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MINIMAL SURFACES IN $\mathbb{M}^2 \times \mathbb{R}$

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ABSTRACT. We study the geometry and topology of properly embedded minimal surfaces in $M \times \mathbb{R}$, where M is a Riemannian surface. When M is a round sphere, we give examples of all genus and we prove such minimal surfaces have exactly two ends or equal $M \times \{t\}$, for some real t. When M has non-negative curvature, we study the conformal type of minimal surfaces in $M \times \mathbb{R}$, and we prove half-space theorems. When M is the hyperbolic plane, we obtain a Jenkins-Serrin type theorem.

Introduction

In this paper we will discuss minimal surfaces Σ in $M \times \mathbb{R}$, where M will be the 2-sphere (with the constant curvature one metric) or a complete Riemannian surface with a metric of non-negative curvature, or M will be the hyperbolic plane. This last case will be treated in detail in a joint paper with Barbara Nelli; here we will state some of the results. The author wishes to thank L. Hauswirth for stimulating conversations.

The theory is naturally inspired by that of minimal surfaces in $\mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$. When M has non-negative curvature, one expects results as in $\mathbb{R}^3$. In $\mathbb{H}^2 \times \mathbb{R}$, the theory is quite different.

In each ambient space, we begin with some examples, and then we prove some theorems. A natural technique to construct global examples is to begin with a geodesic polygon Γ in $N = M \times \mathbb{R}$ and let Σ_0 be a solution to the Plateau problem in N with boundary Γ. When Γ is composed of horizontal and vertical geodesic segments, then Σ_0 can be extended across Γ by symmetry in each edge of Γ. By appropriate choices of Γ this will yield properly embedded minimal surfaces in N, by continually repeating symmetry in the sides of all the polygons obtained.

A variant of this procedure will produce surfaces analogous to Scherk's periodic minimal surfaces. Starting with Γ and its Plateau solution Σ_0, one lets some sides of Γ tend to infinity in length, so that the associated Plateau solutions all pass through a fixed compact region in N (this will be assured

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by certain symmetries in the Γ we use). Then a subsequence of the Plateau solutions will converge to a minimal surface bounded by a geodesic polygon with edges of infinite length. One completes this surface by symmetry in the edges of Γ.

In this paper, we often solve Plateau problems, finding least area surfaces with fixed boundary in a given isotopy class. Some references for doing this in 3-manifolds are [F-H-S] and [M-S-Y]. A reference for finding minimax surfaces of controlled topology is [Rub].

1. Surfaces in $S \times \mathbb{R}$; unduloids

Let S denote the 2-sphere of curvature one, and $S(t) = S \times (t)$. We refer to $S(t)$ as the horizontal sphere at height t, and we denote by h the height function on $S \times \mathbb{R}$, which is the $\mathbb{R}$-coordinate of a point. In a very interesting paper concerning isoperimetric-hypersurfaces in $Q \times \mathbb{R}$, Q an n-dimensional simply connected space-form, Pedrosa and Ritore [P-R] found and studied the minimal hypersurfaces of $S^n \times \mathbb{R}$ invariant under the group of rotations of the first factor. When $n = 2$, they call these surfaces unduloids (embedded) and nodoids, in analogy with the Delaunay surfaces. They are foliated by circles $C(t)$ in the sphere $S(t)$, of radius $r(t)$; the radius function determines the surface.

Before writing the equations of these surfaces found by Pedrosa and Ritore, we describe their existence by Plateau techniques.

Let p denote a fixed point of S (e.g., the north pole) and let r denote distance to p on S. Denote by $C(0)$ the geodesic $r = \pi/2$ of S. Then $\Sigma = C(0) \times \mathbb{R}$ is a totally geodesic minimal annulus in $N = S \times \mathbb{R}$; we will refer to this as a vertical flat annulus. Let D_1, D_2 be the two disks of $S(0)$ bounded by $C(0)$. For $T > 0$, T small, $\Sigma(T)$ (the part of Σ between heights 0 and T) is a stable minimal surface with boundary $C(0) \cup C(T)$. Also, $C(0) \cup C(T)$ bounds another stable surface in $D_1 \times \mathbb{R}$, the union of the two horizontal disks. So there is an unstable surface in $D_1 \times \mathbb{R}$ with boundary $C(0) \cup C(T)$. It is a connected annulus since the only compact minimal surface bounded by a $C(h)$ is a horizontal disk. This annulus can then be extended to an embedded complete minimal annulus by rotation by π about the geodesic boundaries. This rotation about $C(0)$ is the composition of the isometries $(x, t) \mapsto (x, -t)$, and the isometry which is reflection of each $S(t)$ by the geodesic $C(t)$.

Calculations of M. Ritore show that as T tends to 0, these unstable annuli converge to a double cover of the horizontal disk, with a singularity at the center of the disk, just as a catenoid converges to a doubly-covered plane by contraction.

Pedrosa and Ritore call these surfaces unduloids. They deform the flat vertical annulus Σ in the same manner as the Delaunay mean curvature one surfaces in $\mathbb{R}^3$ deform the cylinder of mean curvature one.

They derive the equations for rotationally invariant constant mean curvature H hypersurfaces in $S \times \mathbb{R}$; more generally, in (a space form)$\times\mathbb{R}$. (This is the only place in this paper we change the order of the factors in $S \times \mathbb{R}$. We do this until the end of this section to respect the order chosen by Pedrosa and Ritore.) Here is their solution.

Identify the orbit space with $[0, \pi] \times \mathbb{R}$. An invariant hypersurface is determined by its generating curve γ in the quotient space. Parametrize γ by arc length s and write

$$(x'(s), y'(s)) = (\cos\sigma(s), \sin\sigma(s)).$$

Then Σ has mean curvature H in $S^n \times \mathbb{R}$ is equivalent to γ satisfying the system:

$$\begin{aligned} x' &= \cos\sigma, \\ y' &= \sin\sigma, \\ \sigma' &= H + (n-1)\cot(y)\cos(\sigma). \end{aligned}$$

In addition to the embedded unduloid solutions, they show there are immersed solutions as well; they call them nodoids.

The unduloids are invariant under vertical translation by $2\pi T$, hence yield embedded minimal tori in $S^1(r) \times S^2$.

2. Helicoids in $S \times \mathbb{R}$

We obtain a helicoid by rotating the geodesic $C(0)$ at a constant speed as one rises on the vertical axis at a constant speed. This yields a complete minimal annulus Σ in $S \times \mathbb{R}$ and by passing to the quotient by a suitable vertical translation, an embedded minimal torus in $S \times S^1$.

