THE IDEAL STRUCTURE OF THE C^*-ALGEBRAS OF INFINITE GRAPHS

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ABSTRACT. We classify the gauge-invariant ideals in the C^*-algebras of infinite directed graphs, and describe the quotients as graph algebras. We then use these results to identify the gauge-invariant primitive ideals in terms of the structural properties of the graph, and describe the K-theory of the C^*-algebras of arbitrary infinite graphs.

1. Introduction

There has recently been a great deal of interest in generalisations of the Cuntz-Krieger algebras associated to infinite directed graphs [16], [9] and infinite matrices [6]. The basic theorems of Cuntz and Krieger [3], [2] on uniqueness and ideal structure have elegant extensions to the C^*-algebras of the row-finite graphs in which each vertex emits only finitely many edges [16], [15], [13], [1]. Various authors have investigated the C^*-algebras of arbitrary infinite graphs from different points of view, obtaining satisfactory versions of the uniqueness theorems [9], [20], [26]. However, these articles do not provide a complete description of the ideal structure of graph algebras, as is given in [12] for the Cuntz-Krieger algebras of finite matrices. Indeed, even for row-finite graphs the ideal structure has only been well-understood when the graph satisfies the Condition (K) introduced in [16] (see [1]).

The analysis in [12] shows that to understand the ideal structure of graph algebras we first need to describe the gauge-invariant ideals. The main purpose of this paper is to provide such a description for arbitrary infinite graphs. We give a complete list of the gauge-invariant ideals of $C^*(E)$ for an arbitrary infinite graph E (Theorem 3.6), and then use it to identify all the gauge-invariant primitive ideals (Theorem 4.7). When the graph satisfies Condition

(K) all ideals are gauge-invariant and our results give their complete classification.

The key tool in our approach is a realisation of the quotient $C^*(E)/J$ by a gauge-invariant ideal as the graph algebra of a quotient graph (Proposition 3.4). This result is of considerable interest in its own right, because we are able to explicitly describe the quotient graph. As a further application, we show how to extend the description of $K_*(C^*(E))$ obtained in [20] for row-finite E to arbitrary infinite graphs (Theorem 6.1).

There are several reasons for the current interest in graph algebras apart from the elegance of their theory. First, they provide good test problems in the general theories of groupoid algebras [16], [15], [18], Cuntz-Pimsner algebras [19], [21], [13], [9], [10], and partial crossed products [6], [8]. Second, the simple graph algebras provide a rich family of accessible models for purely infinite simple C^*-algebras. Indeed, Szymański has shown in [25] that every stable, purely infinite, simple and classifiable C^*-algebra with K_1 torsion-free can be realised as a graph algebra. Although there is some debate about what 'purely infinite' should mean for non-simple C^*-algebras [14], there is already considerable interest in their classification, and it is likely that the non-simple graph algebras will again provide an important family of models.

We have had the main results of the present paper for some time (see [11]), and had wanted to include them in a complete analysis of the ideal structure of infinite graph algebras. However, we have received many enquires about this work, and in response have decided to publish it in stages. There is some overlap between the present article and the work of Drinen and Tomforde [4], who describe the primitive ideal space of the C^*-algebras of graphs satisfying Condition (K) by reducing to the row-finite case. Our methods are quite different from theirs: we work directly with quotients of graph algebras rather than algebras Morita equivalent to them. In the sequel, we use these techniques to obtain a complete generalisation of the program of [12] to the C^*-algebras of arbitrary infinite graphs.

2. Preliminaries

Let $E = (E^0, E^1, r, s)$ be a (countable) directed graph, consisting of a set E^0 of vertices, a set E^1 of edges, and range and source maps $r, s : E^1 \to E^0$. A Cuntz-Krieger E-family consists of mutually orthogonal projections $\{P_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ with mutually orthogonal ranges satisfying

(G1) $S_e^* S_e = P_{r(e)}$,
(G2) $S_e S_e^* \leq P_{s(e)}$, and
(G3) $P_v = \sum_{s(e)=v} S_e S_e^*$ if $s^{-1}(v)$ is finite and non-empty.

The C^*-algebra $C^*(E)$ of E is the universal C^*-algebra generated by a Cuntz-Krieger E-family $\{s_e, p_v\}$. If $\{S_e, P_v\}$ is a Cuntz-Krieger E-family, we denote

by $\pi_{S,P}$ the representation of $C^*(E)$ such that $\pi_{S,P}(p_v) = P_v$ and $\pi_{S,P}(s_e) = S_e$.

