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A CHARACTERIZATION OF PRODUCT PRESERVING MAPS WITH APPLICATIONS TO A CHARACTERIZATION OF THE FOURIER TRANSFORM

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ABSTRACT. It is shown that a product preserving bijective (not necessarily real linear or continuous) operator on an appropriate class of complex valued functions must have either the form $[\phi \mapsto \phi \circ u]$ or $[\phi \mapsto \overline{\phi \circ u}]$ where u is a fixed diffeomorphism of the base.

1. Introduction

To state the various results, we need to recall first some well-known definitions and simple observations. For a reference on the standard results stated below, see, for example, [GS], the elementary [S1], or the more advanced [S2].

DEFINITION 1.1. One says that an infinitely smooth function $f : \mathbb{R}^n \to \mathbb{C}$ is *rapidly decreasing* (also called *Schwartz function*) if for any $l \in \mathbb{Z}_+$ and any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ of nonnegative integers one has

$$\sup_{x \in \mathbb{R}^n} \left| \frac{\partial^\alpha f(x)}{\partial x^\alpha} (1 + |x|^l) \right| < \infty,$$

where as usual $\frac{\partial^\alpha f(x)}{\partial x^\alpha} := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$, $|\alpha| := \sum_{i=1}^n \alpha_i$.

The space of all rapidly decreasing complex valued functions on $\mathbb{R}^n$ is denoted by $\mathcal{S}(n)$, and is called the Schwartz space. Below we denote $\mathcal{S}(n) = \mathcal{S}$ when the dependence on n is obvious.

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The space $\mathcal{S} = \mathcal{S}(n)$ becomes a Fréchet space when equipped with the system of norms:

$$\|f\|_N := \sup\left\{ \left| \frac{\partial^\alpha f(x)}{\partial x^\alpha}(1+|x|^N) \right| \,\Big|\, x \in \mathbb{R}^n, |\alpha| \leq N \right\}.$$

One of the main properties of the Schwartz space $\mathcal{S}$ is that the Fourier transform $\mathbb{F} : \mathcal{S} \to \mathcal{S}$, defined by

$$(\mathbb{F}f)(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x,\xi\rangle}\, dx$$

is a linear topological isomorphism.

The space $\mathcal{S}$ has two structures of algebra given by the point-wise product and the convolution

$$\cdot : \mathcal{S} \times \mathcal{S} \to \mathcal{S},$$
$$* : \mathcal{S} \times \mathcal{S} \to \mathcal{S}.$$

Both operations are continuous with respect to both arguments simultaneously.

Let $\mathcal{S}'(n)$ be the topological dual of $\mathcal{S}(n)$. (Again, we will write $\mathcal{S}'(n) = \mathcal{S}'$ whenever there is no possibility for confusion.) It will be equipped with the weak topology. Elements of $\mathcal{S}'$ are called *distributions of tempered growth.* We have the canonical continuous map $\mathcal{S} \to \mathcal{S}'$ given by

$$\langle \phi, f\rangle = \int_{\mathbb{R}^n} f(x)\phi(x)\, dx.$$

This map is injective and has a dense image in the weak topology. We will identify $\mathcal{S}$ with its image in $\mathcal{S}'$: $\mathcal{S} \subset \mathcal{S}'$.

The proof of the following claim is straightforward (see, e.g., [GS], Chapter 2).

Claim 1. (i) *The point-wise product on $\mathcal{S}$ extends (uniquely) to a separately continuous map $\mathcal{S} \times \mathcal{S}' \to \mathcal{S}'$ which is given explicitly by*

$$\langle \phi, \psi \cdot f\rangle = \langle \phi \cdot \psi, f\rangle$$

for any $f \in \mathcal{S}', \phi, \psi \in \mathcal{S}$. Then $\mathcal{S}'$ becomes a module over $\mathcal{S}$.

(ii) *The convolution on $\mathcal{S}$ extends (uniquely) to a separately continuous map $\mathcal{S} \times \mathcal{S}' \to \mathcal{S}'$ which is given explicitly by*

$$\langle \phi, \psi * f\rangle = \langle \phi * (-Id)^*\psi, f\rangle$$

for any $f \in \mathcal{S}', \phi, \psi \in \mathcal{S}$. Then $\mathcal{S}'$ becomes a module over $\mathcal{S}$. (Here $(-Id)^*$ denotes the operator defined by $((-Id)^*\psi)(x) = \psi(-x)$, a special case of the definition of u^* below.)

(iii) *The Fourier transform extends (uniquely) to an isomorphism of linear topological spaces* $\mathbb{F}:\mathcal{S}'\tilde{\to}\mathcal{S}'$. *Moreover,* $\mathbb{F}$ *is an isomorphism of* $\mathcal{S}$*-modules: for any* $\phi\in\mathcal{S}, f\in\mathcal{S}'$

$$\mathbb{F}(\phi\cdot f)=\mathbb{F}(\phi)*\mathbb{F}(f),$$
$$\mathbb{F}(\phi*f)=\mathbb{F}(\phi)\cdot\mathbb{F}(f).$$

Our main result is as follows.

THEOREM 2. *Assume we are given a bijective map* $\mathcal{T}:\mathcal{S}\to\mathcal{S}$ *which admits an extension* $\mathcal{T}':\mathcal{S}'\to\mathcal{S}'$ *and such that for every* $f\in\mathcal{S}$ *and* $g\in\mathcal{S}'$ *we have* $\mathcal{T}'(f\cdot g)=(\mathcal{T}f)\cdot(\mathcal{T}'g)$. *Then there exists a* C^∞*-diffeomorphism* $u:\mathbb{R}^n\to\mathbb{R}^n$ *such that*

$$\begin{aligned} &\textit{either} \qquad \mathcal{T}(f)=f\circ u \quad \textit{for all } f\in\mathcal{S},\\ &\textit{or} \qquad \mathcal{T}(f)=\overline{f\circ u} \quad \textit{for all } f\in\mathcal{S}.\end{aligned}$$

REMARK 3. (1) Let us emphasize that neither real linearity of $\mathcal{T}$ nor continuity are assumed a-priori, but they follow from the theorem a-posteriori.

(2) A version of Theorem 2 for real valued functions was proved in [AAM2].

(3) If one would take the class of continuous functions (even with compact support) instead of $\mathcal{S},\mathcal{S}'$ in the above theorem, then the result would not be true. Indeed the map $C(\mathbb{R}^n)\to C(\mathbb{R}^n)$ given by $[\phi\mapsto|\phi|^2\cdot\arg(\phi)]$ is bijective and multiplicative.

As an immediate corollary, we get the following characterization of the Fourier transform.

THEOREM 4. *Assume we are given a bijective transform* $\mathcal{F}:\mathcal{S}\to\mathcal{S}$ *which admits an extension* $\mathcal{F}':\mathcal{S}'\to\mathcal{S}'$ *which is bijective, and such that for every* $f\in\mathcal{S}$ *and* $g\in\mathcal{S}'$ *we have* $\mathcal{F}'(f\cdot g)=(\mathcal{F}f)*(\mathcal{F}'g)$.

Then, there exists some diffeomorphism $w:\mathbb{R}^n\to\mathbb{R}^n$ *such that either for every* $f\in\mathcal{S}$, $\mathcal{F}f=\mathbb{F}(f\circ w)$, *or, for every* $f\in\mathcal{S}$, $\mathcal{F}f=\overline{\mathbb{F}(f\circ w)}$.

REMARK 5. In [AAM2] the following corollary of Theorem 4 was proved.

