_{u \in \mathcal{A}(S)} \in \mathbb{N}_0^{\mathcal{A}(S)}$ be such that s divides $\prod_{u \in \mathcal{A}(S)} u^{m_u}$ (in S). Then $x = \prod_{u \in \mathcal{A}(S)} u^{m_u} \in Z(sS)$, and there exists some minimal $x^* = \prod_{u \in \mathcal{A}(S)} u^{m_u^*} \in Z(sS)$ with $x^* \mid x$ (in $Z(S)$). Then

$$s \mid \prod_{u \in \mathcal{A}(S)} u^{m_u^*} \mid \prod_{u \in \mathcal{A}(S)} u^{m_u} \quad (\text{in } S)$$

and hence $\omega(S, s) \leq \sum_{u \in \mathcal{A}(S)} m_u^* \leq \sup\{|y| \mid y \in \text{Min}(Z(sS))\}$. □

REMARKS 3.4.

1. Note that $\omega(S, s)$ is finite for all $s \in S$ not only for finitely generated monoids, but more generally for all v -noetherian monoids (see [28, Theorem 4.2]).

2. Let $k, r \in \mathbb{N}_0$ and $n, d_1, \dots, d_k \in \mathbb{N}$. Let $S \subset (\mathbb{N}_0^n, +)$ be the set of all non-negative integer solutions $(x_1, \dots, x_n) \in \mathbb{N}_0^n$ of the following system of equations

$$\begin{aligned} a_{1,1}x_1 + \dots + a_{1,n}x_n &\equiv 0 \pmod{d_1}, \\ &\vdots \\ a_{k,1}x_1 + \dots + a_{k,n}x_n &\equiv 0 \pmod{d_k}, \\ a_{k+1,1}x_1 + \dots + a_{k+1,n}x_n &= 0, \\ &\vdots \\ a_{k+r,1}x_1 + \dots + a_{k+r,n}x_n &= 0, \end{aligned}$$

where all $a_{i,j}$ are integers. Obviously, $S \subset (\mathbb{N}_0^n, +)$ is a submonoid with

$$(*) \quad \mathfrak{q}(S) \cap \mathbb{N}_0^n = S.$$

Let $G = \mathbb{Z}/d_1\mathbb{Z} \times \dots \times \mathbb{Z}/d_k\mathbb{Z} \times \mathbb{Z}^r$ and

$$G_0 = \{(a_{1,i} + d_1\mathbb{Z}, \dots, a_{k,i} + d_k\mathbb{Z}, a_{k+1,i}, \dots, a_{k+r,i}) \in G \mid i \in [1, n]\}.$$

Then S is obviously isomorphic to $\mathcal{B}(G_0)$, the monoid of zero-sum sequences over G_0 . This (or independently, the fact that $S \subset (\mathbb{N}_0^n, +)$ is saturated) show that S is a reduced, finitely generated Krull monoid. If a finitely generated Krull monoid is given in that form, then the characterization of Proposition 3.3 turns out to be extremely useful, as the next corollary illustrates.

COROLLARY 3.5. *Let $S \subset (\mathbb{N}_0^n, +)$ be a saturated submonoid with $\mathcal{A}(S) = \{\mathbf{s}_1, \dots, \mathbf{s}_t\}$, where $n, t \in \mathbb{N}$, and let $A \in M_{n,t}(\mathbb{Z})$ be the matrix whose columns*

are s_1, \dots, s_t . If $s \in S$ and $(x_1, \dots, x_t) \in \mathbb{N}_0^t$, then

$$x_1 s_1 + \dots + x_t s_t \in Z(s + S) \quad \text{if and only if} \quad A \begin{pmatrix} x_1 \\ \vdots \\ x_t \end{pmatrix} \geq s,$$

where \leq denotes the component-wise order on \mathbb{N}_0^n .

Proof. Observe that, by definition of A , we have for s and (x_1, \dots, x_t) as above, that

$$x_1 s_1 + \dots + x_t s_t \in Z(s) \quad \text{if and only if} \quad A \begin{pmatrix} x_1 \\ \vdots \\ x_t \end{pmatrix} = s.$$

If $x_1 s_1 + \dots + x_t s_t \in Z(s + S)$, then there is some $s' \in S$ such that $x_1 s_1 + \dots + x_t s_t = s + s' \in S$, and hence $A(x_1, \dots, x_t)^t = s + s' \geq s$.

Conversely, let $x \in \mathbb{N}_0^t$ (considered as a column) be such that $Ax \geq s$. Then $s' = Ax \in S$ and $s' - s \in \mathfrak{q}(S) \cap \mathbb{N}_0^n = S$. Thus, $s' \in s + S$ and $x_1 s_1 + \dots + x_t s_t \in Z(s') \subset Z(s + S)$. \square

The following example illustrates how Proposition 3.3 and Corollary 3.5 can be used to calculate the ω -invariants. It was developed by using an algorithm due to E. Contejean and H. Devie with slack variables (see [13]; these authors published later a paper to avoid the use of these extra variables).

EXAMPLE 3.6. Let $S \subset (\mathbb{N}_0^3, +)$ be the set of nonnegative integer solutions of

$$\begin{aligned} x + z &= 0 \pmod{2}, \\ y + z &= 0 \pmod{2}. \end{aligned}$$

Then

$$\mathcal{A}(S) = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\},$$

and S is isomorphic to $\mathcal{B}(G_0)$ with $G_0 = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \setminus \{(0, 0)\}$. The set of solutions of

$$\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix} x \geq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

is $\{(0, 0, 0, 1), (1, 1, 1, 0)\} + \mathbb{N}_0^4$, and thus, by Proposition 3.3 and Corollary 3.5, $\omega(S, (1, 1, 1)) = 3$. The set of solutions of

$$\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix} x \geq \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

is $\{(1, 0, 0, 0), (0, 0, 0, 2)\} + \mathbb{N}_0^4$, whence $\omega(S, (2, 0, 0)) = 2$. By symmetry, we get $\omega(S, (0, 2, 0)) = 2 = \omega(S, (0, 0, 2))$. Thus, it follows that $\omega(S) = 3$.

4. The catenary and monotone catenary degrees

DEFINITION 4.1. Let S be atomic and $a \in S$.

1. Let $z, z' \in Z(a)$ be factorizations of a and $N \in \mathbb{N}_0 \cup \{\infty\}$. A finite sequence z_0, z_1, \dots, z_k in $Z(a)$ is called an N -chain of factorizations from z to z' if $z = z_0, z' = z_k$ and $d(z_{i-1}, z_i) \leq N$ for every $i \in [1, k]$. In addition, the chain is called *monotone* if $|z_0| \leq \dots \leq |z_k|$ or $|z_0| \geq \dots \geq |z_k|$.

If there exists a (monotone) N -chain of factorizations from z to z' , we say that z and z' can be *concatenated* by a (monotone) N -chain.

2. We denote by $c(a) \in \mathbb{N}_0 \cup \{\infty\}$ (or by $c_{\text{mon}}(a)$ resp.) the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ such that any two factorizations $z, z' \in Z(a)$ can be concatenated by an N -chain (or by a monotone N -chain).
3. Moreover,

$$c(S) = \sup\{c(b) \mid b \in S\} \in \mathbb{N}_0 \cup \{\infty\}$$

and

$$c_{\text{mon}}(S) = \sup\{c_{\text{mon}}(b) \mid b \in S\} \in \mathbb{N}_0 \cup \{\infty\}$$

denote the *catenary degree* and the *monotone catenary degree* of S .

Whereas the catenary degree is a classic invariant in factorization theory, the monotone catenary degree was introduced only in [15]. However, since then the existence of monotone and of near monotone chains of factorizations have been investigated in various aspects (see [16], [17], [24], [31], [35]). The monotone catenary degree is always (explicitly or implicitly) studied in a two-step procedure.

DEFINITION 4.2. Let S be atomic and $a \in S$.

1. For $k \in \mathbb{Z}$, let $Z_k(a) = \{z \in Z(a) \mid |z| = k\}$ denote the set of factorizations of a having length k . We define

$$c_{\text{adj}}(a) = \sup\{d(Z_k(a), Z_l(a)) \mid k, l \in L(a) \text{ are adjacent}\}$$

and we set

$$c_{\text{adj}}(S) = \sup\{c_{\text{adj}}(b) \mid b \in S\} \in \mathbb{N}_0 \cup \{\infty\}.$$

2. Let $c_{\text{eq}}(a)$ denote the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

For all $z, z' \in Z(a)$ with $|z| = |z'|$ there exists a monotone N -chain concatenating z and z' .

We call

$$c_{\text{eq}}(S) = \sup\{c_{\text{eq}}(b) \mid b \in S\} \in \mathbb{N}_0 \cup \{\infty\}$$

the *equal catenary degree* of S .

Obviously, we have

$$c(a) \leq c_{\text{mon}}(a) = \sup\{c_{\text{eq}}(a), c_{\text{adj}}(a)\} \leq \sup L(a) \quad \text{for all } a \in S,$$

and hence

$$(4.1) \quad c(S) \leq c_{\text{mon}}(S) = \sup\{c_{\text{eq}}(S), c_{\text{adj}}(S)\}.$$

It is well known that the monotone catenary degree $c_{\text{mon}}(S)$ is finite for finitely generated monoids [15, Theorem 3.9], and hence also for Krull monoids with finite class group, because it is stable under transfer homomorphisms; note that, for such Krull monoids, both inequalities $c_{\text{adj}}(S) < c_{\text{eq}}(S) < \infty$ and $c_{\text{eq}}(S) < c_{\text{adj}}(S) < \infty$ can happen (for all this see [31]). Our results will provide a more natural upper bound for $c_{\text{mon}}(S)$, valid among others finitely generated monoids. Inequality (5.1) will show that there is a canonical chain of inequalities involving the set of distances, the ω -invariants, and the catenary and tame degrees. However, there seems to be no obvious relationship between $c_{\text{mon}}(\cdot)$, on the one side and $\omega(\cdot)$, the tame degree or on their canonical upper bound (see Proposition 5.2) on the other side. We study these phenomena by investigating $c_{\text{eq}}(\cdot)$, and $c_{\text{adj}}(\cdot)$ individually, and summarize our discussion after Proposition 5.2.

