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THE AUTOMORPHISM GROUP OF A GRAPH PRODUCT WITH NO SIL

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Abstract. We study the automorphisms of graph products of cyclic groups, a class of groups that includes all right-angled Coxeter and right-angled Artin groups. We show that the group of automorphisms generated by partial conjugations is itself a graph product of cyclic groups providing its defining graph does not contain any separating intersection of links (SIL). In the case that all the cyclic groups are finite, this implies that the automorphism group is virtually CAT(0); it has a finite index subgroup which acts geometrically on a right-angled building.

1. Introduction

Classically, a collection of groups can be combined using free products or direct products. More generally, a graph product of groups is a class of groups which interpolates between these. Let Γ be a finite simplicial graph with vertex set V and let $\{G_v\}_{v\in V}$ be a family of groups. Then the graph product G_Γ is the quotient of the free product of the groups G_v obtained by adding commutator relations between G_v and G_w whenever v, w are adjacent in Γ. A discrete graph Γ gives the free product of the G_v and a complete graph gives the direct product. Graph products encompass several important classes of groups. In particular, one obtains the class of right-angled Coxeter groups by requiring each G_v be isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and the class of right-angled Artin groups when each G_v is isomorphic to $\mathbb{Z}$. In this paper, we require only that the vertex groups be finitely generated abelian groups. Any such graph product is isomorphic to a graph product of cyclic groups, hence we can restrict our attention to the latter.

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The automorphism groups of right-angled Coxeter and right-angled Artin groups have been studied extensively in the literature (see, for example, [2, 4, 6, 11, 13, 18, 20, 23]). Automorphisms of more general graph products were considered by Laurence in his thesis [19]. Building on work of Servatius [21], Laurence describes a finite generating set for $\mathrm{Aut}(G_\Gamma)$ in the case when all vertex groups have the same order, either infinite or a fixed prime p. More recently, in [15], Gutierrez, Piggott, and Ruane begin a unified treatment of the automorphism group of a general graph product of cyclic groups, and in [7], Corredor and Gutierrez extend Laurence's generating set to all such graph products.

The automorphism group of G_Γ is generated by four types of automorphisms: graph symmetries, vertex isomorphisms, transvections, and partial conjugations. The first two types generate a finite subgroup. Transvections, which map $v \mapsto vw$ (or $v \mapsto wv$) for a pair of vertices v, w, are familiar to those who work with free group automorphisms. Indeed, the automorphism group of a free group is entirely generated by these transvections. For graph products, however, the transvections are more restricted (in some cases excluded entirely) and the partial conjugations play an essential role.

Partial conjugations are defined as follows. Given a vertex $v \in V$, let $\mathrm{lk}(v)$ denote the full subgraph of Γ spanned by the vertices adjacent to v and $\mathrm{st}(v)$ the subgraph spanned by v and $\mathrm{lk}(v)$. For each connected component C of $\Gamma \setminus \mathrm{st}(v)$, the *partial conjugation* $\pi_{v,C}$ conjugates all of the generators in C by v and leaves all other generators fixed.

The subgroup of $\mathrm{Aut}(G_\Gamma)$ generated by the partial conjugations is denoted $\mathrm{Aut}^{\mathrm{pc}}(G_\Gamma)$ and will be our main object of study. In the case where all vertex groups have finite order, this subgroup has finite index in the full automorphism group $\mathrm{Aut}(G_\Gamma)$. This is also true for some graph products with infinite vertex groups, namely those for which the structure of Γ does not permit any transvections (e.g., if Γ has no circuits of length less than four and no valence one vertices).

A simplicial graph Γ has a *Separating Intersection of Links (SIL)* if for some pair v, w with $d_\Gamma(v, w) \geq 2$, there is a component of $\Gamma \setminus (\mathrm{lk}(v) \cap \mathrm{lk}(w))$ which contains neither v nor w. Our main theorem, Theorem 3.6, states that if Γ has no SILs, then $\mathrm{Aut}^{\mathrm{pc}}(G_\Gamma)$ is itself a graph product of cyclic groups.

To prove this, we consider the graph $\tilde{\Gamma}$ whose vertices are in one-to-one correspondence with the partial conjugations $\pi_{v,C}$ of G_Γ. Two vertices of $\tilde{\Gamma}$ are connected by an edge if the two partial conjugations commute, thus we have a graph product of cyclic groups $G_{\tilde{\Gamma}}$. In the case where Γ has no SILS, we prove that $\mathrm{Aut}^{\mathrm{pc}}(G_\Gamma)$ is isomorphic to $G_{\tilde{\Gamma}}$. The main technical point is to characterize exactly when two partial conjugations commute. Under the no SILS assumption, we give a simple characterization of when $\pi_{v,C}$ and $\pi_{w,D}$ commute in terms of the relative position of C and D. This is the content of Lemma 3.4.

