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CONDENSER ENERGY UNDER HOLOMORPHIC MOTIONS

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Abstract. We prove that the condenser equilibrium energy is a superharmonic function when one of the plates of a condenser moves under a holomorphic motion and we characterize the cases where harmonicity occurs by showing that this happens if and only if the equilibrium measure on the moved plate is invariant under the holomorphic motion and the equilibrium measure on the fixed plate is unaffected. Also, in many cases, we show that harmonicity of the above function occurs only when the holomorphic motion is related with the level sets of the equilibrium potential of the condenser. In the case where the holomorphic motion is a dilation, we prove that harmonicity occurs if and only if the condenser is essentially an annulus with center at the origin.

1. Introduction

A *condenser* in $\hat{\mathbb{C}}$ is a set $D \setminus K$, where D is a proper subdomain of the extended complex plane and K a compact subset of D such that $D \setminus K$ is connected. The compact sets $\hat{\mathbb{C}} \setminus D$ and K are called the *plates* of the condenser. We denote by $\mathcal{S}(D \setminus K)$ the family of all signed Borel measures $\sigma = \sigma_D - \sigma_K$, where σ_D, σ_K are unit Borel measures on the plates $\hat{\mathbb{C}} \setminus D$ and K, respectively. The *equilibrium energy* of the condenser is the extended positive real number

$$\mathrm{md}(D \setminus K) = \inf_{\sigma \in \mathcal{S}(D \setminus K)} \int \int \log \frac{1}{|z-w|} \, d\sigma(z) \, d\sigma(w)$$

and its capacity is

$$\mathrm{Cap}(D \setminus K) = \frac{2\pi}{\mathrm{md}(D \setminus K)}.$$

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If $\mathrm{md}(D \setminus K) < +\infty$, there exists a unique extremal signed measure $\tau = \tau_D - \tau_K \in \mathcal{S}(D \setminus K)$ such that $\mathrm{md}(D \setminus K)$ is equal to the energy of τ, which is called the *equilibrium measure* of the condenser. The function

$$U_\tau(z) = \int \log \frac{1}{|z-w|}\, d\tau(w)$$

is the *equilibrium potential* of the condenser. See [4].

A classical subject in the theory of condensers is the study of the behavior of the characteristics of a condenser (capacity, potential and equilibrium measure), when its plates are changing via a geometric transformation. We mention here some recent results. V. N. Dubinin proved monotonicity properties of several types of condenser capacities when various types of transformations are applied to its plates such as polarization, Gonchar's standardization, Steiner's and spherical symmetrization; see [8] and references therein. R. Kühnau [14] derived an asymptotic formula for the capacity of a condenser whose plates are arbitrary parallel curves when they approach each other. D. Betsakos [5], [6] proved monotonicity properties of the equilibrium measure and the capacity of some special condensers. V. N. Dubinin [9] derived an asymptotic formula for the capacity of a generalized condenser as some of its plates contract to points. V. N. Dubinin and D. Karp [10], among other related results, proved monotonicity properties of capacity of several plane condensers under translation and rotation of its plates.

The starting point of our work is a result appearing in [16]. In that paper, R. Laugesen considered the case where one of the plates of the condenser is the exterior of an open disk with center at the origin having radius r. Using the explicit formula of the Green function for a disk, he showed that the equilibrium energy is a concave function of $\log \frac{1}{r}$ and is strongly concave unless the other plate is essentially contained in a disk with center at the origin and contains the boundary of that disk. Also, in the same paper, he mentioned (and indicated a possible proof) that if $D \setminus K$ is a condenser and $\lambda D = \{\lambda z : z \in D\}$ is the dilation of D by λ, then the equilibrium energy of the condenser $\lambda D \setminus K$ is a C^1 superharmonic function of λ.

In the present paper, we examine the behavior of equilibrium energy when one of the plates of a condenser moves under a holomorphic motion $f_\lambda(z)$ (that is a function holomorphic in λ and injective in z). Our main result (Theorem 1) is a superharmonicity property of the equilibrium energy under that motion and a characterization of the cases where harmonicity occurs. We prove that the function $\lambda \mapsto \mathrm{md}(f_\lambda(D) \setminus K)$ is superharmonic. Also, we show that $\lambda \mapsto \mathrm{md}(f_\lambda(D) \setminus K)$ is harmonic on an open disk B if and only if the equilibrium measures on the plates $\hat{\mathbb{C}} \setminus f_\lambda(D)$ are invariant under the holomorphic motion and the equilibrium measures on the plate K are the same for all $\lambda \in B$.

In many cases, for example when K has non empty interior, we prove that if $\lambda \mapsto \operatorname{md}(f_\lambda(D) \setminus K)$ is harmonic on an open disk then the holomorphic motion is naturally related with the level sets of the equilibrium potential of the condenser on that disk. That follows from the fact that the equilibrium measures on the plate K of the condensers $f_\lambda(D) \setminus K$ are the same, for all λ on that disk. Another superharmonicity property of holomorphic motions has been proved by T. J. Ransford [18, p. 200].

When $f_\lambda(z)$ is holomorphic on both variables, we prove that condenser energy is a real analytic function of λ. This is the content of Theorem 2. C. J. Earle and S. Mitra [7] proved a similar analyticity property of arbitrary holomorphic motions but only when $D \setminus K$ is doubly connected. Since dilation is a holomorphic motion, Laugesen's claim becomes a special case of the above results. Also, in Theorem 3, we specify for which condensers we get harmonicity: the function $\lambda \mapsto \operatorname{md}(\lambda D \setminus K)$ is harmonic if and only if $D \setminus K$ is essentially an annulus with center at the origin, where "essentially" means except a set of zero logarithmic capacity.

In the following section, we introduce the concepts of the theory of condensers and holomorphic motions that are needed for our results. Theorem 1 is proved in Section 3, Theorem 2 is proved in Section 4 and Theorem 3 is proved in Section 5.