DEFINITION 4.3. Let S be atomic. Then

$$\sim_{S,\text{eq}} = \{(x, y) \in Z(S) \times Z(S) \mid \pi(x) = \pi(y) \text{ and } |x| = |y|\}$$

is called the *monoid of equal-length relations* of S .

Note, if $(x, y) \in \mathcal{A}(\sim_{S,\text{eq}})$, then either $\gcd(x, y) = 1$ or $x = y \in \mathcal{A}(S_{\text{red}})$. Furthermore, we use the convention that $\sup \emptyset = 0$.

PROPOSITION 4.4. Let S be atomic.

1. $\sim_{S,\text{eq}} \subset \sim_S$ is a saturated submonoid, and hence $\sim_{S,\text{eq}}$ is a Krull monoid.
2. If S_{red} is finitely generated, then $\sim_{S,\text{eq}}$ is finitely generated.
3. $c_{\text{eq}}(S) \leq \sup\{|x| \mid (x, y) \in \mathcal{A}(\sim_{S,\text{eq}}) \text{ for some } y \in Z(S) \setminus \{x\}\}$.
4. For $d \in \Delta(S)$ let $A_d = \{x \in Z(S) \mid |x| - d \in L(\pi(x))\}$. Then $c_{\text{adj}}(S) \leq \sup\{|x| \mid x \in \text{Min}(A_d), d \in \Delta(S)\}$.

Proof. 1. Obviously, $\sim_{S,\text{eq}}$ is a submonoid of \sim_S . In order to show that it is saturated, let $(x_1, x_2), (z_1, z_2) \in \sim_{S,\text{eq}}$ be such that (x_1, x_2) divides (z_1, z_2) in \sim_S . Then there exists $(y_1, y_2) \in \sim_S$ such that $x_1 y_1 = z_1$ and $x_2 y_2 = z_2$. This implies that $|y_1| = |z_1| - |x_1| = |z_2| - |x_2| = |y_2|$, and hence $(y_1, y_2) \in \sim_{S,\text{eq}}$. Thus, $\sim_{S,\text{eq}} \subset \sim_S$ is saturated. Since \sim_S is a Krull monoid by [37, Lemma 11], $\sim_{S,\text{eq}}$ is a Krull monoid by [26, Proposition 2.4.4].

2. Let S_{red} be finitely generated. Then \sim_S is finitely generated (as observed in Section 2), and hence $\sim_{S,\text{eq}}$ is finitely generated as a saturated submonoid of a finitely generated monoid [26, Proposition 2.7.5].

3. We set $M = \sup\{|x| \mid (x, y) \in \mathcal{A}(\sim_{S, \text{eq}}) \text{ for some } y \in Z(S)\}$, and have to show that $c_{\text{eq}}(a) \leq M$ for all $a \in S$. Let $a \in S$ and $z, z' \in Z(a)$ with $|z| = |z'|$. Then $(z, z') \in \sim_{S, \text{eq}}$, and we consider a factorization, say

$$(z, z') = (x_1, x'_1) \cdot \dots \cdot (x_l, x'_l), \quad \text{where } (x_i, x'_i) \in \mathcal{A}(\sim_{S, \text{eq}})$$

for all $i \in [1, l]$, $\gcd(x_i, x'_i) = 1$ for all $i \in [1, k]$ and $x_i = x'_i \in \mathcal{A}(S_{\text{red}})$ for all $i \in [k + 1, l]$. Then

$$\begin{aligned} z &= z_0, & z_1 &= x'_1 x_2 \cdot \dots \cdot x_l, & z_2 &= x'_1 x'_2 x_3 \cdot \dots \cdot x_l, \dots, \\ z_k &= x'_1 \cdot \dots \cdot x'_k x_{k+1} \cdot \dots \cdot x_l = x'_1 \cdot \dots \cdot x'_l = z' \end{aligned}$$

is an M -chain of factorizations from z to z' and with $|z_i| = |z|$ for all $i \in [0, k]$.

4. Let $a \in S$ and $k, l \in L(a)$ be adjacent lengths, say $l - k = d \in \Delta(S)$. We pick some $z \in Z_l(a)$. Then there exists some $x \in \text{Min}(A_d)$ such that $x \mid z$. If $x' \in Z(\pi(x))$ with $|x'| = |x| - d$, then $z' = x'(x^{-1}z) \in Z_k(a)$ and $d(Z_k(a), Z_l(a)) \leq d(z, z') \leq |x|$. This shows that $c_{\text{adj}}(a) \leq \sup\{|x| \mid x \in \text{Min}(A_d), d \in \Delta(S)\}$, and hence the assertion follows. \square

The above upper bounds for $c_{\text{eq}}(S)$ and for $c_{\text{adj}}(S)$ are sharp for Krull monoids with small class group (see [31, Section 5]). Moreover, Proposition 4.4.4 will allow us to obtain a more explicit finiteness criterion for $c_{\text{adj}}(S)$ in Proposition 5.2.

DEFINITION 4.5. Let S be atomic.

1. Two elements $z, z' \in Z(S)$ are \mathcal{R} -related if $z = z' = 1$ or if z and z' can be concatenated by a chain of factorizations $z = z_0, \dots, z_k = z'$ such that $\pi(z_i) = \pi(z)$ and $\gcd(z_{i-1}, z_i) \neq 1$ for all $i \in [1, k]$.
2. For $a \in S$, we denote by \mathcal{R}_a the set of \mathcal{R} -(equivalence) classes of $Z(a)$. For $\sigma \in \mathcal{R}_a$ we set $|\sigma| = \min\{|z| \mid z \in \sigma\}$, and we define

$$\mu(a) = \sup\{|\sigma| \mid \sigma \in \mathcal{R}_a\}.$$

3. We set

$$\mu(S) = \sup\{\mu(a) \mid a \in S \text{ with } |\mathcal{R}_a| \geq 2\}.$$

We will need the following result, first proved for finitely generated monoids (see [9, Theorem 3.1]) and then in the general setting (see [37, Corollary 9]).

PROPOSITION 4.6. *If S is atomic, then $c(S) = \mu(S)$.*

There is no analogous result for the monotone catenary degree. Below we will provide the first example of a monoid which is not tame but which has finite monotone catenary degree (indeed the catenary and the monotone catenary degree coincide). On the other side of the spectrum there are tame monoids with infinite monotone catenary degree. We recall the notion of finitely primary monoids, a concept which stems from ring theory. The monoid S is called *finitely primary* if there exist $s, \alpha \in \mathbb{N}$ with the following properties:

S is a submonoid of a factorial monoid $F = F^\times \times [p_1, \dots, p_s]$ with s pairwise non-associated prime elements p_1, \dots, p_s satisfying

$$S \setminus S^\times \subset p_1 \cdot \dots \cdot p_s F \quad \text{and} \quad (p_1 \cdot \dots \cdot p_s)^\alpha F \subset S.$$

If this is the case, then we say that S is finitely primary of rank s and exponent α .

If $s = 1$ and $F^\times = \{1\}$, then S is isomorphic to a numerical monoid. Furthermore, S is tame if and only if it is of rank 1 [26, Theorem 3.1.5].

EXAMPLE 4.7. Let $S = (\mathbb{N} \times \mathbb{N} \cup \{(0, 0)\}, +)$. Then $c(S) = c_{\text{mon}}(S) = 3$ and hence $\Delta(S) = \{1\}$. However, we have $\rho(S) = \infty$ and hence $\omega(S) = \mathbf{t}(S) = \infty$ (for the invariants not defined so far, see the discussion at Inequality (5.1)).

Proof. By definition, S is a finitely primary monoid of rank 2 and exponent 1. It is a special case of the monoid studied in [26, Example 3.1.8], where all assertions have been verified apart from the formula for $c_{\text{mon}}(S)$. Indeed, it is straightforward that $\mathcal{A}(S) = \{(1, m), (m, 1) \mid m \in \mathbb{N}\}$. Furthermore, every element $a = (a_1, a_2) \in S \setminus (\mathcal{A}(S) \cup \{(0, 0)\})$ can be written as a sum of two atoms, namely $(a_1, a_2) = (1, a_2 - 1) + (a_1 - 1, 1)$. These two observations easily imply the assertions on $c(S), \Delta(S)$ and $\rho(S)$. In order to show that $c_{\text{mon}}(S) = 3$ we proceed in two steps.

First, we show that $c_{\text{adj}}(S) = 3$. Let $a \in S$ and $k, l \in \mathbf{L}(a)$ be adjacent lengths, say $k < l$. Since $\Delta(S) = \{1\}$, it follows that $l = k + 1 \geq 3$. Let $z = u_1 \cdot \dots \cdot u_{k+1} \in Z_l(a)$ with $u_1, \dots, u_{k+1} \in \mathcal{A}(S)$. Since $u_1 u_2 u_3$ is a product of two atoms, say $u_1 u_2 u_3 = v_1 v_2$ with $v_1, v_2 \in \mathcal{A}(S)$, we infer that $z' = v_1 v_2 u_4 \cdot \dots \cdot u_{k+1} \in Z_k(a)$, and hence $3 = \mathbf{d}(z, z') = \mathbf{d}(Z_{k+1}(a), Z_k(a))$. Thus have $c_{\text{adj}}(a) = 3$, and hence we get $c_{\text{adj}}(S) = 3$.

