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ON MONOTONICITY OF F-BLOWUP SEQUENCES

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ABSTRACT. For each variety in positive characteristic, there is a series of canonically defined blowups, called F-blowups. We are interested in the question of whether the $(e+1)$th blowup dominates the eth, locally or globally. It is shown that the answer is affirmative (globally for any e) when the given variety is F-pure. As a corollary, we obtain some result on the stability of the sequence of F-blowups. We also give a sufficient condition for local domination.

1. Introduction

The F-blowup introduced in [15] is an interesting notion which relates, for instance, to the desingularization problem, the G-Hilbert scheme and Gröbner bases. The study of it has just started, and there remain various problems. Among them, it seems important to understand the behavior of the sequence consisting of F-blowups.

Consider a variety X in positive characteristic, that is, a separated integral scheme of finite type over an algebraically closed field k of characteristic $p > 0$. Let

$$F_e : X_e \to X, \quad e = 0, 1, 2, \ldots$$

be the e-times iteration of the k-linear Frobenius. Then for each smooth (closed) point $x \in X$, the fiber $F_e^{-1}(x)$ is a fat point of X_e of length $p^{e \cdot \dim X}$. It is considered as a reduced point of the Hilbert scheme of 0-dimensional subschemes of X_e of this length: $F_e^{-1}(x) \in \mathrm{Hilb}_{p^{e\cdot\dim X}}(X_e)$.

DEFINITION 1.1. We define the eth *F-blowup* of X, $\mathrm{FB}_e(X)$, as the closure of the subset

$$\{F_e^{-1}(x) | x \in X(k) \text{ smooth}\} \subset \mathrm{Hilb}_{p^{e\cdot\dim X}}(X_e).$$

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This is indeed a blowup of X, that is birational and projective over X (Proposition 2.1).

It is natural to ask if $\mathrm{FB}_{e+1} X$ dominates $\mathrm{FB}_e X$, that is, if the natural birational map

$$\rho_e : \mathrm{FB}_{e+1} X \dashrightarrow \mathrm{FB}_e X$$

has no indeterminacy. When this holds for all e, we shall say the F-blowup sequence is *monotone*. The answer is generally negative (Example 5.5). One of our main theorems provides a sufficient condition for the monotonicity.

THEOREM 1.2. *Suppose that X is F-pure, that is, the natural morphism $\mathcal{O}_X \to (F_1)_*\mathcal{O}_X$ locally splits as a morphism of $\mathcal{O}_X$-modules. Then the F-blowup sequence of X is monotone.*

The notion of F-purity was introduced by Hochster and Roberts [9], and is now one of important classes of F-singularities (see [11] and the references given there).

We can consider also the local version of the above question: "Is ρ_e defined at a given point of $\mathrm{FB}_{e+1} X$?" We give a sufficient condition for this too.

THEOREM 1.3. *Let $Z \in \mathrm{FB}_{e+1} X$ be a closed point, which is identified with a fat point of X_{e+1}. Suppose that*

(*) *the scheme-theoretic image $\bar{Z}$ of Z by the natural morphism $X_{e+1} \to X_e$ belongs to $\mathrm{FB}_e X$.*

Then ρ_e is defined at Z and $\rho_e(Z) = \bar{Z}$.

Condition (*) means that $Z \in \mathrm{FB}_{e+1} X$ has a natural candidate $\bar{Z}$ of the image in $\mathrm{FB}_e X$. In fact, Theorem 1.3 is a generalization of Theorem 1.2, since (*) always holds if X is F-pure (Proposition 4.1).

We saw in [15] that in some cases, the F-blowup sequence is bounded, that is, all F-blowups of X are dominated by a single blowup of X. (At this point, I do not know of any example where the sequence is unbounded. See Example 6.3.) When both the boundedness and monotonicity hold, the sequence stabilizes (Lemma 6.2).

It is also natural to ask what properties of variety are preserved by F-blowups. We obtain the following result on this issue.