Our main theorem has some interesting geometric implications. Recall that a CAT(0) *metric space* is a proper, complete metric space in which each geodesic triangle is "at least as thin" as the Euclidean triangle with the same side lengths. We say that a finitely generated group G is a CAT(0) *group* if G acts properly, cocompactly by isometries on a CAT(0) metric space (such an action is said to be *geometric*). A group G is *virtually* CAT(0) if some finite index subgroup of G is CAT(0). Note that extending a geometric action from a finite index subgroup to the full group is highly nontrivial. It is unknown if virtually CAT(0) groups are CAT(0).

In Section 2, we show that any graph product of cyclic groups G_Γ acts on a right-angled building. Right-angled buildings are always CAT(0) by a theorem of Davis [9]. If the vertex groups are all finite, then the associated building is locally finite, its automorphism group is a locally compact group, and the graph product sits as a uniform lattice in this group. In particular, G_Γ is a CAT(0) group. (For a discussion of right-angled buildings and their lattices, see [1] and [22].) If some vertex group is infinite cyclic, then the action is no longer proper. However, if all the vertex groups are infinite (the right-angled Artin group case), then there is a different CAT(0) cube complex, the Salvetti complex, which can be used to get a CAT(0) structure on G_Γ.

Our main theorem implies that under the no SILs hypothesis, $\mathrm{Aut}^{\mathrm{pc}}(G_\Gamma)$ is itself a graph product hence also acts on a right-angled building. Moreover, we show that this action extends to the larger group generated by partial conjugations, graph symmetries and vertex isomorphisms. If all of the vertex groups are finite, these generate a finite index subgroup of $\mathrm{Aut}(G_\Gamma)$ and we conclude that $\mathrm{Aut}(G_\Gamma)$ is virtually CAT(0). (This last statement also follows from [15] where they show that under these hypotheses, the inner automorphism group has finite index in $\mathrm{Aut}(G_\Gamma)$. Our construction gives a CAT(0) action of a much larger subgroup, sometimes encompassing the entire automorphism group.)

One would like to know whether, in general, these actions can be extended to the full automorphism group $\mathrm{Aut}(G_\Gamma)$, that is, whether the action can be extended to include transvections. This would almost certainly require a different geometric construction as transvections do not behave well with respect to the geometry of the cube complexes given here. Even in the case of right-angled Coxeter and Artin groups, it is unknown whether the full automorphism groups are CAT(0).

2. Graph products and associated geometries

In this section, we discuss graph products and their associated geometries. We begin with a definition of a graph product.

Definition 2.1. Let Γ be a finite, simplicial graph with vertex set V, together with a labeling of each vertex by a group G_v. Let F_Γ denote the free product of all the vertex groups $G_v, v \in V$. Then the *graph product* G_Γ is the quotient group of F_Γ obtained by adding commutator relations between G_v and G_w whenever v, w are connected by an edge in Γ.

In this paper, we investigate graph products of cyclic groups, that is, graph products for which all of the vertex groups G_v are *cyclic*. More generally, if all of the vertex groups of a graph product G_Γ are finitely generated Abelian groups, then G_Γ is naturally isomorphic to the graph product obtained by replacing each vertex in Γ by a complete graph with vertices labelled by the (indecomposable) cyclic summands of G_v. Thus, our results apply more generally to this class of groups.

Gutierrez and Piggott [14], generalizing work of Laurence [19], have shown that for any graph product of indecomposable cyclic groups, the graph Γ and the vertex groups G_v are uniquely determined by the isomorphism class of G_Γ. Thus, when referring to the graph product G_Γ, we may assume that this data has been specified.

For the remainder of the paper, we assume that all vertex groups are cyclic.

EXAMPLES 2.2. If all of the vertex groups are cyclic of order 2, then we obtain the right-angled Coxeter groups. If all of the vertex groups are infinite cyclic, then we obtain the right-angled Artin groups.

Given $g \in G_\Gamma$, a *reduced word* for g is a minimal length word $g_1 g_2 \ldots g_k$ in F_Γ (with each g_i belonging to some vertex group) representing g. Any word representing g can be reduced by a process of “shuffling” (i.e., interchanging commuting elements) and combining adjacent elements from the same vertex group. Any two reduced words representing g differ only by shuffling [12].

For any subset T of the vertex set V, let G_T denote the graph product associated to the full subgraph of Γ spanned by T. The natural map from G_T into G_Γ splits, hence G_T is isomorphic to its image and we make no distinction between them. By convention, we set $G_\emptyset = 1$.

