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BAER'S EXTENSION EQUIVALENCE

PAUL HILL AND CHARLES MEGIBBEN

This paper is dedicated to the memory of Reinhold Baer

Abstract. We revisit Reinhold Baer's work on equivalent extensions, which can be considered as a forerunner of the authors' series of *equivalence theorems*. Our focus is on a paper entitled *Extension Types of Abelian Groups* published by Baer in 1949. In this paper, the main results were for a rather restrictive class of extensions called little extensions, but the notion of two extensions of A by B being equivalent given there are generally applicable. Our theme here is that Baer's vision and understanding of extensions placed him much ahead of the time in which he studied the subject in the 1930's and '40's.

1. Introduction

As is universally recognized, Reinhold Baer left a large footprint on abelian groups. His interests and investigations were widely based ranging from torsion (primary) groups to torsion-free and mixed groups. For a brief synopsis of his contributions to abelian groups, see for example [3]. Although Baer covered the subject broadly as well as deeply, few, if any, topics in abelian groups were of greater interest to Baer than the extension of one abelian group by another. The special case where the extension is an extension of a torsion group by a torsion-free group is noteworthy. The problem of characterizing the torsion-free groups B for which there is only one extension of each torsion group A by B was posed by Baer and solved by P. Griffith. This problem in a more general context is discussed by Griffith elsewhere in this volume.

In our revisit of Baer's 1949 paper entitled *Extension Types of Abelian Groups* [1], it is remarkable to what extent we find that Baer had already laid the foundation conceptually for our later work on equivalence theorems [5], [6], [7], [8], [10], [11], [12]. It was suggested in [9] that J. Erdos was the first to prove an equivalence theorem of this type, but clearly Baer's work [1] preceded that of Erdos [2]. Moreover, in Appendix II of [1], Baer introduces the notion of the equivalence of two extensions $H \subseteq G$ and $H' \subseteq G'$ being

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manifested by an isomorphism from G onto G' that is induced by a prescribed isomorphism from G/H onto G'/H'. Although Baer only obtained results for an equivalence of this type for finite groups, the idea would prove very valuable to the present authors in [11], [12], and elsewhere.

Unless stated otherwise, all groups herein are assumed to be abelian.

Definition 1. Two subgroups H and H' of G are said to be equivalent if there is an automorphism of G that maps H onto H'. Likewise, the extensions G and G' of a common subgroup H are equivalent if there exists an isomorphism of G onto G' that induces the identity map on H.

This equivalence relation enables us to identify extensions that are distinct in $\text{Ext}(B, A)$ but are structurally the same. For example, if $C(n)$ denotes the cyclic group of order n and p is a prime, then $\text{Ext}(C(p), C(p))$ contains p elements, but structurally speaking there are only two extensions of $C(p)$ by $C(p)$, one of which is the split extension and the other is the natural embedding of $C(p)$ in $C(p^2)$.

As we shall demonstrate, the equivalence of extensions is closely related to the property of being able to extend maps. In turn, being able to extend maps rests on what we shall call a *local extension (of maps) theorem*.

Definition 2. By a local extension theorem we mean a result which states (under suitable hypotheses) that a map from a subgroup H of G into a group G' that either preserves or does not decrease heights, as computed in G and G', can be extended to a map of like kind to any finitely generated extension K of H in G.

Our treatment and philosophy, both here and heretofore, depart from that of Baer [1] in regards to local extensions (of maps). Let us emphasize though that this pertains only to the means rather than the end itself. Indeed, as we have already indicated, Baer's pioneering work on equivalent extensions has served as a model for us. Returning to the means, however, we note that Baer did not establish nor use a local extension theorem in [1]. Instead, he employs a result of N. Steenrod [13] concerning inverse limits of compact spaces. Consequently, Baer was restricted to what he called *little* extensions in proving results such as the following theorem and corollary. First, we define what a little extension is as well as what we shall call a *slim* extension, which is slightly more general.

Definition 3. An extension G of H is called a little extension if: (1) G/H is torsion, and (2) for each relevant prime p of the torsion group G/H the p-primary subgroup of G is a finite group plus a divisible group of finite rank.

We remark that little extensions may be more general than they first appear. For example, any free resolution of a torsion group represents a little extension. More generally, if

$$0 \to A \to B \to C \to 0$$

is a short exact sequence with B torsion-free and C torsion, then B is a little extension of A. The following is a modest generalization of a little extension.