PROPOSITION 1.4. *There exist an F-pure (resp. normal, weakly normal) variety X and $e \in \mathbb{Z}_{>0}$ such that $\mathrm{FB}_e X$ is not F-pure (resp. normal, weakly normal).*

We use the toric geometry in order to construct examples and prove the last proposition. For this purpose, we show that a toric variety is F-pure if and only if it is weakly normal (Proposition 5.3). A similar result has been obtained by Bruns, Li, and Römer [3, Proposition 6.2].

Outline of the paper. In Section 2, we recall some basic facts on F-blowup from [15]. Section 3 is devoted to the proof of Theorem 1.2. In Section 4, we prove that Condition (*) holds whenever X is F-pure and Theorem 1.3. In Section 5, we use the toric geometry to give some examples of F-blowups. In Section 6, we discuss when the F-blowup sequence stabilizes. In Section 7, we prove Proposition 1.4 by using the toric geometry and the nonnormal G-Hilbert scheme found by Craw, Maclagan, and Thomas [4].

Conventions. Throughout the paper, we work over an algebraically closed field k of characteristic $p>0$. A variety means a separated integral scheme of finite type over k. A point of a variety always means a closed point.

2. Preliminaries

In this section, we set up notation and recall some results from [15].[1]

We continue to write X for a given variety over k. All our problems are local on X. So we may suppose X is affine; $X = \operatorname{Spec} R$. Let $e \in \mathbb{Z}_{\geq 0}$ and $q := p^e$. Then we may identify $X_e = \operatorname{Spec} R^{1/q}$ and then $F_e : X_e \to X$ corresponds to the inclusion map $R \hookrightarrow R^{1/q}$. We also have $(F_e)_*\mathcal{O}_X = \mathcal{O}_X^{1/q}$ and $X_e = \mathcal{S}pec_X \mathcal{O}_X^{1/q}$.

The F-blowup can be constructed also with the *relative* Hilbert scheme or the Quot scheme.

Proposition 2.1 ([15, Proposition 2.4]). *The F-blowup* $\mathrm{FB}_e(X)$ *is canonically isomorphic to the irreducible component of* $\mathrm{Hilb}_{q^d}(X_e/X)$ *that dominates* X, *and also to that of* $\mathrm{Quot}_{q^d}(\mathcal{O}_X^{1/q})$.

Moreover, the proof of [15, Proposition 2.4] shows that the isomorphism is the restriction of the natural morphism $\mathrm{Hilb}_{q^d}(X_e/X) \to \mathrm{Hilb}_{q^d}(X_e)$. It follows that each point $Z \in \mathrm{FB}_e(X)$ is included in the fiber $F_e^{-1}(x)$ for some reduced point $x \in X$. Namely the scheme-theoretic image $F_e(Z) \subset X$ is a reduced point. Then the X-scheme structure of $\mathrm{FB}_e(X)$ is given by the map

$$\pi_e : \mathrm{FB}_e(X) \to X, \qquad Z \mapsto \pi_e(Z) := F_e(Z).$$

This is projective and is an isomorphism exactly over the smooth locus of X [15, Corollary 2.5].

Being an irreducible component of the Quot scheme, $\mathrm{FB}_e(X)$ has the following universal property: For a blowup $f : Y \to X$ and a coherent $\mathcal{O}_X$-module $\mathcal{F}$, define the *torsion-free pull-back* $f^\bigstar\mathcal{F}$ the quotient of the usual pull-back $f^*\mathcal{F}$ by the subsheaf of torsions. Then $\pi_e^\bigstar \mathcal{O}_X^{1/q}$ is flat, or equivalently, locally free. Moreover, if for a blowup $f : Y \to X$, $f^\bigstar\mathcal{O}_X^{1/q}$ is flat, then f factors through $\mathrm{FB}_e X$.

[1] In [15], the Frobenius map $R^q \hookrightarrow R$, rather than $R \hookrightarrow R^{1/q}$, is considered. This causes slight notational differences.

More generally, if $\mathcal{G}$ is a coherent sheaf on X and if it is generically locally free of rank r, then its universal (birational) flattening is constructed as the irreducible component of $\mathrm{Quot}_r(\mathcal{G})$ dominating X. See for instance [10, 14] for studies on the universal flattening of a general coherent module.