To a graph product G_Γ, we associate a cubical complex X_Γ as follows. Define two sets, partially ordered by inclusion,

$$\begin{aligned} \mathcal{S}_\Gamma &= \{G_T \mid T \subseteq V,\ G_T \text{ is Abelian}\} \\ &\cong \{T \mid T \subseteq V,\ T \text{ spans a complete subgraph of } \Gamma\}, \\ G\mathcal{S}_\Gamma &= \{gG_T \mid g \in G_\Gamma,\ T \subseteq V,\ G_T \text{ is Abelian}\}. \end{aligned}$$

Let X_Γ be the geometric realization of the poset $G\mathcal{S}_\Gamma$ and let $K \subset X_\Gamma$ be the geometric realization of $\mathcal{S}_\Gamma$. Left multiplication of G_Γ on this poset induces an action of G_Γ on X_Γ. A fundamental domain for this action is K, and hence the action is cocompact. The stabilizer of the vertex gG_T is conjugate to G_T which is finite if and only if all the vertex groups in T are finite. Thus, the action of G_Γ on X_Γ is proper if and only if Γ is a graph of *finite* cyclic groups.

The complexes X_Γ are interesting in their own right. As we will now show, they have the structure of right-angled buildings. These buildings are based on a construction of Davis [9, 10]. In the case of a right-angled Coxeter group, X_G is the well-known Davis complex. For a right-angled Artin group, X_Γ is known as the Deligne complex (or in the terminology of [5], the "modified" Delinge complex). We follow [1] and [8] for basic definitions.

First, recall that a *chamber system over a set* S is a set Φ of *chambers* together with a family of equivalence relations on Φ indexed by S. For $s \in S$, we say two chambers are s*-adjacent* if they are s-equivalent, but not equal. For a word $w = s_1 \dots s_k$, $s_i \in S$, a *gallery* of type w is a sequence of chambers $\phi_0, \phi_1, \dots, \phi_k$ such that ϕ_{i-1} is s_i-adjacent to ϕ_i.

Now suppose S is the generating set of a right-angled Coxeter group W. A W*-valued distance function* on Φ is a function $d : \Phi \times \Phi \to W$ such that, given a reduced word $s_1 \dots s_k$ representing $w \in W$, there exists a gallery of type $s_1 \dots s_k$ from ϕ to ϕ' if and only if $d(\phi, \phi') = w$.

Definition 2.3. Let $W = W_\Gamma$ be a right-angled Coxeter group with generating set S. Then a *right-angled building of type* W is a chamber system Φ over S such that

(1) for all $s \in S$, every s-equivalence class contains at least two chambers,
(2) there exists a W-valued distance function $d : \Phi \times \Phi \to W$.

Let G_Γ be a graph product of cyclic groups. Denote by W_Γ the right-angled Coxeter group obtained by replacing each vertex group G_v by $W_v = \mathbb{Z}/2\mathbb{Z}$. Define a set-theoretic map (not a homomorphism) $\gamma : G_\Gamma \to W_\Gamma$ as follows. For $g \in G_\Gamma$, represent g by a reduced word $g = g_1 \dots g_k$, with $g_i \in G_{v_i}$, and set $\gamma(g) = s_1 s_2 \dots s_k$ where s_i is the generator of W_{v_i}. This is well-defined since any two reduced words for g are related by commutator relations which also hold in W_Γ. Moreover, $s_1 \dots s_k$ is also reduced since no shuffling of $g_1 \dots g_k$ (and hence of $s_1 \dots s_k$) allows two elements of the same vertex group to be combined.

Theorem 2.4. *For any graph product* G_Γ *of cyclic groups*, X_Γ *is a right-angled building of type* W_Γ.

Proof. We take Φ to be the set of translates of K in X_Γ and we say that two chambers gK, hK are s_i-equivalent if $g^{-1}h \in G_{v_i}$. Then every s_i-equivalence class contains q elements where $q = |G_{v_i}|$.

Define $d : \Phi \times \Phi \to W_\Gamma$ by $d(gK, hK) = \gamma(g^{-1}h)$. Then for a reduced word $w = s_1 s_2 \dots s_k$, there exists a gallery of type w from gK to hK if and only if $g^{-1}h = g_1 \dots g_k$ for some $g_i \in G_{v_i}$, or equivalently, $d(gK, hK) = w$. ☐

These buildings and their automorphism groups are studied by Barnhill, Thomas, Haglund, and Paulin [1, 16, 17, 22]. If the vertex groups are all finite, then the (full) automorphism group of the building is a locally compact topological group and G_Γ is a uniform lattice in this group.

Although X_Γ was defined as a simplicial complex, it has a natural cubical structure whose cubes correspond to "intervals". For a pair of subsets $T_1 \subseteq T_2$ in $\mathcal{S}_\Gamma$, the interval $[G_{T_1}, G_{T_2}]$ is the subcomplex of K spanned by the vertices G_T, $T_1 \subseteq T \subseteq T_2$. It is combinatorially a cube of dimension $|T_2 - T_1|$. The translates of these intervals give a cubical structure on all of X_Γ.

The fundamental chamber K is independent of the orders of the vertex groups. Thus, it is isometric to the fundamental chamber in the Davis complex for W_Γ. It was shown by Davis in [9] that any such right-angled building is CAT(0) with respect to the cubical metric described above. (This can also be proved directly for X_Γ using the link condition for cubical complexes.)