Definition 4. An extension G of H is called a slim extension if: (1) G/H is torsion, and (2′) for each relevant prime p of the torsion group G/H the p-primary subgroup of G is a bounded group plus a divisible group.

Theorem 1 (Baer, [1]). *Suppose that G is a little extension of its subgroup H. Then any homomorphism from H into G that does not decrease heights computed in G is induced by an endomorphism of G.*

Corollary 1 (Baer, [1]). *If G is a little extension of its subgroup H and H is pure in G, then the extension splits.*

The preceding theorem serves as a basis for the numerous results in [1] concerning little extensions. This result leads not only to a classification of equivalent extensions of H by G, but also to a classification of equivalent *types* of extensions.

Definition 5. Suppose that G and G' are extensions of H. If there exists a homomorphism from G into G' that leaves the elements of H fixed, then the extension G' of H is said to have type greater than or equal to that of the extension G of H. The types of the two extensions are equivalent (or equal) if each has type greater than or equal to the other.

By proving a local equivalence theorem instead of relying on Steenrod's [13] result cited earlier on inverse limits of compact spaces, we are able to provide an alternate approach to the aforementioned results of Baer and at the same time generalize his results to slim extensions.

2. The local extension theorem

We begin this section with a version of a local extension theorem that first appeared in [4] for primary groups, but the proof is essentially the same for the more general version.

Theorem 2 (Hill, [4]). *Suppose that p is a prime and that the subgroup H of G is nice with respect to p-heights. If x is an element of G and px is contained in H, then any map from H into a group G' that does not decrease heights, as computed in G and G', can be extended to a map from $K = \langle H, x\rangle$ into G' that does not decrease heights.*

The hypothesis of the next theorem is modelled after our definition of a slim extension.

Theorem 3. *Let H be a subgroup of G and let K be a finitely generated extension of H in G. Suppose that ϕ is a homomorphism from H into a group G' that does not decrease heights computed in G and G', respectively. Assume (i) G/H is torsion, and (ii) for each relevant prime p of G/H that the p-primary subgroup of G' is a bounded group plus a divisible group. Then ϕ can be extended to a homomorphism from K into G' that does not decrease heights computed in G and G'.*

Proof. Observe that if H, G, and G' satisfy the hypotheses (i) and (ii), then so do H', G, and G' for any subgroup H' of G containing H. Therefore, since G/H is torsion, it suffices to prove the theorem for the case that K/H has order p for some prime p. Thus, let $K = \langle H, x\rangle$, where $px \in H$.

To show that ϕ can be extended to a homomorphism from K into G' that does not decrease heights, we distinguish two cases. All heights are computed in G and G' as the case may be.

Case I: H is nice in G with respect to p-heights. In this case, an application of the preceding theorem yields the desired result.

Case II: H is not nice in G with respect to p-heights. Since p is a relevant prime for G/H, by hypothesis the reduced part of the p-primary subgroup of G' is bounded by p^m for some positive integer m. Since H is not nice in G with respect to p-heights, there is some element in the coset $x + H$ that has p-height greater than m. Without loss of generality, we may assume that x has p-height greater than m. Thus the p-height of px is at least $m + 2$. By assumption, ϕ does not decrease heights. Therefore the p-height of $\phi(px)$ is at least $m + 2$, so there exists $y \in p^{m+1}G'$ for which $py = \phi(px)$. We claim that we can extend ϕ by mapping x onto y, and that this extension does not decrease heights. The verification of the first claim is routine. To verify the second claim, we need to show that if $h \in H$, then the height of $\phi(x + h)$ is greater than or equal to the height of $x + h$. Since the q-height of x is the same as the q-height of px for any prime q different from p and since ϕ does not decrease q-heights when restricted to H, it suffices to prove that the above extension of ϕ does not decrease p-heights.

Concerned now only with p-heights, we need to show that if $h \in H$, then $y + \phi(h)$ has p-height greater than or equal to that of $x + h$. Clearly, if the p-height of h does not exceed m, then the p-height of $x+h$ is equal to that of h because the p-height of x is greater than m. Further, the p-height of $y + \phi(h)$ is greater than or equal to that of h, so the desired result holds. Finally, we consider the case where the p-height of h exceeds m. In this case, we lose control of the p-height of $x + h$. However, we are saved by the fact that there is no jump in p-heights when we pass from $y + \phi(h)$ to $p(y + \phi(h))$; this is because the reduced part of the p-primary subgroup of G' is bounded by p^m. It now follows that if $x + h$ has p-height equal to $\alpha > m$, then $p(x + h)$ and consequently $p(y + \phi(h))$ has p-height greater than or equal to $\alpha + 1$. Hence, $y + \phi(h)$ must have p-height greater than or equal to α due to the fact that there are no jumps in p-heights in G' at this level. This demonstrates that our extension of ϕ to K still does not decrease heights, and the proof of the theorem is completed. □

Since the hypothetical properties of the preceding theorem are inductive, repeated applications of the theorem yield the following corollaries for slim (and, in particular, little) extensions. The original map in each of these corollaries is the identity on H. It should be noted that the hypotheses imply that the identity on H does not decrease heights when considered as a partial map of G into G'.