Second, we verify that $c_{\text{eq}}(S) = 3$. Let $a = (m, n) \in S$ and $k \in \mathbf{L}(a)$. We have to show that each two factorizations $z, z' \in Z_k(a)$ can be concatenated by a monotone 3-chain of factorizations. By symmetry, we may assume that $m \leq n$, and then we clearly have $k \leq m$. We consider the factorization

$$z^* = (k - 2)(\mathbf{1}, \mathbf{1}) + (\mathbf{m} - \mathbf{k} + \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{n} - \mathbf{k} + \mathbf{1}) \in Z_k(a).$$

Clearly, it is sufficient to show that from every factorization $z \in Z_k(a)$ there is a monotone 3-chain of factorizations to z^* . Let $z \in Z_k(a)$ be given. We proceed by induction on $\mathbf{v}_{(1,1)}(z)$. If $\mathbf{v}_{(1,1)}(z) = k - 2$, then $z = z^*$ and we are done. Suppose that $\mathbf{v}_{(1,1)}(z) < k - 2$. Then there are two atoms $u_1 = (\mathbf{1}, \mathbf{a}_1)$ and $u_2 = (\mathbf{1}, \mathbf{a}_2)$ with $a_1, a_2 \in \mathbb{N}_{\geq 2}$ and $(u_1 + u_2) \mid z$, or there are two atoms $u_1 = (\mathbf{a}_1, \mathbf{1})$ and $u_2 = (\mathbf{a}_2, \mathbf{1})$ with $a_1, a_2 \in \mathbb{N}_{\geq 2}$ and $(u_1 + u_2) \mid z$. By symmetry, we may suppose that the first case holds. Then we define $z' = -u_1 - u_2 + z + (\mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{a}_1 + \mathbf{a}_2 - \mathbf{1})$. Clearly, we have $z' \in Z_k(a)$, $\mathbf{d}(z, z') = 3$, $\mathbf{v}_{(1,1)}(z) < \mathbf{v}_{(1,1)}(z')$, and hence the assertion follows. \square

5. The tame degrees

DEFINITION 5.1. Let S be atomic.

1. For $a \in S$ and $x \in Z(S)$, let $t(a, x) \in \mathbb{N}_0 \cup \{\infty\}$ denote the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

If $Z(a) \cap xZ(S) \neq \emptyset$ and $z \in Z(a)$, then there exists $z' \in Z(a) \cap xZ(S)$ such that $d(z, z') \leq N$.

2. For subsets $S' \subset S$ and $X \subset Z(S)$, we define

$$t(S', X) = \sup\{t(a, x) \mid a \in S', x \in X\} \in \mathbb{N}_0 \cup \{\infty\},$$

and for $a \in S$, we set $t(a, X) = t(\{a\}, X)$.

3. S is said to be *locally tame* if $t(S, u) < \infty$ for all $u \in \mathcal{A}(S_{\text{red}})$. We call

$$t(S) = t(S, \mathcal{A}(S_{\text{red}})) = \sup\{t(S, u) \mid u \in \mathcal{A}(S_{\text{red}})\} \in \mathbb{N}_0 \cup \{\infty\}$$

the *tame degree* of S , and S is called *tame* if $t(S) < \infty$.

4. We set $a(S) = \sup\{|x| \mid (x, y) \in \mathcal{A}(\sim_S) \text{ for some } y \in Z(S)\} \in \mathbb{N}_0 \cup \{\infty\}$.

Let S be reduced and atomic. Local tameness is a central finiteness property in factorization theory, but the finiteness of the tame degree is a rare property (a non-principal order \mathfrak{o} in an algebraic number field is locally tame with finite catenary degree, and it is tame if and only if for every prime ideal \mathfrak{p} containing the conductor there is precisely one prime ideal $\bar{\mathfrak{p}}$ in the principal order $\bar{\mathfrak{o}}$ such that $\bar{\mathfrak{p}} \cap \mathfrak{o} = \mathfrak{p}$). Whereas in v -noetherian monoids (these are monoids satisfying the ascending chain condition for v -ideals) we have $\omega(S, u) < \infty$ for all atoms $u \in \mathcal{A}(S)$, this does not hold for the $t(S, u)$ values (see [29, Corollary 3.6], [28, Theorems 4.2 and 4.4], [27, Theorems 5.3 and 6.7]). However, we have $\omega(S) < \infty$ if and only if $t(S) < \infty$ (see Inequality (5.1) below). A main aim in this section is to show that monoids having a generic presentation satisfy $\omega(S) = t(S)$ (Theorem 5.6 and its corollaries). After that we provide the first examples of numerical monoids S with $\omega(S) < t(S)$.

Let S be atomic and, to avoid trivialities, suppose that S is not factorial. Let us consider $a(S)$. If S is finitely generated, then \sim_S is finitely generated, hence $\mathcal{A}(\sim_S)$ is finite, and thus $a(S) < \infty$. It has been proved that $a(S)$ is an upper bound for a variety of arithmetical invariants, such as the catenary degree (e.g., [37, Proposition 14]; see also the forthcoming Corollary 6.4). However, we have

$$(5.1) \quad 2 + \sup \Delta(S) \stackrel{(1)}{\leq} c(S) \stackrel{(2)}{\leq} \omega(S) \stackrel{(3)}{\leq} t(S) \stackrel{(4)}{\leq} \omega(S)^2 \quad \text{and} \quad \rho(S) \stackrel{(5)}{\leq} \omega(S),$$

where $\rho(S)$ is the elasticity (see Definition 6.1): for (1) see [26, Theorem 1.6.3], for (2), (4) and (5) see [30, Section 3], and (3) can be found in [28, Theorem 3.6]. In the next proposition, we will verify that $a(S)$ is an upper bound for the tame degree $t(S)$, but even this inequality can be strict (even for numerical monoids, see [9, Example 4.4]).

PROPOSITION 5.2. *Let S be atomic.*

1. *For every $u \in \mathcal{A}(S_{\text{red}})$ we have*

$$t(S, u) \leq \sup\{|x|, |y| \mid (x, y) \in \mathcal{A}(\sim_S), x \in uZ(S)\}.$$

2. $t(S) \leq a(S)$.

3. *If $a(S) < \infty$, then $c_{\text{adj}}(S) < \infty$.*

Proof. 1. and 2. Obviously, it is sufficient to prove the first statement. Furthermore, we may assume that S is reduced.

Let $u \in \mathcal{A}(S)$, $a \in uS$ and $z \in Z(a)$. We have to find a factorization $z' \in Z(a) \cap uZ(S)$ such that

$$d(z, z') \leq \sup\{|x|, |y| \mid (x, y) \in \mathcal{A}(\sim_S), x \in uZ(S)\}.$$

Since $a \in uS$, there exists some $\bar{z} \in Z(a) \cap uZ(S)$. We consider a factorization of (z, \bar{z}) in \sim_S , say

$$(z, \bar{z}) = (z_1, \bar{z}_1) \cdot \dots \cdot (z_k, \bar{z}_k),$$

where $k \in \mathbb{N}$, $(z_i, \bar{z}_i) \in \mathcal{A}(\sim_S)$ for all $i \in [1, k]$ and $\bar{z}_1 \in uZ(S)$. Then $z' = \bar{z}_1(z_1^{-1}z) \in Z(a) \cap uZ(S)$ and

$$d(z, z') \leq \max\{|z_1|, |\bar{z}_1|\} \leq \sup\{|x|, |y| \mid (x, y) \in \mathcal{A}(\sim_S), x \in uZ(S)\}.$$

3. Suppose that $a(S) < \infty$. Then

$$B = \{|x| - |y| \mid (x, y) \in \mathcal{A}(\sim_S)\} \subset \mathbb{Z}$$

is finite. Furthermore, by 2. and Inequality (5.1), the set of distances $\Delta(S)$ is finite. By Proposition 4.4.4, it suffices to verify that

$$\sup\{|x| \mid x \in \text{Min}(A_d), d \in \Delta(S)\} < \infty,$$

where $A_d = \{x \in Z(S) \mid |x| - d \in L(\pi(x))\}$. Let $x \in \text{Min}(A_d)$ and $y \in Z(\pi(x))$ such that $|y| = |x| - d$. Consider a factorization

$$(x, y) = (x_1, y_1) \cdot \dots \cdot (x_k, y_k),$$

where $k \in \mathbb{N}$ and $(x_i, y_i) \in \mathcal{A}(\sim_S)$ for all $i \in [1, k]$. There exists a bound $M(B, d)$ with the following property (for the construction of an explicit upper bound, see [23, Lemma 5.1]). There is a subset $I \subset [1, k]$ with $|I| \leq M(B, d)$ such that for

$$x' = \prod_{i \in I} x_i \quad \text{and} \quad y' = \prod_{i \in I} y_i$$

we have $|y'| = |x'| - d$. Thus $x' \in A_d$, and since $x' \mid x$, it follows that $x' = x$ and $|x| = |x'| \leq M(B, d)a(S)$. □

We discuss the relationship between the finiteness of the monotone catenary degree, of the tame degree and of its upper bound $a(S)$. Example 4.7 shows that the monotone catenary degree can be finite even if the monoid is not tame. Conversely, Example 4.5 in [15] provides a finitely primary monoid of rank 1 (hence it is tame and has finite catenary degree) for which $c_{\text{eq}}(S)$ is infinite. In contrast to that example, Theorem 5.1 from [30] shows that a

slightly weaker variant of $c_{\text{adj}}(S)$ is finite for tame monoids. More precisely, it states that in a tame monoid S there is a constant $M \in \mathbb{N}$ with the following property:

For each two adjacent lengths $k, l \in L(a) \cap [\min L(a) + M, \max L(a) - M]$ we have $d(Z_k(a), Z_l(a)) \leq M$.

Proposition 5.2.3 shows that the finiteness of $\mathfrak{a}(S)$ —which is stronger than the finiteness of the tame degree—enforces the finiteness of $c_{\text{adj}}(S)$. It is an interesting open problem whether the finiteness of the tame degree is strong enough to guarantee the finiteness of $c_{\text{adj}}(S)$.

