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STABILITY OF HYPERSURFACES WITH CONSTANT $(r+1)$-TH ANISOTROPIC MEAN CURVATURE

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ABSTRACT. Given a positive function F on S^n which satisfies a convexity condition, we define the r-th anisotropic mean curvature function H_r^F for hypersurfaces in $\mathbb{R}^{n+1}$ which is a generalization of the usual r-th mean curvature function. Let $X: M \to \mathbb{R}^{n+1}$ be an n-dimensional closed hypersurface with $H_{r+1}^F =$ constant, for some r with $0 \le r \le n-1$, which is a critical point for a variational problem. We show that $X(M)$ is stable if and only if $X(M)$ is the Wulff shape.

1. Introduction

Let $F: S^n \to \mathbb{R}^+$ be a smooth function which satisfies the following convexity condition:

$$(D^2F + F1)_x > 0 \quad \forall x \in S^n, \tag{1.1}$$

where S^n denotes the standard unit sphere in $\mathbb{R}^{n+1}$, D^2F denotes the intrinsic Hessian of F on S^n and 1 denotes the identity on T_xS^n, >0 means that the matrix is positive definite. We consider the map

$$\begin{aligned} \phi : S^n &\to \mathbb{R}^{n+1}, \\ x &\mapsto F(x)x + (\operatorname{grad}_{S^n} F)_x, \end{aligned} \tag{1.2}$$

its image $W_F = \phi(S^n)$ is a smooth, convex hypersurface in $\mathbb{R}^{n+1}$ called the Wulff shape of F (see [4], [7]–[9], [11], [14], [18], [19]). We note when $F \equiv 1$, W_F is just the sphere S^n.

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Now let $X: M \to \mathbb{R}^{n+1}$ be a smooth immersion of a closed, orientable hypersurface. Let $\nu: M \to S^n$ denotes its Gauss map, that is ν is the unit inner normal vector of M.

Let $A_F = D^2F + F1$, $S_F = -\mathrm{d}(\phi \circ \nu) = -A_F \circ \mathrm{d}\nu$. S_F is called the F-Weingarten operator, and the eigenvalues of S_F are called anisotropic principal curvatures. Let σ_r be the elementary symmetric functions of the anisotropic principal curvatures $\lambda_1, \lambda_2, \ldots, \lambda_n$:

$$\sigma_r = \sum_{i_1 < \cdots < i_r} \lambda_{i_1} \cdots \lambda_{i_r} \quad (1 \le r \le n). \tag{1.3}$$

We set $\sigma_0 = 1$. The r-th anisotropic mean curvature H_r^F is defined by $H_r^F = \sigma_r / C_n^r$, also see Reilly [16].

For each r, $0 \le r \le n-1$, we set

$$\mathscr{A}_{r,F} = \int_M F(\nu)\sigma_r \, \mathrm{d}A_X. \tag{1.4}$$

The algebraic $(n+1)$-volume enclosed by M is given by

$$V = \frac{1}{n+1} \int_M \langle X, \nu \rangle \, \mathrm{d}A_X. \tag{1.5}$$

We consider those hypersurfaces which are critical points of $\mathscr{A}_{r,F}$ restricted to those hypersurfaces enclosing a fixed volume V. By a standard argument involving Lagrange multipliers, this means we are considering critical points of the functional

$$\mathscr{F}_{r,F;\Lambda} = \mathscr{A}_{r,F} + \Lambda V(X), \tag{1.6}$$

where Λ is a constant. We will show the Euler–Lagrange equation of $\mathscr{F}_{r,F;\Lambda}$ is:

$$(r+1)\sigma_{r+1} - \Lambda = 0. \tag{1.7}$$

So the critical points are just hypersurfaces with $H_{r+1}^F = \text{constant}$.

If $F \equiv 1$, then the function $\mathscr{A}_{r,F}$ is just the functional $\mathscr{A}_r = \int_M S_r \, \mathrm{d}A_X$ which was studied by Alencar, do Carmo, and Rosenberg in [1], where $H_r = S_r / C_n^r$ is the usual r-th mean curvature. For such a variational problem, they call a critical immersion X of the functional $\mathscr{A}_r$ (that is, a hypersurface with $H_{r+1} = \text{constant}$) stable if and only if the second variation of $\mathscr{A}_r$ is nonnegative for all variations of X preserving the enclosed $(n+1)$-volume V. They proved the following theorem.

Theorem 1.1 ([1]). *Suppose $0 \le r \le n-1$. Let $X: M \to \mathbb{R}^{n+1}$ be a closed hypersurface with $H_{r+1} = \text{constant}$. Then X is stable if and only if $X(M)$ is a round sphere.*

Analogously, we call a critical immersion X of the functional $\mathscr{A}_{r,F}$ stable if and only if the second variation of $\mathscr{A}_{r,F}$ (or equivalently of $\mathscr{F}_{r,F;\Lambda}$) is nonnegative for all variations of X preserving the enclosed $(n+1)$-volume V.

In [14], Palmer proved the following theorem (also see Winklmann [19]).

THEOREM 1.2 ([14]). *Let* $X : M \to \mathbb{R}^{n+1}$ *be a closed hypersurface with* $H_1^F = \text{constant}$. *Then* X *is stable if and only if, up to translations and homotheties,* $X(M)$ *is the Wulff shape.*

In this paper, we prove the following theorem.