A conformal parametrization of a helicoid can be obtained as follows. Let

$$X(u, v) = (\cos f(u)\cos v, \cos f(u)\sin v, \sin f(u), v).$$

Then X is conformal in terms of $z = u + iv$ if f satisfies the equation (an elliptic function)

$$f'(u)^2 = 1 + \cos^2 f(u).$$

The mean curvature vector of Σ in $\mathbb{R}^4$ ($S \times \mathbb{R} \subset \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$) is easily calculated in terms of

$$\frac{\partial^2 X}{\partial u^2} + \frac{\partial^2 X}{\partial v^2}\,.$$

A simple calculation then shows the projection of this mean curvature vector onto the unit normal of Σ in $S \times \mathbb{R}$ is zero. Hence Σ is minimal in $S \times \mathbb{R}$.

3. Properties of minimal annuli in $S \times \mathbb{R}$

Let A be a properly immersed minimal annulus in $S \times \mathbb{R}$; A is topologically $D^* = \{z \in \mathbb{C} \mid 0 < |z| \leq 1\}$, with ∂A corresponding to $\{|z| = 1\}$. We will see that A behaves in the same way as when the ambient space is $T \times \mathbb{R}$, T a flat 2-torus [M-R]; i.e., we will see that A is conformally D^* and a subend of A meets each horizontal sphere transversally and in at most one Jordan curve. First we assure the height function is harmonic on A.

Lemma 3.1. *Let Σ be a minimal hypersurface of a Riemannian manifold N. Let X be a Killing vector field on N. If $X = \nabla f$ is the gradient of some function f on N, then f is harmonic on Σ.*

Proof. Let e_1, e_2,..., e_k, n be an orthonormal frame in a neighborhood of a point of Σ, where n is normal to Σ. Since X is a Killing vector field on N, we have:

$$\operatorname{div}(X) = 0 = \langle \nabla_n X, n \rangle.$$

Thus

$$0 = \sum_{i=1}^{k} \langle \nabla_{e_i} X, e_i \rangle + \langle \nabla_n X, n \rangle = \sum_{i=1}^{k} \langle \nabla_{e_i} X, e_i \rangle.$$

Write $X = X^{\perp} + \nabla_{\Sigma} f$, $X^{\perp}$ the normal component of X. Then

$$\begin{aligned} 0 &= \sum_{i=1}^{k} \langle \nabla_{e_i} \nabla_{\Sigma} f, e_i \rangle + \sum_{i=1}^{k} \langle \nabla_{e_i} (X^{\perp}, e_i \rangle \\ &= \Delta_{\Sigma} f - \sum_{i=1}^{k} \langle (X^{\perp}, \nabla_{e_i} e_i \rangle = \Delta_{\Sigma} f - \langle (X^{\perp}, H \rangle = \Delta_{\Sigma} f. \end{aligned}$$ □

Corollary 3.2. *The only compact minimal surfaces (with no boundary) in $S \times \mathbb{R}$ are the $S(t)$.*

Proposition 3.3. *Let A be a properly immersed minimal annulus in $M \times \mathbb{R}$, M a compact surface. Then A is conformally the punctured disk D^*, and a subend of A can be conformally parametrized by D^* so that $h = c\ell n|z|$. In particular, this subend meets each $M(t)$ transversally in at most one Jordan curve.*

Proof. We proceed as in [M-R]. Since A is proper, A must go up or go down, but not both. So we can suppose A goes up, zero is a regular value of h, and $h/\partial A < 0$. Then $\Delta = h^{-1}(-\infty, 0]$ is a smooth compact surface; one component of the boundary of Δ is below zero (namely ∂A), and the others are smooth Jordan curves in $M(0)$. There is no compact minimal surface with boundary in $M(0)$ (other than a part of $M(0)$) since the harmonic function h would have an interior extremum on such a surface. A is an annulus, so it follows there is exactly one component of the boundary of Δ in $M(0)$. By the same reasoning, A meets each $M(t)$, $t > 0$, transversally and in one Jordan curve. Now it is easy to parametrize the subend $h^{-1}[0, \infty)$ so that $h = \ell n|z|$. Use the facts that any compact annulus is conformally $S^1 \times [1, R]$, and a harmonic function on this annulus, constant on each boundary circle, is of the form $a \log|z| + b$, for some constants a, and b. □

Remark. Bill Meeks pointed out to us how these surfaces arise from constant mean curvature annuli and tori in R^3, such as the Delaunay surfaces and Wente tori.

Given a torus M of constant mean curvature in R^3, its Gauss map f to the unit sphere S is a harmonic map. Its holomorhic Hopf quadratic differential is

$$Q(f) = \left[\left(\left| \frac{\partial f}{\partial x} \right| \right)^2 - \left(\left| \frac{\partial f}{\partial y} \right| \right)^2 - 2i \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle \right] dz^2.$$

Since M is a torus, this is constant:

$$Q(f) = c dz^2.$$

After a linear change of coordinates we can assume the constant c is one. Then the map

$$F : S^1 \times \mathbb{R} \longrightarrow S \times \mathbb{R}, F(x, y) = (f(x, y), y)$$

is a conformal harmonic map, i.e., a minimal surface. The Delaunay surfaces yield the unduloids and nodoids.

4. Some surfaces of higher genus

We will now construct properly embedded minimal surfaces of finite topology in $S \times \mathbb{R}$. They will be conformally equivalent to a compact Riemann surface of genus g with two punctures. They have one top end, one bottom end and each is asymptotic to a flat vertical annulus. We will then prove that any properly embedded minimal surface has exactly one top end and one bottom end (or is some $S(t)$).

Recall that $C(0)$ is a fixed geodesic of $S(0)$. Let D_1, D_2 be the two disks of $S(0)$ bounded by $C(0)$. Consider $S \times \mathbb{R}$ as the union of the two vertical solid cylinders $(D_1 \times \mathbb{R}) \cup (D_2 \times \mathbb{R})$, identified along their common boundary; the flat vertical annulus $C(0) \times \mathbb{R}$.