THEOREM 6 ([AAM2]). *Assume we are given a bijective transform* $\mathcal{F}:\mathcal{S}\to\mathcal{S}$ *which admits an extension* $\mathcal{F}':\mathcal{S}'\to\mathcal{S}'$ *which is also bijective, and such that:*

1. *For every* $f\in\mathcal{S}$ *and* $g\in\mathcal{S}'$ *we have* $\mathcal{F}'(f*g)=(\mathcal{F}f)\cdot(\mathcal{F}'g)$.
2. *For every* $f\in\mathcal{S}$ *and* $g\in\mathcal{S}'$ *we have* $\mathcal{F}'(f\cdot g)=(\mathcal{F}f)*(\mathcal{F}'g)$.

Then, $\mathcal{F}$ *is essentially the Fourier transform* $\mathbb{F}$, *that is, for some* $B\in\mathrm{GL}_n(\mathbb{R})$ *with* $|det(B)|=1$, *we have either for all* f *that* $\mathcal{F}(f)=\mathbb{F}(f\circ B)$ *or for all* f *that* $\mathcal{F}(f)=\mathbb{F}(\overline{f\circ B})$.

The main Theorem 2 can be generalized to arbitrary smooth manifolds as follows. Let X be a smooth manifold. Let $\mathcal{D}(X)$ denote the space of complex valued infinitely smooth functions with compact support. The space

$\mathcal{D}(X)$ is equipped with the standard linear topology of (strict) inductive limit of Fréchet spaces. Let $\mathcal{M}(X)$ denote the space of infinitely smooth compactly supported measures on X. The space $\mathcal{M}(X)$ is also equipped with the standard linear topology of (strict) inductive limit of Fréchet spaces. Let $\mathcal{D}'(X)$ denote the topological dual of $\mathcal{M}(X)$. We have the natural linear map $\mathcal{D}(X) \to \mathcal{D}'(X)$ given by

$$f \mapsto \left[\mu \mapsto \int_X f \cdot d\mu\right].$$

This map is injective with dense image. We will identify $\mathcal{D}(X)$ with its image in $\mathcal{D}'(X)$:

$$\mathcal{D}(X) \subset \mathcal{D}'(X).$$

Naturally $\mathcal{D}'(X)$ is a $\mathcal{D}(X)$-module.

THEOREM 7. *Assume we are given a bijective map $\mathcal{T} : \mathcal{D}(X) \to \mathcal{D}(X)$ which admits and extension $\mathcal{T}' : \mathcal{D}'(X) \to \mathcal{D}'(X)$ and such that for every $f \in \mathcal{D}(X)$ and $g \in \mathcal{D}'(X)$ we have $\mathcal{T}'(f \cdot g) = (\mathcal{T} f) \cdot (\mathcal{T}' g)$. Then there exists a C^∞-diffeomorphism $u\colon X \to X$ such that*

$$\text{either} \qquad \mathcal{T}(f) = f \circ u \quad \text{for all } f \in \mathcal{S}, \tag{1}$$

$$\text{or} \qquad \mathcal{T}(f) = \overline{f \circ u} \quad \text{for all } f \in \mathcal{S}. \tag{2}$$

Conversely, any transformation of the form (1) *or* (2) *obviously satisfies the assumptions of the theorem.*

Before we pass to the proofs of our results, let us briefly comment on where these theorems originated from. For reasons connected with the topic of convex analysis, we were interested in the characterization of a very basic concept in convexity: duality and the Legendre transform. In the paper [AM1], it was shown that the Legendre transform can be characterized as follows: up to linear terms, it is the only involution on the class of convex lower semi-continuous functions on $\mathbb{R}^n$ which reverses the (partial) order of functions. Since the Legendre transform has another special property, namely that it exchanges summation of functions with their inf-convolution (for definitions and details see [AM2]), this in fact implied that an involution on lower semi-continuous convex functions which reverses order *must* have this special property. It turns out that also the opposite is true, namely any involutive transform (on this class) which exchanges summation with inf-convolution, must reverse order, and, in fact, be up to linear terms the Legendre transform (see [AM2] for proofs and a discussion). Thus, already at this stage we observed that very minimal basic properties essentially uniquely define some classical transform which traditionally is defined in a concrete, and quite involved form.

It looks very intriguing to determine how far this point of view can be extended. It turns out that also the classical Fourier transform may be defined essentially uniquely by very minimal and basic conditions, namely by the condition of exchanging convolution with product, see [AAM2]. We are going further in this paper and study transformations which are assumed only to preserve product (the above Theorem 6 from [AAM2] is equivalent to understanding maps preserving both product and convolution). Theorem 2 implies that these transforms must be, up to a diffeomorphism of the base, either identity or conjugation. Previously, we could deal with such transforms only in the case of real valued functions, but now we are able also to deal with the case of complex valued functions, which is of course necessary for the study of the Fourier transform. The complex valued case turns out to be much more involved than the real valued case, both algebraically and analytically. Let us mention here only one such point: We use in the proof a 90 years old result of Banach [B] and Sierpiński [S] (from 1920) stating that a measurable and additive function is continuous. This delicate result, perhaps, has never found an application until this paper.

After our papers [AAM1], [AAM2] about the real case have been published, the paper [J], extending the results to the setting of $\mathbb{Z}_n$, has appeared. We thank the anonymous referee for pointing this out to us. However, let us emphasize that the main point of our results is the minimality of the conditions (no linearity or continuity are assumed, but follow aposteriori) which is not the case in the aforementioned paper.

2. Proof of Theorem 2

We present in this section a detailed proof of Theorem 2. The proof of Theorem 7 follows essentially the same lines. We may of course assume without loss of generality that the transformation $\mathcal{T}$ is defined over the whole space $\mathcal{S}'$.

Before starting the proof, a short discussion of what it means to evaluate generalized functions from $\mathcal{S}'$ at a point when needed. Clearly, generalized functions in $\mathcal{S}'$ are defined via their action on functions in $\mathcal{S}$. However, there are subclasses $A \subset \mathcal{S}'$ which are themselves, say, continuous functions, and so evaluating them at a point is meaningful. Consider the following interesting subclass $\mathcal{S}_1 \subset \mathcal{S}'$ consisting of all generalized functions $\phi \in \mathcal{S}'$ such that for every $f \in \mathcal{S}$ we have that $\phi \cdot f \in \mathcal{S}$. For example, all constant functions belong to $\mathcal{S}_1$. It is also easy to see that any function in $\mathcal{S}_1$ is infinitely smooth. In fact, $\mathcal{S}_1$ contains all infinitely smooth functions f with the following growth condition: any partial derivative of f is bounded by $C(1+|x|)^k$ for some constant C and integer k.

Note that for $\phi \in \mathcal{S}_1$ and $\psi \in \mathcal{S}'$, the product $\psi \cdot \phi$ is well defined and belongs to $\mathcal{S}'$. Indeed, for $f \in \mathcal{S}$ we have that $\langle \psi \cdot \phi, f\rangle = \langle \psi, (\phi \cdot f)\rangle$ is the action of a function in $\mathcal{S}'$ with a function in $\mathcal{S}$, which is well defined since the map $\mathcal{S} \to \mathcal{S}$ given by $f \mapsto \phi \cdot f$ is continuous.