Corollary 2. *Suppose that G and G' are slim extensions of a common subgroup H. Then there exists a homomorphism ϕ from G into G' that induces the identity on H if (and only if) $H \cap p^n G \subseteq H \cap p^n G'$ for each (relevant) prime p and positive integer n.*

Corollary 3. *A pure subgroup always splits out of a slim extension.*

3. The relation of the Baer invariants to the relative Ulm invariants

Let G be an extension of its subgroup H. For an arbitrary but fixed prime p, let H^v denote H when considered as a valuated group obtained by computing the p-heights of elements of H in the containing group G. Denoting $H \cap p^\alpha G$ simply by $H(\alpha)$, we see that

$$\dim \big(H(\alpha)[p]/H(\alpha+1)[p]\big)$$

is the α^{th} Ulm invariant of the valuated group H^v. It is hereinafter denoted by $U_\alpha(H^v)$.

Baer [1] calls a descending chain H_n of subgroups of H a *p-Loewy chain* of H provided that $H_0 = H$ and

$$H_i \supseteq H_{i+1} \supseteq pH_i.$$

Thus, the chain of subgroups $H(\alpha)$, defined above, is a p-Loewy chain of H associated to the extension G of H. If p ranges over the (relevant) primes of G/H, then this collection of p-Loewy chains of H yields a Loewy chain of H. Baer shows in [1] that the little extension types of a given group H are in one-to-one correspondence with certain Loewy chains of H and that this correspondence preserves order. If p is a relevant prime of G/H for a little extension G of H, then the Ulm invariants of the p-primary subgroup of G are all finite. Obviously, the Ulm invariants of the valuated group H^v are intimately related to the p-Loewy chain of subgroups $H(\alpha)$. The following cardinal number is also closely related to this p-Loewy chain:

$$\dim\Big(\big(pH(\alpha)\cap p^{\alpha+2}G\big)/pH(\alpha+1)\Big).$$

DEFINITION 6. The preceding number is called the Baer invariant associated to an extension G of H and is denoted by $B_\alpha(G,H)$; if the prime p is not clear from the context, it is denoted by $B_\alpha(G,H)_p$.

Clearly, equivalent extensions have the same Baer invariants.

For a finite ordinal α, Baer established the following lemma in [1]. The lemma will be helpful in verifying our formula that relates the Baer invariants to the relative Ulm invariants. Since the proof of the lemma is routine, it is omitted.

LEMMA 1. $\big(pH(\alpha)\cap p^{\alpha+2}G\big)/pH(\alpha+1)\cong\big(H(\alpha)+p^{\alpha+1}G\big)[p]/\big(H(\alpha)[p]+p^{\alpha+1}G[p]\big)$.

Although Baer's paper [1] was about twenty years prior to the introduction of the relative Ulm invariants in [4], the theorem and corollary that follow once again provide evidence of Baer's insight and vision by showing that he was, in effect, indirectly considering the relative Ulm invariants, at least in the case where the Ulm invariants are finite.

The equivalence theorems referred to in the Introduction involve the extension of a height-preserving map $\phi: H\to H'$ where H and H' are fixed subgroups of the abelian groups G and G', respectively. The desired isomorphism between G and G' that extends ϕ is generally constructed via a local extension theorem, and the cardinal numbers

$$\dim\Big(p^\alpha G[p]/(H+p^{\alpha+1}G)\cap p^\alpha G[p]\Big),$$

for relevant primes p and ordinals α, play a crucial role in the local extension process.

DEFINITION 7. The preceding number is called the α^{th} relative Ulm invariant of G with respect to the subgroup H and is denoted by $U_\alpha(G,H)$; if the prime p is not clear from the context, it is denoted by $U_\alpha(G,H)_p$.