We denote by $\gamma : \mathbb{T} \to \operatorname{Aut} C^*(E)$ the gauge action, which is characterised on generators by $\gamma_z(p_v) = p_v$ and $\gamma_z(s_e) = zs_e$ for $v \in E^0$, $e \in E^1$, $z \in \mathbb{T}$. Existence of the gauge action is equivalent to universality in the definition of $C^*(E)$, as the following gauge-invariant uniqueness theorem shows. This result was proved for finite graphs in [12, Theorem 2.3], for row-finite graphs in [1, Theorem 2.1], and generalised in [20, Theorem 2.7] to the Cuntz-Krieger algebras of infinite matrices and in [10, Theorem 4.1] to Cuntz-Pimsner algebras. Unfortunately, the existing versions do not cover all infinite graphs with sources or sinks.

THEOREM 2.1.   *Let E be an arbitrary directed graph, let $\{S_e, P_v\} \subset B(\mathcal{H}_E)$ be a Cuntz-Krieger E-family, and let $\pi = \pi_{S,P}$ be the representation of $C^*(E)$ such that $\pi(s_e) = S_e$ and $\pi(p_v) = P_v$. Suppose that each P_v is non-zero, and that there is a strongly continuous action β of $\mathbb{T}$ on $C^*(S_e, P_v)$ such that $\beta_z \circ \pi = \pi \circ \gamma_z$ for $z \in \mathbb{T}$. Then π is faithful.*

Proof. If E is an infinite directed graph without sinks or sources (that is, each vertex emits and receives some edges), then $C^*(E)$ is naturally isomorphic to the Cuntz-Krieger algebra of a suitable infinite matrix [9, Theorem 10]. Thus [20, Theorem 2.7] applies to such graphs.

To extend the theorem to graphs with sinks, it suffices to add tails as in [1]. Indeed, let F be the graph obtained by adding a tail (with extra vertices $\{v_i : i = 1, 2, \ldots\}$) to a sink w of E as in [1, §1], let $\mathcal{H}_T = \bigoplus_{i=1}^{\infty} \mathcal{H}_i$ be the direct sum of copies $\mathcal{H}_i$ of $P_w \mathcal{H}_E$, let $\{T_e, Q_v\}$ be the Cuntz-Krieger F-family on $\mathcal{H}_F = \mathcal{H}_E \oplus \mathcal{H}_T$ obtained by extending the Cuntz-Krieger E-family $\{S_e, P_v\}$ as in [1, Lemma 1.2], and let $U : \mathbb{T} \to U(\mathcal{H}_E)$ be a unitary representation such that $(\pi_{S,P}, U)$ is covariant for the gauge action on $C^*(E)$. Then there is a unitary representation $V : \mathbb{T} \to U(\mathcal{H}_F)$ such that $(\pi_{T,Q}, V)$ is covariant for the gauge action on $C^*(F)$. For example, it suffices to set

$$V_z \xi := \begin{cases} U_z \xi & \text{if } \xi \in \mathcal{H}_E \\ z^{-i} U_z \xi & \text{if } \xi \in Q_{v_i} \mathcal{H}_T = \mathcal{H}_i \cong P_w \mathcal{H}_E. \end{cases}$$

The same argument works for graphs with sources: add heads as in [20, §1] and set $V_z \xi := z^i U_z \xi$ instead. $\qquad\square$

We finish this preliminary section by recalling the basic definitions and notation about paths in a directed graph E. If $\alpha_1, \ldots, \alpha_n$ are (not necessarily distinct) edges such that $r(\alpha_i) = s(\alpha_{i+1})$ for $i = 1, \ldots, n-1$, then $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a *path* of length $|\alpha| = n$, with source $s(\alpha) = s(\alpha_1)$ and range $r(\alpha) = r(\alpha_n)$. The set of paths of length n is denoted by E^n, $E^* := \bigcup_{n=0}^{\infty} E^n$ (so that vertices in E^0 are identified with paths of length 0), and the set of infinite paths is denoted E^∞. A *loop* is a path of positive length whose source

and range coincide. A loop α *has an exit* if there exist an edge $e \in E^1$ and index i such that $s(e) = s(\alpha_i)$ but $e \neq \alpha_i$. A graph is said to satisfy Condition (K) if every vertex $v \in E^0$ either lies on no loops, or there are two loops α, β such that $s(\alpha) = s(\beta) = v$ and neither α nor β is an initial subpath of the other [16].