It is not difficult to see that for $\phi \in \mathcal{S}'$ one has: $\phi \in \mathcal{S}_1$ iff $\mathcal{T}\phi \in \mathcal{S}_1$. Indeed

$$\begin{aligned} \mathcal{T}\phi \in \mathcal{S}_1 \quad &\Leftrightarrow \\ \forall f \in \mathcal{S}, \quad (\mathcal{T}\phi)\cdot f \in \mathcal{S} \quad &\Leftrightarrow \\ \forall g \in \mathcal{S}, \quad (\mathcal{T}\phi)\cdot(\mathcal{T}g) \in \mathcal{S} \quad &\Leftrightarrow \\ \forall g \in \mathcal{S}, \quad \mathcal{T}(\phi\cdot g) \in \mathcal{S} \quad &\Leftrightarrow \\ \forall g \in \mathcal{S}, \quad \phi\cdot g \in \mathcal{S} \quad &\Leftrightarrow \quad \phi \in \mathcal{S}_1. \end{aligned}$$

Moreover, this implies that for $\phi \in \mathcal{S}_1$ and $\psi \in \mathcal{S}'$ we have that $\mathcal{T}(\psi\cdot\phi) = (\mathcal{T}\psi)\cdot(\mathcal{T}\phi)$, both sides being well defined: indeed, note that for any $g \in \mathcal{S}$, letting $f = \mathcal{T}^{-1}g$ we have that

$$\mathcal{T}(\psi\cdot\phi)\cdot g = \mathcal{T}(\psi\cdot\phi\cdot f) = \mathcal{T}(\psi)\cdot\mathcal{T}(\phi\cdot f)$$

and likewise

$$(\mathcal{T}\psi\cdot\mathcal{T}\phi)\cdot g = \mathcal{T}(\psi)\cdot\mathcal{T}(\phi\cdot f).$$

Thus, the product of the two with any function $g \in \mathcal{S}$ is the same (and well defined), which easily implies that they are the same element in $\mathcal{S}'$. We will use this observation mainly for ϕ being the constant function.

Finally, note that for functions ϕ in $\mathcal{S}_1$, evaluation at a point is meaningful since as noted before these may be identified with a subclass of C^∞ functions, and in particular are continuous.

Proof of Theorem 2. The proof is divided into 13 steps; several of them are exactly the same as in the case of real valued functions in [AAM2]. We repeat them here for the convenience of the reader.

Step 1. The following identities hold, despite the fact that a-priori both functions in each identity belong to $\mathcal{S}' \setminus \mathcal{S}$. This can be done directly, but follows from the above discussion and the fact that the constant functions belong to $\mathcal{S}_1$.

$$\mathcal{T}(C\delta_x) = \mathcal{T}(\delta_x)\cdot\mathcal{T}(C),$$
$$\mathcal{T}(C_1\cdot C_2) = \mathcal{T}(C_1)\cdot\mathcal{T}(C_2).$$

Step 2. Next, we may determine the image of the constant functions 0 and 1. We show $\mathcal{T}(1) = 1$ and $\mathcal{T}(0) = 0$.

Proof of Step 2. Indeed, for all $f \in \mathcal{S}$, and in fact all $f \in \mathcal{S}'$, we have $1\cdot f = f$, hence $\mathcal{T}(1)\cdot g = g$ for all $g \in \mathcal{S}'$ and hence (taking e.g., $g = 1$) we see that $\mathcal{T}(1) = 1$. Similarly, $0\cdot f = 0$ for all $f \in \mathcal{S}'$, so $\mathcal{T}(0)\cdot g = \mathcal{T}(0)$ for all $g \in \mathcal{S}'$ (e.g., for $g = 0$) and so $\mathcal{T}(0) = 0$.

Definition 2.1. For $f \in \mathcal{S}$ the support of f, denoted $\mathrm{supp}(f)$, is defined as usual as the closure of the set of points where $f \neq 0$. For $\phi \in \mathcal{S}'$, the support of ϕ, denoted $\mathrm{supp}(\phi)$, is defined as follows: $x \notin \mathrm{supp}(\phi)$ if there exists an open neighborhood U with $x \in U$ and such that for any $f \in \mathcal{S}$ with

$\text{supp}(f) \subset U$ one has $\langle \phi, f \rangle = 0$. (When $\phi \in \mathcal{S}$ the second definition coincides with the first.)

Step 3. In this step, we show that any constant multiple of the delta function at a point x is mapped to a function supported at one point.

Proof of Step 3. Indeed, assume that there are two different points $z, y \in \text{supp}(\mathcal{T}\delta_x)$. Take disjoint neighborhoods of y and z, U_y and U_z.

By the definition of the support of a function in $\mathcal{S}'$, we may find two functions, $g_y, g_z \in \mathcal{S}$, supported on U_y and U_z respectively, which satisfy that $\langle \mathcal{T}\delta_x, g_y \rangle \neq 0$, and $\langle \mathcal{T}\delta_x, g_z \rangle \neq 0$. In particular, $(\mathcal{T}\delta_x) \cdot g_y \not\equiv 0$ and $(\mathcal{T}\delta_x) \cdot g_z \not\equiv 0$. However, by disjointness of U_y and U_z, we have $g_y \cdot g_z \equiv 0$.

Assume $g_y = \mathcal{T} f_y$ and $g_z = \mathcal{T} f_z$ for some functions $f_y, f_z \in \mathcal{S}$. Then, by injectivity and Step 2, we see that also $\delta_x \cdot f_y \not\equiv 0$ and $\delta_x \cdot f_z \not\equiv 0$, and therefore $f_y(x) \neq 0$ and $f_z(x) \neq 0$. This in turn implies that $(f_y \cdot f_z)(x) \neq 0$, which means in particular that $(f_y \cdot f_z) \not\equiv 0$, and by multiplication preservation, and injectivity, $\mathcal{T} f_y \cdot \mathcal{T} f_z \not\equiv 0$, contradicting $g_y \cdot g_z \equiv 0$.

We define a function $u : \mathbb{R}^n \to \mathbb{R}^n$ by the following formula: $u(x) = \text{supp}(\mathcal{T}(\delta_x))$.

Step 4. In this step, we show that if $f \in \mathcal{S}$ satisfies that $f(x) = 1$ then $(\mathcal{T} f)(u(x)) = 1$, and if $f(x) = 0$ then $(\mathcal{T} f)(u(x)) = 0$.

Proof of Step 4. It is well known that the generalized functions which are supported on a single point u are only the functions which are the sum of a finite number of various derivatives of the generalized function δ_u.

Thus, we have by Step 3 that for every x there exist an M and constants $\gamma_\alpha \in \mathbb{C}$ for $\alpha \in (\mathbb{N} \cup 0)^n$ with $|\alpha| \leq M$ such that

$$\mathcal{T}(\delta_x) = \gamma_0 \delta_{u(x)} + \sum_\alpha \gamma_\alpha \frac{\partial^\alpha}{\partial t^\alpha} \delta_{u(x)}$$

(where as before, α is a multi-index, $|\alpha| = \sum_i \alpha_i$ and $\frac{\partial^\alpha}{\partial t^\alpha} = \frac{\partial^{|\alpha|}}{\prod \partial t_i^{\alpha_i}}$). Note that to be precise we must write $M(x)$ and $\gamma_\alpha(x)$ for each α, since these coefficients may, a-priori, depend on the point x which we fixed. We may assume that for some α with $|\alpha| = M$ we have $\gamma_\alpha \neq 0$.

Consider a function $f \in \mathcal{S}$, then $f \cdot \delta_x = f(x)\delta_x$. Therefore, $(\mathcal{T} f) \cdot (\mathcal{T}\delta_x) = \mathcal{T}(f(x)\delta_x)$. Assume $f(x) = 0$, then we have that $(\mathcal{T} f) \cdot (\mathcal{T}\delta_x) \equiv 0$. Plugging in the formula for $\mathcal{T}\delta_x$, we see that in this case

$$(\mathcal{T} f) \cdot \left(\gamma_0 \delta_{u(x)} + \sum_\alpha \gamma_\alpha \frac{\partial^\alpha}{\partial t^\alpha} \delta_{u(x)} \right) \equiv 0.$$

We may use the formula for multiplication of a function by a derivative of the delta function, and get many equations. The equations corresponding the derivatives α with $|\alpha| = M$ are easily seen to be $(\mathcal{T} f)(u(x)) \gamma_\alpha \frac{\partial^\alpha}{\partial t^\alpha} \delta_{u(x)} = 0$. Thus, since for one of them $\gamma_\alpha \neq 0$, we get $(\mathcal{T} f)(u(x)) = 0$.