Consider geodesic coordinates (r, θ) in D_1, where the center of D_1 is $r = 0$, and $C(0)$ is $r = \pi/2$. The rays $r(\theta) = \{\theta = \text{constant}, 0 \leq r \leq \pi/2\}$ are geodesics, and the circular arcs of $C(0)$ between two θ values, θ_1 and θ_2, we denote by $C(\theta_1, \theta_2)$. When $|\theta_1 - \theta_2| < \pi$, $C(\theta_1, \theta_2)$ denotes the arc of $C(0)$ of length less than π. In these coordinates, the end points of $C(\theta_1, \theta_2)$ are $(r = \pi/2, \theta_1)$ and $(r = \pi/2, \theta_2)$. Let T be a fixed number, and denote by $\Gamma(T, \theta_1, \theta_2)$ the geodesic polygon in $S \times \mathbb{R}$, with the five sides, $r(\theta_1)$, $r(\theta_2)$, the two vertical geodesic segments $\{(\pi/2, \theta_1, t) \mid 0 \le t \le T\}$, $\{(\pi/2, \theta_2, t) \mid 0 \le t \le T\}$, and the arc $(C(\theta_1, \theta_2), T)$ of $C(T)$; cf. Figure 1 below.

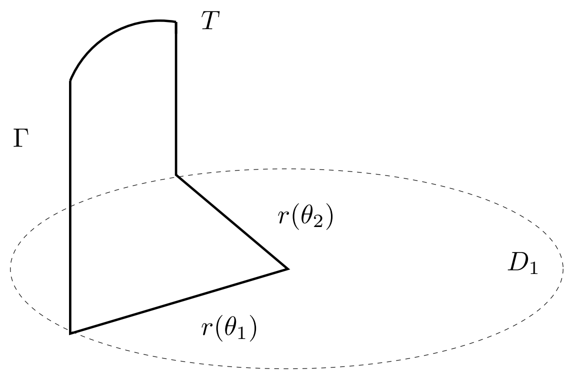

Figure 1

Let Σ be a least area compact minimal surface with boundary $\Gamma = \Gamma(T, \theta_1, \theta_2)$. We claim Σ is an embedded disk and $\mathrm{int}(\Sigma)$ is a vertical graph over the domain in D_1 bounded by $r(\theta_1) \cup r(\theta_2) \cup C(\theta_1, \theta_2)$.

To see this notice that Rado's theorem is true for minimal surfaces in $D_1 \times \mathbb{R}$: if $\Gamma \subset D_1 \times \mathbb{R}$ is a Jordan curve which has a convex projection to D_1 then any compact minimal surface in $D_1 \times \mathbb{R}$ bounded by Γ is an embedded disk and its interior is a vertical graph over the domain in D_1 bounded by the projection of Γ. Vertical translation in $D_1 \times \mathbb{R}$ is an isometry, and the height function is harmonic on a minimal surface, so the usual proof of Rado's theorem goes through.

In our case, $\partial D_1 \times \mathbb{R}$ is a good barrier for solving the Plateau problem in $D_1 \times \mathbb{R}$, so $\Sigma \subset D_1 \times \mathbb{R}$ and Rado's theorem applies.

Next observe that Σ can be continued by rotation by π about each edge in its boundary. Given a geodesic C in some $S(t)$, rotation by π about C is the ambient isometry which is the composition of symmetry of $S(t)$ by C, and symmetry of $S \times \mathbb{R}$ by $S(t)$. Given a vertical geodesic B of $S \times \mathbb{R}$, rotation by π about B is the symmetry of each $S(t)$ through the point $S(t) \cap B$. Notice that when $B \subset \partial D_1 \times \mathbb{R}$, the rotation by π about B permutes $D_1 \times \mathbb{R}$ and $D_2 \times \mathbb{R}$. On the other hand, rotation about an $r(\theta)$ in $D_1 \times \mathbb{R}$ sends Σ into $D_1 \times \mathbb{R}$. Consider the rotation of Σ about $r(\theta_1)$. The polygon Γ has image a polygon $\Gamma(1)$ as depicted in Figure 2 below, and the image of Σ is easy to understand.

Continuing to reflect across the rays, the resulting surface will close-up after $2k$ successive reflections when $\theta_2 - \theta_1 = \pi/k$ (for some integer $k = 1, 2, \dots$). Then the $(2k)$ images of Σ yield an embedded minimal disk $\Sigma(k)$ whose interior is a smooth vertical graph over D_1, and whose boundary is a geodesic polygon on $\partial D_1 \times \mathbb{R}$, composed of vertical and horizontal geodesics. The vertical geodesics go from height $-T$ to height T; the case $k = 2$ is depicted in Figure 3 below.

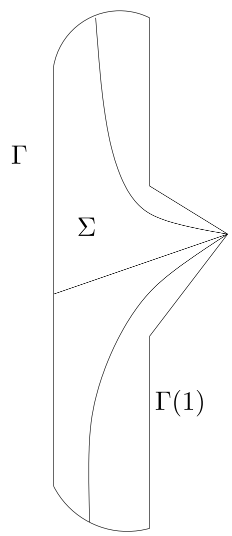

Figure 2

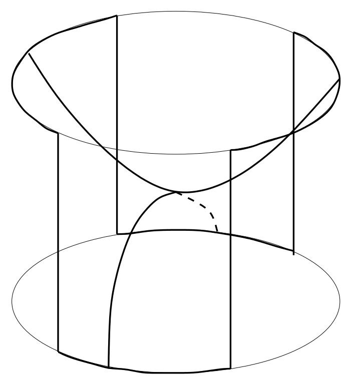

Figure 3

Notice that the height function on $\Sigma(k)$ has a critical point at the center of D_1, which is on $\Sigma(k)$, of index $1 - k$, and no critical points elsewhere on $\Sigma(k)$.

There are two natural ways to proceed now to obtain properly embedded minimal surfaces from $\Sigma(k)$. We can let $T \to \infty$, or we can do all symmetries of $\Sigma(k)$ across the geodesic boundaries.

First consider the surface obtained by fixing T and doing all the symmetries in the sides of $\Sigma = \Sigma(k, T)$. This yields a properly embedded minimal surface in $S \times \mathbb{R}$ which is invariant by vertical translation by $2T$. In the quotient $S \times \mathbb{R}/2T = S \times S^1$, one obtains a compact surface of genus k (just count the indices of the critical points of h). Notice that the surface $k = 1$ is an embedded minimal annulus in $S \times \mathbb{R}$; in fact, it is the helicoid we introduced previously.