3. Gauge-invariant ideals

For a row-finite graph E, the gauge-invariant ideals in the graph algebra $C^*(E)$ are in one-to-one correspondence with the saturated hereditary subsets of E^0 [1, Theorem 4.1]; indeed, if H is saturated and hereditary, then the corresponding ideal I_H is generated by $\{p_v : v \in H\}$, and if I is a gauge-invariant ideal, then $H := \{v \in E^0 : p_v \in I\}$ is saturated and hereditary with $I = I_H$. When some vertices emit infinitely many edges, not every gauge-invariant ideal arises this way: in the graph
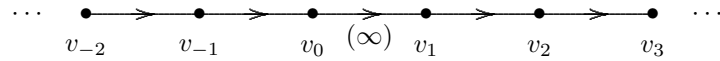
(where the symbol (∞) indicates infinitely many edges from v_0 to v_1) the projections p_{v_i} associated to $H := \{v_i : i > 0\}$ generate a gauge-invariant ideal I_H with $H = \{v : p_v \in I_H\}$, but H is not saturated in the sense of [1]. So we have to adjust the notion of saturation.

Let E be a directed graph which is not necessarily row-finite. As usual, we write $v \geq w$ when there is a path from v to w, and say that a subset H of E^0 is *hereditary* if $v \in H$ and $v \geq w$ imply $w \in H$. Now we say that a subset X of E^0 is *saturated* if every vertex v which satisfies $0 < |s^{-1}(v)| < \infty$ and $s(e) = v \implies r(e) \in X$ itself belongs to X. (With this definition, the set $H = \{v_i : i > 0\}$ in the above example is saturated.) The *saturation* $\Sigma(X)$ of a set X is the smallest saturated set containing X, and $\Sigma H(X)$ denotes the smallest saturated hereditary subset of E^0 containing X. If $v \in \Sigma(X) \backslash X$, then $0 < |s^{-1}(v)| < \infty$; otherwise, $\Sigma(X) \backslash \{v\}$ is a smaller saturated set containing X. If $v \in \Sigma(X)$, then there is a path α with $s(\alpha) = v$ and $r(\alpha) \in X$. To see this, note that the elements of $\Sigma(X)$ with this property form a saturated set containing X.

If H is hereditary, so is its saturation $\Sigma(H)$. To see this, suppose $v \in \Sigma(H)$ and $v \geq w$, so that there is a path $\alpha = (\alpha_1, \ldots, \alpha_r)$ with $s(\alpha) = v$ and $r(\alpha) = w$. If the path enters H, then it stays there. So suppose $r(\alpha_i) \notin H$ for all i. Since $0 < |s^{-1}(v)| < \infty$, the saturation property implies that $r(\alpha_1) \in \Sigma(H)$, for otherwise $\Sigma(H) \backslash \{r(\alpha_1)\}$ would be a smaller saturated set containing H. Since $r(\alpha_1) \in \Sigma(H)$, it also satisfies $0 < |s^{-1}(r(\alpha_1))| < \infty$; repeating this argument shows that $r(\alpha_i) \in \Sigma(H)$ for all i, and in particular that $w = r(\alpha) \in \Sigma(H)$.

REMARK 3.1.   For $X \subset E^0$ we can construct $\Sigma H(X)$ as the union of the sequence $\Sigma_n(X)$ of subsets of E^0 defined inductively as follows:

$$\Sigma_0(X) := X \cup \{w \in E^0 : \text{ there is a path from a vertex in } X \text{ to } w\},$$

$$\Sigma_{n+1}(X) := \Sigma_n(X) \cup \{w \in E^0 : 0 < |s^{-1}(w)| < \infty, \; r(s^{-1}(w)) \subset \Sigma_n(X)\}$$

The next lemma provides evidence that our notion of saturation is the right one.

LEMMA 3.2.   *Suppose E is a directed graph and I is an ideal in $C^*(E)$. Then*

$$H_I := \{v \in E^0 : p_v \in I\}$$

is a saturated hereditary subset of E^0.