Similarly, assume that $f(x)=1$, then $(\mathcal{T}f)\cdot(\mathcal{T}\delta_x)=\mathcal{T}\delta_x$, and we may plug in the formula for $\mathcal{T}\delta_x$, getting the equation

$$(\mathcal{T}f)\cdot\left(\gamma_0\delta_{u(x)}+\sum_\alpha\gamma_\alpha\frac{\partial^\alpha}{\partial t^\alpha}\delta_{u(x)}\right)=\gamma_0\delta_{u(x)}+\sum_\alpha\gamma_\alpha\frac{\partial^\alpha}{\partial t^\alpha}\delta_{u(x)}.$$

Comparing the terms next to $\frac{\partial^\alpha}{\partial t^\alpha}\delta_{u(x)}$, we get this time that $(\mathcal{T}f)(u(x))=1$.

We remark that in the exact same way, for a function $f\in\mathcal{S}$ with $f(x)=C$, what one would get is that $(\mathcal{T}f)(u(x))=(\mathcal{T}C)(u(x))$. (This uses the fact, mentioned at the beginning, that evaluation of $\mathcal{T}C$ at a point has meaning.) In other words, this step actually shows that the transform, on $\mathcal{S}$, is local: its value at $u(x)$ only depends on the value of f at x.

Step 5. We show here that the generalized function $\mathcal{T}\delta_x$ is some constant multiple of $\delta_{u(x)}$. We may thus write

$$\mathcal{T}(\delta_x)=c_x\delta_{u(x)}.$$

Proof of Step 5. From Step 3, we know that for every x there exist constants $\gamma_\alpha(x)$ and $M(x)$ such that

$$\mathcal{T}(\delta_x)=\gamma_0(x)\delta_{u(x)}+\sum_\alpha\gamma_\alpha(x)\frac{\partial^\alpha}{\partial t^\alpha}\delta_{u(x)},$$

and so that there exists some α with $|\alpha|=M$ and $\gamma_\alpha(x)\neq 0$, and with $\gamma_\alpha(x)=0$ for $|\alpha|>M(x)$.

Let $f\in\mathcal{S}$, and $g=\mathcal{T}f$, and compute

$$g\cdot\left(\gamma_0(x)\delta_{u(x)}+\sum_{0<|\alpha|\le M}\gamma_\alpha(x)\frac{\partial^\alpha}{\partial t^\alpha}\delta_{u(x)}\right)$$

which is obviously itself a function supported on $u(x)$, that is, it is a-priori equal to some expression of the form

$$g\cdot(\mathcal{T}\delta_x)=\sum_\alpha a_\alpha\frac{\partial^\alpha}{\partial t^\alpha}\delta_{u(x)}, \tag{3}$$

where a_α depend on both x and g.

To compute these terms, we integrate by parts to get

$$\begin{aligned}&\int\left(g(t)\cdot\frac{\partial^\alpha}{\partial t^\alpha}\delta_y(t)\right)z(t)\,dt\\&\quad=\int\frac{\partial^\alpha}{\partial t^\alpha}\delta_y(t)\big(g\cdot z(t)\big)\,dt\\&\quad=(-1)^{|\alpha|}\frac{\partial^\alpha}{\partial t^\alpha}(g\cdot z)(y)=(-1)^{|\alpha|}\sum_{\alpha_1+\alpha_2=\alpha}\left(\frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}}g(y)\cdot\frac{\partial^{\alpha_2}}{\partial t^{\alpha_2}}z\right)(y)\\&\quad=(-1)^{|\alpha|}\sum_{\alpha_1+\alpha_2=\alpha}\left(\frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}}g(y)\right)\cdot\left(\int\left(\delta_y(t)\frac{\partial^{\alpha_2}}{\partial t^{\alpha_2}}z\right)(t)\,dt\right)\end{aligned}$$

$$= (-1)^{|\alpha|} \sum_{\alpha_1+\alpha_2=\alpha} \left(\frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}} g(y) \right) \cdot (-1)^{|\alpha_2|} \left(\int \left(\frac{\partial^{\alpha_2}}{\partial t^{\alpha_2}} \delta_y \right)(t) \cdot z(t)\, dt(t)\, dt \right)$$
$$= \int \left(\sum_{\alpha_1+\alpha_2=\alpha} (-1)^{|\alpha_1|} \left(\frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}} g \right)(y) \cdot \left(\frac{\partial^{\alpha_2}}{\partial t^{\alpha_2}} \delta_y \right) \right)(t) \cdot z(t)\, dt.$$

Thus, we know what is $g \cdot \frac{\partial^\alpha}{\partial t^\alpha} \delta_y$ for every α. In particular, it equals a linear combination of derivatives of δ_y of order α and lower, with coefficients depending on derivatives of order α and lower of g at y.

Return to equation (3), and consider the coefficient, on the left hand side, of $\frac{\partial^\alpha}{\partial t^\alpha} \delta_{u(x)}$ for α with $|\alpha| = M$. It appears only as a by-product of $\frac{\partial^\alpha}{\partial t^\alpha} \delta_{u(x)}$ from the part $\mathcal{T}(\delta_x)$ on the left hand side (so, with $\alpha_1 = 0$ and $\alpha_2 = \alpha$) and hence its coefficient is exactly $g(u(x))\gamma_\alpha$. So we have determined a_α for $|\alpha| = M$.

Assume that $M \neq 0$, and consider the coefficient of some $\frac{\partial^\alpha}{\partial t^\alpha} \delta_{u(x)}$ with $|\alpha| = M - 1$. Thus, letting $\alpha = (\alpha_1, \ldots, \alpha_n)$, it can be generated either by simply letting $\alpha_1 = 0$ and $\alpha_2 = \alpha$, or by letting $\alpha_1 = e_i$ and $\alpha_2 = \alpha$. Therefore, we get the equation (for $|\alpha| = M - 1$) that

$$a_\alpha = g(u(x))\gamma_\alpha - \sum_{i=1}^n \gamma_{\alpha+e_i} \left(\frac{\partial}{\partial t_i} g \right)(u(x)).$$

Next, assume that $f \in \mathcal{S}$ satisfies that $f(x) = 1$, and as before, $g = \mathcal{T} f$. Step 4 implies that $g(u(x)) = 1$, and in fact we know that this is equivalent, since all conditions are symmetric with respect to $\mathcal{T}$ and $\mathcal{T}^{-1}$, therefore if $(\mathcal{T} f)(u(x)) = 1$ then $f(x) = 1$.

Under this assumption, we have that $\mathcal{T}(f \cdot \delta_x) = \mathcal{T}\delta_x$, so that the coefficients of $\frac{\partial^\alpha}{\partial t^\alpha} \delta_{u(x)}$ in each side of this equation are the same. Comparing the coefficients for $|\alpha| = M - 1$, we get the equation, for every α (after cancelation of the term $g(u(x))\gamma_\alpha$)

$$0 = \sum_{i=1}^n \gamma_{\alpha+e_i} \left(\frac{\partial}{\partial t_i} g \right)(u(x)).$$

Note that at least one of these equations is nontrivial, since we know that there is some α' with $|\alpha'| = M$ and $\gamma_{\alpha'} \neq 0$, and so α' has some coordinate i with $\alpha'_i \neq 0$, so that $\alpha = \alpha' - e_i$ gives a nontrivial equation above.

To get the contradiction, we choose $g \in \mathcal{S}$ with $g(u(x)) = 1$ but for which one of the above equations fails. This is clearly possible, since the condition is only that some linear combination of the directional derivatives be equal to 0 at the chosen point x, and that $g(u(x)) = 1$. We may even take g to be linear in some neighborhood of $u(x)$, and smoothed out outside to belong to $\mathcal{S}$.