THEOREM 1.3. *Suppose* $0 \le r \le n-1$. *Let* $X : M \to \mathbb{R}^{n+1}$ *be a closed hypersurface with* $H_{r+1}^F = \text{constant}$. *Then* X *is stable if and only if, up to translations and homotheties,* $X(M)$ *is the Wulff shape.*

REMARK 1.4. In the case $F \equiv 1$, Theorem 1.3 becomes Theorem 1.1. Theorem 1.3 gives an affirmative answer to the problem proposed in [8]. We also note that in the case $F \equiv 1$, our result here gives a new and geometric proof of Theorem 1.1, which is different from [1].

2. Preliminaries

Let $X : M \to R^{n+1}$ be a smooth closed, oriented hypersurface with Gauss map $\nu : M \to S^n$, that is, ν is the unit inner normal vector field. Let X_t be a variation of X, and $\nu_t : M \to S^n$ be the Gauss map of X_t. We define

$$\psi = \left\langle \frac{\mathrm{d}X_t}{\mathrm{d}t}, \nu_t \right\rangle, \qquad \xi = \left(\frac{\mathrm{d}X_t}{\mathrm{d}t} \right)^{\top}, \tag{2.1}$$

where $\top$ represents the tangent component and ψ, ξ are dependent of t. The corresponding first variation of the unit normal vector is given by (see [8], [11], [14], [19])

$$\nu_t' = -\operatorname{grad}\psi + \mathrm{d}\nu_t(\xi), \tag{2.2}$$

the first variation of the volume element is (see [2], [3], or [10])

$$\partial_t \mathrm{d}A_{X_t} = (\operatorname{div}\xi - nH\psi)\,\mathrm{d}A_{X_t}, \tag{2.3}$$

and the first variation of the volume V is

$$V'(t) = \int_M \psi\,\mathrm{d}A_{X_t}, \tag{2.4}$$

where grad, div, H represents the gradients, the divergence, the mean curvature with respect to X_t, respectively.

Let $\{E_1, \dots, E_n\}$ be a local orthogonal frame on S^n, let $e_i = e_i(t) = E_i \circ \nu_t$, where $i = 1, \dots, n$ and ν_t is the Gauss map of X_t, then $\{e_1, \dots, e_n\}$ is a local orthogonal frame of $X_t : M \to \mathbb{R}^{n+1}$.

The structure equations of $x : S^n \to \mathbb{R}^{n+1}$ are:

$$\begin{cases} \mathrm{d}x = \sum_i \theta_i E_i, \\ \mathrm{d}E_i = \sum_j \theta_{ij} E_j - \theta_i x, \\ \mathrm{d}\theta_i = \sum_j \theta_{ij} \wedge \theta_j, \\ \mathrm{d}\theta_{ij} - \sum_k \theta_{ik} \wedge \theta_{kj} = \frac{1}{2} \sum_{k,l} \tilde{R}_{ijkl} \theta_k \wedge \theta_l = -\theta_i \wedge \theta_j, \end{cases} \tag{2.5}$$

where $\theta_{ij}+\theta_{ji}=0$ and $\tilde{R}_{ijkl}=\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl}$.

The structure equations of X_t are (see [12], [13]):

$$\begin{cases} \mathrm{d}X_t=\sum_i \omega_i e_i, \\ \mathrm{d}\nu_t=-\sum_{i,j} h_{ij}\omega_j e_i, \\ \mathrm{d}e_i=\sum_j \omega_{ij}e_j+\sum_j h_{ij}\omega_j\nu_t, \\ \mathrm{d}\omega_i=\sum_j \omega_{ij}\wedge\omega_j, \\ \mathrm{d}\omega_{ij}-\sum_k \omega_{ik}\wedge\omega_{kj}=\frac{1}{2}\sum_{k,l}R_{ijkl}\theta_k\wedge\theta_l, \end{cases} \tag{2.6}$$

where $\omega_{ij}+\omega_{ji}=0$, $R_{ijkl}+R_{ijlk}=0$, and R_{ijkl} are the components of the Riemannian curvature tensor of $X_t(M)$ with respect to the induced metric $\mathrm{d}X_t\cdot\mathrm{d}X_t$. Here, we have omitted the variable t for some geometric quantities.

From $\mathrm{d}e_i=\mathrm{d}(E_i\circ\nu_t)=\nu_t^*\,\mathrm{d}E_i=\sum_j \nu_t^*\theta_{ij}e_j-\nu_t^*\theta_i\nu_t$, we get

$$\begin{cases} \omega_{ij}=\nu_t^*\theta_{ij}, \\ \nu_t^*\theta_i=-\sum_j h_{ij}\omega_j, \end{cases} \tag{2.7}$$

where $\omega_{ij}+\omega_{ji}=0$, $h_{ij}=h_{ji}$.

Let $F: S^n\to\mathbb{R}^+$ be a smooth function, we denote the coefficients of covariant differential of F, $\mathrm{grad}_{S^n}F$ with respect to $\{E_i\}_{i=1,\ldots,n}$ by F_i, F_{ij} respectively.