2. Background material

2.1. Condenser capacity. We denote by $C_l(\cdot)$ the logarithmic capacity (see e.g. [1, p. 151] or [18, p. 127]). If two planar sets A, B differ only on a set of zero logarithmic capacity (namely, $C_l(A \setminus B) = C_l(B \setminus A) = 0$), then we say that A, B are *nearly everywhere equal* and write $A \overset{\text{n.e.}}{=} B$. Nearly everywhere equal sets have the same potential theoretic behavior.

The energy of $\sigma \in \mathcal{S}(D \setminus K)$ is defined by

$$I(\sigma) = \int\int \log \frac{1}{|z-w|}\, d\sigma(z)\, d\sigma(w),$$

whenever

$$\int\int \left| \log \frac{1}{|z-w|} \right| d|\sigma|(z)\, d|\sigma|(w) < +\infty, \tag{1}$$

and its potential by

$$U_\sigma(z) = \int \log \frac{1}{|z-w|}\, d\sigma(w).$$

Then $I(\sigma) > 0$, for all $\sigma \in \mathcal{S}(D \setminus K)$ such that (1) holds (see [15, p. 80]).

Let $D \setminus K$ be a condenser with finite equilibrium energy and equilibrium measure $\tau = \tau_D - \tau_K$. By the *condenser theorem* (see [4, p. 321]), there exist finite constants $V_D \geq 0$ and $V_K \leq 0$ such that

(i) $V_K \le U_\tau(z) \le V_D$, for all $z \in \hat{\mathbb{C}}$,
(ii) $U_\tau = V_D$ in $\hat{\mathbb{C}} \setminus D$, except on a subset of zero logarithmic capacity,
(iii) $U_\tau = V_K$ in K, except on a subset of zero logarithmic capacity,
(iv) $\operatorname{supp}(\tau_D) \subset \partial D$ and $\operatorname{supp}(\tau_K) \subset \partial K$,
(v) $\operatorname{md}(D \setminus K) = V_D - V_K$.

Also the equilibrium energy is invariant under Möbius transformations (see [4, p. 318]).

The capacity of a condenser $D \setminus K$ can be expressed by a Dirichlet integral: Let

$$V(z) = \frac{V_D - U_\tau(z)}{\operatorname{md}(D \setminus K)}, \quad z \in D \setminus K.$$

Then V is the solution of the Generalized Dirichlet Problem on $D \setminus K$ with boundary values 0 on ∂D and 1 on ∂K. Moreover,

$$\int_{D\setminus K} |\nabla V|^2 = \frac{1}{\operatorname{md}(D \setminus K)^2} \int_{D\setminus K} |\nabla U_\tau|^2 = \frac{2\pi}{\operatorname{md}(D \setminus K)} = \operatorname{Cap}(D \setminus K),$$

where we used the formula ([15, p. 97])

$$\operatorname{md}(D \setminus K) = I(\tau) = \frac{1}{2\pi} \int_{D\setminus K} |\nabla U_\tau|^2.$$

In the following theorem, taken from [17], we describe the relation of two domains D_1, D_2 relative to which the equilibrium measures on the plate K of the condensers $D_1 \setminus K$ and $D_2 \setminus K$ are the same. We shall need this result when we examine harmonicity of condenser energy under a holomorphic motion.

THEOREM 2.1 ([17, Theorem 1 and Remark 2.3]). *Let K be a compact subset of $\hat{\mathbb{C}}$ that has a connected component that is neither a singleton nor a piecewise analytic arc. Let $D_1 \setminus K$ and $D_2 \setminus K$ be two condensers with finite equilibrium energy and denote by $\tau_1 = \tau_K^1 - \tau_{D_1}$ and $\tau_2 = \tau_K^2 - \tau_{D_2}$ their equilibrium measures, respectively. Also, denote by V_{D_2} the constant value of the equilibrium potential U_{τ_2} on the plate $\hat{\mathbb{C}} \setminus D_2$. Then:*

(i) $\tau_K^1 = \tau_K^2$ *and* $\operatorname{md}(D_1 \setminus K) = \operatorname{md}(D_2 \setminus K)$ *if and only if* $D_1 \overset{n.e.}{=} D_2$;
(ii) *If* $\operatorname{md}(D_1 \setminus K) < \operatorname{md}(D_2 \setminus K)$ *and the set*

$$\tilde{D_2} = \left\{ z \in D_2 : U_{\tau_2}(z) < V_{D_2} - \big(\operatorname{md}(D_2 \setminus K) - \operatorname{md}(D_1 \setminus K)\big) \right\}$$

contains K, we have

$$\tau_K^1 = \tau_K^2 \quad \text{if and only if} \quad D_1 \overset{n.e.}{=} \tilde{D}_2.$$

2.2. Holomorphic motions. Let A be a subset of $\hat{\mathbb{C}}$ and let N be a domain in $\mathbb{C}$.

DEFINITION 2.2. A holomorphic motion of A, parameterized by N with basepoint $\lambda_b \in N$ is a map

$$f : N \times A \mapsto \hat{\mathbb{C}}$$

such that

(i) for any fixed $z \in A$, the map $\lambda \mapsto f(\lambda, z)$ is holomorphic in N,
(ii) for any fixed $\lambda \in N$, the map $z \mapsto f(\lambda, z) = f_\lambda(z)$ is an injection,
(iii) the mapping $f(\lambda_b, \cdot)$ is the identity on A.

There is no assumption regarding the continuity of f as a function of z in A or as a function of (λ, z) in $N \times A$. That such continuity occurs is a consequence of the following fundamental theorem on the extendability of holomorphic motions (see [2], [19]). We denote by $\mathbb{D}$ the unit disk.