The proof of this step is thus complete, and we see that $M = 0$, and the image of a delta function $\mathcal{T}\delta_x$ is some constant multiple of $\delta_{u(x)}$.

Step 6. In this step, we note that $u : \mathbb{R}^n \to \mathbb{R}^n$ is invertible, and also that multiples of the delta functions at x are mapped by $\mathcal{T}$ to multiples of the delta function at $u(x)$.

Proof of Step 6. Indeed, first note that $\mathcal{T}(c\delta_x) = \mathcal{T}(c) \cdot \mathcal{T}(\delta_x) = (\mathcal{T}c)(u(x)) \times C_x\delta_{u(x)}$, so that multiples of the delta function δ_x are also mapped to multiples of $\delta_{u(x)}$.

To construct u^{-1} simply note that all considerations above are also true for $\mathcal{T}^{-1}$, we may define $v(y) = \mathrm{supp}(\mathcal{T}^{-1}\delta_y)$, and by the equation $\mathcal{T}^{-1}(\mathcal{T}\delta_x) = \delta_x$ and $\mathcal{T}(\mathcal{T}^{-1}\delta_y) = \delta_y$, we get that $u(v(y)) = y$ and $v(u(x)) = x$ so that u is invertible.

Step 7. In this step, we determine the general form of the transform, namely that for any $f \in \mathcal{S}_1$ and $y = u(x)$,

$$(\mathcal{T}f)(y) = (\mathcal{T}(f(x)))(y).$$

(In fact, this was done already in the proof of Step 4.)

Proof of Step 7. For any $f \in \mathcal{S}$ we have that, letting $y = u(x)$, that

$$(\mathcal{T}f)(y) \cdot C_x\delta_y = \mathcal{T}(f \cdot \delta_x) = \mathcal{T}\big(f(x) \cdot \delta_x\big) = \mathcal{T}(f(x)) \cdot C_x\delta_y = \mathcal{T}(f(x))(y)C_x\delta_y.$$

Therefore, $(\mathcal{T}f)(y) = \mathcal{T}(f(x))(y)$. That is,

$$(\mathcal{T}f)(y) = \mathcal{T}(f(u^{-1}(y)))(y). \tag{4}$$

The same is true for functions $f \in \mathcal{S}_1$, since these can be multiplied by δ_x in exactly the same way, by the discussion at the beginning of the section.

Step 8. In this step, we show that $u : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, locally bounded, and open.

Proof of Step 8. From Step 7, we see that for any $f \in \mathcal{S}$, if $f(x) = 1$ then $(\mathcal{T}f)(u(x)) = 1$, and if $f(x) = 0$ then $(\mathcal{T}f)(u(x)) = 0$ (actually, we already saw this in Step 4). Let A be some bounded open subset of $\mathbb{R}^n$, then we may take f to equal 0 outside A and $\neq 0$ in A. Then for $\mathcal{T}f$, we have that $(\mathcal{T}f) \neq 0$ exactly on $u(A)$, and we get that $u(A)$ is open as well. Since $\mathcal{T}f \in \mathcal{S}$, $u(A)$ is bounded. (Recall that u is bijective, which eases these considerations.) The same applies to u^{-1}, using $\mathcal{T}^{-1}$.

Step 9. Define, for every $x \in \mathbb{R}^n$, the function $\alpha_x : \mathbb{C} \to \mathbb{C}$, by $\alpha_x(C) = \mathcal{T}(C)(x)$. We first deal with $|C| = 1$, and show that either $\alpha_x(C) = C$ for every x and $|C| = 1$, or $\alpha_x(C) = \overline{C}$ for every x and $|C| = 1$.

Proof of Step 9. By Step 7, we have that

$$(\mathcal{T}f)(y) = \alpha_y(f(u^{-1}(y))). \tag{5}$$

Similarly,

$$(\mathcal{T}^{-1}f)(y) = \beta_y(f(u(y))),$$

so that, looking at $\mathcal{T}^{-1}\mathcal{T}f$ we have that $\beta_y(\alpha_{u(y)}(C)) = C$ (and $\alpha_y(\beta_{u^{-1}(y)}(C)) = C$). We thus know that for every x, α_x is multiplicative, and by the above, also invertible.

Fix some $m \in \mathbb{N}$ and consider the constant function $e^{2\pi i/m}$. For every x, by multiplicativity, $\alpha_x(e^{2\pi i/m})$ is again a root of unity of degree m, so for some $\omega = \omega_{m,x} \in \{0, 1, \dots, m-1\} = \mathbb{Z}_m$ we have

$$\alpha_x(e^{2\pi i/m}) = e^{2\pi i \omega_{m,x}/m}.$$

By (5), considering the constant function $e^{2\pi i/m}$ which is in $\mathcal{S}_1$, we have that $\mathcal{T}(e^{2\pi i/m})(x) = e^{2\pi i \omega_{m,x}/m}$, and since it must be continuous we get that $\omega_{m,x} = \omega_m$ does not depend on x. Moreover, since $e^{2\pi i/m}$ is primitive, so must be its image, that is, $\omega_m \in \mathbb{Z}_m^*$ (invertible).

Let us first consider the one-dimensional case, $x \in \mathbb{R}$. The general case will be very similar. Fixing $\theta \in \mathbb{R}$, we look at the function $\phi_\theta(\cdot) = e^{2\pi i(\cdot+\theta)} \in C(\mathbb{R})$. By (5),

$$\varphi_\theta(x) := (\mathcal{T}\phi_\theta)(x) = \alpha_x\left(e^{2\pi i(u^{-1}(x)+\theta)}\right) \in C(\mathbb{R}).$$

Consider also $\psi_\theta := \varphi_\theta \circ u$. Since u is continuous by Step 8,

$$\psi_\theta(x) = \alpha_{u(x)}\left(e^{2\pi i(x+\theta)}\right) \in C(\mathbb{R}).$$

Finally, define $\eta_\theta(y) = \psi_\theta(y - \theta)$, so that

$$\eta_\theta(y) = \alpha_{u(y-\theta)}(e^{2\pi i y}) \in C(\mathbb{R}).$$

We notice several things regarding η_θ. First, for $y \in \mathbb{Q}$, we have that $\eta_\theta(y) \in S^1$ (it is even a root of unity). Therefore, by continuity, $\eta_\theta \in C(\mathbb{R} \to S^1)$. Secondly, for $y \in \mathbb{Q}$, the root of unity which is $\eta_\theta(y)$ does not depend on θ. Therefore, by continuity, η_θ does not depend on θ at all. We thus denote it by η. The third fact is that for $y_1, y_2 \in \mathbb{Q}$ we have that

$$\eta(y_1 + y_2) = \eta(y_1) \cdot \eta(y_2).$$

However, by continuity again, this fact extends to $\mathbb{R}$ and we get that $\eta : \mathbb{R} \to S^1$ is a multiplicative and continuous function, that is, a continuous character, which implies that there is some $a \in \mathbb{R}$ such that

$$\eta(y) = e^{2\pi i a y}.$$

Because of the first fact, namely that roots of unity are mapped to roots of unity, we see that $a \in \mathbb{Q}$. Moreover, since primitive roots are mapped to primitive roots of the same order, we see that $a \in \{\pm 1\}$.

Going back to the definition of η, we see that for every θ

$$e^{2\pi i a y} = \alpha_{u(y-\theta)}(e^{2\pi i y}).$$

Since $\theta \in \mathbb{R}$ can be anything, and u is onto, we conclude that there exists $a \in \{\pm 1\}$ such that for any x and any y

$$\alpha_x(e^{2\pi i y}) = e^{2\pi i a y},$$

which completes the proof that on S^1, α_x is conjugation for all x or α_x is identity for all x. Let us next prove that the same is true for higher dimensions, $x \in \mathbb{R}^n$.