From (2.7), $\mathrm{d}(F(\nu_t))=\nu_t^*\,\mathrm{d}F=\nu_t^*(\sum_i F_i\theta_i)=-\sum_{i,j}(F_i\circ\nu_t)h_{ij}\omega_j$, thus,

$$\mathrm{grad}(F(\nu_t))=-\sum_{i,j}(F_i\circ\nu_t)h_{ij}e_j=\mathrm{d}\nu_t(\mathrm{grad}_{S^n}F). \tag{2.8}$$

Through a direct calculation, we easily get

$$\mathrm{d}\phi=(D^2F+F1)\circ\mathrm{d}x=\sum_{i,j}A_{ij}\theta_i E_j, \tag{2.9}$$

where A_{ij} is the coefficient of A_F, that is, $A_{ij}=F_{ij}+F\delta_{ij}$.

Taking exterior differential of (2.9) and using (2.5), we get

$$A_{ijk}=A_{jik}=A_{ikj}, \tag{2.10}$$

where A_{ijk} denotes coefficient of the covariant differential of A_F on S^n.

We define $(A_{ij}\circ\nu_t)_k$ by

$$\mathrm{d}(A_{ij}\circ\nu_t)+\sum(A_{kj}\circ\nu_t)\omega_{ki}+\sum_k(A_{ik}\circ\nu_t)\omega_{kj}=\sum_k(A_{ij}\circ\nu_t)_k\omega_k. \tag{2.11}$$

By a direct calculation by using (2.7) and (2.11), we have

$$(A_{ij}\circ\nu_t)_k=-\sum_l h_{kl}A_{ijl}\circ\nu_t. \tag{2.12}$$

We define L_{ij} by

$$\left(\frac{\mathrm{d}e_i}{\mathrm{d}t}\right)^{\top} = -\sum_j L_{ij}e_j, \tag{2.13}$$

where $\top$ denote the tangent component, then $L_{ij} = -L_{ji}$ and we have (see [2], [3], or [10])

$$h'_{ij} = \psi_{ij} + \sum_k \{h_{ijk}\xi_k + \psi h_{ik}h_{jk} + h_{ik}L_{kj} + h_{jk}L_{ki}\}. \tag{2.14}$$

Let $s_{ij} = \sum_k (A_{ik}\circ\nu)h_{kj}$, then from (2.7) and (2.9), we have

$$\mathrm{d}(\phi\circ\nu_t) = \nu_t^*\,\mathrm{d}\phi = -\sum_{i,j} s_{ij}\omega_j e_i. \tag{2.15}$$

We define S_F by $S_F = -\mathrm{d}(\phi\circ\nu) = -A_F\circ\mathrm{d}\nu$, then we have $S_F(e_j) = \sum_i s_{ij}e_i$. We call S_F the F-Weingarten operator. From the positive definiteness of (A_{ij}) and the symmetry of (h_{ij}), we know the eigenvalues of (s_{ij}) are all real. We call them anisotropic principal curvatures, and denote them by $\lambda_1,\dots,\lambda_n$.

Taking exterior differential of (2.15) and using (2.6), we get

$$s_{ijk} = s_{ikj}, \tag{2.16}$$

where s_{ijk} denotes coefficient of the covariant differential of S_F.

We have n invariants, the elementary symmetric function σ_r of the anisotropic principal curvatures:

$$\sigma_r = \sum_{i_1<\cdots<i_r} \lambda_{i_1}\cdots\lambda_{i_n} \quad (1\le r\le n). \tag{2.17}$$

For convenience, we set $\sigma_0 = 1$ and $\sigma_{n+1} = 0$. The r-th anisotropic mean curvature H_r^F is defined by

$$H_r^F = \sigma_r/C_n^r, \quad C_n^r = \frac{n!}{r!(n-r)!}. \tag{2.18}$$

We have, by use of (2.2) and (2.6),

$$\begin{aligned}\sum_{i,j}\frac{\mathrm{d}((A_{ij}E_i\otimes E_j)\circ\nu_t)}{\mathrm{d}t} &= \sum_{i,j}\big\langle \big(D(A_{ij}E_i\otimes E_j)\big)_{\nu_t}, \nu'_t\big\rangle \\ &= -\sum_{i,j,k} A_{ijk}\bigg(\psi_k + \sum_l h_{kl}\xi_l\bigg)e_i\otimes e_j,\end{aligned} \tag{2.19}$$

where D is the Levi–Civita connection on S^n.

On the other hand, we have

$$\sum_{i,j}\frac{\mathrm{d}((A_{ij}E_i\otimes E_j)\circ\nu_t)}{\mathrm{d}t} = \sum_{i,j}\left\{A'_{ij}e_i\otimes e_j + A_{ij}\left(\frac{\mathrm{d}e_i}{\mathrm{d}t}\right)^{\top}\otimes e_j + A_{ij}e_i\otimes\left(\frac{\mathrm{d}e_j}{\mathrm{d}t}\right)^{\top}\right\}. \tag{2.20}$$

By use of (2.13), we get from (2.19) and (2.20)

$$\frac{\mathrm{d}(A_{ij}\circ\nu_t)}{\mathrm{d}t} = A'_{ij}(t) = \sum_k\left\{-A_{ijk}\psi_k - \sum_l A_{ijk}h_{kl}\xi_l + A_{ik}L_{kj} + A_{jk}L_{ki}\right\}. \tag{2.21}$$

By (2.12), (2.14), (2.21) and the fact $L_{ij}=-L_{ji}$, through a direct calculation, we get the following lemma.