THEOREM 2.3 ([2, p. 34]). *If $f : \mathbb{D} \times A \mapsto \hat{\mathbb{C}}$ is a holomorphic motion with basepoint* 0, *then f has an extension to $F : \mathbb{D} \times \hat{\mathbb{C}} \mapsto \hat{\mathbb{C}}$ such that*

(i) *F is a holomorphic motion of $\hat{\mathbb{C}}$,*
(ii) *each $F_\lambda(\cdot) : \hat{\mathbb{C}} \mapsto \hat{\mathbb{C}}$ is a quasiconformal self homeomorphism,*
(iii) *F is jointly continuous in (λ, z).*

It is clear that the conclusions of the above theorem are valid if we replace $\mathbb{D}$ by an arbitrary disk. This can be done using an auxiliary Möbius transformation.

3. Condenser energy under holomorphic motions

We consider a condenser $D \setminus K$ with finite equilibrium energy. Also we consider a holomorphic motion

$$f : N \times \hat{\mathbb{C}} \mapsto \hat{\mathbb{C}}.$$

We assume that for all $\lambda \in N$, we have $K \subset f_\lambda(D)$.

Let V be a subset of N and $\sigma_{\lambda_0} = \sigma_D^{\lambda_0} - \sigma_K \in \mathcal{S}(f_{\lambda_0}(D) \setminus K)$, $\lambda_0 \in V$. For every $\lambda \in V$ we consider the measure

$$\sigma_D^{\lambda}(E) = \sigma_D^{\lambda_0}\big(f_{\lambda_0}\big(f_\lambda^{-1}(E)\big)\big) \quad \text{for every } E \subset \hat{\mathbb{C}} \setminus f_\lambda(D) \text{ Borel measurable,}$$

and the signed measure $\sigma_\lambda = \sigma_D^\lambda - \sigma_K \in \mathcal{S}(f_\lambda(D) \setminus K)$. That is, we transfer the measure σ_{λ_0} to the condensers $f_\lambda(D) \setminus K$ via f_λ without changing it on the plate K. The family of signed measures

$$\big\{\sigma_\lambda = \sigma_D^\lambda - \sigma_K \in \mathcal{S}\big(f_\lambda(D) \setminus K\big) : \lambda \in V\big\}$$

will be called the *transmission* of σ_{λ_0} to the condensers $f_\lambda(D) \setminus K$ under the holomorphic motion f_λ. The next lemma states that a holomorphic motion f_λ transfers a signed measure in such a way that the energy is a harmonic function of λ.

LEMMA 3.1. *Let D, K, f_λ and N be as above. Consider an open subset V of N, a point $\lambda_0 \in V$, a signed measure $\sigma_{\lambda_0} = \sigma_D^{\lambda_0} - \sigma_K \in \mathcal{S}(f_{\lambda_0}(D) \setminus K)$ with compact support and finite energy and the transmission $\{\sigma_\lambda : \lambda \in V\}$ of σ_{λ_0}. Then the function*

$$V \ni \lambda \mapsto I(\sigma_\lambda) \in \mathbb{R}$$

is harmonic on V.

Proof. The functions

$$\lambda \mapsto \log \frac{1}{|f_\lambda(f_{\lambda_0}^{-1}(x)) - f_\lambda(f_{\lambda_0}^{-1}(y))|}, \quad x, y \in \operatorname{supp}\big(\sigma_D^{\lambda_0}\big) \tag{2}$$

and

$$\lambda \mapsto \log \frac{1}{|f_\lambda(f_{\lambda_0}^{-1}(x)) - y|}, \quad x \in \operatorname{supp}\big(\sigma_D^{\lambda_0}\big), y \in \operatorname{supp}(\sigma_K) \tag{3}$$

are harmonic in V for $x \neq y$ as real parts of holomorphic functions.

Since σ_{λ_0} has finite energy, the integral

$$\int\int \log \frac{1}{|x-y|}\, d\sigma_D^{\lambda_0}(x)\, d\sigma_D^{\lambda_0}(y)$$

is finite. Therefore, since $\log \frac{1}{|x-y|}$ is ∞ on the diagonal

$$diag = \big\{(x,x) : x \in \operatorname{supp}\big(\sigma_D^{\lambda_0}\big)\big\},$$

$\sigma_D^{\lambda_0} \times \sigma_D^{\lambda_0}(diag) = 0$.

Let $\lambda \in V$. Then, because of the harmonicity of the functions in (2), (3) and Fubini's theorem, for all $r > 0$ such that $\overline{B(\lambda, r)} \subset V$,

$$\begin{aligned}
&\frac{1}{\pi r^2} \int_{B(\lambda,r)} I(\sigma_\zeta)\, d\zeta \\
&\quad = \frac{1}{\pi r^2} \int_{B(\lambda,r)} \Big[I(\sigma_K) \\
&\qquad + \int\int_{(\hat{\mathbb{C}} \setminus f_{\lambda_0}(D))^2 \setminus diag} \log \frac{1}{|f_\zeta(f_{\lambda_0}^{-1}(x)) - f_\zeta(f_{\lambda_0}^{-1}(y))|}\, d\sigma_D^{\lambda_0}(x)\, d\sigma_D^{\lambda_0}(y) \\
&\qquad - 2 \int\int_{(\hat{\mathbb{C}} \setminus f_{\lambda_0}(D)) \times K} \log \frac{1}{|f_\zeta(f_{\lambda_0}^{-1}(x)) - y|}\, d\sigma_D^{\lambda_0}(x)\, d\sigma_K(y) \Big]\, d\zeta \\
&\quad = I(\sigma_K) + \int\int \log \frac{1}{|f_\lambda(f_{\lambda_0}^{-1}(x)) - f_\lambda(f_{\lambda_0}^{-1}(y))|}\, d\sigma_D^{\lambda_0}(x)\, d\sigma_D^{\lambda_0}(y) \\
&\qquad - 2 \int\int \log \frac{1}{|f_\lambda(f_{\lambda_0}^{-1}(x)) - y|}\, d\sigma_D^{\lambda_0}(x)\, d\sigma_K(y) \\
&\quad = I(\sigma_\lambda).
\end{aligned}$$

Therefore, the function $\lambda \mapsto I(\sigma_\lambda)$ is locally integrable on V and satisfies the averaging property (with respect to area measure) on all disks $B(\lambda, r)$ such that $\overline{B(\lambda, r)} \subset V$. By [3, Theorem 1.25, p. 18], the function $\lambda \mapsto I(\sigma_\lambda)$ is continuous and therefore harmonic on V. $\square$

We proceed to prove our main result.