For a fixed prime p, let $U_\alpha(G)$ and $U_\alpha(G,H)$, respectively, denote the α^{th} Ulm invariant of G and the α^{th} relative Ulm invariant of G with respect to a subgroup H. Recall that $U_\alpha(H^v)$ denotes the α^{th} Ulm invariants of H when considered as a valuated group obtained by taking the p-heights computed in G of the elements of H.

THEOREM 4. *Let H be a subgroup of G. Then, for each ordinal α,*

$$U_\alpha(G)=B_\alpha(G,H)+U_\alpha(G,H)+U_\alpha(H^v).$$

Proof. As before, let $H(\alpha)$ denote $H \cap p^\alpha G$. Note that

$$\dim\big((H(\alpha)[p]+p^{\alpha+1}G[p])/p^{\alpha+1}G[p]\big) = \dim\big(H(\alpha)[p]/H(\alpha+1)[p] = U_\alpha(H^v).$$

Also, the preceding lemma yields

$$B_\alpha(G,H) = \dim\big((H(\alpha)+p^{\alpha+1}G\big)[p])/\big(H(\alpha)[p]+p^{\alpha+1}G[p])\big).$$

For simplicity of notation, set

$$\begin{aligned} A &= \dim\big(p^\alpha G[p]/p^{\alpha+1}G[p]\big) = U_\alpha(G),\\ B &= \dim\big((H(\alpha)+p^{\alpha+1}G)[p]/p^{\alpha+1}G[p]\big),\\ C &= \dim\big((H(\alpha)[p]+p^{\alpha+1}G[p])/p^{\alpha+1}G[p]\big). \end{aligned}$$

Using the relations established above, we conclude that $B = B_\alpha(G,H)+C$, and that $C = U_\alpha(H^v)$. Hence, $B = B_\alpha(G,H)+U_\alpha(H^v)$. Moreover, $U_\alpha(G) = U_\alpha(G,H)+B$. Therefore,

$$U_\alpha(G) = B_\alpha(G,H)+U_\alpha(G,H)+U_\alpha(H^v). \qquad \square$$

Corollary 4. *Let p be an arbitrary but fixed prime. Suppose that H is a subgroup of G and that the p-Ulm invariants of G are finite, as they are for a little extension and a relevant prime p of G/H. If the Ulm invariants of G and of the valuated group H^v are known, then the relative Ulm invariants of G with respect to H can be computed from the Baer invariants.*

4. Baer's construction

For readers for whom Baer's notion of a p-Loewy chain H_n of H has little appeal, it may help to observe that these are simply valuations in disguise. Indeed if we define a function $v_p : H \to \omega \cup \{\infty\}$ by $v_p(x) = \sup\{n : x \in H_n\}$, then the following properties are satisfied: (a) $v_p(x+y) \geq \min\{v_p(x), v_p(y)\}$, (b) $v_p(px) > v_p(x)$ where $\infty > \infty$, and (c) $v_p(nx) = v_p(x)$ when $p \nmid n$. Notice that $v_p(x) = \infty$ if and only if $x \in H_\infty = \bigcap_{n<\omega} H_n$. Conversely, given a function $v_p : H \to \omega \cup \{\infty\}$ satisfying these conditions, the subgroups $H_n = \{x \in H : v(x) \geq n\}$ form a p-Loewy chain of H. Indeed when constructing a p-Loewy chain H_n in a given group, it is useful to first define a p-valuation. Also notice that the fact that $H_\infty = \{x \in H : v_p(x) = \infty\}$ need not be 0 explains the existence of nonzero divisible p-groups in slim extensions of H even when the latter is reduced.

Baer establishes in [1] a remarkable embedding theorem of a p-valuated group H in a group G where G/H is a p-group and the given valuation on H is induced by the p-height function on G.

Theorem 5 (Baer, [1]). *If H_n is a p-Loewy chain of H, then there exists an extension G of H with the following properties:*

(i) *$H_n = H \cap p^n G$ for each nonnegative integer n.*

(ii) $G[p] = H[p] \oplus X_0 \oplus X_1 \oplus \cdots X_n \oplus \cdots$ *where* $X_n \cong (pH_n \cap H_{n+2})/pH_{n+1}$ *for each* n.

(iii) G/H *is a* p*-group.*

If the p-Loewy chain of H is *trivial* in the sense that $H_n = p^n H$ for all $n < \omega$, then $pH_n \cap H_{n+2}/pH_{n+1} = 0$ for all n and therefore in Baer's construction, $G[p] = H[p]$ and $H \cap p^n G = p^n H$ for all $n < \omega$. These two conditions coupled with the fact that G/H is a p-group imply that $G = H$.