Let S be atomic but not factorial. We discuss the relationship between $\omega(S)$ and $\mathfrak{t}(S)$. If $u \in S$ is an atom but not prime, then

$$\mathfrak{t}(S, uS^\times) = \max\{\omega(S, u), 1 + \tau(S, u)\} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$$

and thus

$$(5.2) \quad \mathfrak{t}(S) = \max\{\omega(S), 1 + \tau(S)\}$$

(see [28, Theorem 3.6]; since it is not needed again, we do not repeat the definition of the τ -invariant). For a large class of Krull monoids it was determined that $\mathfrak{t}(S) = 1 + \tau(S)$ [28, Corollary 4.6]. In contrast to that result, M. Omidali recently proved that $\omega(S) = \mathfrak{t}(S)$ for numerical monoids generated by almost arithmetical progressions (see [33, Theorem 3.10]). Theorem 5.6 provides a result of this type in a more general setting. We will frequently make use of the following fact: if $\mathfrak{t}(S) < \infty$, then there is an $a \in S$ and a $u \in \mathcal{A}(S_{\text{red}})$ such that $\mathfrak{t}(a, u) = \mathfrak{t}(a, \mathcal{A}(S_{\text{red}})) = \mathfrak{t}(S)$.

LEMMA 5.3. *Let S be a reduced and finitely generated. Then for every subset $X \subset S$ there exists a finite set $E \subset X$ such that $X \subset ES$. Clearly, E can be chosen to be minimal.*

Proof. This is a special case of [26, Proposition 2.7.4]. □

The next lemma is a generalization of Lemma 5 in [9].

LEMMA 5.4. *Let S be reduced and atomic, $a \in S$, $z \in Z(a)$ and $u, v \in \mathcal{A}(S)$.*

1. *Suppose that $\mathfrak{t}(a, \mathcal{A}(S)) = d(z, Z(a) \cap uZ(S)) > 0$, $z \in vZ(S)$ and $a \in uvS$. Then $\mathfrak{t}(v^{-1}a, \mathcal{A}(S)) \geq \mathfrak{t}(a, \mathcal{A}(S))$.*
2. *Let $a \in S$ be minimal such that $\mathfrak{t}(a, \mathcal{A}(S)) = \mathfrak{t}(S) > 0$ (this means that no proper divisor b of a satisfies $\mathfrak{t}(b, \mathcal{A}(S)) = \mathfrak{t}(S)$) and let $z' \in Z(a) \cap uZ(S)$ such that $d(z, z') = d(z, Z(a) \cap uZ(S)) = \mathfrak{t}(S)$. Then $z \in \text{Min}(Z(uS))$.*

Proof. 1. By definition, we have $\mathfrak{t}(v^{-1}a, \mathcal{A}(S)) \geq d(v^{-1}z, Z(v^{-1}a) \cap uZ(S))$. If $z' \in Z(v^{-1}a) \cap uZ(S)$ such that $d(v^{-1}z, z') = d(v^{-1}z, Z(v^{-1}a) \cap uZ(S))$, then

$$\begin{aligned} \mathfrak{t}(v^{-1}a, \mathcal{A}(S)) &\geq d(v^{-1}z, z') = d(z, vz') = d(z, v(Z(v^{-1}a) \cap uZ(S))) \\ &\geq d(z, Z(a) \cap uZ(S)) = \mathfrak{t}(a, \mathcal{A}(S)). \end{aligned}$$

2. Assume to the contrary that $z \notin \text{Min}(Z(uS))$. Then there exists an atom, say v , such that $v^{-1}z \in Z(uS)$. Since $z' \in uZ(S)$ with $d(z, z') = t(S) > 0$, it follows that $u \nmid z$ and hence $d(z, Z(a) \cap uZ(S)) > 0$. Thus, 1. implies that $t(v^{-1}a, \mathcal{A}(S)) \geq t(a, \mathcal{A}(S)) = t(S)$, a contradiction to the minimality of a . \square

We say that S has a *unique minimal presentation* if it has a minimal presentation σ and for each minimal presentation τ we have $\sigma \cup \sigma^{-1} = \tau \cup \tau^{-1}$. If this holds then σ is called a *unique minimal presentation* of S . Next, we show that every generic presentation is a unique minimal presentation. We would like to remind the reader that the term *generic presentation* (in semigroup theory) has been chosen to reflect the origins of this concept in *generic lattice ideals* (see the work by I. Peeva and B. Sturmfels [34]).

PROPOSITION 5.5. *Every generic presentation of S is a unique minimal presentation.*

Proof. We may suppose that S is reduced. Then S is finitely generated. Recall that any minimal presentation is constructed by choosing pairs of elements in different \mathcal{R} -classes of elements with more than one \mathcal{R} -class (see [38, Chapter 9]).

Let $\sigma \subset \sim_S$ be a generic presentation. Then every pair $(x, y) \in \sigma$ has full support, and x and y are in different \mathcal{R} -classes. Thus, if $x, y \in Z(s)$, then $Z(s)$ can consist of only two \mathcal{R} -classes, and the union of their support is the set of all atoms. So for every $s \in S$ with $|Z(s)| \geq 2$, the set of factorizations $Z(s)$ consists of precisely two \mathcal{R} -classes, and σ is unique if and only if each such \mathcal{R} -class contains precisely one factorization.

Assume to the contrary that there is an $s \in S$ such that $Z(s)$ consists of two \mathcal{R} -classes and that two distinct factorizations $z, z' \in Z(s)$ are in the same \mathcal{R} -class. By definition, z and z' can be concatenated by a chain of factorizations $z = z_0, \dots, z_k = z'$ such that $\pi(z_i) = s$ and $\gcd(z_{i-1}, z_i) \neq 1$ for all $i \in [1, k]$.

We set $z_1 = xy_1$, $z_2 = xy_2$ and $s_1 = \pi(y_1) \in S$ where $x = \gcd(z_1, z_2)$ and $y_1, y_2 \in Z(S)$. Note that s_1 is a proper divisor of $s_0 = s$. Since $\text{supp}(y_1) \subset \text{supp}(z_1)$ and $\text{supp}(y_2) \subset \text{supp}(z_2)$, y_1 and y_2 are in the same \mathcal{R} -class, because otherwise $Z(s_1)$ would consist of two \mathcal{R} -classes and there would be a relation without full support.

Iterating this construction we obtain an infinite sequence $(s_i)_{i \geq 0}$ where s_{i+1} is a proper divisor of s_i for all $i \in \mathbb{N}_0$, a contradiction to S being finitely generated. \square

THEOREM 5.6. *Let S be atomic, $P \subset S$ a set of representatives of the set of primes of S and T the set of all $a \in S$ such that $p \nmid a$ for all $p \in P$. Suppose that $T = \coprod_{i \in I} T_i$, $T \neq T^\times$ and that there is an $i^* \in I$ such that T_{i^*} has a generic presentation and $t(T_{i^*}) = t(T)$. Then $c(S) = \omega(S) = t(S)$.*

REMARK. Since $t(T) = \sup\{t(T_i) \mid i \in I\}$ [26, Proposition 1.6.8], the assumption is of course satisfied if all T_i have generic presentations.

Proof. By [26, Theorem 1.2.3], $T \subset S$ is an atomic submonoid and $S = \mathcal{F}(P) \times T$. Note that neither S nor T are factorial because $T \neq T^\times$. Since

$$c(T_{i^*}) \leq c(T) = c(S) \leq \omega(S) \leq t(S) = \sup\{t(T_i) \mid i \in I\} = t(T_{i^*}),$$

it suffices to show that $c(T_{i^*}) = t(T_{i^*})$. Thus, after a change of notation, we may assume that S is reduced, not factorial and has a generic presentation. Let $\sigma \subset \sim_S$ denote this generic presentation of S . We start with the following assertion.

A. For every $u \in \mathcal{A}(S)$, we have

$$\text{Min}(Z(uS)) = \{u\} \cup \{x \in Z(S) \mid (x, y) \in \sigma \cup \sigma^{-1} \text{ for some } y \in uZ(S)\}.$$

Proof of A. Since $u \in \text{Min}(Z(uS))$, we may focus on the elements different from u . Let $x = \prod_{v \in \mathcal{A}(S)} v^{k_v} \in \text{Min}(Z(uS)) \setminus \{u\}$ with $k_v \in \mathbb{N}_0$ for all $v \in \mathcal{A}(S)$, and set $a = \pi(x) \in S$. Since $x \neq u$, it follows that $k_u = 0$. There exists a $y \in Z(a) \cap uZ(S)$, and the pair (x, y) belongs to the congruence generated by σ . Hence, there exists $(x', y') \in \sigma \cup \sigma^{-1}$ with $x' \mid x$. Since $u \nmid x$, we get that $u \nmid x'$ and because σ is generic, it follows that $u \mid y'$ and hence $y' \in Z(uS)$. Since $x \in \text{Min}(Z(uS)) \setminus \{u\}$, we infer that $x = x'$. The uniqueness property of a generic presentation implies that $Z(a) = \{x', y'\}$. Thus, we get that $y = y'$ and hence $(x, y) = (x', y') \in \sigma$.

Conversely, let $x \in Z(S)$ and $y \in uZ(S)$ be such that $(x, y) \in \sigma \cup \sigma^{-1}$. Then we clearly have $x \in Z(uS)$, and assume to the contrary that it is not minimal. Then there is an $x' = \prod_{v \in \mathcal{A}(S)} v^{k'_v} \in Z(uS)$, where $k'_v \in \mathbb{N}_0$ for all $v \in \mathcal{A}(S)$, and with $x' \mid x$ and $x' \neq x$. Note that $k'_u = 0$, because $u \mid y$ and the supports of x and y are disjoint. Since σ is generic, there exists a $y' = \prod_{v \in \mathcal{A}(S)} v^{l'_v}$, where all $l'_v \in \mathbb{N}_0$, with $l'_u \neq 0$ and $\pi(x') = \pi(y')$. Since the pair (x', y') is in the congruence generated by σ , there exist $(x'', y'') \in \sigma \cup \sigma^{-1}$ such that $x'' \mid x'$. This implies that $x'' \mid x$ and $x'' \neq x$. This contradicts the fact, that elements whose factorizations appear in a generic presentation are not comparable [21, Corollary 6]. □

Since σ is a generic presentation, for every $u \in \mathcal{A}(S)$ and every $(x, y) \in \sigma$ we have $u \mid x$ or $u \mid y$. Thus, **A** and Proposition 3.3 imply that

$$\omega(S) = \max\{\max\{|x|, |y|\} \mid (x, y) \in \sigma\}.$$

Now the minimality property of a generic presentation (see Proposition 5.5), the above formula for $\omega(S)$ together with Proposition 4.6 imply that $\omega(S) = c(S)$.