THEOREM 1. *Let $D\setminus K$ be a condenser with finite equilibrium energy and f_λ be a holomorphic motion of $\overline{D}$ such that $K\subset f_\lambda(D)$ for all $\lambda\in N$. Then the function*

$$\lambda\mapsto \mathrm{md}\big(f_\lambda(D)\setminus K\big)$$

is superharmonic on N. The above function is harmonic on a disk $B(\lambda_0,r_0)\subset N$ if and only if the equilibrium measure of the condenser $f_\lambda(D)\setminus K$ is the transmitted measure of the equilibrium measure of the condenser $f_{\lambda_0}(D)\setminus K$ under the holomorphic motion f_λ, for every $\lambda\in B(\lambda_0,r_0)$.

Proof. We denote by $\tau_\lambda=\tau_D^\lambda-\tau_K^\lambda$ the equilibrium measure of the condenser $f_\lambda(D)\setminus K$.

Let $\lambda_m\to\lambda$ and choose a subsequence λ_{m_k} such that

$$\lim_{k\to\infty} I(\tau_{\lambda_{m_k}})=\liminf_{m\to\infty} I(\tau_{\lambda_m}).$$

We can assume that $\infty\notin\partial f_\lambda(D)$ because of the invariance of equilibrium energy under Möbius transformations. So we can assume that the supports of the measures $\tau_{\lambda_{m_k}}$ lie in a compact subset of $\mathbb{C}$. By the Riesz Representation theorem ([11, pp. 212, 223]) and Alaoglu's Compactness theorem ([11, p. 169]), there exist a subsequence $\tau_{\lambda_{m_{k_l}}}$ and a Borel measure ν such that

$$\tau_{\lambda_{m_{k_l}}}\xrightarrow{w^*}\nu.$$

By the lower-semicontinuity of energy in measure (see [15, pp. 78–79])

$$I(\nu)\le\liminf_{l\to\infty} I(\tau_{\lambda_{m_{k_l}}})=\liminf_{m\to\infty} I(\tau_{\lambda_m}).$$

Also by the fact that $\tau_{\lambda_{m_{k_l}}}\in\mathcal{S}(f_{\lambda_{m_{k_l}}}(D)\setminus K)$ and the definition of the equilibrium measure, we have $\nu\in\mathcal{S}(f_\lambda(D)\setminus K)$ and $I(\tau_\lambda)\le I(\nu)$. So

$$\begin{aligned}\mathrm{md}\big(f_\lambda(D)\setminus K\big)&=I(\tau_\lambda)\\&\le I(\nu)\\&\le\liminf_{m\to\infty} I(\tau_{\lambda_m})\\&=\liminf_{m\to\infty}\mathrm{md}\big(f_{\lambda_m}(D)\setminus K\big),\end{aligned}$$

and thus the function $\lambda\mapsto\mathrm{md}(f_\lambda(D)\setminus K)$ is lower semicontinuous on N.

Let $\lambda_0\in N$ and $B(\lambda_0,r_0)\subset N$. We denote by s the arc length on $\partial B(\lambda_0,r_0)$. We denote by $\{\sigma_\lambda=\sigma_D^\lambda-\sigma_K\}$ the transmission of τ_{λ_0} to the condensers $f_\lambda(D)\setminus K$ under the holomorphic motion f_λ. Then, by the definition of the equilibrium measure and by Lemma 3.1, for all $0<r<r_0$,

$$\begin{aligned}\frac{1}{2\pi r}\int_{\partial B(\lambda_0,r)}\mathrm{md}\big(f_\zeta(D)\setminus K\big)\,ds(\zeta)&=\frac{1}{2\pi r}\int_{\partial B(\lambda_0,r)} I(\tau_\zeta)\,ds(\zeta)\\&\le\frac{1}{2\pi r}\int_{\partial B(\lambda_0,r)} I(\sigma_\zeta)\,ds(\zeta)\end{aligned}$$

$$
\begin{aligned}
&= I(\sigma_{\lambda_0}) \\
&= I(\tau_{\lambda_0}) \\
&= \mathrm{md}\big(f_{\lambda_0}(D) \setminus K\big).
\end{aligned}
$$

So, $\lambda \mapsto \mathrm{md}(f_\lambda(D) \setminus K)$ is superharmonic on N.

Suppose now that $\lambda \mapsto \mathrm{md}(f_\lambda(D) \setminus K)$ is harmonic on $B(\lambda_0, r_0)$. Then, as above,

$$
\begin{aligned}
\mathrm{md}\big(f_{\lambda_0}(D) \setminus K\big) &= \frac{1}{2\pi r} \int_{\partial B(\lambda_0, r)} \mathrm{md}\big(f_\zeta(D) \setminus K\big)\, ds(\zeta) \\
&= \frac{1}{2\pi r} \int_{\partial B(\lambda_0, r)} I(\tau_\zeta)\, ds(\zeta) \\
&\leq \frac{1}{2\pi r} \int_{\partial B(\lambda_0, r)} I(\sigma_\zeta)\, ds(\zeta) \\
&= I(\sigma_{\lambda_0}) \\
&= I(\tau_{\lambda_0}) \\
&= \mathrm{md}\big(f_{\lambda_0}(D) \setminus K\big).
\end{aligned}
$$

Therefore,

$$
\frac{1}{2\pi r} \int_{\partial B(\lambda_0, r)} I(\tau_\zeta)\, ds(\zeta) = \frac{1}{2\pi r} \int_{\partial B(\lambda_0, r)} I(\sigma_\zeta)\, ds(\zeta)
$$

for all $0 < r < r_0$ and integrating with respect to r from 0 to r_0 we obtain

$$
\frac{1}{\pi r^2} \int_{B(\lambda_0, r_0)} \big[I(\tau_\lambda) - I(\sigma_\lambda)\big]\, d\lambda = 0.
$$