Not being aware of the significance of the relative Ulm invariants, Baer not surprisingly fails to note one of the most striking aspects of the extension G of H given in Theorem 5; namely, that all the Ulm invariants of G relative to H vanish, including the invariant

$$U_\infty(G, H) = \dim\big(p^\infty G[p]/(H \cap p^\infty G[p])\big).$$

Theorem 6. *Given a* p*-Loewy chain* H_n *of* H*, there exists an extension* G *of* H *with* G/H *a* p*-group,* $H \cap p^n G = H_n$ *for all* $n < \omega$ *and* $U_\alpha(G, H) = 0$ *for* $\alpha = \infty$ *and for all* $\alpha < \omega$*. In particular,* $U_n(G) = U_n(H^v) + B_n(G, H)$ *for all* $n < \omega$*.*

Proof. We require a closer look at Baer's construction. First we note that G is obtained as the ascending union of groups G_n where $G_1[p] = H[p]$, $p^n G_n = H_n$, $p^{n-1} G_n \cap H = H_{n-1}$ and $G_{n+1}[p]/G_n[p] \cong (pH_{n-1} \cap H_{n+1})/pH_n$ for all positive integers n. Moreover, the construction of the subgroups X_{n-1} is quite explicit. Starting with a fixed basis B of $pH_{n-1} \cap H_{n+1}$ modulo pH_n, generators of G_{n+1} are selected among which there corresponds to each $b \in B$ an element b' subject to the relation $p^{n+1} b' = b$. Then the basis for X_{n-1} consists of all elements $b^* = p^n b' - p^{n-1} b''$ where b'' is any element of G_n satisfying $p^n b'' = b$ — recall that, by choice, $b \in pH_{n-1} \subseteq H_n = p^n G_n$.

The verification that $U_{n-1}(G, H) = 0$ for all positive integers n hinges on observing that all the basis elements $b^* = p^n b' - p^{n-1} b''$ for X_{n-1} may be selected with $p^{n-1} b'' \in H$. To see that this is possible, recall that $b \in pH_{n-1}$ and consequently we may write $b = pw$ where $w \in H_{n-1} = p^{n-1} G_n \cap H$. Therefore there is a $z \in G_n$ with $p^{n-1} z = w \in H$ and so $p^n z = pw = b$. Taking $b'' = z$ yields the desired conclusion. It is now clear that $X_{n-1} \subseteq (H + p^n G) \cap p^{n-1} G[p]$.

More generally, $\bigoplus_{i \geq n} X_i \subseteq H + p^{n+1} G$ for all nonnegative integers n and consequently $U_n(G, H) = 0$ will follow once we establish that

$$p^n G[p] = H_n[p] \oplus X_n \oplus X_{n+1} \oplus \cdots$$

holds for all n. Proceeding by induction, assume that $p^{n-1} G[p] = H_{n-1}[p] \oplus X_{n-1} \oplus X_n \oplus \cdots$ and choose a subgroup Y_{n-1} of $H[p]$ such that $H_{n-1}[p] = Y_{n-1} \oplus H_n[p]$. Then, since $H_n[p] \subseteq p^n G$, the desired direct decomposition of $p^n G[p]$ will follow once we show that $(Y_{n-1} \oplus X_{n-1}) \cap p^n G = 0$. Suppose then that $y + x \in p^n G$ where $y \in Y_{n-1}$ and $x \in X_{n-1}$. From the description above of a basis for X_{n-1}, we have a representation $x = \sum_{b\in B} t_b(p^n b' - p^{n-1}b'')$ where almost all the t_b are zero. Thus $y - \sum_{b\in B} t_b p^{n-1}b''$ is in $H \cap p^n G = H_n$ and since $py = 0$,

$$\sum_{b\in B} t_b b = p\left(\sum_{b\in B} t_b p^{n-1} b'' - y\right) \in pH_n.$$

But in view of the fact that B is a basis for $pH_{n-1} \cap H_{n+2}$ modulo pH_n, each t_b is divisible by p and therefore $y + x = y \in Y_{n-1} \cap p^n G = Y_{n-1} \cap H_n[p] = 0$, as desired.

Finally, by part (ii) of Theorem 5, $p^\infty G[p] \subseteq H[p]$, and consequently we also have $U_\infty(G, H) = 0$. □

5. An existence theorem for slim extensions

We begin with a simple observation about slim extensions.