3. Proof of Theorem 1.2

We may suppose that X is affine. Then for each e, we have an isomorphism of $\mathcal{O}_X$-modules

$$\mathcal{O}_X^{1/p^{e+1}} \cong \mathcal{O}_X^{1/p^e} \oplus \mathcal{M}_e$$

for some $\mathcal{O}_X$-module $\mathcal{M}_e$. Then the torsion-free pull-back by π_{e+1}

$$\pi_{e+1}^{\bigstar}\mathcal{O}_X^{1/p^{e+1}} \cong \pi_{e+1}^{\bigstar}\mathcal{O}_X^{1/p^e} \oplus \pi_{e+1}^{\bigstar}\mathcal{M}_e$$

is flat and locally free. From the characterization of flat module as a summand of a free module [5, Corollary 6.6], $\pi_{e+1}^{\bigstar}\mathcal{O}_X^{1/p^e}$ is flat. From the universality of $\mathrm{FB}_e\, X$, we have a natural morphism $\mathrm{FB}_{e+1}\, X \to \mathrm{FB}_e\, X$. We have proved the theorem.

4. On local domination by $\mathrm{FB}_{e+1}\, X$ over $\mathrm{FB}_e\, X$

Proposition 4.1. *Suppose that X is F-pure. Then Condition (*) in Theorem 1.3 holds for every $e \geq 0$ and every $Z \in \mathrm{FB}_{e+1}\, X$.*

Proof. The identity map of $\pi_{e+1}^{\bigstar}\mathcal{O}_X^{1/p^e}$ can be factored as

$$\pi_{e+1}^{\bigstar}\mathcal{O}_X^{1/p^e} \to \pi_{e+1}^{\bigstar}\mathcal{O}_X^{1/p^{e+1}} \to \pi_{e+1}^{\bigstar}\mathcal{O}_X^{1/p^{e+1}}/\pi_{e+1}^{\bigstar}\mathcal{M}_e \cong \pi_{e+1}^{\bigstar}\mathcal{O}_X^{1/p^e}.$$

Taking the fibers of these sheaves at Z, we obtain

$$\mathrm{id}_{k[Z']} : k[Z'] \to k[Z] \to k[Z'].$$

Here $Z' \in \mathrm{FB}_e\, X$ is the image of Z by the natural map $\mathrm{FB}_{e+1}\, X \to \mathrm{FB}_e\, X$, which exists from Theorem 1.2, and $k[Z]$ and $k[Z']$ are the coordinate rings of fat points $Z \subset X_{e+1}$ and $Z' \subset X_e$, respectively. Hence, the map $k[Z'] \to k[Z]$, which is the ring homomorphism defining the natural morphism $Z \to Z'$, is injective. This shows that $Z' = \bar{Z}$ and the proposition follows. □

Proof of Theorem 1.3. We write $Z_{e+1} := Z$ and $Z_e := \bar{Z}$. Let $G \subset \mathrm{FB}_{e+1}(X) \times_k \mathrm{FB}_e(X)$ be the closure of the graph of $\rho_e : \mathrm{FB}_{e+1}(X) \dashrightarrow \mathrm{FB}_e(X)$ and $\psi_i : G \to \mathrm{FB}_i(X)$, $i = e, e+1$, the projections. We have to show that ψ_{e+1} is an isomorphism around $a := (Z_{e+1}, Z_e) \in G$.

We shall first show that set-theoretically $\psi_{e+1}^{-1}(Z_{e+1}) = \{a\}$. For $i = e, e+1$, let $W_i \subset G \times_k X_i$ be the family of fat points over G. More precisely, this is the pull-back of the universal family over $\mathrm{FB}_i(X)$, which is a closed subscheme of $\mathrm{FB}_i(X) \times_k X_i$, by the projection ψ_i. Then W_i is isomorphic to the associated reduced scheme of $G \times_X X_i$. In other words, $\mathcal{O}_{W_i}$ is identified with the torsion-free pull-back of $\mathcal{O}_{X_i} = \mathcal{O}_X^{1/p^i}$ by the natural map $G \to X$. In particular,

W_i is reduced. Hence, W_e is the scheme-theoretic image of W_{e+1} by the natural morphism $G \times_k X_{e+1} \to G \times_k X_e$. If $b = (Y_{e+1}, Y_e) \in G$, then the fiber of $W_{e+1} \to G$ (resp. $W_e \to G$) over b is Y_{e+1} (resp. Y_e). It follows that the scheme-theoretic image $\bar{Y}_{e+1}$ of Y_{e+1} in X_e is included in Y_e. Now if $Y_{e+1} = Z_{e+1}$, then $Z_e := \bar{Y}_{e+1} \subset Y_e$. But by assumption, both Z_e and Y_e have length p^{ed}. Hence, $Z_e = Y_e$. This shows that $\psi_{e+1}^{-1}(Z_{e+1}) = \{a\}$.