Since $\omega(S) \leq t(S)$, it remains to show that converse inequality. Let $a \in S$ be minimal such that $t(a, \mathcal{A}(S)) = t(S)$, and let $u \in \mathcal{A}(S)$, $z \in Z(a)$ and $z' \in Z(a) \cap uZ(S)$ such that

$$t(S) = t(a, \mathcal{A}(S)) = d(z, Z(a) \cap uZ(S)) = d(z, z').$$

By Lemma 5.4, it follows that $z \in \text{Min}(Z(uS))$. Thus, by **A**, there exist $(x, y) \in \sigma \cup \sigma^{-1}$ such that $y = z$. Therefore x and y are factorizations of a , which appear in the unique presentation of S . This implies that $Z(a) = \{x, y\}$, and thus $z' = x$ and $\mathfrak{t}(S) = \mathfrak{d}(z, z') \leq \max\{|x|, |y|\} \leq \omega(S)$. \square

Let R be an integral domain. We denote by $R^\bullet = R \setminus \{0\}$ its multiplicative monoid of non-zero elements, by $\mathfrak{X}(R)$ the set of all minimal nonzero prime ideals of R , by \widehat{R} its complete integral closure, and by $(R:\widehat{R}) = \{f \in R \mid f\widehat{R} \subset R\}$ the conductor of R in \widehat{R} .

COROLLARY 5.7. *Let R be a weakly Krull domain, $\mathfrak{f} = (R:\widehat{R}) \neq \{0\}$, $\mathcal{P}^* = \{\mathfrak{p} \in \mathfrak{X}(R) \mid \mathfrak{p} \supset \mathfrak{f}\}$ and $S = \mathcal{I}_v^*(R)$ the monoid of v -invertible v -ideals equipped with v -multiplication. If for every $\mathfrak{p} \in \mathcal{P}^*$, the monoid $R_{\mathfrak{p}}^\bullet$ has a generic presentation, then $\mathfrak{c}(S) = \omega(S) = \mathfrak{t}(S)$.*

Proof. By [26, Theorem 3.7.1], the monoid S is isomorphic to $\mathcal{F}(P) \times T$, where

$$P = \{\mathfrak{p} \in \mathfrak{X}(R) \mid \mathfrak{p} \not\supset \mathfrak{f}\} \quad \text{and} \quad T = \prod_{\mathfrak{p} \in \mathcal{P}^*} (R_{\mathfrak{p}}^\bullet)_{\text{red}}.$$

Thus, the assertion follows from Theorem 5.6. \square

Let all notations be as in Corollary 5.7. It is easy to point out explicit examples where the assumptions hold (for details see [26, Section 3.7]). Every one-dimensional noetherian domain, in particular every order in a Dedekind domain, is weakly Krull. Let $\mathfrak{p} \in \mathcal{P}^*$. Then $R_{\mathfrak{p}}^\bullet$ is finitely primary, and $R_{\mathfrak{p}}^\bullet$ is tame if and only if there exists precisely one prime ideal $\widehat{\mathfrak{p}} \in \mathfrak{X}(\widehat{R})$ satisfying $\widehat{\mathfrak{p}} \cap R = \mathfrak{p}$. Suppose this holds true, and set $H = (R_{\mathfrak{p}}^\bullet)_{\text{red}}$. Then $H \subset F = F^\times \times [p]$ where p is a prime element of the factorial monoid F , and its value monoid $v_p(H) = \{v_p(a) \mid a \in H\} \subset (\mathbb{N}_0, +)$ is a numerical monoid. If R is a non-principal order in an algebraic number field, then F^\times is finite.

COROLLARY 5.8. *Let S be a numerical monoid with $\mathcal{A}(S) = \{n_1, n_2, n_3\}$. Define*

$$c_i = \min\{k \in \mathbb{N} \mid kn_i \in \langle n_j, n_k \rangle\} \quad \text{and} \quad c_i n_i = r_{i,j} n_j + r_{i,k} n_k,$$

where $\{i, j, k\} = \{1, 2, 3\}$ and $r_{i,j} \in \mathbb{N}_0$. Then S has a generic presentation if and only if $r_{i,j} > 0$ for all $i, j \in \{1, 2, 3\}$. If this is the case, then

$$\mathfrak{c}(S) = \omega(S) = \mathfrak{t}(S) = \max\{c_1, c_2, c_3, r_{12} + r_{1,3}, r_{2,1} + r_{2,3}, r_{3,1} + r_{3,2}\}.$$

Proof. The first assertion follows from [39, Example 8.23]. The formula follows from the proof of Theorem 5.6 by taking into account that in this setting, the minimal presentation for S is $\{(c_i n_i, r_{i,j} n_j + r_{i,k} n_k) \mid \{i, j, k\} = \{1, 2, 3\}\}$ (see again [39, Example 8.23]). \square

The catenary degree of numerical monoids with embedding dimension three has been also described (with a different approach) in [1]. Let S be atomic. Obviously, the requirement that S has a generic presentation (enforcing $c(S) = t(S)$) is a strong assumption, and also the general philosophy in factorization theory confirms the idea that the equality of the catenary and the tame degree should be an exceptional phenomenon (see also Corollary 5.10). On the other hand, all types of numerical monoids studied so far share this exceptional phenomenon. This contrast will become more clear in the following remark, where we also construct the first infinite family of numerical monoids whose ω -invariants are strictly smaller than the tame degrees.

REMARKS 5.9. 1. Let S be a numerical monoid with $\mathcal{A}(S) = \{n_1, \dots, n_t\}$, where $t \in \mathbb{N}$ and $1 < n_1 < \dots < n_t$, and let $s \in S$. Consider a factorization $z = a_1 \mathbf{n}_1 + \dots + a_t \mathbf{n}_t \in \text{Min}(Z(s + S))$. Then $z \in Z(a)$ where $a = a_1 n_1 + \dots + a_t n_t$, and $a = s + u$ for some $u \in S$. Pick some $j \in [1, t]$ such that $\mathbf{n}_j \mid z$. The minimality of z implies that $z - \mathbf{n}_j \notin Z(s + S)$, and thus $a - s - n_j = u - n_j \notin S$, whence $u \in \text{Ap}(S, n_j)$. Thus, if we want to compute the elements in $\text{Min}(Z(s + S))$ (which in view of Proposition 3.3 enables us to determine $\omega(S)$), we only have to find the factorizations of the elements of the form $s + u$ with u in the Apéry set of some atom.

We implemented this procedure in GAP by using the `numericalsgps` package (see [14]). We did an exhaustive search computing all numerical monoids with Frobenius number up to 20. That makes 3515 numerical monoids, and the only monoids S in this set fulfilling $\omega(S) < t(S)$ are $\langle 5, 6, 9 \rangle$, $\langle 5, 8, 12 \rangle$ and $\langle 6, 8, 9 \rangle$.

2. The minimal presentations of the above three numerical monoids are very similar. Playing around with the Smith normal form of the matrix whose rows are the differences of the relators of these monoids, one can find even wilder examples. The monoid $S = \langle 19, 46, 391 \rangle$ has $\omega(S) = 23 < 39 = t(S)$.

3. We present an infinite family of numerical monoids whose ω -invariants are strictly smaller than the tame degrees. Let q be a prime, $p_1, p_2 \in \mathbb{N}$ with $p_1 < p_2$, $p_1 + p_2 = q$ and $\text{gcd}(p_1, p_2) = 1$, and $k \in \mathbb{N}_{\geq 2} \setminus q\mathbb{N}$ such that $p_1 k < q < p_2 k$. We define

$$S_k = \langle p_1 k, q, p_2 k \rangle,$$

and set $n_1 = p_1 k$, $n_2 = q$, $n_3 = p_2 k$ and $c_i = \min\{m \in \mathbb{N} \mid mn_i \in \langle n_j, n_k \rangle\}$ with $\{i, j, k\} = \{1, 2, 3\}$ (note that c_1, c_2 and c_3 are as in Corollary 5.8).

- (a) The Diophantine equation $qx + p_2 ky = p_2 kt$ has general solution $x = kt - p_2 ks$, $y = -t + qs$, $s \in \mathbb{Z}$. The first t for which x and y can be non-negative is $t = p_2$. This in particular means that $p_1 k$ is not in $\langle q, p_2 k \rangle$ and that $c_1 = p_2$. In fact, $p_2 n_1 = p_1 n_3$, and $(p_2 + 1)n_1 = kn_2 + (p_1 - 1)n_3$.
- (b) Analogously one proves that $c_2 = k$; $kn_2 = n_1 + n_3$.
- (c) It is also easy to show that $c_3 = p_1$: $p_1 n_3 = p_2 n_1$. Moreover, $(p_1 + 1)n_3 = (p_2 - 1)n_1 + kn_2$.