THEOREM 7. *Suppose G is a slim extension of H and let $\mathcal{P}$ consist of the relevant primes of the torsion group G/H. Then for each $p \in \mathcal{P}$, the reduced part of the p-primary subgroup of H is bounded and the subgroups $H_n^p = H \cap p^n G$ form a nontrivial p-Loewy chain of H with $(pH_n^p \cap H_{n+2}^p)/pH_{n+1}^p = 0$ for almost all $n < \omega$.*

Proof. By Definition 4, the reduced part of the p-primary subgroup of G is bounded when $p \in \mathcal{P}$. This implies that H has the same property and furthermore that $U_n(G)_p = 0$ for almost all $n < \omega$ when $p \in \mathcal{P}$. But then

$$\begin{aligned} \dim\left((pH_n^p \cap H_{n+2}^p)/pH_{n+1}^p\right) &= \dim\left((pH_n^p \cap p^{n+2}G)/pH_{n+1}^p\right) \\ &= B_n(G, H)_p = 0 \end{aligned}$$

for almost all $n < \omega$. □

Applying Baer's construction in Theorem 5, we readily see that the conditions satisfied by H in the preceding theorem are also sufficient for the existence of a slim extension G of H.

THEOREM 8. *Let $\mathcal{P}$ be a set of primes and H an abelian group with a nontrivial p-Loewy chain H_n^p for each prime $p \in \mathcal{P}$. Then there exists a slim extension G of H with $\mathcal{P}$ the set of relevant primes of the torsion group G/H if and only if, for each prime $p \in \mathcal{P}$, (1) the reduced part of the p-primary subgroup of H is bounded and (2) $(pH_n^p \cap H_{n+2}^p)/pH_{n+1}^p = 0$ for almost all $n < \omega$. Moreover, when these conditions are satisfied, G can be constructed with $U_\alpha(G, H)_p = 0$ for $\alpha = \infty$ and for all $\alpha < \omega$.*

Proof. For each $p \in \mathcal{P}$, let G^p be the extension of H as given in Theorem 5 and observe, by the hypotheses on H and by part (ii) of Theorem 5, that G^p is a slim extension of H with G^p/H a p-group. Furthermore, Theorem 6 implies that $U_\alpha(G,H)_p = 0$ for $\alpha = \infty$ and for all $\alpha < \omega$. Finally, the "free" sum over H of all these G^p with $p \in \mathcal{P}$ yields the desired slim extension G of H. □

6. The equivalence criterion for slim extensions

Exploiting the relative Ulm invariants and the technique of local extensions, we establish an equivalence criterion for slim extensions sharper than the one given by Baer [1] for little extensions.

THEOREM 9. *Let G and G' be slim extensions of H and H', respectively, where the Ulm invariants of G relative to H are equal to the Ulm invariants of G' relative to H'. If $\phi : H \to H'$ is an isomorphism and K is a finite extension of H in G, then there is a subgroup K' of G' and a height-preserving isomorphism $\psi : K \to K'$ that extends ϕ. Moreover, the Ulm invariants of G relative to K are equal to the Ulm invariants of G' relative to K'.*

Proof. As in the proof of Theorem 3, it suffices to consider the case where $K = \langle H, x\rangle$ and $px \in H$ for some prime p. To show that ϕ can be extended to a height-preserving isomorphism ψ, we distinguish two cases. All heights, of course, are computed in G and G' as the case may be.

Case I: H is nice in G with respect to p-heights. As in Proposition 2.4 of Hill [4], if x has finite p-height then the equality of the relative Ulm invariants allows us to select an $x' \in G'$ with the same p-height and such that $\phi(px) = px'$. We then need only take $\psi(x) = x'$. In case x has p-height ∞, we require the hypothesis that $U_\infty(G,H) = U_\infty(G',H')$. Indeed a routine argument shows that the equality of these invariants implies the existence of an isomorphism $\theta : D_p \to D'_p$ with $\theta\,|H \cap D_p = \phi\,|H \cap D_p$ where D_p and D'_p are the maximal divisible subgroups of the p-primary subgroups of G and G', respectively. In this special case, we take $x' = \theta(x)$. That the Ulm invariants of G relative to K equal the Ulm invariants of G' relative to K' is as in Hill [4].

Case II. The proof here is exactly as in Case II of Theorem 3, noting that when ϕ preserves heights, the p-height of the element $y \in G'$ selected there equals the p-height of x in G. In this case, the relative Ulm invariants are not affected. □

As we shall explain below, the following Equivalence Criterion is a consequence of the preceding theorem.