Let R be the coordinate ring of X as before, $\mathfrak{z}_i \subset R^{1/p^i}$ the defining ideals of Z_i and

$$\phi_i : T_a G \to T_{Z_i} \mathrm{FB}_i(X) \hookrightarrow \mathrm{Hom}(\mathfrak{z}_i, R^{1/p^i}/\mathfrak{z}_i)$$

the maps of Zariski tangent spaces (for the identification of the tangent space to the Hilbert scheme with $\mathrm{Hom}(\mathfrak{z}_i, R^{1/p^i}/\mathfrak{z}_i)$, see for instance [6, Proof of Theorem VI-29]). To show that ψ_{e+1} is an isomorphism around a, it is enough to show that ϕ_{e+1} is injective. Take $0 \neq v \in T_a G$. If $\phi_e(v) = 0$, then $\phi_{e+1}(v) \neq 0$. So we may suppose that $\phi_e(v) \neq 0$. Let

$$W_e^v \subset \operatorname{Spec} R^{1/p^e}[t]/(t^2) \quad \text{and} \quad W_{e+1}^v \subset \operatorname{Spec} R^{1/p^{e+1}}[t]/(t^2)$$

be the pull back of W_e and W_{e+1} by our tangent vector

$$v : \operatorname{Spec} k[t]/(t^2) \to G.$$

Since $\phi_e(v) \neq 0$, the defining ideal of W_e^v does contain an element of the form $f + gt$, $f \in \mathfrak{z}_e$, $g \in R^{1/p^e} \setminus \mathfrak{z}_e$ so that $\phi_e(v) \in \mathrm{Hom}(\mathfrak{z}_e, R^{1/p^e}/\mathfrak{z}_e)$ maps f to the class of g modulo $\mathfrak{z}_e$, which is nonzero.

Such an element $f + gt$ is also contained in the defining ideal of W_{e+1}^v and $\phi_{e+1}(v)$ maps f to g modulo $\mathfrak{z}_{e+1}$. Since by assumption $\mathfrak{z}_e = \mathfrak{z}_{e+1} \cap R^{1/p^e}$, we have $g \notin \mathfrak{z}_{e+1}$. Hence $\phi_{e+1}(v) \neq 0$ and ϕ_{e+1} is injective, which completes the proof. $\square$

5. The toric case

Let $M = \mathbb{Z}^d$ be a free Abelian group of rank d and $A \subset M$ a finitely generated submonoid which generates M as a group. We associate to A and M the monoid algebras $k[A] \subset k[M] = \bigoplus_{m \in M} k \cdot x^m$ and the affine toric varieties $X := \operatorname{Spec} k[A] \supset T := \operatorname{Spec} k[M]$. We shall make an additional assumption that A contains no nontrivial group or equivalently the cone $A_\mathbb{R} \subset M_\mathbb{R}$ spanned by A has a vertex. This involves no loss of generality.[2]

Let $A_\mathbb{R}^\vee \subset M_\mathbb{R}^\vee$ be the dual cone of $A_\mathbb{R}$, which is d-dimensional since $A_\mathbb{R}$ is strongly convex. The F-blowup $\mathrm{FB}_e X$ is a (possibly nonnormal) toric