The functions $\lambda \mapsto I(\tau_\lambda) = \mathrm{md}(f_\lambda(D) \setminus K)$ and $\lambda \mapsto I(\sigma_\lambda)$ are harmonic on $B(\lambda_0, r_0)$, therefore the difference $I(\tau_\lambda) - I(\sigma_\lambda)$ is a continuous and non-positive function of λ. So we must have $I(\tau_\lambda) = I(\sigma_\lambda)$ for all $\lambda \in B(\lambda_0, r_0)$ and by the uniqueness of the condenser equilibrium measure, we must have $\tau_\lambda = \sigma_\lambda$; that is, the equilibrium measure of the condenser $f_\lambda(D) \setminus K$ is the transmitted measure of the equilibrium measure of the condenser $f_{\lambda_0}(D) \setminus K$ under the holomorphic motion f_λ.

The converse follows from Lemma 3.1. □

As a consequence, we obtain subharmonicity of condenser capacity under holomorphic motions.

Corollary 3.2. *Let D, K, f_λ and N be as above. Then the function*

$$
\lambda \mapsto \mathrm{Cap}\big(f_\lambda(D) \setminus K\big)
$$

is subharmonic on N.

Proof. From Theorem 1, we have that $\lambda \mapsto \operatorname{md}(f_\lambda(D) \setminus K)$ is superharmonic on N. So $\lambda \mapsto -\operatorname{md}(f_\lambda(D) \setminus K)$ is subharmonic on N. Since $\phi(x) = \frac{-2\pi}{x}$ is convex and increasing on $(-\infty, 0)$ and

$$\phi\big(-\operatorname{md}\big(f_\lambda(D) \setminus K\big)\big) = \frac{2\pi}{\operatorname{md}(f_\lambda(D) \setminus K)} = \operatorname{Cap}\big(f_\lambda(D) \setminus K\big),$$

the function $\lambda \mapsto \operatorname{Cap}(f_\lambda(D) \setminus K)$ is subharmonic on N (see [18, Theorem 2.6.3, p. 43] or [1, Theorem 3.4.3, p. 73]). $\square$

In the following corollary, we describe a property that must be satisfied by a holomorphic motion, if the condenser energy is a harmonic function under that motion, for a large class of condensers. In particular, we show that the holomorphic motion must be related to the level sets of the equilibrium potential of the condenser.

For any $\alpha \in (V_K, V_D)$, we denote by D_α the sets

$$D_\alpha = \big\{\xi \in D : U_\tau(\xi) < V_D - \alpha\big\},$$

where τ is the equilibrium measure of the condenser $D \setminus K$. Then each D_α is open because the equilibrium potential U_τ is upper semicontinuous on D and the boundaries of the sets D_α are the level sets of the condenser equilibrium potential on $D \setminus K$.

COROLLARY 3.3. *Let D, K, f_λ and N be as above and let λ_b be the base-point of f_λ. Also suppose that $\hat{\mathbb{C}} \setminus K$ is regular for the Dirichlet problem and K has a connected component that is neither a singleton neither a piecewise analytic arc. If $\lambda \mapsto \operatorname{md}(f_\lambda(D) \setminus K)$ is harmonic on a disk $B(\lambda_b, r) \subset N$, then every connected component of ∂D that is a nondegenerate continuum is a union of piecewise analytic arcs and the holomorphic motion f_λ is essentially a parametrization of the sets D_α, $\alpha \in (V_K, V_D)$. That is, for each $\lambda \in B(\lambda_b, r)$ with*

$$\operatorname{md}\big(f_\lambda(D) \setminus K\big) < \operatorname{md}(D \setminus K),$$

there exists $\alpha \in (V_K, V_D)$ such that $f_\lambda(D) \overset{n.e.}{=} D_\alpha$.

Proof. Let $\tau = \tau_D - \tau_K$ be the equilibrium measure of the condenser $D \setminus K$ and $\tau_\lambda = \tau_D^\lambda - \tau_K^\lambda$ be the equilibrium measure of the condenser $f_\lambda(D) \setminus K$, for all $\lambda \in B(\lambda_b, r)$. Suppose that $\lambda \mapsto \operatorname{md}(f_\lambda(D) \setminus K)$ is harmonic on $B(\lambda_b, r)$. From Theorem 1, $\tau_K^\lambda = \tau_K$, for all $\lambda \in B(\lambda_b, r)$. Choose $\lambda_0 \in B(\lambda_b, r)$ such that $\operatorname{md}(f_{\lambda_0}(D) \setminus K) > \operatorname{md}(D \setminus K)$. Since $\hat{\mathbb{C}} \setminus K$ is regular for the Dirichlet problem, the open set

$$\Omega = \big\{z \in f_{\lambda_0}(D) : U_{\tau_{\lambda_0}}(z) < V_{D_{\lambda_0}} - \big(\operatorname{md}\big(f_{\lambda_0}(D) \setminus K\big) - \operatorname{md}(D \setminus K)\big)\big\}$$

contains K. Then, from Theorem 2.1, $D \overset{\text{n.e.}}{=} \Omega$. Since $\partial\Omega \subset f_{\lambda_0}(D) \setminus \operatorname{supp}(\tau_K)$ and $U_{\tau_{\lambda_0}}$ is harmonic therein, $\partial\Omega$ is a union of piecewise analytic arcs. Therefore every connected component of ∂D that is a nondegenerate continuum is a union of piecewise analytic arcs. Also, for each $\lambda \in B(\lambda_b, r)$ with $\mathrm{md}(f_\lambda(D) \setminus K) < \mathrm{md}(D \setminus K)$ from Theorem 2.1,

$$f_\lambda(D) \overset{\text{n.e.}}{=} D_\alpha,$$

where $\alpha = \mathrm{md}(D \setminus K) - \mathrm{md}(f_\lambda(D) \setminus K)$. □

REMARK 3.4. Suppose that K is the union of two disjoint continua. Then, for sufficiently small α, the open sets D_α will not be connected; in particular they will constitute of two connected components each one containing a connected component of K. Also, since D is a domain and f_λ is a homeomorphism, it is not possible to have $f_\lambda(D) = D_\alpha$ for sufficiently small α. So, in order to have harmonicity of condenser energy on N, the plate K must be nearly everywhere equal to a nondegenerate continuum.