THEOREM 10. *The slim extensions G and G' of H are equivalent if and only if the following two conditions are satisfied:*

(i) *$H \cap p^nG = H \cap p^nG'$ for all prime powers p^n.*

(ii) *The Ulm invariants of G relative to H equal the Ulm invariants of G' relative to H.*

Observe, by Corollary 2, that condition (i) yields maps $\psi : G \to G'$ and $\psi' : G' \to G$ that leave the elements of H fixed. But condition (ii) is required to construct an isomorphism ψ of G onto G' that induces the identity map ϕ on H.

We consider all triples (ψ, K, K') where K and K' are extensions of H in G and G', respectively, and $\psi : K \to K'$ is a height-preserving isomorphism that induces the identity map ϕ on H. We place a further restriction on the subgroups K and K' that will insure that G and G' are slim extensions of K and K', respectively. Namely, if $\mathcal{P}$ is the set of relevant primes for G/H, we say that K is "closed" provided $(G/K)[q] = 0$ for all primes $q \notin \mathcal{P}$. The operative facts here are that an ascending chain of "closed" subgroups is "closed" and $M = \langle K, x\rangle$ is "closed" when K is "closed" and $px \in K$ for $p \in \mathcal{P}$. In particular, the finitely generated extensions K and K' in Theorem 9 are "closed." Exploiting the symmetry involved in the equality of the relative Ulm invariants, we map back and forth between G and G' to insure that all elements in the two groups are captured in a maximal triple (ψ, G, G'), yielding thereby the desired isomorphism that leaves the elements of H fixed.

7. Minimal extensions and direct decompositions

Following Baer, we say that an extension G of H is a *minimal extension* if $G = G_1 \oplus G_2$ with $H \subseteq G_1$ implies $G_2 = 0$. In [1], Baer presents a charming compactness argument establishing that for each little extension G of H there exists a direct decomposition $G = G_1 \oplus G_2$ where $H \subseteq G_1$ and G_1 is a minimal extension of H. This suggests that such direct decompositions are an artifact of Baer's reliance on [13] to extend maps and leaves the status of minimal extensions quite mysterious. But, as we shall see below, a little extension G of H is a minimal extension precisely when the Ulm invariants of G relative to H vanish and moreover the canonical decomposition $G = G_1 \oplus G_2$ with G_1 a minimal extension of H also holds for slim extensions. Of course, we must rely on structure theory rather than on compactness.

First, we show that if G is a slim extension of H where the Ulm invariants of G relative to H vanish, then G is a minimal extension of H. Indeed if we were to have a direct decomposition $G = G_1 \oplus G_2$ where $H \subseteq G_1$ and $G_2 \neq 0$, then there would be some prime p with $G_2[p] \neq 0$ since G/H is a torsion group. If there were an element $x \in G_2[p]$ having finite p-height n, then we would quickly obtain a contradiction to the fact that $p^nG[p] \subseteq H + p^{n+1}G$. Similarly, if there were a nonzero element $x \in G_2[p]$ having p-height ∞, this would contradict the fact that $p^\infty G[p] \subseteq H \cap p^\infty G[p]$. Conversely, if the slim extension G of H is a minimal extension, then an inspection of the proof of our next theorem will confirm that the Ulm invariants of G relative to H must vanish.

THEOREM 11. *If G is a slim extension of H, then there exists a direct decomposition $G = G_1 \oplus G_2$ where $H \subseteq G_1$, G_1 is a minimal extension of H and G_2 is a torsion group with the reduced part of each of its primary components bounded.*

Proof. Let $\mathcal{P}$ and the H_n^p be as in Theorem 7 and apply Theorem 8 to construct a slim extension G_1' of H such that (a) $H \cap p^n G_1' = H_n^p$ for all $p \in \mathcal{P}$ and all $n < \omega$ and (b) $U_\alpha(G_1', H)_p = 0$ for $\alpha = \infty$ and for all $\alpha < \omega$. Thus from the analysis of the Ulm invariants in Section 3, $U_n(G)_p = U_n(H^v)_p + B_n(G, H)_p + U_n(G, H)_p$ and $U_n(G_1')_p = U_n(H^v)_p + B_n(G_1', H)_p$ for all $n < \omega$ and all $p \in \mathcal{P}$. Notice that the invariants $U_n(H^v)_p$ are the same when computed in G or in G' since $H \cap p^n G = H_n^p = H \cap p^n G_1'$ for all prime powers p^n with $p \in \mathcal{P}$. Now choose a torsion group G_2' with $U_n(G_2')_p = U_n(G, H)_p$ for all $n < \omega$ and all $p \in \mathcal{P}$ and observe that $G' = G_1' \oplus G_2'$ is still a slim extension of H. Since clearly $H \cap p^n G = H_n^p = H \cap p^n G'$ for all prime powers p^n, Theorem 10 applies to yield an isomorphism $\psi : G' \to G$ that leaves the elements of H fixed. Therefore $G = G_1 \oplus G_2$ where $G_1 = \psi(G_1')$, $G_2 = \psi(G_2')$ and obviously, when $p \in \mathcal{P}$, $U_\alpha(G_1, H)_p = 0$ for $\alpha = \infty$ and for all $\alpha < \omega$. That G_1 is a minimal extension of H follows from the discussion preceding the statement of the theorem. □