LEMMA 2.1.

$$\frac{\mathrm{d}s_{ij}}{\mathrm{d}t} = s'_{ij}(t) = \sum_k\{(A_{ik}\psi_k)_j + s_{ijk}\xi_k + \psi s_{ik}h_{kj} + s_{kj}L_{ki} + s_{ik}L_{kj}\}.$$

As M is a closed oriented hypersurface, one can find a point where all the principal curvatures with respect to ν are positive. By the positiveness of A_F, all the anisotropic principal curvatures are positive at this point. Using the results of Gårding [5], we have the following lemma.

LEMMA 2.2. *Let $X: M\to\mathbb{R}^{n+1}$ be a closed, oriented hypersurface. Assume $H^F_{r+1}>0$ holds at every point of M, then $H^F_k>0$ holds on every point of M for every $k=1,\dots,r$.*

Using the characteristic polynomial of S_F, σ_r is defined by

$$\det(tI-S_F) = \sum_{r=0}^{n}(-1)^r\sigma_r t^{n-r}. \tag{2.22}$$

So, we have

$$\sigma_r = \frac{1}{r!}\sum_{i_1,\dots,i_r;j_1,\dots,j_r}\delta^{j_1\cdots j_r}_{i_1\cdots i_r}s_{i_1j_1}\cdots s_{i_rj_r}, \tag{2.23}$$

where $\delta^{j_1\cdots j_r}_{i_1\cdots i_r}$ is the usual generalized Kronecker symbol, i.e., $\delta^{j_1\cdots j_r}_{i_1\cdots i_r}$ equals $+1$ (resp. -1) if $i_1\cdots i_r$ are distinct and $(j_1\cdots j_r)$ is an even (resp. odd) permutation of $(i_1\cdots i_r)$ and in other cases it equals zero.

We introduce two important operators P_r and T_r by

$$P_r = \sigma_r I - \sigma_{r-1}S_F + \cdots + (-1)^r S_F^r, \quad r=0,1,\dots,n, \tag{2.24}$$

$$T_r = P_r A_F, \quad r=0,1,\dots,n-1. \tag{2.25}$$

Obviously, $P_n = 0$ and we have

$$P_r = \sigma_r I - P_{r-1} S_F = \sigma_r I + T_{r-1}\,\mathrm{d}\nu, \quad r = 1, \dots, n. \tag{2.26}$$

From the symmetry of A_F and $\mathrm{d}\nu$, $S_F A_F$ and $\mathrm{d}\nu \circ S_F$ are symmetric, so $T_r = P_r A_F$ and $\mathrm{d}\nu \circ P_r$ are also symmetric for each r.

LEMMA 2.3. *The matrix of P_r is given by:*

$$(P_r)_{ij} = \frac{1}{r!} \sum_{i_1,\dots,i_r;j_1,\dots,j_r} \delta^{j_1\cdots j_r i}_{i_1\cdots i_r j} s_{i_1 j_1} \cdots s_{i_r j_r}. \tag{2.27}$$

Proof. We prove Lemma 2.3 inductively. For $r = 0$, it is easy to check that (2.27) is true.

Assume (2.27) is true for $r = k$, then from (2.26),

$$\begin{aligned}
(P_{k+1})_{ij} &= \sigma_{k+1}\delta^i_j - \sum_l (P_k)_{il} s_{lj} \\
&= \frac{1}{(k+1)!} \sum \left(\delta^{j_1\cdots j_{k+1}}_{i_1\cdots i_{k+1}} \delta^i_j - \sum_l \delta^{j_1\cdots j_{l-1} i j_{l+1}\cdots j_{k+1}}_{i_1\cdots i_{l-1} i_l i_{l+1}\cdots i_{k+1}} \delta^{j_l}_j \right) \\
&\quad \times s_{i_1 j_1} \cdots s_{i_{k+1} j_{k+1}} \\
&= \frac{1}{(k+1)!} \sum \delta^{j_1\cdots j_{k+1} i}_{i_1\cdots i_{k+1} j} s_{i_1 j_1} \cdots s_{i_{k+1} j_{k+1}}.
\end{aligned}$$

□

LEMMA 2.4. *For each r, we have*

(i) $\sum_j (P_r)_{jij} = 0$,
(ii) $\mathrm{tr}(P_r S_F) = (r+1)\sigma_{r+1}$,
(iii) $\mathrm{tr}(P_r) = (n-r)\sigma_r$,
(iv) $\mathrm{tr}(P_r S_F^2) = \sigma_1 \sigma_{r+1} - (r+2)\sigma_{r+2}$.

Proof. We only prove (i), the others are easily obtained from (2.23), (2.26), and (2.27).

Noting $s_{i_1 j_1} \cdots s_{i_r j_r j} = s_{i_1 j_1} \cdots s_{i_r j j_r}$ by (2.16) and $\delta^{j_1\cdots j_r j}_{i_1\cdots i_r i} = -\delta^{j_1\cdots j j_r}_{i_1\cdots i_r i}$, we have

$$\sum_j (P_r)_{jij} = \frac{1}{(r-1)!} \sum_{i_1,\dots,i_r;j_1,\dots,j_r;j} \delta^{j_1\cdots j_r j}_{i_1\cdots i_r i} s_{i_1 j_1} \cdots s_{i_r j_r j} = 0.$$

□

REMARK 2.5. When $F = 1$, Lemma 2.4 was a well-known result (for example, see Barbosa–Colares [2], Reilly [15], or Rosenberg [17]).