By using this information it easily follows that

$$\begin{aligned} \text{Min}(Z(n_1 + S_k)) &= \{\mathbf{n}_1, k\mathbf{n}_2, p_1\mathbf{n}_3\}, \\ \text{Min}(Z(n_2 + S_k)) &= \{(p_2 + 1)\mathbf{n}_1, \mathbf{n}_1 + \mathbf{n}_3, \mathbf{n}_2, (p_1 + 1)\mathbf{n}_3\}, \\ \text{Min}(Z(n_3 + S_k)) &= \{p_2\mathbf{n}_1, k\mathbf{n}_2, \mathbf{n}_3\}. \end{aligned}$$

Therefore, Proposition 3.3 implies that

$$\omega(S_k) = \max\{k, p_2 + 1\}.$$

Note that $Z((p_1 + 1)n_3) = \{(p_1 + 1)\mathbf{n}_3, (p_2 - 1)\mathbf{n}_1 + k\mathbf{n}_2, p_2\mathbf{n}_1 + \mathbf{n}_3\}$. Thus, analyzing the factorizations of the elements in $\pi(\text{Min}(Z(n_i + S_k)))$ for $i \in [1, 3]$, and by using 1., we obtain that

$$\begin{aligned} \mathfrak{t}(S_k) &= \max\{\mathfrak{t}((p_2 + 1)e_1, \mathcal{A}(S_k)), \mathfrak{t}((p_1 + 1)e_3, \mathcal{A}(S_k))\} \\ &= \max\{p_2 + 1, k + p_2 - 1\} = k + p_2 - 1, \end{aligned}$$

which is strictly larger than $\omega(S_k)$. Furthermore, if $k \geq p_2 + 1$, then

$$1 < \frac{\mathfrak{t}(S_k)}{\omega(S_k)} = 1 + \frac{p_2 - 1}{k} \leq 1 + \frac{p_2 - 1}{p_2 + 1} < 2.$$

We end this section with a brief glance at Krull monoids. For them the equivalence of the catenary and the tame degree is an even rarer phenomenon than it is for numerical monoids. Let S be a Krull monoid with class group G and let $G_P \subset G$ denote the set of classes containing prime divisors. If $D(G_P) < \infty$, then S is tame, and the converse holds—among others—if S the multiplicative monoid of nonzero elements of a domain (see [30, Theorem 4.2]). If G is finite with $|G| \geq 3$ and $G_P = G$, then [26, Corollary 3.4.12] shows that

$$c(S) = c(G) \leq D(G) = \omega(S) \leq \mathfrak{t}(G) \leq \mathfrak{t}(S),$$

where the final inequality can be strict.

COROLLARY 5.10. *Let G be a finite abelian group with $|G| \geq 3$.*

1. $c(G) = \mathfrak{t}(G)$ if and only if $G \in \{C_3, C_4, C_2^2, C_2^3\}$.
2. The monoid of zero-sum sequences $\mathcal{B}(G^\bullet)$ has a generic presentation if and only if $G \in \{C_3, C_2^2\}$.

Proof. 1. See [26, Corollary 6.5.7].

2. By 1. and by Theorem 5.6, we have to check only the groups in $\{C_3, C_4, C_2^2, C_2^3\}$. We recall the following facts (for details see [38, Chapter 9]). If σ is a minimal presentation for $\mathcal{B}(G^\bullet)$ and $(a, b) \in \sigma$, then a and b are in different \mathcal{R} -classes. In fact, any minimal presentation is constructed by choosing pairs of elements in different \mathcal{R} -classes of elements with more than one \mathcal{R} -class.

If $G = C_3 = \{0, g, 2g\}$, then $\mathcal{A}(\mathcal{B}(G^\bullet)) = \{U_1 = g^3, U_2 = (2g)^3, V = g(2g)\}$ and $\sigma = \{(U_1U_2, V^3)\} \subset \sim_{\mathcal{B}(G^\bullet)}$ is a generic presentation.

If $G = C_2 \oplus C_2 = \{0, e_1, e_2, e_1 + e_2\}$, then $\mathcal{A}(\mathcal{B}(G^\bullet)) = \{U_1 = e_1^2, U_2 = e_2^2, U_3 = (e_1 + e_2)^2, V = e_1 e_2 (e_1 + e_2)\}$ and $\sigma = \{(U_1 U_2 U_3, V^2)\} \subset \sim_{\mathcal{B}(G^\bullet)}$ is a generic presentation.

Let $G = C_4 = \{0, g, 2g, -g\}$. Then $\mathcal{A}(\mathcal{B}(G^\bullet)) = \{U_1 = g^4, U_2 = (2g)^2, U_3 = (-g)^4, U_4 = (-g)g, U_5 = g^2(2g), U_6 = ((2g)(-g)^2)\}$ and $(U_1 U_3, U_4^4) \in \sim_{\mathcal{B}(G^\bullet)}$. Since $\mathcal{Z}(U_4^4) = \{U_1 U_3, U_4^4\}$, the set of factorizations of U_4^4 has only two \mathcal{R} -classes, where each consists of precisely one factorization. Thus, $(U_4^4, U_1 U_3) \in \sigma U \sigma^{-1}$, for every minimal presentation σ . Obviously, this pair does not have full support, and hence $\mathcal{B}(G^\bullet)$ has no generic presentation.

Let $G = C_2 \oplus C_2 \oplus C_2 = \{0, e_1, e_2, e_3, e_1 + e_2, e_1 + e_3, e_2 + e_3, e_1 + e_2 + e_3\}$. Then $U_1 = e_1^2, U_2 = e_2^2, U_3 = (e_1 + e_2)^2, V = e_1 e_2 (e_1 + e_2) \in \mathcal{A}(\mathcal{B}(G^\bullet))$ and $(U_1 U_2, V^3) \in \sim_{\mathcal{B}(G^\bullet)}$. Since every minimal presentation σ contains the relation $(U_1 U_2, V^3)$ which does not have full support, it follows that $\mathcal{B}(G^\bullet)$ has no generic presentation. □

6. Unions of sets of lengths

DEFINITION 6.1. Let S be atomic and $k \in \mathbb{N}$.

1. If $S = S^\times$, we set $\mathcal{U}_k(S) = \{k\}$. If $S \neq S^\times$, let $\mathcal{U}_k(S)$ denote the set of all $m \in \mathbb{N}$ for which there exist $u_1, \dots, u_k, v_1, \dots, v_m \in \mathcal{A}(S)$ with $u_1 \cdot \dots \cdot u_k = v_1 \cdot \dots \cdot v_m$.
2. We define

$$\rho_k(S) = \sup \mathcal{U}_k(S) \in \mathbb{N} \cup \{\infty\} \quad \text{and} \quad \lambda_k(S) = \min \mathcal{U}_k(S) \in [1, k].$$

3. For $a \in S$, $\rho(a) = \rho(\mathcal{L}(a))$ is called the *elasticity* of a , and

$$\rho(S) = \sup \{ \rho(L) \mid L \in \mathcal{L}(S) \} \in \mathbb{R}_{\geq 1} \cup \{\infty\}$$

is called the *elasticity* of S . We say that S has *finite accepted elasticity* if there exists some $a \in S$ with $\rho(a) = \rho(S) < \infty$.

Let $k, l \in \mathbb{N}$. Then $k \in \mathcal{U}_k(S)$, $\mathcal{U}_k(S) + \mathcal{U}_l(S) \subset \mathcal{U}_{k+l}(S)$,

$$\lambda_{k+l}(S) \leq \lambda_k(S) + \lambda_l(S) \leq k + l \leq \rho_k(S) + \rho_l(S) \leq \rho_{k+l}(S),$$

$$\rho(S) = \sup \left\{ \frac{\rho_k(S)}{k} \mid k \in \mathbb{N} \right\} = \lim_{k \rightarrow \infty} \frac{\rho_k(S)}{k}$$

and

$$\frac{1}{\rho(S)} = \inf \left\{ \frac{\lambda_k(S)}{k} \mid k \in \mathbb{N} \right\} = \lim_{k \rightarrow \infty} \frac{\lambda_k(S)}{k},$$

(see [26, Proposition 1.4.2] and [20, Section 3]). Moreover, if $S \neq S^\times$, then

$$\mathcal{U}_k(S) = \bigcup_{k \in L, L \in \mathcal{L}(S)} L$$

is the union of all sets of lengths containing k . These unions were introduced by S. T. Chapman and W. W. Smith in [12]. It was proved only recently that a v -noetherian monoid, which satisfies $\rho_k(S) < \infty$ for all $k \in \mathbb{N}$, is locally

tame (see [28, Corollary 4.3]). For Krull monoids with finite class group, the invariants $\rho_k(S)$ are studied in [25].

The first part of this section is devoted to the invariant $\rho_k(S)$ in a more general setting, and after that we study the structure of the unions of sets of lengths for numerical monoids.

PROPOSITION 6.2. *Let S be atomic with $S \neq S^\times$.*

1. *If S has finite accepted elasticity, then the sets*

$$M = \left\{ k \in \mathbb{N} \mid \frac{\rho_k(S)}{k} = \rho(S) \right\} \cup \{0\}$$

and

$$M' = \left\{ k \in \mathbb{N} \mid \frac{\lambda_k(S)}{k} = \frac{1}{\rho(S)} \right\} \cup \{0\}$$

are submonoids of $(\mathbb{N}_0, +)$, distinct from $\{0\}$.

2. *Let $a \in S$, $z = u_1 \cdots u_l \in \mathcal{Z}(a)$ and $z' = v_1 \cdots v_\rho \in \mathcal{Z}(a)$ where $l \in \mathbb{N}$, $\rho = \rho_l(S)$ and $u_1, \dots, u_l, v_1, \dots, v_\rho \in \mathcal{A}(S_{\text{red}})$. If there is no $k \in [1, l - 1]$ such that $\rho_k(S) + \rho_{l-k}(S) = \rho_l(S)$, then $(z, z') \in \mathcal{A}(\sim_S)$.*

Proof. We may suppose that S is reduced.

1. Suppose that S has finite accepted elasticity. First, we consider the set M . By definition, there is an $a \in S$ such that $\rho(S) = \rho(a)$. If $k = \min L(a)$ and $\rho = \max L(a)$, then

$$\rho(S) = \frac{\rho}{k} \leq \frac{\rho_k(S)}{k} \leq \rho(S),$$

and hence $k \in M$. Let $i \in [1, 2]$ and $k_i \in M$. Since $(k_1 + k_2)\rho(S) = \rho_{k_1}(S) + \rho_{k_2}(S) \leq \rho_{k_1+k_2}(S)$, it follows that

$$\rho(S) \geq \frac{\rho_{k_1+k_2}(S)}{k_1 + k_2} \geq \frac{\rho_{k_1}(S) + \rho_{k_2}(S)}{k_1 + k_2} = \rho(S).$$

Thus equality holds, and $k_1 + k_2 \in M$. To verify the assertion on M' , we choose an $l \in \mathbb{N}$ such that $\rho_l(S)/l = \rho(S)$. Then $\lambda_{\rho_l(S)}(S) \leq l$, and since

$$\frac{1}{\rho(S)} \leq \frac{\lambda_{\rho_l(S)}(S)}{\rho_l(S)} \leq \frac{l}{\rho_l(S)} = \frac{1}{\rho(S)},$$

it follows that $\rho_l(S) \in M'$. Let $i \in [1, 2]$ and $k_i \in M'$. Since $(k_1 + k_2)/\rho(S) = \lambda_{k_1}(S) + \lambda_{k_2}(S) \geq \lambda_{k_1+k_2}(S)$, it follows that

$$\frac{1}{\rho(S)} \leq \frac{\lambda_{k_1+k_2}(S)}{k_1 + k_2} \leq \frac{\lambda_{k_1}(S) + \lambda_{k_2}(S)}{k_1 + k_2} = \frac{1}{\rho(S)}.$$

Thus, equality holds and $k_1 + k_2 \in M'$.