[2] Conversely, suppose that A contains a nontrivial group. Let $B \subset A$ be the maximal group and let $a_1, \dots, a_m, b_1, \dots, b_n$ be generators of A such that $a_i \notin B$ and $b_i \in B$. Then there exists a subset of $\{b_1, \dots, b_n\}$, say $\{b_1, \dots, b_l\}$, $l \le n$, which generates a monoid containing no nontrivial group but still generates B as a group. Let A' be the monoid generated by $a_1, \dots, a_m, b_1, \dots, b_l$, which contains no nontrivial group. Then $k[A]$ is a localization of $k[A']$ by an element. Indeed if we put $b := \sum_{i=1}^l b_i$, then $k[A] = k[A']_{x^b}$. Thus, the toric variety associated to A is an open subvariety of the one associated to A'.

variety and determines a fan Δ_e which is a subdivision of $A^\vee_{\mathbb{R}}$. For each d-dimensional cone $\sigma \in \Delta_e$, there exists a corresponding affine toric open subvariety $U_\sigma \subset X$. By the inclusion $T \subset U_\sigma$, the coordinate rings of each U_σ is naturally embedded in $k[M]$. It is expressed as follows: Fix a d-dimensional $\sigma \in \Delta_e$ and an interior point $w \in \sigma$. Then put

$$B_\sigma := \left\{ m \in \frac{1}{q}A \middle| \exists m' \in \frac{1}{q}A, m - m' \in M, \langle m, w\rangle > \langle m', w\rangle \right\}$$

and

$$C_\sigma := \left\{ m - m' \middle| m \in \frac{1}{q}A, m' \in \frac{1}{q}A \setminus B_\sigma, m - m' \in M, \langle m, w\rangle > \langle m', w\rangle \right\}.$$

THEOREM 5.1 ([15, Proposition 3.8]). *The coordinate ring of U_σ is generated by x^c, $c \in C_\sigma$ as a k-algebra.*

Now we recall the notion of weak normality in the sense of Andreotti and Bombieri [1].

DEFINITION 5.2. An affine variety $\operatorname{Spec} R$ with function field K is said to be *weakly normal* if $R = R^{1/p} \cap K$.

We easily see that the monoid algebra $k[A]$ is weakly normal if and only if $A = \frac{1}{p}A \cap M$.

It has been known to experts that the F-purity implies the weak normality (for example, see [2, Proposition 1.2.5]). For the monoid algebra, the converse is also true.

PROPOSITION 5.3. *The ring $k[A]$ is F-pure if and only if it is weakly normal.*

Proof. Although I do not know of any reference, maybe this result is known to experts. Bruns, Li, and Römer [3, Proposition 6.2] have proved a similar result. Suppose that $k[A]$ is weakly normal, so $A = \frac{1}{p}A \cap M$. We define a k-linear map $\phi : k[\frac{1}{p}A] \to k[A]$ by

$$\phi(x^m) = \begin{cases} x^m, & m \in A, \\ 0, & m \notin A. \end{cases}$$

We claim that it is a $k[A]$-module homomorphism and hence the inclusion map $k[A] \hookrightarrow k[\frac{1}{p}A]$ splits. To see this, it is enough to show that for any $m \in \frac{1}{p}A$ and $n \in A$,

$$\phi(x^{m+n}) = x^n \phi(x^m). \tag{1}$$

When $m \in A$, this is obvious. If $m \notin A$, then $m + n \notin A$. (Conversely, if $m + n \in A$, then $m + n \in M$ and $m \in M \cap \frac{1}{p}A = A$, a contradiction.) Hence, (1) holds in this case too. $\square$

As a corollary, we recover [3, Corollary 6.3].

COROLLARY 5.4. *$k[A]$ is normal, which is of course independent of the characteristic, if and only if it is weakly normal (equivalently F-pure) in arbitrary positive characteristic.*

Proof. $k[A]$ is normal if and only if for any $m \in M$ and $n \in \mathbb{Z}_{>0}$ with $nm \in A$, we have $m \in A$ if and only if for any $m \in M$ and every prime number p with $pm \in A$, we have $m \in A$. The last condition is equivalent to that $k[A]$ is weakly normal in every positive characteristic. $\square$

EXAMPLE 5.5 (An example where the monotonicity fails). Suppose k has Characteristic 2 and $A \subset \mathbb{Z}_{\geq 0}$ is the monoid generated by $8, 9, 10, 11$.

$$A = \{0, 8, 9, 10, 11, 16, 17, 18, \ldots\}.$$

Then the associated 1-dimensional toric variety X is not weakly normal, nor F-pure. We claim that $\mathrm{FB}_1(X)$ is smooth but $\mathrm{FB}_2(X)$ is not. In particular, $\mathrm{FB}_2(X)$ does not dominate $\mathrm{FB}_1(X)$.