Since $P_{r-1} S_F$ is symmetric and L_{ij} is anti-symmetric, we have

$$\sum_{i,j,k} (P_{r-1})_{ji} (s_{kj} L_{ki} + s_{ik} L_{kj}) = 0. \tag{2.28}$$

From (2.16), (2.26), and (i) of Lemma 2.4, we get

$$(\sigma_r)_k = \sum_j (\sigma_r \delta_{jk})_j = \sum_j (P_r)_{jkj} + \sum_{j,l} [(P_{r-1})_{jl} s_{lk}]_j \\ = \sum_{i,j} (P_{r-1})_{ji} s_{ijk}. \tag{2.29}$$

3. First and second variation formulas of $\mathscr{F}_{r,F;\Lambda}$

Define the operator $L_r : C^\infty(M) \to C^\infty(M)$ as follows:

$$L_r f = \sum_{i,j} [(T_r)_{ij} f_j]_i. \tag{3.1}$$

LEMMA 3.1.

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}t} = \sigma_r'(t) = L_{r-1}\psi + \psi \langle T_{r-1} \circ \mathrm{d}\nu_t, \mathrm{d}\nu_t\rangle + \langle \operatorname{grad} \sigma_r, \xi\rangle.$$

Proof. Using (2.23), (2.28), (2.29), Lemma 2.1, Lemma 2.3, and (i) of Lemma 2.4, we have

$$\begin{aligned}
\sigma_r' &= \frac{1}{(r-1)!} \sum_{i_1,\dots,i_r;j_1,\dots,j_r} \delta^{j_1\cdots j_r}_{i_1\cdots i_r} s_{i_1 j_1} \cdots s_{i_{r-1} j_{r-1}} s'_{i_r j_r} \\
&= \sum_{i,j} (P_{r-1})_{ji} s'_{ij} \\
&= \sum_{i,j,k} (P_{r-1})_{ji} [(A_{ik}\psi_k)_j + \psi s_{ik} h_{kj} + s_{ijk}\xi_k + s_{kj} L_{ki} + s_{ik} L_{kj}] \\
&= \sum_{i,j,k} [(P_{r-1})_{ji} A_{ik} \psi_k]_j + \psi \sum_{i,j,k,l} (P_{r-1})_{ji} A_{il} h_{lk} h_{kj} + \sum_k (\sigma_r)_k \xi_k \\
&= \sum_{j,k} [(T_{r-1})_{jk} \psi_k]_j + \psi \sum_{i,j,k} (T_{r-1})_{ji} h_{ik} h_{kj} + \sum_k (\sigma_r)_k \xi_k \\
&= L_{r-1}\psi + \psi\langle T_{r-1} \circ \mathrm{d}\nu_t, \mathrm{d}\nu_t\rangle + \langle \operatorname{grad}\sigma_r, \xi\rangle.
\end{aligned}$$

□

LEMMA 3.2. *For each $0 \le r \le n$, we have*

$$\operatorname{div}\big(P_r(\operatorname{grad}_{S^n} F) \circ \nu_t\big) + F(\nu_t) \operatorname{tr}(P_r \circ \mathrm{d}\nu_t) = -(r+1)\sigma_{r+1} \tag{3.2}$$

and

$$\operatorname{div}(P_r X^\top) + \langle X, \nu_t\rangle \operatorname{tr}(P_r \circ \mathrm{d}\nu_t) = (n-r)\sigma_r. \tag{3.3}$$

Proof. From (2.6), (2.15), and Lemma 2.4,

$$\begin{aligned}
\operatorname{div}\big(P_r(\operatorname{grad}_{S^n} F) \circ \nu_t\big) &= \operatorname{div}\big(P_r(\phi \circ \nu_t)^\top\big) \\
&= \sum_{i,j} \big((P_r)_{ji} \langle \phi \circ \nu_t, e_i\rangle\big)_j
\end{aligned}$$

$$\begin{aligned}
&= -\sum_{i,j}(P_r)_{ji}s_{ij} + F(\nu_t)\sum_{i,j}(P_r)_{ji}h_{ij}\\
&= -\operatorname{tr}(P_rS_F) - F(\nu_t)\operatorname{tr}(P_r\circ \mathrm{d}\nu_t)\\
&= -(r+1)\sigma_{r+1} - F(\nu_t)\operatorname{tr}(P_r\circ \mathrm{d}\nu_t),\\
\operatorname{div}(P_rX^\top) &= \sum_{i,j}((P_r)_{ji}\langle X, e_i\rangle)_j\\
&= \sum_{i,j}(P_r)_{ji}\delta_{ij} + \sum_{i,j}(P_r)_{ji}h_{ij}\langle X,\nu_t\rangle\\
&= \operatorname{tr}(P_r) - \operatorname{tr}(P_r\circ \mathrm{d}\nu_t)\langle X,\nu_t\rangle\\
&= (n-r)\sigma_r - \operatorname{tr}(P_r\circ \mathrm{d}\nu_t)\langle X,\nu_t\rangle.
\end{aligned}$$

Thus, the conclusion follows. □

THEOREM 3.3 (First variational formula of $\mathscr{A}_{r,F}$).

$$\mathscr{A}'_{r,F}(t) = -(r+1)\int_M \psi\sigma_{r+1}\,\mathrm{d}A_{X_t}. \tag{3.4}$$