Next consider letting $T \to \infty$. Recall that the Plateau solution $\Sigma(T)$ with boundary $\Gamma(T, \theta_1, \theta_2)$ is a vertical graph over the geodesic triangle $\Delta = r(\theta_1) \cup r(\theta_2) \cup C(\theta_1, \theta_2)$, of the function $u(T)$ with boundary values zero on the sides $r(\theta_1) \cup r(\theta_2)$ and the value T on $C(\theta_1, \theta_2)$. The function $u(T)$ is continuous at all points of Δ except the two endpoints of $C = C(\theta_1, \theta_2)$. We will prove shortly that the functions $u(T)$ converge uniformly (on compact subsets of $\Delta - C$), to a function $u(\infty)$, defined on $\Delta - C$, provided $\theta_1 - \theta_2$ is strictly less then π. The graph of $u(\infty)$ is a minimal surface with boundary $\Gamma(\infty, \theta_1, \theta_2)$, and its gradient approaches infinity as one approaches C from the interior of Δ (cf. Theorem 4.1).

Thus, when $\theta_2 - \theta_1 = \pi/k$ (for some integer $k = 2, 3, \dots$), the surfaces $\Sigma(k)$ converge to a minimal surface $\Sigma(\infty)$ bounded by the complete vertical geodesics $B(i\,\pi/k)$, $i = 1, \dots, 2k$, the vertical lines over the points on ∂D_1, given by ($r = \pi/2$, $\theta_i = i\pi/k$). Clearly $\Sigma(\infty) = \Sigma(k, \infty)$ is a graph over D_1 (i.e., its interior).

Now do rotation by π about the vertical geodesic $B(\pi/k)$. This induces a diffeomorphism from $\partial\Sigma(k, \infty)$ to itself and extends $\Sigma(k, \infty)$ to a complete properly embedded surface M with no boundary. The reader can verify that there is one top end, one bottom end, and each of these ends is asymptotic to a flat vertical annulus. The height function has exactly two critical points, each of index $1 - k$. They are the centers of D_1 and D_2. Since the top and bottom ends each give rise to a critical point of index one at the punctures, it follows that M is conformally diffeomorphic to a closed Riemann surface of genus $k - 1$ punctured in two points.

The Gauss-Bonnet theorem yields the total curvature of $\Sigma(k, T)$ to be $2\pi(1 - k)$. Since this does not depend on T, the total curvature of $\Sigma(k, \infty)$ is also $2\pi(1-k)$. Hence the total curvature of M is $4\pi(1-k)$, which is $2\pi\mathfrak{X}(M)$.

Now we will prove the existence of the Scherk-type surface we discussed.

Assume $0 < \theta_1 < \theta_2 < \pi$, and $\Sigma(T)$ is the least area Plateau solution with boundary $\Gamma(T, \theta_1, \theta_2)$. We know that $\Sigma(T)$ is the minimal graph of a function $u(T)$ with boundary values equal to zero on the two sides of the triangle $\{\theta = \theta_1, \ \ 0 \le r < 1\}$, $\{\theta = \theta_2, \ \ 0 \le r < 1\}$, and equal to T on the third side of the triangle $C = C(\theta_1, \theta_2) = \{r = 1, \ \ \theta_1 \le \theta \le \theta_2\}$.

THEOREM 4.1. *As $T \to \infty$, $u(T)$ converges to the function $u(\infty)$ defined on the triangle with boundary values zero on the sides of the triangle $r(\theta_1)$ and $r(\theta_2)$ and the value infinity on the third side $C = C(\theta_1, \theta_2)$. Moreover the gradient of $u(\infty)$ diverges as one approaches the third side from the interior of the triangle.*

Proof. To show that $u(\infty)$ exists we will prove that for any compact set K of the triangle minus the third side C, the functions $u(T)$ are all bounded above on K, with the bound independent of T. We will construct a barrier over the graph of the $u(T)$ on K.

Let ε and δ be small positive numbers (to be determined) and define a geodesic quadrilateral in D_1 with sides $A(\delta)$, $B(\delta)$, $C(\delta)$, $D(\delta)$ defined as follows:

$$A(\delta) = \{(r, \theta) | \varepsilon \le r \le 1,\ \theta = \theta_1 - \delta\}\,,$$
$$B(\delta) = \{(r, \theta) | \varepsilon \le r \le 1,\ \theta = \theta_2 + \delta\}\,,$$
$$C(\delta) = \{(r, \theta) | r = 1,\ \theta_1 - \delta \le \theta \le \theta_2 + \delta\}\,,$$

and $D(\delta)$ is the minimizing geodesic joining $(\varepsilon, \theta_1 - \delta)$ to $(\varepsilon, \theta_2 + \delta)$, whose length we call ε_1. Let F denote the convex domain on $D(1)$ bounded by this quadrilateral.

Let $h > 0$ and denote by $R(1,h)$ and $R(2,h)$ the Jordan curves which are the boundary of $A(\delta) \times [0,h]$ and $B(\delta) \times [0,h]$, respectively. The area of each of these disks is $(\pi/2 - \varepsilon)h$. Consider the piecewise smooth annulus with boundary $R(1,h) \cup R(2,h) : F \cup F(h) \cup (C \times [0,h]) \cup (D \times [0,h])$ (we omitted δ in C and D). The area of this annulus is at most $\pi + \pi + l(\delta)h + \varepsilon_1 h$, where $l(\delta) = (\theta_2 + \delta) - (\theta_1 - \delta)$. Clearly one can choose ε small so that for all δ sufficiently small and h sufficiently large, this annulus has less area than the two disks $R(1,h) \cup R(2,h)$. By the Douglas criteria for the Plateau problem, there exists a least area minimal annulus $a(\delta, h)$ with boundary $R(1,h) \cup R(2,h)$. Henceforth, we assume h large enough so that $a(\delta,h)$ exists.

Observe that for each $T > 0$, the surface $a(\delta, h)$ is above the graph of $u(T)$, in the following sense. Vertically translate $a(\delta,h)$ a height T (so every point of $a(\delta,h)$ is then above height T). Now continuously lower the translated $a(\delta,h)$ back to height zero. By the maximum principle there is no point of contact between the surfaces as one goes from height T to height zero; we chose $\delta > 0$, so the boundary of $a(\delta,h)$ never touches the graph of $u(T)$. Thus $a(\delta)$ is above $u(T)$ in the sense that if a vertical line meets both surfaces, then the point of $u(T)$ is below the points of $a(\delta,h)$. Now we can let δ tend to zero to conclude $a(h) = a(0,h)$ is also above the graph of $u(T)$, and by the boundary maximum principle, at each interior point of the vertical lines on $\Gamma(T, \theta_1, \theta_2)$, the tangent plane to $a(h)$ is "outside" the tangent plane to the graph of $u(T)$.