4. Analyticity of condenser energy

Again we assume that f_λ is a holomorphic motion parameterized by the domain N.

We proceed to prove that the functions $\lambda \mapsto \mathrm{md}(f_\lambda(D) \setminus K)$ and $\lambda \mapsto \mathrm{Cap}(f_\lambda(D) \setminus K)$ are real analytic when $f_\lambda(z)$ is holomorphic on both variables.

THEOREM 2. *Let $D \setminus K$ be a condenser with finite equilibrium energy and f_λ be a holomorphic motion of an open neighborhood of $\overline{D}$ such that $K \subset f_\lambda(D)$ for every $\lambda \in N$. We assume that the function $z \mapsto f_\lambda(z)$ is holomorphic, for every $\lambda \in N$. Then the functions*

$$\lambda \mapsto \mathrm{Cap}\big(f_\lambda(D) \setminus K\big)$$

and

$$\lambda \mapsto \mathrm{md}\big(f_\lambda(D) \setminus K\big)$$

are real analytic on N.

Proof. Since $z \mapsto f_\lambda(z)$ is an injection, it is a conformal mapping.

Let V be the solution of the Generalized Dirichlet Problem on $D \setminus K$ with boundary values 0 on ∂D and 1 on ∂K. Then the function

$$z \mapsto \phi_\lambda(z) = V \circ f_\lambda^{-1}(z)$$

is the solution of the Generalized Dirichlet Problem on $f_\lambda(D) \setminus K$ with boundary values 0 on $\partial f_\lambda(D)$ and 1 on ∂K, for all $\lambda \in N$. Then

$$\mathrm{Cap}\big(f_\lambda(D) \setminus K\big) = \int_{f_\lambda(D) \setminus K} \big|\nabla \phi_\lambda(z)\big|^2 \, dz$$

for all $\lambda \in N$. Since V is harmonic on $D \setminus K$ and since $f_\lambda(z) = f(\lambda, z)$ is holomorphic on both variables λ and z from the hypothesis, we obtain that $\phi_\lambda(z) = \phi(\lambda, z)$ is harmonic on both variables λ and z. By [3, Theorem 1.28,

p. 21], $\phi_\lambda(z)=\phi(\lambda,z)$ is real analytic on both variables λ and z. Therefore the functions

$$\lambda \mapsto \left|\nabla\phi_\lambda(z)\right|^2 = \left(\frac{\partial\phi_\lambda}{\partial x}(z)\right)^2 + \left(\frac{\partial\phi_\lambda}{\partial y}(z)\right)^2, \quad z=x+iy$$

and

$$\lambda \mapsto \int_{f_\lambda(D)\setminus K} \left|\nabla\phi_\lambda(z)\right|^2 dz$$

are real analytic on N (see e.g. [13, Propositions 2.2.2, 2.2.3 and 2.2.8, pp. 29–33]). So

$$\lambda \mapsto \operatorname{Cap}\bigl(f_\lambda(D)\setminus K\bigr)$$

is real analytic on N. Also, since

$$\operatorname{md}\bigl(f_\lambda(D)\setminus K\bigr) = \frac{2\pi}{\operatorname{Cap}(f_\lambda(D)\setminus K)}$$

and $\operatorname{Cap}(f_\lambda(D)\setminus K)>0$, the function

$$\lambda \mapsto \operatorname{md}\bigl(f_\lambda(D)\setminus K\bigr)$$

is real analytic on N (see e.g. [13, Proposition 2.2.2, p. 29]). □

5. Harmonicity of condenser energy under dilation

We consider the holomorphic motion $f:(\mathbb{C}\setminus\{0\})\times\hat{\mathbb{C}}\mapsto\hat{\mathbb{C}}$,

$$f_\lambda(z)=\lambda z,$$

with basepoint 1, which will be called *dilation*. The dilation

$$\{\lambda z : z\in E\}$$

of a set E will be denoted by λE.

Let $D\setminus K$ be a condenser. We consider the set

$$N=\{\lambda\in\mathbb{C} : K\subset\lambda D\}.$$

THEOREM 5.1. *Let $D\setminus K$ be a condenser with finite equilibrium energy. Then $\lambda\mapsto\operatorname{md}(\lambda D\setminus K)$ is a real analytic superharmonic function on N and $\lambda\mapsto\operatorname{Cap}(\lambda D\setminus K)$ is a real analytic subharmonic function on N.*

Proof. Dilation is a holomorphic motion which is conformal on both variables. Therefore, by Theorem 2, the functions

$$\lambda\mapsto\operatorname{md}(\lambda D\setminus K) \quad\text{and}\quad \lambda\mapsto\operatorname{Cap}(\lambda D\setminus K)$$

are real analytic functions on N. Also, by Theorem 1 and Corollary 3.2, we obtain superharmonicity and subharmonicity of the functions $\lambda\mapsto\operatorname{md}(\lambda D\setminus K)$ and $\lambda\mapsto\operatorname{Cap}(\lambda D\setminus K)$, respectively. □

It is clear from Corollary 3.3 that, if we fix a holomorphic motion f_λ of the plane, it is not always possible to find a condenser $D\setminus K$ such that $\lambda\mapsto \operatorname{md}(f_\lambda(D)\setminus K)$ is harmonic. We shall examine this problem in the case where the holomorphic motion is dilation.