Proof. We have $(F(\nu_t))' = \langle \operatorname{grad}_{S^n} F, \nu'_t\rangle$, so by use of Lemma 3.1, Lemma 3.2, (2.2), (2.3), (2.8), (2.26), and Stokes formula, we have

$$\begin{aligned}
\mathscr{A}'_{r,F}(t) &= \int_M \big(F(\nu_t)\sigma'_r + (F(\nu_t))'\sigma_r\big)\,\mathrm{d}A_{X_t} + F(\nu_t)\sigma_r\,\partial_t\mathrm{d}A_{X_t}\\
&= \int_M \{F(\nu_t)\operatorname{div}(T_{r-1}\operatorname{grad}\psi) + F(\nu_t)\psi\langle T_{r-1}\circ \mathrm{d}\nu_t, \mathrm{d}\nu_t\rangle\\
&\quad + F(\nu_t)\langle \operatorname{grad}\sigma_r, \xi\rangle + \langle \sigma_r(\operatorname{grad}_{S^n} F)\circ\nu_t, -\operatorname{grad}\psi + \mathrm{d}\nu_t(\xi)\rangle\\
&\quad + F(\nu_t)\sigma_r(-nH\psi + \operatorname{div}\xi)\}\,\mathrm{d}A_{X_t}\\
&= \int_M \big\{-\langle \operatorname{grad}(F(\nu_t)), T_{r-1}\operatorname{grad}\psi\rangle + F(\nu_t)\psi\langle T_{r-1}\circ \mathrm{d}\nu_t, \mathrm{d}\nu_t\rangle\\
&\quad + \langle F(\nu_t)\operatorname{grad}\sigma_r, \xi\rangle + \psi\operatorname{div}\big(\sigma_r(\operatorname{grad}_{S^n} F)\circ\nu_t\big)\\
&\quad + \langle \sigma_r\operatorname{grad}(F(\nu_t)), \xi\rangle - nH\psi F(\nu_t)\sigma_r + F(\nu_t)\sigma_r\operatorname{div}\xi\big\}\,\mathrm{d}A_{X_t}\\
&= \int_M \big\{-\langle T_{r-1}\operatorname{grad}(F(\nu_t)), \operatorname{grad}\psi\rangle + F(\nu_t)\psi\langle T_{r-1}\circ \mathrm{d}\nu_t, \mathrm{d}\nu_t\rangle\\
&\quad + \psi\operatorname{div}\big(\sigma_r(\operatorname{grad}_{S^n} F)\circ\nu_t\big) - nH\psi F(\nu_t)\sigma_r\big\}\,\mathrm{d}A_{X_t}\\
&= \int_M \psi\big\{\operatorname{div}\big(\sigma_r(\operatorname{grad}_{S^n} F)\circ\nu_t\big) + \operatorname{div}(T_{r-1}\operatorname{grad}(F(\nu_t)))\\
&\quad + F(\nu_t)\langle T_{r-1}\circ \mathrm{d}\nu_t, \mathrm{d}\nu_t\rangle - nHF(\nu_t)\sigma_r\big\}\,\mathrm{d}A_{X_t}\\
&= \int_M \psi\{\operatorname{div}[(\sigma_r + T_{r-1}\circ \mathrm{d}\nu_t)(\operatorname{grad}_{S^n} F)\circ\nu_t]\\
&\quad + F(\nu_t)\operatorname{tr}[(T_{r-1}\circ \mathrm{d}\nu_t + \sigma_r I)\circ \mathrm{d}\nu_t]\}\,\mathrm{d}A_{X_t}
\end{aligned}$$

$$= \int_M \psi\big\{\operatorname{div}\big(P_r(\operatorname{grad}_{S^n} F)\circ\nu_t\big) + F(\nu_t)\operatorname{tr}(P_r\circ \mathrm{d}\nu_t)\big\}\,\mathrm{d}A_{X_t}$$
$$= -(r+1)\int_M \psi\sigma_{r+1}\,\mathrm{d}A_{X_t}. \qquad \square$$

REMARK 3.4. When $F=1$, Lemma 4.1 and Theorem 3.3 were proved by R. Reilly [15] (also see [2], [3]).

From (1.6), (2.4), and (3.4), we get

PROPOSITION 3.5 (The first variational formula). *For all variations of X, we have*

$$\mathscr{F}'_{r,F;\Lambda}(t) = -\int_M \psi\{(r+1)\sigma_{r+1} - \Lambda\}\,\mathrm{d}A_{X_t}. \tag{3.5}$$

Hence, we obtain the Euler–Lagrange equation of $\mathscr{F}_{r,F;\Lambda}$:

$$(r+1)\sigma_{r+1} - \Lambda = 0. \tag{3.6}$$

THEOREM 3.6 (The second variational formula). *Let $X: M \to R^{n+1}$ be an n-dimensional closed hypersurface, which satisfies (3.6), then for all variations of X preserving V, the second variational formula of $\mathscr{A}_{r,F}$ at $t=0$ is given by*

$$\mathscr{A}''_r(0) = \mathscr{F}''_{r,F;\Lambda}(0) = -(r+1)\int_M \psi\{L_r\psi + \psi\langle T_r\circ \mathrm{d}\nu, \mathrm{d}\nu\rangle\}\,\mathrm{d}A_X, \tag{3.7}$$

where ψ satisfies

$$\int_M \psi\,\mathrm{d}A_X = 0. \tag{3.8}$$

Proof. Differentiating (3.5), we get (3.7) by use of (3.6) and Lemma 3.1. $\square$

We call $X: M\to R^{n+1}$ to be a stable critical point of $\mathscr{A}_{r,F}$ for all variations of X preserving V, if it satisfies (3.6) and $\mathscr{A}''_r(0)\geq 0$ for all ψ with condition (3.8).