This barrier $a(h)$ shows that $u(T)$ is uniformly bounded over some compact domain of $\Delta \setminus C$: the domain covered by $a(h)$. The idea is now to show that these compact domains exhaust $\Delta \setminus C$ as $h \to \infty$.

For $h_2 > h_1$, one can use $a(h_1)$ as a barrier to solve the Plateau problem to find a least area annulus $a(h_2)$ with boundary $R(1,h_2) \cup R(2,h_2)$. So as one translates $a(h_1)$ vertically a height $h_2 - h_1$, there is no point where the two surfaces touch, interior or boundary. Thus as $h_2 \to \infty$, the angle the tangent plane of $a(h_2)$ makes along the vertical boundary segments is controlled by that of $a(h_1)$.

For each positive integer n, let $M(n)$ be the surface $a(2n)$ translated down a distance n. A subsequence of the $M(n)$ converges to a minimal surface $M(\infty)$. Notice that $a(h_1)$ can be translated up to $+\infty$, and down to $-\infty$, without ever touching $M(\infty)$. So there is some component M of $M(\infty)$ whose boundary equals the vertical lines L_1, L_2 passing through the endpoints of C; $L_1 \cup L_2 = \partial(C \times \mathbb{R})$. Moreover the maximum distance between M and $C \times \mathbb{R}$ is strictly less than $\pi/2$. To complete the proof of Theorem 4.1, it suffices to show $M = C \times \mathbb{R}$.

Recall that D_1 is the hemisphere of $S(0)$ containing the spherical triangle Δ. Choose a point $p \in \partial D_1 \setminus C$ and let $\alpha(t)$ denote the family of geodesic arcs of D_1 joining p to $-p$, such that $\alpha(0) = \alpha$, $\alpha(1)$ is the geodesic arc of ∂D_1 joining p to $-p$ that contains the arc C, and let $\alpha(t)$ foliate the half-disk E of D_1 between $\alpha(0)$ and $\alpha(1)$, $0 \leq t \leq 1$. Denote by $F(t)$ the minimal surfaces $\alpha(t) \times \mathbb{R}$, $0 \leq t \leq 1$. The $F(t)$ foliate the region $E \times \mathbb{R}$; the foliation is singular at $\{p\} \times \mathbb{R}$ and $\{-p\} \times \mathbb{R}$.

Now $M \subset \Delta \times \mathbb{R} \subset E \times \mathbb{R}$. As t goes from 0 to 1, the family of surfaces $F(t)$ can not touch M at some first $t < 1$, since M would then equal $F(t)$ by the maximum principle. So either $M = C \times \mathbb{R}$ or there is a smallest positive $t < 1$ such that M is asymptotic to $F(t)$ at infinity. The latter case is impossible. If not, let $x_n \in M$ be such that $\operatorname{dist}(x_n, F(t))$ tends to zero as $n \to \infty$. Let $\Sigma(n)$ be the minimal surface M vertically translated so the height of x_n becomes zero. A subsequence of $\Sigma(n)$ converges to a minimal surface Σ that touches $F(t)$ at some point (at height zero), so $\Sigma = F(t)$.

Consider a compact domain K of $F(t)$, K a positive distance from $\partial F(t)$, and choose K so that the vertical projection on D_1 contains points of $E \setminus \Delta$. Domains of $M(n)$ converge uniformly to K as $n \to \infty$, so there are also points of $M(n)$ whose vertical projection is in $E \setminus \Delta$. This is impossible since $M(n)$ and M have the same vertical projection. This completes the proof of Theorem 4.1. $\square$

Now we will see that the end structure of the surfaces we constructed is typical.

THEOREM 4.2. *Let M be a properly embedded minimal surface in $S \times \mathbb{R}$ of finite topology. Then M has exactly one top end and one bottom end, or $M = S(t)$ for some t.*

Proof. M of finite topology means M is homeomorhic to a compact surface minus a finite number of points. A neighborhood of each such point in M can be chosen homeomorhic to an annulus. If M is bounded above or below, then the height function would have an extremum on M and then M equals some $S(t)$ by the maximum principle for harmonic functions. So we can assume M has at least one annular end going up and another annular end going down. It suffices to prove that there can not be more than one end (going up say). Suppose on the contrary that A_1 and A_2 are annular ends going up.

By Proposition 3.3, we can assume A_1 and A_2 both meet each $S(t)$ transversally in exactly one Jordan curve $C_1(t)$ and $C_2(t)$, respectively, for each $t \geq 0$.

Denote by $E(t)$ the annular region of $S(t)$ bounded by $C_1(t) \cup C_2(t)$. For each integer n, let $B(n)$ be the union of the $E(t)$, $0 \leq t \leq n$. Notice that $\partial B(n)$ is a good barrier for solving a Plateau problem in $B(n)$. Also, $C_1(0)$ and $C_1(n)$ are homologous in $B(n)$, but neither $C_1(0)$ nor $C_1(n)$ is homologous to zero in $B(n)$. Thus there is a least area connected annulus $\Sigma(n)$ in $B(n)$ with boundary $C_1(0) \cup C_1(n)$.

By standard curvature estimates, a subsequence of the $\Sigma(n)$ converges to a complete stable minimal annulus Σ, with $\partial\Sigma = C_1(0)$.

As before, we can assume Σ meets each $S(t)$ transversally in one Jordan curve $\gamma(t)$.

Now we observe that the area of Σ is infinite. Let ν be the upward pointing conormal vector along $\gamma(t)$. The height function h is harmonic on Σ, hence has a constant non-zero flux across each $\gamma(t)$. This flux is

$$\int_{\gamma(t)} |\nabla h| \, ds,$$

where s is arc length along $\gamma(t)$.

By the coarea formula, the area of Σ is

$$\int_{t=0}^{\infty} \left(\int_{\gamma(t)} \frac{ds}{|\nabla h|} \right) dt \geq \int_{t=0}^{\infty} \left(\int_{\gamma(t)} |\nabla h| ds \right) dt = \infty.$$

However by the work of Doris Fischer-Colbrie and Silveira [DF-C], [Sil], there is no stable minimal surface in $S \times \mathbb{R}$ of infinite area. The stability operator is $L = \Delta - K + q$, where K is the intrinsic curvature of Σ, $q = T + |A|^2/2$, T the scalar curvature of $S \times \mathbb{R}$ (which is one) and A the second fundamental form. Stability yields a positive function u satisfying $L(u) = 0$. The metric $u \, ds = d\tilde{s}$ is then a complete metric on Σ whose curvature $\widetilde{K}$ is non negative and given by

$$\widetilde{K} = \frac{1}{u^2} \left(q + \frac{|\nabla u|^2}{u^2} \right).$$

Then

$$\int_{\Sigma} q dA \leq \int_{\Sigma} \widetilde{K} \, d\widetilde{A} < \infty,$$

so $\int_{\Sigma} T dA < \infty$, which contradicts infinite area. □

The techniques used in Theorem 4.2 also give information about intersection of minimal submanifolds.