2. Assume to the contrary that $(z, z') \notin \mathcal{A}(\sim_S)$. Then there exists $(x, x') \in \sim_S$ such that $(x, x') \mid (z, z')$ with $1 \neq (x, x') \neq (z, z')$. After renumbering, if

necessary, we may suppose that $x = u_1 \cdot \dots \cdot u_k$ and $x' = v_1 \cdot \dots \cdot v_\psi$ where $k \in [1, l - 1]$ and $\psi \in [1, \rho - 1]$. Then $u_{k+1} \cdot \dots \cdot u_l = v_{\psi+1} \cdot \dots \cdot v_\rho$ and

$$\rho_l(S) = \rho = \psi + (\rho - \psi) \leq \rho_k(S) + \rho_{l-k}(S) \leq \rho_l(S),$$

a contradiction. □

COROLLARY 6.3. *Let S be a numerical monoid with $\mathcal{A}(S) = \{n_1, \dots, n_t\}$ where $t \in \mathbb{N}$ and $1 < n_1 < \dots < n_t$.*

1. *Then*

$$\rho(S) = \frac{n_t}{n_1} \quad \text{and} \quad \min \Delta(S) = \gcd(n_2 - n_1, \dots, n_t - n_{t-1}).$$

2. $\{k \in \mathbb{N} \mid \frac{\rho_k(S)}{k} = \rho(S)\} \cup \{0\} = \frac{\text{lcm}(n_1, n_t)}{n_t} \mathbb{N}_0$.
3. $\{k \in \mathbb{N} \mid \frac{\lambda_k(S)}{k} = \frac{1}{\rho(S)}\} \cup \{0\} = \frac{\text{lcm}(n_1, n_t)}{n_1} \mathbb{N}_0$.

Proof. 1. See [11, Theorem 2.1] and [8, Proposition 2.9].

2. Let $a \in \mathbb{N}$ be a multiple of $\text{lcm}(n_1, n_t)$. We show that a/n_t is in the set on the left-hand side. We have

$$a = \frac{a}{n_1} \mathbf{n}_1 = \frac{a}{n_t} \mathbf{n}_t, \quad \min \mathbf{L}(a) \leq \frac{a}{n_t}, \quad \max \mathbf{L}(a) \geq \frac{a}{n_1}$$

and

$$\frac{n_t}{n_1} = \rho(S) \geq \rho(a) = \frac{\max \mathbf{L}(a)}{\min \mathbf{L}(a)} \geq \frac{n_t}{n_1}.$$

This shows that $\min \mathbf{L}(a) = a/n_t$, $\max \mathbf{L}(a) = a/n_1$ and

$$\frac{n_t}{n_1} = \rho(a) \leq \frac{\rho_{\min \mathbf{L}(a)}(S)}{\min \mathbf{L}(a)} \leq \rho(S) = \frac{n_t}{n_1}.$$

Thus equality holds and $\min \mathbf{L}(a) = a/n_t$ has the required property.

Conversely, let $k \in \mathbb{N}$ with $\rho_k(S)/k = \rho(S) = n_t/n_1$. We choose $a \in \mathcal{U}_k(S)$ with $\max \mathbf{L}(a) = \rho_k(S)$. Then $\min \mathbf{L}(a) \leq k$ and

$$\frac{n_t}{n_1} = \frac{\rho_k(S)}{k} \leq \frac{\max \mathbf{L}(a)}{\min \mathbf{L}(a)} = \rho(a) \leq \frac{n_t}{n_1}$$

implies that $\min \mathbf{L}(a) = k$. Since $a/n_t \leq \min \mathbf{L}(a)$, $\max \mathbf{L}(a) \leq a/n_1$ and

$$\frac{n_t}{n_1} = \rho(a) \leq \frac{a/n_1}{a/n_t} = \frac{n_t}{n_1},$$

it follows that $n_1 \mid a$ and $n_t \mid a$. Therefore, $\text{lcm}(n_1, n_t) \mid a$ and

$$k = \min \mathbf{L}(a) = \frac{a}{n_t} \in \frac{\text{lcm}(n_1, n_t)}{n_t} \mathbb{N}_0.$$

3. Let $a \in \mathbb{N}$ be a multiple of $\text{lcm}(n_1, n_t)$. We show that a/n_1 is in the set on the left-hand side which runs along the lines of 2. Conversely, let $k \in \mathbb{N}$

with $\lambda_k(S)/k = 1/\rho(S) = n_1/n_t$. We choose $a \in \mathcal{U}_k(S)$ with $\min L(a) = \lambda_k(S)$. Again arguing as in 2., we infer that

$$k = \max L(a) = \frac{a}{n_1} \in \frac{\text{lcm}(n_1, n_t)}{n_1} \mathbb{N}_0. \quad \square$$

COROLLARY 6.4. *Let S be a reduced Krull monoid, $F = \mathcal{F}(P)$ a free monoid such that $S \subset F$ is a saturated and cofinal submonoid, $G = F/S$ and $G_P = \{pq(S) \mid p \in P\} \subset G$ the set of classes containing prime divisors. Suppose that $G_P = -G_P$ and that $D(G_P) < \infty$.*

1. We have $\rho(S) = D(G_P)/2$ and $2\mathbb{N} \subset \{k \in \mathbb{N} \mid \frac{\rho_k(S)}{k} = \rho(S)\}$.
2. Let $m \in \mathbb{N}$ be minimal such that

$$\rho_{2m+1}(G_P) - mD(G_P) = \max\{\rho_{2k+1}(G_P) - kD(G_P) \mid k \in \mathbb{N}\}.$$

Then $\rho_{2m+1}(S) \leq a(\mathcal{B}(G_P))$.

Proof. 1. See [26, Theorem 3.4.10].

2. For every $k \in \mathbb{N}$, we set (as it is usual) $\rho_k(G_P) = \rho_k(\mathcal{B}(G_P))$, and by [26, Theorem 3.4.10] we have $\rho_k(S) = \rho_k(G_P)$. Thus, it suffices to verify that $\rho_{2m+1}(G_P)$ has the asserted upper bound. Let $U_1, \dots, U_{2m+1}, V_1, \dots, V_\rho \in \mathcal{A}(\mathcal{B}(G_P))$ with $U_1 \cdots U_{2m+1} = V_1 \cdots V_\rho$ and $\rho = \rho_{2m+1}(G_P)$. We assert that there is no $k \in [1, 2m]$ such that $\rho_k(G_P) + \rho_{2m+1-k}(G_P) = \rho$. If this holds, then Proposition 6.2 implies that $(z = U_1 \cdots U_{2m+1}, z' = V_1 \cdots V_\rho) \in \mathcal{A}(\sim_{\mathcal{B}(G_P)})$ and hence

$$\begin{aligned} \rho_{2m+1}(G_P) &= \rho = \max\{2m + 1, \rho\} = \max\{|z|, |z'|\} \\ &\leq \sup\{|x| \mid (x, y) \in \mathcal{A}(\sim_{\mathcal{B}(G_P)}) \text{ for some } y \in Z(\mathcal{B}(G_P))\} \\ &= a(\mathcal{B}(G_P)). \end{aligned}$$

Assume to the contrary, that there is a $k \in [1, 2m]$ such that $\rho_k(G_P) + \rho_{2m+1-k}(G_P) = \rho_{2m+1}(G_P)$. Then either k or $2m + 1 - k$ are odd, say $k = 2s + 1$ with $s \in \mathbb{N}_0$. Since, by 1., we have $\rho_{2(m-s)}(G_P) = (m - s)D(G_P)$, we infer that

$$\begin{aligned} \rho_{2s+1}(G_P) - sD(G_P) &= \rho_{2m+1}(G_P) - \rho_{2(m-s)}(G_P) - sD(G_P) \\ &= \rho_{2m+1}(G_P) - mD(G_P), \end{aligned}$$

a contradiction. □

Let all notations be as Corollary 6.4, and suppose in addition that $G_P = G$ is finite Abelian but non-cyclic. In all situations studied so far, the set

$$M = \left\{ k \in \mathbb{N} \mid \frac{\rho_k(S)}{k} = \rho(S) \right\} \cup \{0\}$$

contains an odd element, and hence (by Proposition 6.2 and by Corollary 6.4.1) M is a numerical monoid. The standing conjecture is that this holds for all finite non-cyclic Abelian groups G (see [25]).

Next, we deal with the structure of the unions of sets of lengths. Suppose S is a Krull monoid such that every class contains a prime divisor. Then it was shown only recently that, for all $k \in \mathbb{N}$, the unions $\mathcal{U}_k(S)$ are arithmetical progressions with difference 1 (see [18, Theorem 4.1], [22] for a simpler proof, and also [20]). In [36], unions of sets of lengths are studied for non-principal order in number fields, and in [10], for domains of the form $V + XB[X]$, where V is a discrete valuation domain and B the ring of integers in a finite extension field over the quotient field of V . In [2], S. T. Chapman et al. showed that in numerical monoids, generated by arithmetical progressions, all unions are arithmetical progressions. We now generalize this result. Clearly, monoids generated by arithmetical progressions satisfy the assumption in Proposition 6.5 and Theorem 6.6. A numerical monoid, for which the assumption and the statement fail, is discussed in Remark 6.7.