The action of G_Γ takes intervals to intervals, hence preserves the cubical metric and the quotient by G_Γ is just the fundamental chamber K. Thus G_Γ acts faithfully (the stabilizer of $G_\emptyset$ is trivial), cocompactly by isometries on X_Γ. As noted above, however, the action is proper if and only if every vertex group is finite.

COROLLARY 2.5. *For all graph products of cyclic groups, the cubical metric on X_Γ is* CAT(0). *If the vertex groups are all finite, then G_Γ is a* CAT(0) *group.*

REMARK 2.6. For use later in the paper, we remark that this action can be extended to a slightly larger group. Let Σ_Γ be the (finite) group of automorphisms of G_Γ generated by symmetries of the graph Γ (which permute the generators of G_Γ) and automorphisms of a single vertex group. This group acts on the poset $G\mathcal{S}_\Gamma$ in the obvious way, $\sigma \cdot gG_T = \sigma(gG_T)$, and hence it acts on X_Γ. Combining this with the G_Γ-action gives an action of the semi-direct product $G_\Gamma \rtimes \Sigma_\Gamma$ on X_Γ. This action is again proper, cocompact, isometric, and faithful.

3. Automorphism groups and separating intersections of links

In this section, we introduce the no SILs condition on Γ and study automorphism groups of graph products of cyclic groups G_Γ satisfying this condition.

Servatius [21] and Laurence [18, 19] described a finite generating set for $\mathrm{Aut}(G_\Gamma)$ for certain classes of graph products, such as right-angled Artin groups. This result has recently been extended to all graph products of cyclic groups by Corredor and Gutierrez in [7]. We now describe this generating set.

In order to simplify notation, we will think of the vertex v as the generator of the cyclic group G_v, so that the vertex set V generates G_Γ. Denote the order of v (and hence of G_v) by $|v|$. Associated to a vertex v in Γ are two subgraphs: the link of v, $\mathrm{lk}(v)$, is the full subgraph spanned by the vertices adjacent to v and the star of v, $\mathrm{st}(v)$, is the subgraph spanned by v and $\mathrm{lk}(v)$.

THEOREM 3.1 ([7, 18]). *If G_Γ is a graph product of cyclic groups, then $\mathrm{Aut}(G_\Gamma)$ is generated by automorphisms of the following types*:

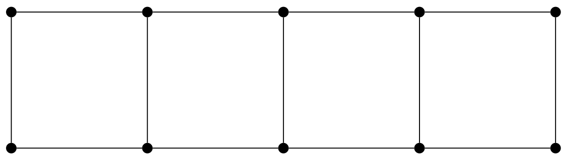

FIGURE 1. A graph with separating stars but no SILs.

(1) Symmetries: *induced by symmetries of* Γ, *permute the generators.*
(2) Vertex isomorphisms: *automorphisms of a single vertex group* G_v.
(3) Partial conjugations: *conjugate all of the generators in one connected component* C *of* $\Gamma \setminus \mathrm{st}(v)$ *by* v.
(4) Transvections: *map* $v \mapsto vw^k$ *or* $v \mapsto w^k v$ *where one of the following holds*
    (a) $|v| = \infty$, $k = 1$, *and* $\mathrm{lk}(v) \subseteq \mathrm{st}(w)$, *or*
    (b) $|v| = p^i$, $|w| = p^j$, $k = \max\{1, p^{j-i}\}$, *and* $\mathrm{st}(v) \subseteq \mathrm{st}(w)$.

We are interested primarily in the partial conjugations. Denote by $\pi_{v,C}$ the partial conjugation by v of the component C, and let $\mathrm{Aut}^{\mathrm{pc}}(W_\Gamma)$ denote the group generated by all partial conjugations.

It follows from Lemma 2.8 of [15] that when the vertex groups are all finite, $\mathrm{Aut}^{\mathrm{pc}}(W_\Gamma)$ has finite index in the full automorphism group $\mathrm{Aut}(G_\Gamma)$. This is also the case when there are no permissible transvections (for example, when Γ has no cycles of length less than 5 and no vertices of valence less than 2).

The interaction between two partial conjugations $\pi_{v,C}$ and $\pi_{w,D}$ depends on the relative position of the components C and D. A crucial role will be played by the following.

DEFINITION 3.2. A simplicial graph Γ has a *Separating Intersection of Links (SIL)* if for some pair (v, w), with $d_\Gamma(v, w) \geq 2$, there is a component of $\Gamma \setminus (\mathrm{lk}(v) \cap \mathrm{lk}(w))$ which contains neither v nor w.