THEOREM 4.3. *Let Σ_1, Σ_2 be properly embedded minimal submanifolds of $S \times \mathbb{R}$. Then $\Sigma_1 \cap \Sigma_2 \neq \emptyset$ or $\Sigma_1 = S(t_1)$, $\Sigma_2 = S(t_2)$ for some t_1, t_2.*

Proof. We know that a properly immersed minimal submanifold is either some $S(t)$, or meets each $S(t)$ in a non empty compact set. We can assume the latter case holds for both Σ_1 and Σ_2. We will assume $\Sigma_1 \cap \Sigma_2 = \emptyset$ and arrive at a contradiction.

Elementary separation properties imply $\Sigma_1 \cup \Sigma_2 = \partial B$, B a domain of $S \times \mathbb{R}$. Then $\Sigma_1(t) \cup \Sigma_2(t) = \partial B(t)$ for each t such that Σ_1 and Σ_2 meet

$S(t)$ transversally. $\Sigma_1(t)$ is homologous to $\Sigma_1(0)$ in $B \cap [0,t]$ and $\Sigma_1(t)$ is not homologous to zero in $B \cap [0,t]$. Thus $\Sigma_1(t) \cup \Sigma_1(0)$ bounds a connected least area minimal surface $\Sigma(t)$ in $B \cap [0,t]$. A subsequence of the $\Sigma(t)$, as $t \to \infty$, converges to a stable minimal surface Σ with $\partial\Sigma = \Sigma_1(0)$. As in the proof of Theorem 4.2, no such stable surface exists. This proves the theorem. □

Remark. Notice that the above argument shows one need not assume finite topology in Theorem 4.2.

We can say something for properly immersed surfaces.

Theorem 4.4. *Let Σ be a properly immersed minimal surface in $S \times \mathbb{R}$. Then Σ meets every flat vertical annulus.*

Proof. Let $A = C(0) \times \mathbb{R}$ be a flat vertical annulus and assume $A \cap \Sigma = \emptyset$. We can assume (after a possible rotation of the S factor) that $\mathrm{dist}(A, \Sigma) = 0$; so some sequence of points in Σ is converging to A at infinity.

Let F be a (small) compact piece of an unduloid, chosen so that $\partial F \subset A$ and $F \cap \Sigma = \emptyset$. Such an F can be found since Σ is properly immersed and unduloids exist arbitrarily close to A.

Now translate F vertically. Since Σ is asymptotic to A at infinity, there will be a first point of contact of the translated F with Σ. Then Σ equals this translated unduloid by the maximum principle. This contradicts $\Sigma \cap A = \emptyset$. □

We finish this section with a conjecture: a properly embedded minimal annulus in $S \times \mathbb{R}$ meets each $S(t)$ in a circle. There is a 2-parameter family of such annuli, and each properly embedded minimal annular end is asymptotic to the end of a surface in this family.

5. Non-negative curvature

Now let M be a complete surface of non-negative curvature and consider minimal surfaces in $M \times \mathbb{R}$.

One has a Bernstein-type theorem: a minimal (vertical) graph over all of M is totally geodesic. This follows since such a minimal graph Σ is stable (vertical translation is an isometry of $M \times \mathbb{R}$ so Σ is a leaf of a minimal foliation. In [R], we proved that a limit leaf of a minimal lamination is stable, and R. Schoen [Sch] has proved that a complete stable minimal surface in a 3-manifold of non-negative Ricci curvature, is totally geodesic. In $\mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$ there are results on the conformal geometry of minimal surfaces [C-K-M-R]. We will see that they generalize to the present situation.

For the next result we assume the geodesic curvature of all geodesic circles of M (from some fixed point p) of radius at least one is bounded by some constant C.

Proposition 5.1. *Let r be the distance in M to the point p, and let $f = \ell n\, r$ be the natural extension of r to $M \times \mathbb{R}$ (independent of the height), defined where $r \neq 0$. Let Σ be a minimal surface in $M \times \mathbb{R}$. Then, for $r \geq 1$,*

$$\Delta_\Sigma(f) \leq \frac{c}{r}\,|\nabla_\Sigma h|^2.$$

Proof. Let Δ denote the Laplacian of $M \times \mathbb{R}$. Since f does not depend on the height, we have

$$\Delta_M(f) = \Delta(f), \quad \text{and}$$
$$\Delta_M(f) = \frac{\Delta_M(r)}{r} - \frac{|\nabla_M(r)|^2}{r^2} = \frac{\Delta_M(r)}{r} - \frac{1}{r^2}\,.$$

By the Laplacian comparison theorem, and since M has non-negative curvature, we have

$$\Delta_M(r) \leq \frac{1}{r}, \quad \text{and}$$
$$\Delta_M(f) \leq 0.$$

Let e_1, e_2, n be an orthonormal frame in a neighborhood of a point of Σ, where e_1, e_2 are tangent to Σ, and n is normal to Σ. Then

$$\Delta_M(f) = \sum_{i=1}^{2}\langle \nabla_{e_i}\nabla f, e_i\rangle + \langle \nabla_n \nabla f, n\rangle,$$

where ∇f is the gradient in $M \times \mathbb{R}$.

Write ∇f as its tangent and normal part to Σ:

$$\nabla f = (\nabla f)^\perp + \nabla_\Sigma f.$$

Then

$$\langle \nabla_{e_i}\nabla f, e_i\rangle = \langle \nabla_{e_i}\nabla_\Sigma f, e_i\rangle + df(n)\langle \nabla_{e_i} n, e_i\rangle.$$

Hence $\Delta_M f = \Delta_\Sigma f + df(n)H + \langle \nabla_n \nabla f, n\rangle$, where H is the mean curvature of Σ. Since Σ is minimal, this last equation becomes

$$\Delta_M f = \Delta_\Sigma f + \mathrm{Hess}(f)(n,n),$$

where Hess is the hessian in $M \times \mathbb{R}$.