LEMMA 6.5. *Let S be a numerical monoid with $\mathcal{A}(S) = \{n_1, \dots, n_t\}$, where $t \in \mathbb{N}$, $1 < n_1 < \dots < n_t$, and $d = \gcd(n_2 - n_1, \dots, n_t - n_{t-1})$. Suppose that the Diophantine equations*

$$(n_2 - n_1)x_2 + \dots + (n_t - n_1)x_t = dn_1$$

and

$$(n_t - n_1)y_1 + \dots + (n_t - n_{t-1})y_{t-1} = dn_t$$

have solutions in the non-negative integers. Then there exists an element $a^* \in S$ such that $\rho(a^*) = \rho(S)$ and $L(a^*)$ is an arithmetical progression with difference d .

Proof. We proceed in several steps.

1. Let $a \in \mathbb{N}$ be a multiple of n_1 and of n_t . Then

$$z = \frac{a}{n_1} \mathbf{n}_1 \quad \text{and} \quad z' = \frac{a}{n_t} \mathbf{n}_t$$

are factorizations of a . Obviously, we have $\min L(a) = a/n_t$, $\max L(a) = a/n_1$ and hence $\rho(a) = n_1^{-1}n_t$. By Corollary 6.3.1 it follows that $\rho(a) = \rho(S)$.

2. Since S is finitely generated, Proposition 5.2.2 and Equation (5.1) imply that S is locally tame with finite set of distances $\Delta(S)$, and $\Delta(S) \neq \emptyset$ because $\rho(S) > 1$. Thus, [26, Theorem 4.3.6.1] implies that there is an $\bar{a} \in S$ with the following property: for every $b \in S$ we have

$$L(\bar{a}b) = y + (L' \cup L^* \cup L'') \subset y + d\mathbb{Z},$$

where $y \in \mathbb{Z}$, L^* is an arithmetical progression with difference d , $\min L^* = 0$, $L' \subset [-\mathfrak{t}(S, \mathbb{Z}(\bar{a})), -1]$ and $L'' \subset \max L^* + [1, \mathfrak{t}(S, \mathbb{Z}(\bar{a}))]$.

3. Let $(\alpha_2, \dots, \alpha_t) \in \mathbb{N}_0^{t-1}$ and $(\beta_1, \dots, \beta_{t-1}) \in \mathbb{N}_0^{t-1}$ be solutions of the given Diophantine equations, and set

$$\alpha_1 = \alpha_2 + \dots + \alpha_t \quad \text{and} \quad \beta_t = -(\beta_1 + \dots + \beta_{t-1}).$$

Now let $a^* \in \mathbb{N}$ be a multiple of $\text{lcm}(\bar{a}, n_1, n_t)$ such that

$$\frac{a^*}{n_1} \geq \gamma(d + \alpha_1) \quad \text{and} \quad \frac{a^*}{n_t} \geq \gamma|d + \beta_t|, \quad \text{where } \gamma = \left\lceil \frac{\mathbf{t}(S, \mathbf{Z}(\bar{a}))}{d} \right\rceil.$$

We assert that a^* has the required properties. By 1., it follows that

$$\min L(a^*) = \frac{a^*}{n_t}, \quad \max L(a^*) = \frac{a^*}{n_1} \quad \text{and} \quad \rho(a^*) = \rho(S).$$

We set $a^* = \bar{a}b$ with $b \in S$, and write $L(a^*)$ in the form $L(a^*) = y + (L' \cup L^* \cup L'') \subset y + d\mathbb{Z}$ with all properties as in 2. (note that such a representation need not be unique).

Let $\nu \in [0, \gamma]$. Then

$$x_\nu = \left(\frac{a^*}{n_1} - \nu(d + \alpha_1) \right) \mathbf{n}_1 + \nu\alpha_2 \mathbf{n}_2 + \cdots + \nu\alpha_t \mathbf{n}_t$$

is a factorization of a^* of length

$$|x_\nu| = \frac{a^*}{n_1} - \nu d - \nu(\alpha_1 - \alpha_2 - \cdots - \alpha_t) = \max L(a^*) - \nu d \in L(a^*).$$

Similarly,

$$y_\nu = \left(\frac{a^*}{n_t} + \nu(d + \beta_t) \right) \mathbf{n}_t + \nu\beta_1 \mathbf{n}_1 + \cdots + \nu\beta_{t-1} \mathbf{n}_{t-1}$$

is a factorization of a^* of length

$$|y_\nu| = \frac{a^*}{n_t} + \nu d + \nu(\beta_1 + \cdots + \beta_{t-1} + \beta_t) = \min L(a^*) + \nu d \in L(a^*).$$

This reveals that $L(a^*)$ starts and ends with arithmetical progressions having difference d and $(\gamma + 1)$ elements. Thus it follows that L' and L'' are (possibly empty) arithmetical progressions with difference d , and thus $L(a^*)$ is an arithmetical progression with difference d . □

THEOREM 6.6. *Let S be a numerical monoid with $\mathcal{A}(S) = \{n_1, \dots, n_t\}$ where $t \in \mathbb{N}$, $1 < n_1 < \cdots < n_t$, and $d = \text{gcd}(n_2 - n_1, \dots, n_t - n_{t-1})$. Suppose that the Diophantine equations*

$$(n_2 - n_1)x_2 + \cdots + (n_t - n_1)x_t = dn_1$$

and

$$(n_t - n_1)y_1 + \cdots + (n_t - n_{t-1})y_{t-1} = dn_t$$

have solutions in the non-negative integers. Then there exists a $k^* \in \mathbb{N}$ such that $\mathcal{U}_k(S)$ is an arithmetical progression with difference d for all $k \geq k^*$, and

$$\lim_{k \rightarrow \infty} \frac{|\mathcal{U}_k(S)|}{k} = \frac{1}{d} \left(\frac{n_t}{n_1} - \frac{n_1}{n_t} \right).$$

Proof. By Lemma 6.5, all assumptions in [18, Theorem 3.1] are satisfied, and hence this result implies the assertion. □

REMARKS 6.7. 1. If $\mathcal{A}(S)$ is an arithmetical progression, then all sets $\mathcal{U}_k(S)$ are arithmetical progressions (see [2, Theorem 2.7]). However, in general, we have $k^* > 2$. Indeed, $S = \langle 4, 5, 13, 14 \rangle$ satisfies the assumptions of Theorem 6.6, but since $\mathcal{U}_2(S) = \{2, 6, 7\}$ is not an arithmetical progression, it follows that $k^* > 2$.

2. Unions of sets of lengths in finitely generated monoids are almost arithmetical progressions (see [20, Theorems 3.5 and 4.2]). But even in a numerical monoid, there may exist infinitely many $k \in \mathbb{N}$, for which these unions are not arithmetical progressions, as the following example shows.

Let $S = \langle 4, 10, 21 \rangle$ and $k \in \mathbb{N}$. Then $d = \gcd(6, 11) = 1$. We assert that $\mathcal{U}_k(S)$ is not an arithmetical progression with difference 1. We set $S_k = \{a \in S \mid k \in \mathcal{L}(a)\}$ and observe that $S_k = \{a\mathbf{4} + b\mathbf{10} + c\mathbf{21} \mid a, b, c \in \mathbb{N}_0 \text{ with } a + b + c = k\}$, $\min S_k = 4k$ and $\max S_k = 21k$. In particular, we see that $S_k = \{4k, \dots, 21k - 28, 21k - 22, 21k - 17, 21k - 11, 21k\}$, where the elements are written down in increasing order. The element $21k$ has a unique factorization of maximal length, namely

$$\begin{cases} 21t\mathbf{4} & \text{if } k = 4t, \\ 21t\mathbf{4} + \mathbf{21} & \text{if } k = 4t + 1, \\ (21t + 8)\mathbf{4} + \mathbf{10} & \text{if } k = 4t + 2, \\ (21t + 8)\mathbf{4} + \mathbf{10} + \mathbf{21} & \text{if } k = 4t + 3. \end{cases}$$

Setting $l = \max \mathcal{L}(21k)$ we assert that there is no $s \in S_k$ with $l - 1 \in \mathcal{L}(s)$. If this holds, then $\mathcal{U}_k(S)$ is not an arithmetical progression with difference 1. To verify our assertion we distinguish four cases.

- If $k = 4t$, then $l = 21t$. An element with a factorization of length $21t - 1$ is greater than or equal to $(21t - 1)4 = (21t)4 - 4 > 21k - 11$, and thus it does not belong to S_k .
- If $k = 4t + 1$, then $l = 21t + 1$. Elements having a factorization of length $21t$ are $(21t)4$, $(21t - 1)4 + 10 = (21t)4 + 6$, $(21t - 2)4 + 2 \cdot 10 = 21t + 12$, $(21t - 1)4 + 21 = (21t)4 + 17$, $(21t - 2)4 + 10 + 21 = (21t)4 + 23, \dots$. In this setting the four largest elements of S_k are $(21t)4 + 21$, $(21t)4 + 10$, $(21t)4 + 4$ and $(21t)4 - 1$. Hence also in this case, there is no element in S_k having a factorization of length $l - 1$.
- If $k = 4t + 2$, then $l = 21t + 9$. The set of elements having a factorization of length $21t + 8$ is $\{(21t + 8)4 = (21t)4 + 32, (21t + 7)4 + 10 = (21t)4 + 39, \dots, (21t + 8)21\}$, and the two largest elements of S_k are $(21t)4 + 42$ and $(21t)4 + 31$. Again we see that no element in S_k can have a factorization of length $l - 1$.
- If $k = 4t + 3$, then $l = 21t + 10$. Arguing as above one easily checks that no elements in S_k have factorizations of length $l - 1$.

In view of Theorem 6.6 and the Remarks 6.7, we end this paper with the formulation of the following problem.

OPEN PROBLEM. Characterize the numerical monoids S for which there exists a $k^* \in \mathbb{N}$ such that the unions of sets of lengths $\mathcal{U}_k(S)$ are arithmetical progressions for all $k \geq k^*$.

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