LEMMA 5.2. *Let D be a Greenian subdomain of $\hat{\mathbb{C}}$ such that $\infty\notin\partial D$ or $0\notin\partial D$. Suppose that*

$$\lambda D \overset{n.e.}{=} \kappa D$$

for all κ,λ which lie on a nondegenerate continuum γ. Then there exist positive numbers $r_1<r$ such that either

$$D \overset{n.e.}{=} \left\{z\in\hat{\mathbb{C}} : r_1<|z|<r\right\}$$

or

$$D \overset{n.e.}{=} \left\{z\in\hat{\mathbb{C}} : |z|<r\right\}$$

or

$$D \overset{n.e.}{=} \left\{z\in\hat{\mathbb{C}} : |z|>r\right\}.$$

Proof. Let $G(x,y)$ be the Green function of D. Then, if G_λ is the Green function of λD, $\lambda\in\gamma$,

$$G_\lambda(x,y)=G\left(\frac{x}{\lambda},\frac{y}{\lambda}\right)$$

([18, Theorem 4.4.4, p. 107]). Since $\lambda D \overset{\text{n.e.}}{=} \kappa D$, we must have $G_\lambda(x,y)=G_\kappa(x,y)$ for all $x,y\in\lambda D\cap\kappa D$ and $\lambda,\kappa\in\gamma$ ([1, Corollary 5.2.5, p. 128]). Let

$$R=\left\{\zeta\in\partial D : \lim_{D\ni y\to\zeta} G(x,y)=0, x\in D\right\}.$$

Then $C_l(\partial D\setminus R)=0$ (see [12, Theorem 8.30, p. 176] or [1, Theorem 5.7.4, p. 148]). Also, for each $\lambda\in\gamma$, let

$$R_\lambda=\left\{\zeta\in\partial\lambda D : \lim_{\lambda D\ni y\to\zeta} G_\lambda(x,y)=0, x\in\lambda D\right\}.$$

Then, since $G_\lambda(x,y)=G(\frac{x}{\lambda},\frac{y}{\lambda})$, we have $R_\lambda=\lambda R$ for all $\lambda\in\gamma$ and since $G_\lambda=G_\kappa$ we have $\lambda R=\kappa R$ for all $\lambda,\kappa\in\gamma$.

Suppose that $\infty\notin\partial D$. Set $M:=\sup\{|\zeta| : \zeta\in R\}<+\infty$. Then

$$|\lambda|M=\sup\left\{|\zeta| : \zeta\in\lambda R\right\}=\sup\left\{|\zeta| : \zeta\in\kappa R\right\}=|\kappa|M,$$

and hence $|\lambda|=|\kappa|$, for all $\lambda,\kappa\in\gamma$.

Suppose that $0\notin\partial D$. Set $m:=\inf\{|\zeta| : \zeta\in R\}>0$. Then

$$|\lambda|m=\inf\left\{|\zeta| : \zeta\in\lambda R\right\}=\inf\left\{|\zeta| : \zeta\in\kappa R\right\}=|\kappa|m,$$

and hence $|\lambda|=|\kappa|$, for all $\lambda,\kappa\in\gamma$. So, in any case, $|\lambda|=|\kappa|$ for all $\lambda,\kappa\in\gamma$.

If $\zeta_0\in R\setminus\{0,\infty\}$, then $\{\kappa\zeta_0 : \kappa\in\gamma\}\subset\lambda R$, for all $\lambda\in\gamma$ and $\{\kappa\zeta_0 : \kappa\in\gamma\}$ is an arc of a circle with center at the origin. So, for all $\lambda\in\gamma$,

$$\left\{\frac{\kappa}{\lambda}\zeta_0 : \kappa\in\gamma\right\}\subset R$$

and R contains an arc of a circle with center at the origin which has length greater than

$$\varepsilon := \left| \left\{ \frac{\kappa}{\lambda}\zeta_0 : \kappa, \lambda \in \gamma \right\} \right| > 0$$

and contains ζ_0 in its interior. Since the same is true for all the points of the above arc, the connected component of R that contains ζ_0 must be a circle with center at the origin. Since D is connected, R can have at most two connected components. If $Z = \partial D \setminus R$, then we have that $\partial D \setminus Z$ is one or two circles and $C_l(\partial D \setminus R) = 0$. The conclusion follows immediately. □

We proceed to describe the case where $\lambda \mapsto \mathrm{md}(\lambda D \setminus K)$ is a harmonic function.

THEOREM 3. *Let $D \setminus K$ be a condenser such that at least one connected component of K is a nondegenerate continuum and at least one connected component of ∂D is a nondegenerate continuum. Then the following are equivalent*:

(i) *The function $\lambda \mapsto \mathrm{md}(\lambda D \setminus K)$ is harmonic on an open subset of N.*

(ii) *There exist positive numbers $r < s$ such that either*

$$D \overset{n.e.}{=} \{z \in \hat{\mathbb{C}} : |z| < s\} \quad \textit{and} \quad K \overset{n.e.}{=} \{z \in \hat{\mathbb{C}} : |z| \le r\}$$

or

$$D \overset{n.e.}{=} \{z \in \hat{\mathbb{C}} : |z| > r\} \quad \textit{and} \quad K \overset{n.e.}{=} \{z \in \hat{\mathbb{C}} : |z| \ge s\}.$$

(iii) *The function $\lambda \mapsto \mathrm{md}(\lambda D \setminus K)$ is harmonic on N.*

Proof. (i) ⇒ (ii). Suppose that $\lambda \mapsto \mathrm{md}(\lambda D \setminus K)$ is harmonic on the disk $B(\lambda_0, r) \subset N$.