Now evaluate $\mathrm{Hess}(f)$ using an orthonormal frame v_1, v_2, e where v_1, v_2 are tangent to M, $e = \dfrac{\partial}{\partial t} = \nabla h$, and h is the height function.

Since f depends only on the M coordinates, we have

$$\mathrm{Hess}(f)(e,e) = 0, \quad \text{and}$$
$$\mathrm{Hess}(f)(v) = \mathrm{Hess}_M(f)(v),$$

for v tangent to M. Let π be projection of $M \times \mathbb{R}$ to the M factor and let A be the endomorphism of the tangent space of M defined by $A(Y) = \nabla_Y(\nabla f)$. So

$$\begin{aligned}\operatorname{Hess} f(Y,Y) &= \langle \nabla_Y(\nabla f), Y\rangle \\ &= \langle \nabla_{\pi(Y)}(\nabla f), \pi(Y)\rangle \\ &= \langle A(\pi(Y)), \pi(Y)\rangle.\end{aligned}$$

Since $\Delta_M(f) \leq 0$, we have

$$\Delta_\Sigma(f) \leq -\operatorname{Hess} f(n,n) \leq |A|\,|\pi(n)|^2.$$

A simple calculation shows $|\pi(n)| = |\nabla_\Sigma h|$; it remains to estimate $|A|$. Let $q \in M$, $d(p,q) = r$, and let v be a unit tangent vector to M at q. Then

$$\begin{aligned}\operatorname{Hess} f(v,v) &= \left\langle \nabla_v \left(\frac{\nabla r}{r}\right), v\right\rangle \\ &= \frac{1}{r}\,\langle \nabla_v(\nabla r), v\rangle + v\left(\frac{1}{r}\right)\langle \nabla r, v\rangle.\end{aligned}$$

When $v = \nabla r$, the first term is zero, so

$$\operatorname{Hess} f(v,v) = -\frac{1}{r^2}\,\langle \nabla r, v\rangle.$$

When $v = (\nabla r)^\perp = T$, this is

$$\frac{1}{r}\,\langle \nabla_T(\nabla r), T\rangle.$$

Since T is the unit tangent vector to the geodesic circle of radius r through the point q, this last term equals $\dfrac{1}{r}\,kg(q)$. This proves Proposition 5.1. □

A Riemann surface Σ is parabolic if a bounded harmonic function is determined by its boundary values. If $\partial\Sigma = \emptyset$ the condition is that a bounded harmonic function is constant. It is well known that a proper subdomain of a parabolic surface is parabolic and removing a compact domain does not alter parabolicity. Also, if a bounded harmonic function is positive on the boundary of a parabolic surface Σ, then it is positive on Σ. Finally, it is easy to see that if Σ admits a proper non-negative superharmonic function, then Σ is parabolic.

Proposition 5.2. *Let M be a surface satisfying the hypothesis of Proposition 5.1. Then a properly immersed minimal surface Σ in $M \times \mathbb{R}^+$ is parabolic.*

Proof. As in $\mathbb{R}^3$, we first show $\Sigma(T) = \Sigma \cap (M \times [0,T])$ is parabolic. Consider the function

$$\varphi = \ell n(r) + (T - h^2).$$

Clearly φ is proper and non-negative on the part of Σ_T outside the cylinder $r \geq 1$. By Proposition 5.1, φ is superharmonic outside of some larger cylinder $r \geq r_0$. Since Σ is proper, the part of $\Sigma(T)$ inside this cylinder is compact. Hence $\Sigma(T)$ is parabolic.

Now let u be any test function on Σ, i.e., u is harmonic, $0 \leq u \leq 1$, and u is zero on $\partial\Sigma$ (if $\partial\Sigma \neq \emptyset$).

Then for any $T > 0$, we have

$$0 \leq u \leq \frac{h}{T},$$

on $\partial\Sigma(T)$. So for any $p \in \Sigma(T)$,

$$0 \leq u(p) \leq \frac{h(p)}{T},$$

since $\Sigma(T)$ is parabolic. But this also holds for any $T_0 > T$. Hence $u \equiv 0$. □

Corollary 5.3 (Half-space theorem). *Under the above hypothesis on M, a properly immersed minimal surface in a half-space $M \times [0, \infty)$ is some $M(T)$.*

Proof. Σ is parabolic and h is harmonic and bounded on each $\Sigma(T)$, so h is determined by its boundary values on $\Sigma(T)$. Hence if $\Sigma \cap M(T) \neq \emptyset$, then $h = T$ on Σ. □

Proposition 5.4. *With the previous hypothesis on the geodesic curvature of the geodesic circles of M, let Σ be a properly immersed minimal surface in the region of $M \times \mathbb{R}$ defined by $|h| \leq c\ \ell n(r)$, for some $c > 0$ and $r \geq 1$. Then Σ is parabolic and if Σ has a compact boundary, then Σ has quadratic area growth.*

The proof of Proposition 5.4 is as in $\mathbb{R}^3$; we refer the reader to [C-K-M-R].

6. Negative curvature

Now we discuss $M \times \mathbb{R}$, where M is the hyperbolic plane. We take as model for M the unit disk $\{x_1^2 + x_2^2 < 1\}$ with the constant curvature -1 metric

$$ds^2 = \frac{dx_1^2 + dx_2^2}{F}, \qquad F = \left(\frac{1 - x_1^2 - x_2^2}{2}\right)^2.$$

In this section we will announce some results obtained in collaboration with Barbara Nelli [N-R].

The minimal surface equation of a vertical graph over M, $u = u(x_1, x_2)$, is

$$\operatorname{div}\left(\frac{\nabla u}{\tau}\right) = 0,$$

where

$$\tau = \sqrt{1 + F|\nabla u|^2}.$$

One can explicitly solve this equation to find the catenoidal surfaces of revolution about the x_3-axis. Rather than write the elliptic functions parametrizing the trace curves of these catenoids, I describe some of their properties. Let $C_{\pm}(t)$ be the two circles at infinity of $M \times \mathbb{R}$ defined by

$$\{x_1^2 + x_2^2 = 1,\ x_3 = \pm t\}.$$

Then for each $t > 0$, there exists a catenoid $\Sigma(t)$ whose asymptotic boundary is $C_+(t) \cup C_-(t)$. As $t \to 0$, $\Sigma(t)$ converges to a doubly covered disk M, with one singularity at the origin. As $t \to \infty$, the surfaces $\Sigma(t)$ diverge in $M \times \mathbb{R}$. The limiting asymptotic angle of the $\Sigma(t)$ varies from 0 to $\pi/2$ as t goes from 0 to infinity.