For each e, Δ_e contains only one 1-dimensional cone, say σ_e. Then

$$B_{\sigma_1} = \frac{1}{2}A \setminus \left\{0, \frac{9}{2}\right\} = \left\{4, 5, \frac{11}{2}, 8, \frac{17}{2}, 9 \ldots\right\},$$
$$B_{\sigma_2} = \frac{1}{4}A \setminus \left\{0, \frac{9}{4}, \frac{5}{2}, \frac{11}{4}\right\} = \left\{2, 4, \frac{17}{4}, \frac{9}{2}, \ldots\right\}.$$

Since $1 = 11/2 - 9/2 \in C_{\sigma_1}$, $\mathrm{FB}_1(X) = \operatorname{Spec} k[x]$. On the other hand, since none of

$$0 + 1, \frac{9}{4} + 1, \frac{5}{2} + 1, \frac{11}{4} + 1$$

belong to $\frac{1}{4}A$, $1 \notin C_{\sigma_2}$. Indeed $C_{\sigma_2} = \langle 2, 3 \rangle$, and $\mathrm{FB}_2(X) = \operatorname{Spec} k[x^2, x^3]$.

EXAMPLE 5.6 (An example where the monotonicity holds, but Condition (*) fails). Suppose again k has Characteristic 2 and $A = \langle 2, 3 \rangle$. Then for every $e > 0$, $\mathrm{FB}_e X \cong \operatorname{Spec} k[x]$, because it is the only nontrivial blowup of X. In particular, the monotonicity holds. We have

$$B_{\sigma_1} = \frac{1}{2}A \setminus \left\{0, \frac{3}{2}\right\} = \left\{1, 2, \frac{5}{2}, \ldots\right\},$$
$$B_{\sigma_2} = \frac{1}{4}A \setminus \left\{0, \frac{1}{2}, \frac{3}{4}, \frac{5}{4}\right\} = \left\{1, \frac{3}{2}, \frac{7}{4}, \ldots\right\}.$$

Then $B_{\sigma_1} \neq B_{\sigma_2} \cap \frac{1}{2}A$. Indeed $3/2$ only belongs to the right-hand side. Therefore, Condition (*) in Theorem 1.3 fails.

6. Stability of F-blowup sequences

Definition 6.1. Let $X_1, X_2, \ldots$ be a sequence of blowups of some variety X. Then we say that the sequence *stabilizes* if $\exists e_0$, $\forall e \geq e_0$, the natural birational map $X_{e+1} \dashrightarrow X_e$ extends to an isomorphism.

We say that the sequence is *bounded* if there exists a blowup Y of X which dominates all the X_i, $i \geq 1$.

The stability obviously implies the boundedness. Conversely, from the following lemma, the boundedness together with the monotonicity implies the stability.

Lemma 6.2. *Let*

$$X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} \cdots$$

be a sequence of proper surjective morphisms of varieties, and $g_i : Y \to X_i$, $i \geq 0$, *surjective morphisms of varieties such that for every* i, $f_i \circ g_i = g_{i-1}$. *Then for sufficiently large* i, f_i *is an isomorphism.*

Proof. (Though this fact is perhaps well-known, we include a proof for the sake of completeness.) Let $\Gamma_i \subset Y \times_k X_i$ be the graph of g_i and $H_i \subset Y \times_k Y$ its inverse image by

$$\mathrm{id}_Y \times g_i : Y \times_k Y \to Y \times_k X_i.$$

Then we have

$$H_i = \bigsqcup_{y \in Y} \{y\} \times g_i^{-1}(g_i(y)).$$

Clearly, $H_{i-1} \supset H_i$. Since $Y \times_k Y$ has the Noetherian underlying topological space, for sufficiently large i, $H_{i-1} = H_i$ and so f_i is injective and finite.