8. Extension types

Once again, we follow Baer [1] and say that the *type* of the extension G' of H is greater than or equal to the type of the extension G of H if there is a homomorphism from G into G' that induces the identity map on H. We denote this by $\mathrm{Ext}[H < G] \leq \mathrm{Ext}[H < G']$. Obviously, this introduces a partial order among the extension types of H. If we have two extensions G and G' of H and if the type of each extension is greater than or equal to the other, then the two extensions are said to have equivalent extension types and we write $\mathrm{Ext}[H < G] = \mathrm{Ext}[H < G']$.

The following result was proved by Baer ([1], Theorem 2) for little extensions.

THEOREM 12. *The following properties of slim extensions G and G' of H are equivalent.*

(i) $\mathrm{Ext}[H < G] \leq \mathrm{Ext}[H < G']$.
(ii) *The height of each element of H computed in G' is greater than or equal to its height computed in G.*
(iii) *There exists a slim extension G'' of G with the property that $\mathrm{Ext}[H < G''] = \mathrm{Ext}[H < G']$.*

Proof. That (i) implies (ii) is a trivial consequence of the fact that homomorphisms do not decrease heights.

We now want to show that (ii) implies (iii). Thus assume that (ii) holds. Let $T = \bigoplus T_p$ be a torsion group with the property that its p-primary part T_p is a divisible plus a bounded group. Then $G'' = G' \bigoplus T$ remains a slim extension of H. Moreover, if T_p is suitably chosen and sufficiently large (to provide the necessary relative invariants), then the identity on H can be extended to an isomorphism from G into G'' since it does not decrease heights computed in G and G', respectively. Thus, G'' can be considered an extension of G. Since both the inclusion map of G' into G'' and the projection map of G'' onto G' are the identity when restricted to H, it follows that $\mathrm{Ext}[H < G''] = \mathrm{Ext}[H < G']$, and (ii) implies (iii).

Finally, we need to verify that (iii) implies (i). Suppose that $\mathrm{Ext}[H < G''] = \mathrm{Ext}[H < G']$. We know, in particular, that there is a homomorphism ϕ from G'' into G' that induces the identity map on H. By restricting ϕ to G, we obtain a homomorphism from G into G' that remains the identity map on H, so (i) is an immediate consequence of (iii). □

We conclude with the following clarification of the relationship between equivalent extensions and equivalent types of extension.

Theorem 13. *Suppose that G and G' are slim extensions of a common subgroup H. Then $\mathrm{Ext}[H < G] = \mathrm{Ext}[H < G']$ if and only if there exists a torsion group B such that each of its primary subgroups B_p is a divisible plus a bounded group having the property that $G \oplus B$ and $G' \oplus B$ are equivalent extensions of H.*

Proof. In one direction the proof is trivial. If such a group B exists for which there is an isomorphism ϕ from $G \oplus B$ onto $G' \oplus B$ that induces the identity on H, then certainly there are homomorphisms in both directions between G and G' that restrict to the identity map on H.

For the proof in the other direction, suppose that $\mathrm{Ext}[H < G] = \mathrm{Ext}[H < G']$. Then the elements of H must have the same height in G as in G'. For a suitable group B satisfying the conditions of the theorem, the Ulm invariants of $G \oplus B$ relative to H are equal to the Ulm invariants of $G' \oplus B$ relative to H. An application of Theorem 10 establishes the fact that $G \oplus B$ and $G' \oplus B$ are equivalent extensions of H. □

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Paul Hill, Western Kentucky University, Bowling Green, KY 42101, USA
E-mail address: `paul.hill@wku.edu`

Charles Megibben, Vanderbilt University, Nashville, TN 37240, USA
E-mail address: `charles.megibben@vanderbilt.edu`