The helicoid

$$X(u, v) = (v \cos au, v \sin au, u)$$

is a minimal surface in $M \times \mathbb{R}$. Here $v \in (-1, 1)$, $u \in \mathbb{R}$, and $a \neq 0$.

There are many minimal graphs over M, non-constant, so there is no Bernstein theorem here. We prove:

Theorem 6.1. *Let Γ be a rectifiable Jordan curve at infinity of $M \times \mathbb{R}$, Γ a vertical graph over $\{x_1^2 + x_2^2 = 1,\ x_3 = 0\}$. Then there is a minimal graph over M whose asymptotic boundary is Γ.*

We prove a removable singularities theorem for the minimal surface equation.

Theorem 6.2. *Let u be a solution of the minimal surface equation over a punctured disk in M. Then u extends smoothly to the puncture.*

Also we establish a Jenkins-Serrin type theorem for minimal graphs over domains in M bounded by geodesic polygons.

Theorem 6.3. *Let Γ be a convex geodesic polygon in M, with sides labelled $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$, as one traverses Γ once. Let P be a simple closed polygon strictly inscribed in Γ, whose vertices are chosen among the vertices of Γ. A necessary and sufficient condition that there exist a minimal graph u in the domain bounded by Γ taking the values $+\infty$ on $a_1, \ldots, a_n$ and $-\infty$ on $b_1, \ldots, b_n$ is that*

$$2 \sum_{a_i \in P} |a_i| < |P|, \quad 2 \sum_{b_i \in P} |b_i| < |P|$$

for any polygon P chosen as above and

$$\sum_{i=1}^{n} |a_i| = \sum_{j=1}^{n} |b_j|.$$

Here $|a|$ means the length of a and $|P|$ the perimeter of P.

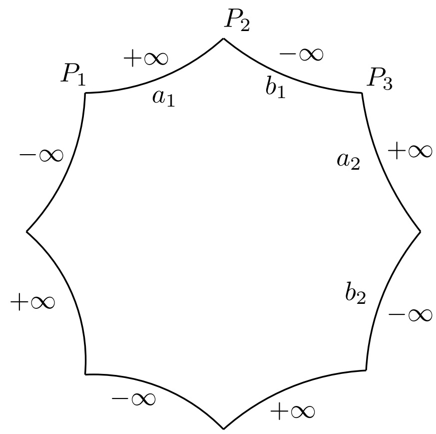

Figure 4

Just as in classical Jenkins-Serrin theorem we can also prescribe continuous data on convex arcs of the boundary.

Let us illustrate this theorem when Γ is a regular octogon in M. When Γ is chosen so the interior angles are all $\pi/2$ then the minimal graph u extends to a complete embedded minimal surface in $M \times \mathbb{R}$ (a Scherk-type surface). The graph of u is bounded by the vertical geodesics over the vertices of Γ, so one extends the graph by rotation by π about all the vertical edges that arise.

We will now explain why the graph u exists, taking the values plus and minus infinity on alternating sides of the octogon Γ.

Label the vertices of Γ, $p_1, p_2, \ldots, p_8$, in order and let a_1 be the edge bounded by p_1, p_2, b_1 the edge between p_2 and p_3, etc.; cf. Figure 4.

Let n be a positive integer and define $\Gamma(n)$ to be the compact geodesic polygon obtained by raising all the a_j to height n, descending all the b_j to height $-n$, and then joining all the vertices of the raised a_i to the lowered b_i by the vertical geodesics between the vertices. The vertical projection of $\Gamma(n)$ to $M(0)$ is Γ.

Solve the Plateau problem for $\Gamma(n)$ to obtain a least area disk $\Sigma(n)$ with boundary $\Gamma(n)$. As before, Rado's theorem is true in $M \times \mathbb{R}$ since vertical translation is an isometry; so $\Sigma(n)$ is the graph of a function u_n defined in the domain bounded by Γ. On a_i, u_n takes the value n, and on b_j, u_n takes the value $-n$. Moreover, by symmetry of $\Gamma(n)$, $u_n(\sigma) = \sigma$, where σ is the center of Γ.

The proof of Theorem 4.1 works in the geodesic triangles here in hyperbolic space. Hence the sequence u_n converges to a minimal graph u whose boundary values are $\pm\infty$ as desired.

Notice that the completed minimal surface obtained from extending the graph of u, is invariant by many Fuchsian groups. For example, one obtains

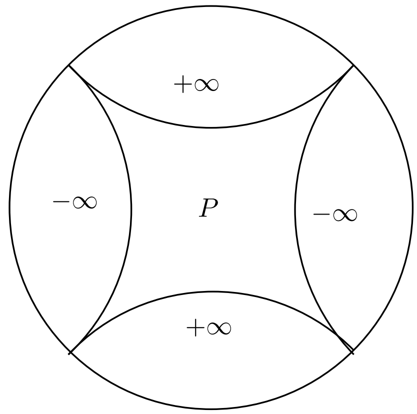

Figure 5

a (Scherk-type) surface which is a sphere with four top ends and four bottom ends as follows. Let P be the regular octogon in the hyperbolic plane with $\pi/2$ angles, and let $Q(j)$ be the 8 domains obtained by rotating P about its vertices. Consider the 4 translations identifying alternate sides of P, and consider the graph over the union of P and the $Q(j)$. The translation identifying two alternate sides is the translation along the edge between the sides. Quotient this graph by the squares of the 4 translations. This gives the 8-punctured sphere in the quotient; its total curvature is -12π.

The same construction works with any regular k-gon, with vertex angle $\pi/2$. If one is not concerned about extending to an embedded complete surface, the vertex angles need not be $\pi/2$.

Another interesting example is obtained by putting the vertices of Γ at infinity; cf. Figure 5.

We leave it to the reader to construct a minimal graph u defined in the interior of P and taking the values $\pm\infty$ on ∂P as indicated in Figure 5.

Another technique to construct surfaces is by desingularizing intersections of minimal surfaces intersecting in a reasonable fashion. For example, desingularizing the intersection of a flat vertical plane (or annulus) with a totally geodesic horizontal surface produces a singly-periodic Scherk-type surface. Desingularizing the intersection of a catenoid with its plane of symmetry produces a Costa-Hoffman-Meeks surface. These desingularization theorems have not yet been proved.

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