Now we may suppose that the f_i are finite and the X_i are affine, say $X_i = \operatorname{Spec} R_i$. If S denotes the integral closure of R_0, then $(R_i)_{i \in \mathbb{Z}_{\geq 0}}$ is an ascending chain of R_0-submodules of S. Since S is a Noetherian R_0-module, the chain stabilizes. □

Example 6.3.

(1) If X is a 1-dimensional variety, then for sufficiently large e, $\mathrm{FB}_e(X)$ is the normalization of X [15, Corollary 3.19]. In particular, the F-blowup sequence stabilizes.
(2) If $G \subset GL_d(k)$ is a finite subgroup of order prime to p and $X = \mathbb{A}_k^d/G$, then for sufficiently large e, $\mathrm{FB}_e(X)$ is isomorphic to the G-Hilbert scheme $\mathrm{Hilb}^G(\mathbb{A}_k^d)$ [15, Theorem 4.4], [13, Theorem 1.3]. Hence, the F-blowup sequence stabilizes.
(3) The F-blowup sequence of a toric variety is bounded [15, Theorem 3.13]. Hence, the F-blowup sequence of a weakly normal toric variety stabilizes. For the normal case, this has been already proved in [15, Theorem 3.12].

(4) Let R be a Noetherian complete local domain over k and $X = \operatorname{Spec} R$. We can define the eth F-blowup of X as the universal flattening of R^{1/p^e}, see [15, Section 2.3.2]. We say that X has *finite F-representation type* if there appear only finitely many indecomposable R-modules, say $M_1, \dots, M_n$, up to isomorphism in R^{1/p^i}, $i \geq 0$, as direct summands, see [12, Definition 3.1.1]. If X has finite F-representation type, then the F-blowup sequence of X is bounded. Indeed if a blowup $Y \to X$ is a flattening of $N := \bigoplus_{i=1}^n M_i$, then Y dominates all the F-blowups. Moreover, if X is F-pure, then for sufficiently large e, R^{1/p^e} has exactly $M_1, \dots, M_n$ as indecomposable direct summands. Then $\mathrm{FB}_e(X)$ is the universal flattening of N. In particular, the F-blowup sequence stabilizes. For instance, every simple singularity has finite F-representation type. See [8] for simple singularities in positive characteristic. Moreover, as in the following lemma, every simple singularity of dimension ≥ 3 is F-pure.

LEMMA 6.4. *Let $R = k[[x_0, \dots, x_n]]/(f)$ be a simple hypersurface singularity of dimension $n \geq 3$. Then R is F-pure.*

Proof. From the classification [8], we may suppose that f is of the form

$$f(x_0, \dots, x_n) = g(x_0, \dots, x_{n-2}) + x_{n-1}x_n.$$

Then the monomial $x_{n-1}^{p-1}x_n^{p-1}$ appears in the expansion of f^{p-1}. Hence, $f^{p-1} \notin (x_0, \dots, x_n)^{[p]}$. From Fedder's criterion [7, Proposition 2.1], R is F-pure. □

7. Proof of Proposition 1.4

We start with an example of Craw–Maclagan–Thomas [4, Example 5.7]. If $\mathrm{char}(k) \neq 5$, then there exists a finite Abelian subgroup $G \subset GL_k(6)$ of order 5^4 such that the associated G-Hilbert scheme $\mathrm{Hilb}^G(\mathbb{A}_k^6)$ is nonnormal. Let $X := \mathbb{A}_k^6/G$, which is a normal toric (hence F-pure) variety.

As in Example 6.3, the F-blowup sequence of X stabilizes. For sufficiently large e, we have

$$\mathrm{FB}_\infty(X) := \mathrm{FB}_e(X) \cong \mathrm{Hilb}^G(\mathbb{A}^6).$$

In particular, $\mathrm{FB}_\infty(X)$ is nonnormal.

But $\mathrm{FB}_\infty(X)$ is independent of the base field [15, Theorem 3.12]: No matter what the base field is, the combinatorial data defining the toric variety $\mathrm{FB}_\infty(X)$ does not change. In particular, $\mathrm{FB}_\infty(X)$ is well defined and nonnormal also in Characteristic 5. From Corollary 5.4, it is not weakly normal nor F-pure in every positive characteristic.

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