REMARK 3.3. The no SILs condition is most interesting for connected graphs. For if Γ has more than two connected components, then it necessarily has a SIL. If it has two components Γ_1 and Γ_2, one of which is not a complete graph, then it also has a SIL since if v and w are vertices in Γ_1 with $d(v, w) \geq 2$, then $\Gamma_2 \subset \Gamma \setminus (\mathrm{lk}(v) \cap \mathrm{lk}(w))$ is a component containing neither v nor w. Thus, a graph with no SILs is either connected or it is the disjoint union of two complete graphs.

As illustrated in Figure 1, a graph with no SILs may still have separating stars, giving rise to partial conjugations. In the case where Γ has no SILs, we will prove that $\mathrm{Aut}^{\mathrm{pc}}(G_\Gamma)$ is itself a graph product of cyclic groups $G_{\tilde{\Gamma}}$ where the vertex set of $\tilde{\Gamma}$ corresponds to the set of partial conjugations. The edges of $\tilde{\Gamma}$ will correspond to the partial conjugations that commute, and are prescribed by the following lemma.

Lemma 3.4. *Suppose Γ is a connected simplicial graph which does not contain any SILs and let v and w be vertices of Γ. Suppose $d(v,w) \geq 2$, and let C_0 be the component of $\Gamma \setminus \mathrm{st}(v)$ containing w, and D_0 be the component of $\Gamma \setminus \mathrm{st}(w)$ containing v. Then*

(1) *Every component of $\Gamma \setminus \mathrm{st}(v)$, except C_0, lies entirely in D_0, and every component of $\Gamma \setminus \mathrm{st}(w)$, except D_0, lies entirely in C_0.*
(2) *The partial conjugations $\pi_{v,C}$ and $\pi_{w,D}$ commute unless $C = C_0$ and $D = D_0$.*

Proof. (1) Let C be a component of $\Gamma \setminus \mathrm{st}(v)$. If C contains any vertex of $\mathrm{lk}(w)$, then it also contains w, so $C = C_0$. If not, then $C \cap \mathrm{st}(w) = \emptyset$, so C lies completely in some component D of $\Gamma \setminus \mathrm{st}(w)$. We claim that $D = D_0$.

Let $\bar{C}$ denote the graph generated by C and the vertices adjacent to C. Clearly $\bar{C} \setminus C \subset \mathrm{lk}(v)$. On the other hand, $\bar{C} \setminus C \not\subset \mathrm{lk}(v) \cap \mathrm{lk}(w)$, since this would imply that C was a component of $\Gamma \setminus \mathrm{lk}(v) \cap \mathrm{lk}(w)$ which did not contain v or w. It follows that C and v are adjacent to a vertex which is not in the link of w. Hence, C and v are in the same component of $\Gamma \setminus \mathrm{st}(w)$, that is, $D = D_0$ as claimed.

(2) First, we note that π_{v,C_0} and π_{w,D_0} do not commute. By direct computation,

$$\pi_{v,C_0} \circ \pi_{w,D_0}(v) = \pi_{v,C_0}(wvw^{-1}) = vwvw^{-1}v^{-1},$$
$$\pi_{w,D_0} \circ \pi_{v,C_0}(v) = \pi_{w,D_0}(v) = wvw^{-1}.$$

Next, consider the case where $C \neq C_0$ and $D \neq D_0$. By (1) $C \cap D = \emptyset$, and we do another direct computation.

$$\pi_{v,C} \circ \pi_{w,D}(x) = \pi_{w,D} \circ \pi_{v,C}(x) = \begin{cases} vxv^{-1}, & x \in C, \\ wxw^{-1}, & x \in D, \\ x, & x \notin (C \cup D). \end{cases}$$

Now suppose $C \neq C_0$ and $D = D_0$. Then by (1), we know that $C \subset D$, $v \in D$, and $w \notin C$. We can once again check this by direct computation.

$$\pi_{v,C} \circ \pi_{w,D}(x) = \pi_{w,D} \circ \pi_{v,C}(x) = \begin{cases} wvxv^{-1}w^{-1}, & x \in C, \\ wxw^{-1}, & x \in D \setminus C, \\ x, & x \notin D. \end{cases}$$

The remaining case where $C = C_0$ and $D \neq D_0$ is similar. □

We now construct the graph $\tilde{\Gamma}$. The vertices of $\tilde{\Gamma}$ are in one-to-one correspondence with the partial conjugations $\pi_{v,C}$, and are denoted by $\tilde{V} = \{p_{v,C}\}$. Any two vertices $p_{v,C}$ and $p_{w,D}$ are connected by an edge unless $d(v,w) \geq 2$,

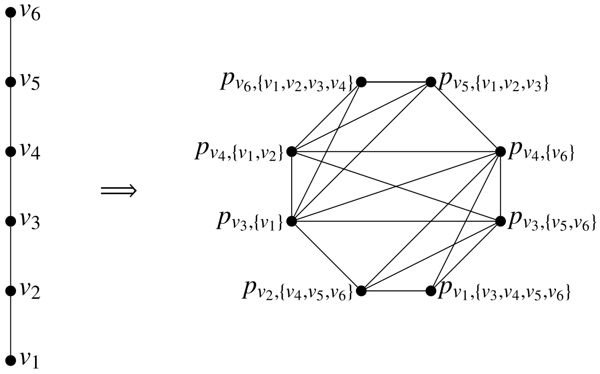

FIGURE 2. The graphs Γ and $\tilde{\Gamma}$.