Fixing again $\theta \in \mathbb{R}$, we look at the function $\phi_\theta(x) = e^{2\pi i(x_1+\theta)} \in \mathcal{S}_1$, where for $x \in \mathbb{R}^n$ we denote by x_1 its first coordinate. By (5),

$$\varphi_\theta(x) := (\mathcal{T}\phi_\theta)(x) = \alpha_x\left(e^{2\pi i((u^{-1}(x))_1+\theta)}\right) \in C(\mathbb{R}^n \to \mathbb{R}).$$

Consider also $\psi_\theta = \varphi_\theta \circ u$, so that

$$\psi_\theta(x) = \alpha_{u(x)}\left(e^{2\pi i(x_1+\theta)}\right) \in C(\mathbb{R}^n \to \mathbb{R}).$$

Finally, define $\eta_\theta(y) = \psi_\theta(y - (\theta, 0, \ldots, 0))$, so that

$$\eta_\theta(y) = \alpha_{u(y-(\theta,0,\ldots,0))}(e^{2\pi i y_1}) \in C(\mathbb{R}^n \to \mathbb{R}).$$

Again we notice several things regarding η_θ. First, for $y \in \mathbb{R}^n$ with $y_1 \in \mathbb{Q}$, we have that $\eta_\theta(y) \in S^1$ (is even a root of unity). Therefore, by continuity, $\eta_\theta \in C(\mathbb{R}^n \to S^1)$. Secondly, for $y_1 \in \mathbb{Q}$, the root of unity which is $\eta_\theta(y)$ does not depend on θ. Therefore, by continuity, η_θ does not depend on θ at all. We may thus denote it by η. The third fact is that for z, w with $z_1, w_1 \in \mathbb{Q}$ we have that

$$\eta(z+w) = \eta(z) \cdot \eta(w),$$

a fact which again extends to $\mathbb{R}^n$ by continuity, and we get that $\eta : \mathbb{R}^n \to S^1$ is a multiplicative and continuous function, that is, a continuous character, and further, it depends only on the first coordinate. This implies that there is some $a \in \mathbb{R}$ such that

$$\eta(y) = e^{2\pi i a y_1}.$$

As before, since primitive roots of unity are mapped to primitive roots of unity of the same order, we see that $a = \pm 1$.

Going back to the definition of η, we see that for every θ and every y

$$e^{2\pi i a y_1} = \alpha_{u(y-(\theta,0,\ldots,0))}(e^{2\pi i y_1}).$$

Rewriting this we get that for all $x \in \mathbb{R}^n$ and $\theta \in \mathbb{R}$,

$$e^{2\pi i a(x_1+\theta)} = \alpha_{u(x)}\left(e^{2\pi i(x_1+\theta)}\right),$$

which in turn implies that for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$

$$\alpha_x(e^{2\pi i y}) = e^{2\pi i a y}$$

which is what we wanted to show.

Step 10. In this step, we show that for every $x \in \mathbb{R}^n$ there is some $a(x) \in \mathbb{R}$ such that $|(\mathcal{T}C)(x)| = |C|^{a(x)}$. Moreover, $a(x)$ is a C^∞ function of x, and nowhere 0.

Proof of Step 10. Clearly, $\mathcal{T}C$ is multiplicative in C, so that for every x, $(\mathcal{T}C)(x)$ is multiplicative as well. We also know that as a function of C, $T_x(C) := |(\mathcal{T}C)(x)|$ is onto, because $\mathcal{T}$ is onto and because of formula (4).

Given a multiplicative function on $\mathbb{R}$, very little is needed in order to establish the existence of $a(x)$ as above. Indeed, one may consider the function $t_x(C) = \log|T_x(C)|$, which is additive, and one only needs to show that it is linear, which by the fact that T_x is onto, and $T_x(1)=1$, implies the formula with $a(x)$.

For example, if we show that T_x is monotone on $\mathbb{R}_+$, this is enough (a non-linear, additive function cannot be monotone). Assume by contradiction that there are $0<C_1<C_2$ such that $(\mathcal{T}C_1)(x)>(\mathcal{T}C_2)(x)$. By multiplicativity, this means that $(\mathcal{T}c)(x)>1$ for $c=C_1/C_2<1$.

From Step 4 (applied to $\mathcal{T}^{-1}$), we know that this implies (since the constant functions c is never equal to 1) that for all $y\in\mathbb{R}^n$ we have $(\mathcal{T}c)(y)>1$.

Construct a function $f\in\mathcal{S}$ which equals to c at x, equals 0 at some other x', and assumes values only between 0 and c (for example, take such a linear function on the interval $[x',x]$ and smooth it so as to belong to $\mathcal{S}$ and have compact support).

Then by Step 4 again, $(\mathcal{T}f)(z)\neq 1$ for any z, however by (4) we have that

$$(\mathcal{T}f)(u(x))=\mathcal{T}(f(x))(u(x))=(\mathcal{T}c)(u(x))>1$$

and

$$(\mathcal{T}f)(u(x'))=\mathcal{T}(f(x'))(u(x'))=(\mathcal{T}0)(u(x'))=0.$$

By the mean value theorem, there is some $z\in[u(x),u(x')]$ for which $(\mathcal{T}f)(z)=1$, a contradiction.

We conclude that T_x is monotone increasing on $\mathbb{R}_+$, which (together with onto and $T_x(1)=1$) implies that it is of the form $T_x(C)=|C|^{a(x)}$.

The fact that $a(x)$ is in C^∞ follows from the fact that $|\mathcal{T}(C)|=|C|^{a(x)}$ is a C^∞ function of x for every C (for example, enough just for say $C=2$), which in turn follows from the fact that $C\in\mathcal{S}_1$. The fact that $a(x)$ is never equal to 0 follows from the fact that for every x, the function T_x is onto, and for $|C|=1$ one has $|\mathcal{T}(C)|=1$ by Step 9.

Step 11. We claim that $u(x)\in C^\infty$ as well.

Proof of Step 11. By Steps 7, 9 and 10,

$$|(\mathcal{T}f)(y)|=|f(u^{-1}(y))|^{a(y)} \tag{6}$$

for any non-vanishing $f\in\mathcal{S}$. Fixing a point y_0, to find the derivative of $\partial_\xi u^{-1}$ in some direction ξ, simply define a function $f_i\in\mathcal{S}$ to be, in a neighborhood of y_0, equal to $f_i(x)=e^{\langle x,e_i\rangle}$. Then (6) becomes

$$|(\mathcal{T}f_i)(y)|=\exp[\langle u^{-1}(y),e_i\rangle a(y)].$$

Since $|(\mathcal{T}f_i)(y)|$ and $a(y)$ are C^∞-smooth and non-vanishing, the function $\frac{\log|(\mathcal{T}f_i)(y)|}{a(y)}=\langle u^{-1}(y),e_i\rangle$ is C^∞-smooth for any i. Hence $u^{-1}\in C^\infty$. Finally, the same arguments can be applied to $\mathcal{T}^{-1}$, so also u is continuously differentiable.

Step 12. Let us prove the following lemma.

LEMMA. *Let $v:\mathbb{R}^n\to\mathbb{R}^n$ be a homeomorphism, and also assume that $B:\mathbb{R}^n\times\mathbb{R}\to\mathbf{S}^1\subset\mathbb{C}$ satisfies the following two conditions*:

(1) *$B(x, z_1 + z_2) = B(x, z_1)B(x, z_2)$ for all $x \in \mathbb{R}^n$, $z_1, z_2 \in \mathbb{R}$.*

(2) *For all $g(x) \in C^\infty(\mathbb{R}^n, \mathbb{R})$, the function $B(x, g(v(x)))$ is continuous.*

Then $B(x, z) = e^{ic(x)z}$ with $c(x) \in C(\mathbb{R}^n, \mathbb{R})$.