4. Proof of Theorem 1.3

Firstly, we prove that if $X(M)$ is, up to translations and homotheties, the Wulff shape, then X is stable.

From $\mathrm{d}\phi = (D^2F + F1)\circ \mathrm{d}x$, $\mathrm{d}\phi$ is perpendicular to x. So $\nu = -x$ is the unit inner normal vector. We have

$$\mathrm{d}\phi = -A_F\circ \mathrm{d}\nu = \sum_{ijk} A_{jk}h_{ki}\omega_i e_j. \tag{4.1}$$

On the other hand,

$$\mathrm{d}\phi = \sum_i \omega_i e_i, \tag{4.2}$$

so we have

$$s_{ij} = \sum_k A_{ik} h_{kj} = \delta_{ij}. \tag{4.3}$$

From this, we easily get $\sigma_r = C_n^r$ and $\sigma_{r+1} = C_n^{r+1}$, thus, the Wulff shape satisfies (3.6) with $\Lambda = (r+1)C_n^{r+1}$. Through a direct calculation, we easily know for Wulff shape,

$$\mathscr{A}_r''(0) = -(r+1)C_{n-1}^r \int_M [\operatorname{div}(A_F \operatorname{grad}\psi) + \psi\langle A_F \circ \mathrm{d}\nu, \mathrm{d}\nu\rangle]\, \mathrm{d}A_X, \tag{4.4}$$

and ψ satisfies

$$\int_M \psi\, \mathrm{d}A_X = 0. \tag{4.5}$$

From Palmer [14] (also see Winklmann [19]), we know $\mathscr{A}_r''(0) \geq 0$, that is the Wulff shape is stable.

Next, we prove that if X is stable, then up to translations and homotheties, $X(M)$ is the Wulff shape. We recall the following lemmas.

Lemma 4.1 ([7], [8]). *For each $r = 0, 1, \ldots, n-1$, the following integral formulas of Minkowski type hold*:

$$\int_M \big(H_r^F F(\nu) + H_{r+1}^F \langle X, \nu\rangle\big)\, \mathrm{d}A_X = 0, \quad r = 0, 1, \ldots, n-1. \tag{4.6}$$

Lemma 4.2 ([7], [8], [14]). *If $\lambda_1 = \lambda_2 = \cdots = \lambda_n = const \neq 0$, then up to translations and homotheties, $X(M)$ is the Wulff shape.*

From Lemmas 4.1 and (3.8), we can choose $\psi = \alpha F(\nu) + H_{r+1}^F\langle X, \nu\rangle$ as the test function, where $\alpha = \int_M F(\nu) H_r^F\, \mathrm{d}A_X / \int_M F(\nu)\, \mathrm{d}A_X$. For every smooth function $f : M \to \mathbb{R}$, and each r, we define:

$$I_r[f] = L_r f + f\langle T_r \circ \mathrm{d}\nu, \mathrm{d}\nu\rangle. \tag{4.7}$$

Then we have from (3.7)

$$\mathscr{A}_r''(0) = -(r+1)\int_M \psi I_r[\psi]\, \mathrm{d}A_X. \tag{4.8}$$

Lemma 4.3. *For each $0 \leq r \leq n-1$, we have*

$$I_r[F\circ\nu] = -\langle \operatorname{grad}\sigma_{r+1}, (\operatorname{grad}_{S^n} F)\circ\nu\rangle + \sigma_1\sigma_{r+1} - (r+2)\sigma_{r+2} \tag{4.9}$$

and

$$I_r[\langle X, \nu\rangle] = -\langle \operatorname{grad}\sigma_{r+1}, X^\top\rangle - (r+1)\sigma_{r+1}. \tag{4.10}$$

Proof. From (2.8) and (2.26),

$$\begin{aligned} I_r[F\circ\nu] &= \operatorname{div}\{T_r \operatorname{grad}(F(\nu))\} + F(\nu)\langle T_r \circ \mathrm{d}\nu, \mathrm{d}\nu\rangle \\ &= \operatorname{div}\big(T_r \circ \mathrm{d}\nu(\operatorname{grad}_{S^n} F)\circ\nu\big) + F(\nu)\langle T_r \circ \mathrm{d}\nu, \mathrm{d}\nu\rangle \\ &= \operatorname{div}\big(P_{r+1}(\operatorname{grad}_{S^n} F)\circ\nu\big) + F(\nu)\operatorname{tr}(P_{r+1}\,\mathrm{d}\nu) \end{aligned}$$

$$
\begin{aligned}
&\quad - \langle \operatorname{grad} \sigma_{r+1}, (\operatorname{grad}_{S^n} F)\circ\nu\rangle \\
&\quad - \sigma_{r+1}\big\{\operatorname{div}\big(P_0(\operatorname{grad}_{S^n} F)\circ\nu\big) + F(\nu)\operatorname{tr}(P_0\,\mathrm{d}\nu)\big\},\\
I_r[\langle X,\nu\rangle] &= \operatorname{div}(T_r \operatorname{grad}\langle X,\nu\rangle) + \langle X,\nu\rangle\langle T_r\circ \mathrm{d}\nu,\, \mathrm{d}\nu\rangle\\
&= \operatorname{div}(T_r\circ \mathrm{d}\nu\, X^\top) + \langle X,\nu\rangle\langle T_r\circ \mathrm{d}\nu, \mathrm{d}\nu\rangle\\
&= \operatorname{div}(P_{r+1}X^\top) + \langle X,\nu\rangle \operatorname{tr}(P_{r+1}\,\mathrm{d}\nu) - \langle \operatorname{grad}\sigma_{r+1}, X^\top\rangle\\
&\quad - \sigma_{r+1}\{\operatorname{div}(P_0 X^\top) + \langle X,\nu\rangle\operatorname{tr}(P_0\,\mathrm{d}\nu)\}.
\end{aligned}
$$