$v \in D$, and $w \in C$. We assign to the vertex $p_{v,C}$ the cyclic group of order $|v|$. An example is shown in Figure 2.

By the lemma, there is a homomorphism $\phi : G_{\tilde{\Gamma}} \to \mathrm{Aut}^{\mathrm{pc}}(G_\Gamma)$ which takes $p_{v,C} \mapsto \pi_{v,C}$. This homomorphism is clearly surjective; our goal is to prove that if Γ contains no SILs, then ϕ is an isomorphism. To do this, we will pass to the outer automorphism group.

The outer automorphism group of G_Γ is the quotient of $\mathrm{Aut}(G_\Gamma)$ by the subgroup $\mathrm{Inn}(G_\Gamma)$ of inner automorphsims of G_Γ. The inner automorphism by a vertex v is the product of all the partial conjugations by v, hence this subgroup lies in $\mathrm{Aut}^{\mathrm{pc}}(G_\Gamma)$ and we can define $\mathrm{Out}^{\mathrm{pc}}(G_\Gamma)$ accordingly. We would like to define a corresponding quotient for $G_{\tilde{\Gamma}}$.

The inner automorphism group is isomorphic to the group modulo its center. In the case of G_Γ, the center is generated by the vertices (if any) which are connected to every other vertex in Γ. Let Δ be the (possibly empty) graph generated by these vertices and let $\Gamma_0 = \Gamma \setminus \Delta$. Then G_Γ decomposes as the direct product of G_Δ and G_{Γ_0}, so $\mathrm{Inn}(G_\Gamma)$ is isomorphic to G_{Γ_0}.

We denote by p_v the product $p_v = \prod p_{v,C}$ over all components C of $\Gamma \setminus \mathrm{st}(v)$, so that $\phi(p_v)$ is the inner automorphism by v.

LEMMA 3.5. *The correspondence $v \mapsto p_v$ induces a homomorphism $\tilde{f} : G_{\Gamma_0} \to G_{\tilde{\Gamma}}$ and the image of $\tilde{f}$ is a normal subgroup of $G_{\tilde{\Gamma}}$.*

Proof. The first statement follows by definition of $\tilde{\Gamma}$ since if $d(v, w) \leq 1$ in Γ, then $p_{v,C}$ commutes with $p_{w,D}$ for all C, D, so commuting relations are preserved and the order of p_v is $|v|$.

To prove that the image is normal, we will show that for any $v \in \Gamma_0$ and for any generator of $p_{w,D}$ of $G_{\tilde{\Gamma}}$, the following equation holds.

$$p_{w,D}p_v p_{w,D}^{-1} = \begin{cases} p_v, & \text{if } v \notin D, \\ p_w p_v p_w^{-1}, & \text{if } v \in D. \end{cases}$$

Note that in either case, $p_{w,D}p_v p_{w,D}^{-1}$ is in $\tilde{f}(G_{\Gamma_0})$.

Case 1: $v \notin D$. Then by Lemma 3.4 $p_{w,D}$ commutes with $p_{v,C}$ for every C.

Case 2: $v \in D$. Consider the expression $p_w p_v p_w^{-1}$. Then for each connected component D' of $\Gamma \setminus \mathrm{st}(w)$ with $D \neq D'$, the partial conjugation $p_{w,D'}$ commutes with p_v by Lemma 3.4. Simplifying the expression, we get the desired result. □

In light of the lemma, we can now form the quotient group, $Q = G_{\tilde{\Gamma}}/\tilde{f}(G_{\Gamma_0})$. If f denotes the inclusion of the inner automorphisms into $\mathrm{Aut}^{\mathrm{pc}}(G_\Gamma)$, then the diagram below clearly commutes.

$$\begin{tikzcd} G_{\Gamma_0} \arrow[r, "\tilde{f}"] \arrow[d, "\cong"] & G_{\tilde{\Gamma}} \arrow[d, "\phi"] \\ \mathrm{Inn}(G_\Gamma) \arrow[r, "f"] & \mathrm{Aut}^{\mathrm{pc}}(G_\Gamma) \end{tikzcd}$$

It follows that $\tilde{f}$ is injective and that ϕ induces a map on the quotient groups, so we have a commutative diagram of exact sequences,

$$\begin{tikzcd} 1 \arrow[r] & G_{\Gamma_0} \arrow[r, "\tilde{f}"] \arrow[d, "\cong"] & G_{\tilde{\Gamma}} \arrow[r, "\tilde{g}"] \arrow[d, "\phi"] & Q \arrow[r] \arrow[d, "\psi"] & 1 \\ 1 \arrow[r] & \mathrm{Inn}(G_\Gamma) \arrow[r, "f"] & \mathrm{Aut}^{\mathrm{pc}}(G_\Gamma) \arrow[r, "g"] & \mathrm{Out}^{\mathrm{pc}}(G_\Gamma) \arrow[r] & 1 \end{tikzcd}$$

We are now ready to state and prove our main result.