Proof. Let $A = \{x \in \mathbb{R}^n | B(x, \cdot) \text{ is discontinuous}\}$. First, we prove that A has no accumulation points.

Assume the contrary, that is, $A \ni x_k \to x_\infty$, $x_m \neq x_k$ for $m \neq k$, and $x_k \neq x_\infty$. Take

$$\psi(x) = \begin{cases} e^{1-\frac{1}{1-|x|^2}}, & |x| < 1, \\ 0, & |x| \geq 1. \end{cases}$$

This is a smooth cut-off function on $\mathbb{R}^n$. Denote $c_\alpha = \max |\frac{\partial^\alpha \psi}{\partial x^\alpha}|$.

Let

$$\delta_k = \min\left(\frac{1}{2} \min\{|v(x_m) - v(x_k)| : 1 \leq m \leq \infty, m \neq k\}, \frac{1}{2^k}\right).$$

Now by property (1), $B(x_k, \cdot)$ is a discontinuous character, and therefore attains values $e^{i\phi}$ with $\phi \geq \pi/2$ in $(0, \varepsilon)$ for all $\varepsilon > 0$ (otherwise, $B(x_k, (0, \varepsilon)) \subset (e^{-i\pi/2}, e^{i\pi/2})$, and thus for all $j \in \mathbb{N}$, $B(x_k, (0, \varepsilon/2^j)) \subset (e^{-i\pi/2^j}, e^{i\pi/2^j})$ implying continuity of $B(x_k, \cdot)$ at 0, and therefore continuity everywhere).

Hence, we can choose $0 < z_k < e^{-1/\delta_k}$ such that $|B(x_k, z_k) - 1| > 1/2$. Also, define

$$g_k(x) = z_k \psi\left(\frac{x - v(x_k)}{\delta_k}\right).$$

Then $g_k(x) \in C^\infty(\mathbb{R}^n)$ is a family of cut-off functions such that $g_k(v(x_k)) = z_k$, and g_k vanishes outside $|x - v(x_k)| < \delta_k$. Also, $|\frac{\partial^\alpha g_k}{\partial x^\alpha}| \leq z_k \frac{c_\alpha}{\delta_k^{|\alpha|}}$. Now we define $g(x) = \sum_{k=1}^\infty g_k(x)$. For all $\alpha = (j_1, \dots, j_n)$, we have

$$\sum_{k=1}^\infty \left|\frac{\partial^\alpha g_k}{\partial x^\alpha}\right| \leq c_\alpha \sum_{k=1}^\infty \frac{e^{-1/\delta_k}}{\delta_k^{|\alpha|}} \ll \sum_{k=1}^\infty \delta_k \leq \sum_{k=1}^\infty \frac{1}{2^k} = 1$$

and therefore $g(x)$ is a $C^\infty(\mathbb{R}^n)$ function. Note also that $g(v(x_k)) = z_k$ and $g(v(x_\infty)) = 0$. By property (2), $B(x, g(v(x)))$ is a continuous function, and hence $B(x_k, g(v(x_k))) \to B(x_\infty, g(v(x_\infty))) = B(x_\infty, 0) = 1$.

But $|B(x_k, g(v(x_k))) - 1| = |B(x_k, z_k) - 1| > 1/2$, a contradiction. We conclude that A has no accumulation points, and therefore its complement is dense in $\mathbb{R}^n$.

Next, we claim that A is empty. Assume the contrary, and take $x_0 \in A$ and $x_n \in \mathbb{R}^n \backslash A$, $x_n \to x_0$. For all $z_0 \in \mathbb{R}$, $B(x, z_0)$ is a continuous function on $\mathbb{R}^n$ by property (2), and therefore $B(x_n, z_0) \to B(x_0, z_0)$. By assumption, $B(x_n, \cdot) \to B(x_0, \cdot)$ pointwise. This implies $B(x_0, \cdot)$ is a measurable function, as the pointwise limit of continuous functions. By a theorem of Banach, $B(x_0, \cdot)$ must be continuous (by property (1)), a contradiction.

We conclude that $B(x,z)$ is continuous for every fixed x, which implies that $B(x,z)=e^{ic(x)z}$ for some function $c:\mathbb{R}^n\to\mathbb{R}$. We are left to show that $c(x)$ is continuous.

Let $x_n\to x_0$. Then

$$B(x_n,z)B(x_0,z)^{-1}=e^{i(c(x_n)-c(x_0))z}=e^{ic_nz}\to 1$$

for all $z\in\mathbb{R}$, where we denoted $c_n=c(x_n)-c(x_0)$. We will show that $c_n\to 0$ (which means continuity of c at x_0). Take $f\in\mathcal{S}$. By the dominated convergence theorem,

$$\int_{-\infty}^{\infty} f(z)e^{ic_nz}\,dz\to\int_{-\infty}^{\infty} f(z)\,dz$$

i.e. $\hat{f}(-c_n)\to\hat{f}(0)$ for all $f\in\mathcal{S}$, or equivalently, $f(-c_n)\to f(0)$ for all $f\in\mathcal{S}$. It follows that $c_n\to 0$, which completes the proof. $\square$

Step 13. In this step, we complete the proof of the theorem by showing that for all $C\in\mathbb{C}$, either $\alpha_x(C)=C$ for every x and C, or $\alpha_x(C)=\overline{C}$ for every x and C.

Proof of Step 13. To determine α_x on $\mathbb{C}$, we must show that $\alpha_x|_{\mathbb{R}^+}$ is the identity function for every x, since then we would have, by the previous step, that for a fixed $a\in\{\pm 1\}$ and all θ

$$\alpha_x(Re^{i\theta})=\alpha_x(R)\alpha_x(e^{i\theta})=Re^{ia\theta}.$$

By Step 10 for $R>0$

$$|\alpha_x(R)|=R^{a(x)},$$

where $a\in C^\infty(\mathbb{R}^n,\mathbb{R})$, a nowhere vanishes.

Let us define $B:\mathbb{R}^n\times\mathbb{R}\to S^1\subset\mathbb{C}$ by

$$B(x,z):=\frac{\alpha_x(e^z)}{|\alpha_x(e^z)|}.$$

By Step 12, $B(x,z)=e^{ib(x)z}$ with $b(x)\in C(\mathbb{R}^n,\mathbb{R})$.

Thus, we deduce that for $R>0$

$$\alpha_x(R)=e^{ib(x)\log(R)}R^{a(x)}=R^{a(x)+ib(x)},$$

where $a\in C^\infty$, $b\in C$.

This means that the general form of the transform is either

$$(\mathcal{T}f)(x)=(|f(u^{-1}(x))|)^{a(x)+ib(x)}\cdot\frac{f(u^{-1}(x))}{|f(u^{-1}(x))|}$$

or

$$(\mathcal{T}f)(x)=(|f(u^{-1}(x))|)^{a(x)+ib(x)}\cdot\frac{\overline{f(u^{-1}(x))}}{|f(u^{-1}(x))|}.$$

Let us assume without loss of generality that it is of the first form and show that $a(x)\equiv 1$ and $b(x)\equiv 0$ (the other case will follow from this since we may always conjugate the expression to get another transform satisfying the conditions). Let us show that $b\in C^\infty$. For any $R>0$ the function $\mathcal{T}(R)\in C^\infty$. But $(\mathcal{T}R)(x)=R^{a(x)+ib(x)}$. Since $a\in C^\infty$ by Step 10, the function $R^{ib(x)}$ is C^∞ for any $R>0$. This readily implies that $b\in C^\infty$.