So the conclusions follow from Lemma 3.2. □

As H^F_{r+1} is a constant, from (4.9) and (4.10), we have

$$
\begin{aligned}
I_r[\psi] &= \alpha I_r[F\circ\nu] + H^F_{r+1} I_r[\langle X,\nu\rangle]\\
&= \alpha\big(\sigma_1\sigma_{r+1} - (r+2)\sigma_{r+2}\big) - (r+1)H^F_{r+1}\sigma_{r+1}\\
&= C_n^{r+1}\{\alpha[nH^F_1 H^F_{r+1} - (n-r-1)H^F_{r+2}] - (r+1)(H^F_{r+1})^2\}.
\end{aligned}
\tag{4.11}
$$

Therefore, we obtain from Lemma 4.1 (recall H^F_{r+1} is constant and $\int_M \psi\,\mathrm{d}A_X = 0$)

$$
\begin{aligned}
&\frac{1}{r+1}\mathscr{A}_r''(0)\\
&\quad = -\int_M \psi I_r[\psi]\,\mathrm{d}A_X\\
&\quad = -\int_M \psi C_n^{r+1}\{\alpha[nH^F_1H^F_{r+1} - (n-r-1)H^F_{r+2}] - (r+1)(H^F_{r+1})^2\}\,\mathrm{d}A_X\\
&\quad = -\alpha C_n^{r+1}\int_M [\alpha F(\nu) + H^F_{r+1}\langle X,\nu\rangle][nH^F_1H^F_{r+1} - (n-r-1)H^F_{r+2}]\,\mathrm{d}A_X\\
&\quad = -\alpha^2 C_n^{r+1}\int_M F(\nu)[nH^F_1H^F_{r+1} - (n-r-1)H^F_{r+2}]\,\mathrm{d}A_X\\
&\qquad - \alpha C_n^{r+1}H^F_{r+1}\int_M \langle X,\nu\rangle[nH^F_1H^F_{r+1} - (n-r-1)H^F_{r+2}]\,\mathrm{d}A_X\\
&\quad = -\alpha^2 C_n^{r+1}\int_M F(\nu)[nH^F_1H^F_{r+1} - (n-r-1)H^F_{r+2}]\,\mathrm{d}A_X\\
&\qquad + \alpha C_n^{r+1}H^F_{r+1}\int_M F(\nu)[nH^F_{r+1} - (n-r-1)H^F_{r+1}]\,\mathrm{d}A_X\\
&\quad = -\alpha^2(n-r-1)C_n^{r+1}\int_M F(\nu)(H^F_1H^F_{r+1} - H^F_{r+2})\,\mathrm{d}A_X\\
&\qquad - \frac{\alpha(r+1)C_n^{r+1}(H^F_{r+1})^2}{\int_M F(\nu)\,\mathrm{d}A_X}\\
&\qquad \times\left\{\int_M F(\nu)H^F_1\,\mathrm{d}A_X\int_M F(\nu)\frac{H^F_r}{H^F_{r+1}}\,\mathrm{d}A_X - \left(\int_M F(\nu)\,\mathrm{d}A_X\right)^2\right\},
\end{aligned}
$$

where we used $\alpha = \int_M F(\nu) H_r^F \,\mathrm{d}A_X / \int_M F(\nu)\,\mathrm{d}A_X$ in the last equality of the above formula.

As H_{r+1}^F is a constant, it must be positive by the compactness of M. Thus, by Lemma 2.2, $H_1^F, \dots, H_r^F$ are all positive. So, from [6] or [20], we have:

(i) for each $0 \le r < n-1$,

$$H_1^F H_{r+1}^F - H_{r+2}^F \ge 0, \tag{4.12}$$

with the equality holds if and only if $\lambda_1 = \cdots = \lambda_n$, and

(ii) for each $1 \le r \le n-1$,

$$\begin{aligned} &\int_M F(\nu) H_1^F \,\mathrm{d}A_X \int_M F(\nu) \frac{H_r^F}{H_{r+1}^F} \,\mathrm{d}A_X - \left(\int_M F(\nu)\,\mathrm{d}A_X \right)^2 \\ &\quad \ge \int_M F(\nu) H_1^F \,\mathrm{d}A_X \int_M F(\nu)/H_1^F \,\mathrm{d}A_X - \left(\int_M F(\nu)\,\mathrm{d}A_X \right)^2 \\ &\quad \ge 0, \end{aligned} \tag{4.13}$$

with the equality holds if and only if $\lambda_1 = \cdots = \lambda_n$.

From (4.12) and (4.13), we easily obtain that, for each $0 \le r \le n-1$,

$$\mathscr{A}_r''(0) \le 0,$$

with the equality holds if and only if $\lambda_1 = \cdots = \lambda_n$. Thus, from Lemma 4.2, up to translations and homotheties, $X(M)$ is the Wulff shape. We complete the proof of Theorem 1.3.

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