THEOREM 3.6. *Let G_Γ be a graph product of cyclic groups whose defining graph Γ contains no SILs. Then the map $\phi : G_{\tilde{\Gamma}} \to \mathrm{Aut}^{\mathrm{pc}}(G_\Gamma)$ is an isomorphism. In particular,* $\mathrm{Aut}^{\mathrm{pc}}(G_\Gamma)$ *is a graph product of cyclic groups of the same order(s) as the vertex groups of G_Γ.*

Proof. In light of the exact sequence above, it suffices to prove that the map ψ on the quotient groups is an isomorphism. The theorem will then follow from the 5-lemma. Since ϕ and g are both surjective, $\psi \circ \tilde{g} = g \circ \phi$ is surjective, and so ψ is as well.

Suppose Γ is connected. We first argue that Q is Abelian. Take two generators $p_{v,C}$ and $p_{w,D} \in G_{\tilde{\Gamma}}$ which *do not* commute. By Lemma 3.4, we know that $p_{v,C}$ *does* commute with $p_{w,D'}$ for every connected component D' of

$\Gamma \setminus \mathrm{st}(w)$ with $D' \neq D$, hence it commutes with the product $p' = \prod_{D' \neq D} p_{w,D'}$. But p' and $p_{w,D}$ represent inverse elements in Q, so the images of $p_{v,C}$ and $p_{w,D}$ commute in Q.

Since Q is Abelian, for any $\bar{x} \in Q$, we can write $\bar{x}$ as a product of generators $p_{v,C}$ in which all occurrences of a vertex v appear together. That is, we can choose a representative $x \in G_{\tilde{\Gamma}}$ of the form

$$x = \prod_v \prod_C p_{v,C}^{k_{v,C}}.$$

Now suppose $\bar{x} \in \ker(\psi)$. By the commutivity of the relevant diagram, $\phi(x) = \prod_v \prod_C \pi_{v,C}^{k_{v,C}}$ lies in $\mathrm{Inn}(W_\Gamma)$. A product of this form is an inner automorphism if and only if, for fixed v, the power $k_{v,C}$ is the same for every component C of $\Gamma \setminus \mathrm{st}(v)$. This means that x has the form $\prod_v p_v^{k_v}$, which is clearly in G_{Γ_0}, thus $\bar{x}$ is trivial in Q. We conclude that the kernel of ψ is trivial, so ψ is an isomorphism.

It remains to consider the case when Γ is not connected. By Remark 3.3, this occurs only when Γ is the disjoint union of two complete graphs. In this case, every partial conjugation is an inner automorphism, so $\Gamma = \tilde{\Gamma}$, and since G_Γ has trivial center, $\mathrm{Aut}^{\mathrm{pc}}(G_\Gamma) = \mathrm{Inn}(G_\Gamma) \cong G_\Gamma$. □

As noted in Section 2, any graph product of finitely generated Abelian groups is isomorphic to a graph product of cyclic groups obtained by "blowing up" a vertex v with group G_v into a complete graph with vertices labeled by the indecomposable cyclic summands of G_v. Applying Theorem 3.6 to this new graph product, we see that $\mathrm{Aut}^{\mathrm{pc}}(G_\Gamma)$ is again a graph product of cyclic groups. Moreover, this graph product is just the blow-up of $G_{\tilde{\Gamma}}$ (where $\tilde{\Gamma}$ is defined as above, but the vertex groups G_v are no longer cyclic). Thus, we may restate the theorem as follows.

Theorem 3.7. *Let G_Γ be a graph product of finitely generated Abelian groups whose defining graph Γ contains no SILs. Then the map $\phi : G_{\tilde{\Gamma}} \to \mathrm{Aut}^{\mathrm{pc}}(G_\Gamma)$ is an isomorphism. In particular, $\mathrm{Aut}^{\mathrm{pc}}(G_\Gamma)$ is also a graph product of finitely generated Abelian groups.*

We remark that the proof of the theorem above also gives an independent proof of the following result of [15].