Let l be a connected component of K that is a nondegenerate continuum and let ϕ be a conformal mapping of $\hat{\mathbb{C}} \setminus l$ onto $\hat{\mathbb{C}} \setminus \overline{B(0,1)}$. We consider the condenser $\phi(D \setminus K)$, with plates $\phi(\hat{\mathbb{C}} \setminus D)$ and $\overline{B(0,1)} \cup \phi(K \setminus l)$. If V is the solution of the Generalized Dirichlet Problem on $D \setminus K$ with boundary values 0 on ∂D and 1 on ∂K, then $V \circ \phi^{-1}$ is the solution of the Generalized Dirichlet Problem on $\phi(D \setminus K)$ with boundary values 0 on $\phi(\partial D)$ and 1 on $\partial(\overline{B(0,1)} \cup \phi(K \setminus l))$. By the conformal invariance of the Dirichlet integral,

$$\begin{aligned} \mathrm{Cap}(D \setminus K) &= \int_{D \setminus K} |\nabla V(x)|^2 \, dx = \int_{\phi(D \setminus K)} \left|\nabla V\big(\phi^{-1}(x)\big)\right|^2 dx \\ &= \mathrm{Cap}\big(\phi(D \setminus K)\big). \end{aligned}$$

So

$$\mathrm{md}(D \setminus K) = \mathrm{md}\big(\phi(D \setminus K)\big).$$

In a similar way we can show that for all $\lambda \in N$,

$$\mathrm{md}(\lambda D \setminus K) = \mathrm{md}\big(\phi(\lambda D \setminus K)\big).$$

So $\lambda \mapsto \mathrm{md}(\phi(\lambda D \setminus K))$ is harmonic on the disk $B(\lambda_0, r) \subset N$. The mapping

$$B(\lambda_0, r) \times \phi(D \setminus K) \ni (\lambda, x) \mapsto \phi\big(\lambda \phi^{-1}(x)\big)$$

is a holomorphic motion which, by Theorem 2.3, can be extended to $B(\lambda_0, r) \times \hat{\mathbb{C}}$. The energy of the condenser $\phi(D \setminus K)$ is a harmonic function under the above holomorphic motion. So, by Theorem 1, the equilibrium measure of the condenser $\phi(\lambda D \setminus K)$ is the transmitted measure of the equilibrium measure of the condenser $\phi(\lambda_0 D \setminus K)$ under the above holomorphic motion, for every $\lambda \in B(\lambda_0, r)$. Since the interior of the set $\overline{B(0,1)} \cup \phi(K \setminus l)$ is non empty, by Theorem 2.1,

$$\phi(\lambda D \setminus l) \overset{\text{n.e.}}{=} \phi(\kappa D \setminus l)$$

for all λ, κ that lie in the level set

$$\big\{\zeta \in B(\lambda_0, r) : \mathrm{md}\big(\phi(\zeta D \setminus K)\big) = \mathrm{md}\big(\phi(\lambda_0 D \setminus K)\big)\big\},$$

of the harmonic function $\zeta \mapsto \mathrm{md}(\phi(\zeta D \setminus K))$. So $\lambda D \overset{\text{n.e.}}{=} \kappa D$, for all λ, κ that lie in the level set

$$\big\{\lambda \in B(\lambda_0, r) : \mathrm{md}(\lambda D \setminus K) = \mathrm{md}(\lambda_0 D \setminus K)\big\},$$

of the harmonic function $\lambda \mapsto \mathrm{md}(\lambda D \setminus K)$.

Suppose that $0 \notin \partial D$ or $\infty \notin \partial D$. By Lemma 5.2, there exist positive numbers $r_1 < r$ such that

$$D \overset{\text{n.e.}}{=} \big\{z \in \hat{\mathbb{C}} : r_1 < |z| < r\big\}$$

or

$$D \overset{\text{n.e.}}{=} \big\{z \in \hat{\mathbb{C}} : |z| < r\big\}$$

or

$$D \overset{\text{n.e.}}{=} \big\{z \in \hat{\mathbb{C}} : |z| > r\big\}.$$

Since $K \subset D$, in any of the above cases it is true that $0 \notin K$ or $\infty \notin K$. Because of the invariance of the condenser equilibrium energy under Möbius transformations, we have that

$$\mathrm{md}(\lambda D \setminus K) = \mathrm{md}\left(\frac{1}{\lambda}\lambda D \setminus \frac{1}{\lambda}K\right) = \mathrm{md}\left(D \setminus \frac{1}{\lambda}K\right).$$

So, the function $\lambda \mapsto \mathrm{md}(D \setminus \lambda K)$ is harmonic on $\{\frac{1}{\lambda} : \lambda \in B(\lambda_0, r)\}$. Since $\hat{\mathbb{C}} \setminus K$ is connected, as before, by Lemma 5.2 we have that there exist positive numbers $s_1 < s$ such that

$$\hat{\mathbb{C}} \setminus K \overset{\text{n.e.}}{=} \big\{z \in \hat{\mathbb{C}} : s_1 < |z| < s\big\} \tag{4}$$

or

$$\hat{\mathbb{C}} \setminus K \overset{\text{n.e.}}{=} \big\{z \in \hat{\mathbb{C}} : |z| < s\big\} \tag{5}$$

or

$$\hat{\mathbb{C}} \setminus K \overset{\text{n.e.}}{=} \big\{z \in \hat{\mathbb{C}} : |z| > s\big\}. \tag{6}$$

Since $D \setminus K$ is connected, (ii) follows immediately.

Suppose that $0, \infty \in \partial D$. Then $0, \infty \notin \partial K$ and, as before, one of the relations (4), (5) and (6) must hold. But (4) cannot be true because $D \setminus K$ is connected. If (5) or (6) were true, we would have $\infty \in K \subset D$ or $0 \in K \subset D$, respectively, which is impossible since $0, \infty \in \partial D$. Therefore, the assumption $0, \infty \in \partial D$ cannot be true.

(ii) $\Rightarrow$ (iii). Suppose that $D \setminus K$ is nearly everywhere equal to an annulus

$$\{z \in \hat{\mathbb{C}} : r < |z| < s\}.$$

Then

$$\operatorname{md}(\lambda D \setminus K) = \log \frac{|\lambda| s}{r}$$

and $\lambda \mapsto \operatorname{md}(\lambda D \setminus K)$ is harmonic on N.

(iii) $\Rightarrow$ (i). This is obvious. □

REMARK 5.3. I do not know whether the assumptions that at least one connected component of K is a nondegenerate continuum and at least one connected component of ∂D is a nondegenerate continuum are necessary.

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