Since $u\in C^\infty$ by Step 11, we may and will assume, for convenience of writing mainly, that $u=Id$. The reason that we may do this without losing generality is that in what follows we will only use the conditions of the theorem for functions in $\mathcal{S}$ with compact support (namely functions in $\mathcal{D}$). The transform $\mathcal{T}\circ u$ satisfies the multiplicativity assumptions on $\mathcal{D}$ and it is easy to check that $\mathcal{T}$ is bijective also on $\mathcal{D}$, a fact remaining valid for $\mathcal{T}\circ u$, which we shall use.

Next, we will use the smoothness to show that $a(x)\in\mathbb{Z}^+$. Fix $y\in\mathbb{R}^n$ and let $f(x)=x_1-y_1$ in a neighborhood of y, and smoothed to be in $\mathcal{D}$. Then (with the assumption $u=Id$) we have that in the neighborhood of y

$$(\mathcal{T}f)(x)=|x_1-y_1|^{a(x)+ib(x)}\operatorname{sign}(x_1-y_1)\in C^\infty.$$

Restrict the function to the ray $\{(x_1,y_2,y_3,\dots,y_n):x_1\in\mathbb{R}\}$. From smoothness, we may make a Taylor series for this function: let $k\in\mathbb{Z}^+$ be the order of the first nonzero derivative of the function at y (we will show the finiteness of k right away). Then we know that as $x_1\to y_1$ we have (for some fixed $\gamma\in\mathbb{C}$ depending on y)

$$|x_1-y_1|^{\hat a(x_1)+i\hat b(x_1)}\operatorname{sign}(x_1-y_1)=\gamma(x_1-y_1)^k\big(1+o(1)\big),$$

where we have denoted $\hat a(x_1)=a(x_1,y_2,\dots,y_n)$ and similarly for $\hat b$. We may also decompose $\hat a(x)$ and $\hat b(x)$ into Taylor series around y_1, and get that

$$\hat a(x_1)=a(y)+O(|x_1-y_1|)\quad\text{and}\quad\hat b(x_1)=b(y)+O(|x_1-y_1|).$$

From the fact that $a(y)\neq 0$ for all y we see that as $x_1\to y_1$, on the ray, the function in absolute value behaves like

$$|x_1-y_1|^{a(x)}=|x_1-y_1|^{a(y)+O(|x_1-y_1|)}=|x_1-y_1|^{a(y)}\big(1+o(1)\big)$$

and so converges to 0 at the rate $|x_1-y_1|^{a(y)}$, which means that $a(y)=k$ is a positive integer. This also explains why k must be finite, because we know that $a(y)\neq 0$ and if the function had all its derivatives equal to 0 it could not converge to 0 at such a slow rate $|x_1-y_1|^{a(y)}$. We conclude that $a(y)=k\in\mathbb{Z}^+$, and from continuity of $a(y)$ it does not depend on y.

Next, we turn to showing that $b(y)=0$ for all y, which follows from the same argument, because if $b(y)=b_0\neq 0$ then we would get that the as $x_1\to y_1+$, say, the function behaves like

$$|x_1-y_1|^{k+ib_0}\big(1+o(1)\big)$$

which must be the same behavior, as $x_1\to y_1$, as $\gamma(x_1-y_1)^k$ for a fixed γ, however if $b_0\neq 0$ then the term oscillates, and this contradicts γ being constant. Thus, $b_0=b(y)=0(\bmod 2\pi)$ for all y.

Finally, we use the bijectivity of $\mathcal{T}$ on $\mathcal{D}$ to show that $k=1$. Indeed, if $k>1$, we claim this implies that some functions in $\mathcal{D}$ are not attained as images of functions under $\mathcal{T}$. Indeed, if $(\mathcal{T}f)(x) = |f(x)|^k(\frac{f(x)}{|f(x)|})$ then we have that whenever $(\mathcal{T}f)(x)=0$ then also $\nabla(\mathcal{T}f)(x)=0$, which clearly need not happen for all $g\in\mathcal{D}$, so there are some $g\in\mathcal{D}$ which are not of the form $\mathcal{T}f$ for any $f\in\mathcal{D}$, a contradiction. We conclude that $a(y)\equiv 1$.

There was no loss in assuming that $u=Id$, and the same is true in the general case. Also, the same argument works if the transform is complex conjugated, so we have actually shown here that under the conditions of the theorem, there exists $u\in C^\infty(\mathbb{R}^n\to\mathbb{R}^n)$ such that either for all f and x, we have $(\mathcal{T}f)(x)=f(u^{-1}(x))$, or, for all f and x, we have $(\mathcal{T}f)(x)=\overline{f(u^{-1}(x))}$. □

3. Proof of Theorem 4

We recall the following.

THEOREM 4. *Assume we are given a bijective transform $\mathcal{F}:\mathcal{S}\to\mathcal{S}$ which admits an extension $\mathcal{F}':\mathcal{S}'\to\mathcal{S}'$ which is bijective, and such that for every $f\in\mathcal{S}$ and $g\in\mathcal{S}'$ we have $\mathcal{F}'(f\cdot g)=(\mathcal{F}f)*(\mathcal{F}'g)$.*

Then, there exists some diffeomorphism $w:\mathbb{R}^n\to\mathbb{R}^n$ such that either for every $f\in\mathcal{S}$, $\mathcal{F}f=\mathbb{F}(f\circ w)$, or, for every $f\in\mathcal{S}$, $\mathcal{F}f=\overline{\mathbb{F}(f\circ w)}$.

Proof of Theorem 4. Again we may assume that $\mathcal{F}$ is defined over the whole space $\mathcal{S}'$. Denote $\mathcal{T}=\mathbb{F}\circ\mathcal{F}$. then we have that $\mathcal{T}:\mathcal{S}'\to\mathcal{S}'$ satisfies $\mathcal{T}(\mathcal{S})=\mathcal{S}$ and for every $f\in\mathcal{S}$ and $g\in\mathcal{S}'$ we have $\mathcal{T}(f\cdot g)=\mathbb{F}((\mathcal{F}f)*(\mathcal{F}g))=\mathcal{T}f\cdot\mathcal{T}g$. Therefore, by Theorem 2, we have that there exists some diffeomorphism $u:\mathbb{R}^n\to\mathbb{R}^n$ such that, either for every $f\in\mathcal{S}$, $\mathcal{T}f=f\circ u$, in which case $\mathcal{F}f=\mathbb{F}^{-1}(f\circ u)=\mathbb{F}(f\circ w)$, or, $\mathcal{T}f=\overline{f\circ u}$, in which case $\mathcal{F}f=\mathbb{F}^{-1}\overline{f\circ u}=\overline{\mathbb{F}(f\circ w)}$. □

Similarly, replacing in the proof above $\mathbb{F}\circ\mathcal{F}$ by $\mathcal{F}\circ\mathbb{F}$, we get a proof of the following corollary.

COROLLARY 8. *Assume we are given a bijective transform $\mathcal{F}:\mathcal{S}\to\mathcal{S}$ which admits an extension $\mathcal{F}':\mathcal{S}'\to\mathcal{S}'$ and such that for every $f\in\mathcal{S}$ and $g\in\mathcal{S}'$ we have $\mathcal{F}'(f*g)=(\mathcal{F}f)\cdot(\mathcal{F}'g)$.*

Then, there exists some diffeomorphism $w:\mathbb{R}^n\to\mathbb{R}^n$ such that either for every $f\in\mathcal{S}$, $\mathcal{F}f=(\mathbb{F}f)\circ w$, or, for every $f\in\mathcal{S}$, $\mathcal{F}f=\overline{(\mathbb{F}f)\circ w}$.

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