Corollary 3.8 ([15]). *Assume Γ is connected. Then $\mathrm{Out}^{\mathrm{pc}}(G_\Gamma)$ is Abelian if and only if Γ contains no SILs.*

Proof. In the proof of the main theorem, we showed that if Γ has no SILs, then Q, and hence $\mathrm{Out}^{\mathrm{pc}}(G_\Gamma)$, is Abelian. If Γ has a SIL, that is a component C of $\Gamma \setminus (\mathrm{lk}(v) \cap \mathrm{lk}(w))$ containing neither v nor w where $d(v,w) \geq 2$, then it is straightforward to check that the commutator $[\pi_{v,C}, \pi_{w,C}]$ is not an inner automorphism. Hence, $\mathrm{Out}^{\mathrm{pc}}(G_\Gamma)$ is not Abelian. □

4. Geometric implications

Recall that a group G is a CAT(0) group if it acts geometrically (i.e., properly, cocompactly by isometries) on a complete CAT(0) space. It is an open question whether automorphism groups of graph products, even in the Coxeter group case, are CAT(0) groups. Theorem 3.6 gives some partial answers.

In the case where all vertex groups are finite, $\mathrm{Aut}^{\mathrm{pc}}(G_\Gamma)$ has finite index in $\mathrm{Aut}(G_\Gamma)$ so by Theorems 2.4 and 3.6 we obtain

Corollary 4.1. *Let G_Γ be a graph product of finite cyclic groups whose defining graph has no SILs. Then the automorphism group $\mathrm{Aut}(G_\Gamma)$ is virtually $\mathrm{CAT}(0)$. More precisely, there is a faithful, geometric action of $\mathrm{Aut}^{\mathrm{pc}}(G_\Gamma)$ on the right-angled building $X_{\tilde{\Gamma}}$.*

It would be nice to extend this action to the whole automorphism group. Recall from Theorem 3.1 that $\mathrm{Aut}(G_\Gamma)$ is generated by four types of automorphisms: symmetries, vertex isomorphisms, partial conjugations, and transvections. Letting Σ_Γ denote the group generated by symmetries and vertex isomorphisms, the subgroup of $\mathrm{Aut}(G_\Gamma)$ generated by the first three types of automorphisms is a semi-direct product, $\mathrm{Aut}^{\mathrm{pc}}(G_\Gamma) \rtimes \Sigma_\Gamma$. We can easily extend the action of $\mathrm{Aut}^{\mathrm{pc}}(G_\Gamma)$ on $X_{\tilde{\Gamma}}$ to this larger group.

Corollary 4.2. *Let G_Γ be a graph product of finite cyclic groups whose defining graph has no SILs. Then the action of $\mathrm{Aut}^{\mathrm{pc}}(G_\Gamma)$ on $X_{\tilde{\Gamma}}$ extends to a faithful, geometric action of $\mathrm{Aut}^{\mathrm{pc}}(G_\Gamma) \rtimes \Sigma_\Gamma$.*

Proof. By Remark 2.6, the action of $G_{\tilde{\Gamma}}$ on $X_{\tilde{\Gamma}}$ extends to a faithful, geometric action of the semi-direct product $G_{\tilde{\Gamma}} \rtimes \Sigma_{\tilde{\Gamma}}$. The group Σ_Γ embeds naturally in $\Sigma_{\tilde{\Gamma}}$ (an isomorphism of G_v goes to the product of the corresponding isomorphisms of $G_{p_{v,C}}$ for all components C). Combining this embedding with the isomorphism ϕ^{-1}, we get an inclusion $\mathrm{Aut}^{\mathrm{pc}}(G_\Gamma) \rtimes \Sigma_\Gamma \hookrightarrow G_{\tilde{\Gamma}} \rtimes \Sigma_{\tilde{\Gamma}}$, and hence an induced action on $X_{\tilde{\Gamma}}$. $\square$

If some of the vertex groups are infinite cyclic, then the action of $G_{\tilde{\Gamma}}$ on $X_{\tilde{\Gamma}}$ is not proper. However, if *all* of the vertex groups are infinite, then G_Γ and $G_{\tilde{\Gamma}}$ are right-angled Artin groups and we can use a different geometric construction, the Salvetti complex (see [3]), to get an action on a CAT(0) space.

Corollary 4.3. *Let Γ be a simplicial graph with no SILs, and suppose G_Γ is a right-angled Artin group. Then the subgroup of $\mathrm{Aut}(G_\Gamma)$ generated by partial conjugations, inversions and graph symmetries acts faithfully and geometrically on a $\mathrm{CAT}(0)$ cube complex, the Salveti complex of G_Γ.*

Proof. It is easy to show that the action of $A_{\tilde{\Gamma}}$ on its Salvetti complex extends to an action of $A_{\tilde{\Gamma}} \rtimes \Sigma_{\tilde{\Gamma}}$. The proof then proceeds as above. $\square$

We close by remarking that some graph products G_Γ of cyclic groups do not permit transvections in which case the subgroup in Corollaries 4.2 and 4.3 constitutes the entire automorphism group. This is the case, for example, if Γ has no triangles and no vertices of valence less than two, or if every pair of adjacent vertex groups have relatively prime order. For those that do permit transvections, the action described above does not extend in any natural way to an isometric action of the transvections. In this case, proving that the full automorphism group is CAT(0) will almost certainly require a different space.

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