Proof. Suppose first that $v \in H_I$ and $v \geq w$. Then there is a path α with $s(\alpha) = v$ and $r(\alpha) = w$, and then $s_\alpha^* s_\alpha = p_w$, $s_\alpha s_\alpha^* \leq p_v$. So $p_v \in I$ implies $s_\alpha s_\alpha^* \in I$, and $p_w = s_\alpha^*(s_\alpha s_\alpha^*)s_\alpha \in I$. Thus H_I is hereditary. If $v \in E^0$ has $0 < |s^{-1}(v)| < \infty$ and $p_{r(e)} \in I$ for all $e \in E^1$ with $s(e) = v$, then $s_e = s_e p_{r(e)} \in I$ for all $e \in E^1$ with $s(e) = v$, and the Cuntz-Krieger relation (G3) at v implies that $p_v \in I$; thus H_I is saturated. $\qquad\qquad\square$

For a hereditary subset H of E^0, we let I_H be the ideal of $C^*(E)$ generated by $\{p_v : v \in H\}$; since Lemma 3.2 implies that $\{v \in E^0 : p_v \in I_H\}$ is saturated and contains H, we immediately have that $I_H = I_{\Sigma(H)}$. Since the projections generating I_H are fixed by the gauge action it follows that the ideal itself is gauge-invariant. As in [1, Lemma 4.3], we can verify that

$$(1) \qquad I_H = \overline{\operatorname{span}}\{s_\alpha p_v s_\beta^* : \alpha, \beta \in E^*, : v \in H, : r(\alpha) = r(\beta) = v\}.$$

Suppose H is a saturated hereditary subset of E^0. When E is row-finite, the ideal I_H is Morita equivalent to the C^*-algebra $C^*(H)$ of the graph $(H, s^{-1}(H), r, s)$, and the quotient $C^*(E)/I_H$ is naturally isomorphic to the C^*-algebra of the graph $F := (E^0 \setminus H, r^{-1}(E^0 \setminus H), r, s)$ [1, Theorem 4.1]. In general, to realise the quotient $C^*(E)/I_H$ as a graph algebra we have to add extra vertices to F. The problem occurs when a vertex v sends infinitely many edges into H but also finitely many into $E^0 \setminus H$, in which case the image of the projection

$$(2) \qquad\qquad p_{v,H} := \sum_{s(e)=v, \; r(e) \notin H} s_e s_e^*$$

will be strictly smaller in $C^*(E)/I_H$ than the image of p_v. To get round this, we add a new sink $\beta(v)$ to F^0 and extra edges $\beta(e)$ with $r(\beta(e)) = \beta(v)$ for each edge e with $r(e) = v$.

Formally, we define H_∞^{fin} to be the set of such vertices; thus
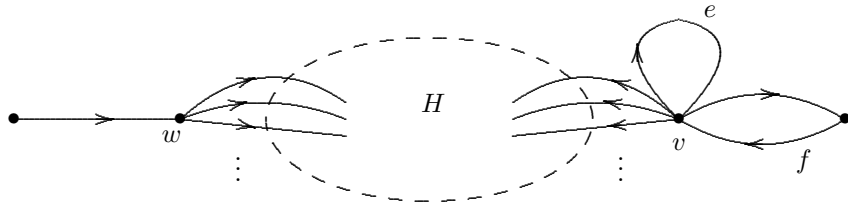
$$H_\infty^{\mathrm{fin}} := \{v \in E^0 \setminus H : |s^{-1}(v)| = \infty \text{ and } 0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus H)| < \infty\}.$$
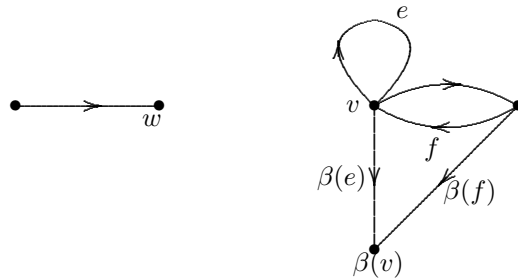
We then define a graph E/H by

$$(E/H)^0 := (E^0 \setminus H) \cup \{\beta(v) : v \in H_\infty^{\mathrm{fin}}\},$$
$$(E/H)^1 := r^{-1}(E^0 \setminus H) \cup \{\beta(e) : e \in E^1, r(e) \in H_\infty^{\mathrm{fin}}\},$$

with r, s extended by $s(\beta(e)) = s(e)$ and $r(\beta(e)) = \beta(r(e))$.

EXAMPLE 3.3.   In the following graph,

we have $H_\infty^{\mathrm{fin}} = \{v\}$, and the graph E/H looks like

PROPOSITION 3.4.   *Let H be a hereditary subset of a directed graph E. Then the ideal I_H defined in (1) is Morita equivalent to the C^*-algebra of the graph $(H, s^{-1}(H), r, s)$. Let $\pi : C^*(E) \to C^*(E)/I_H$ be the quotient map, let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger E-family, and write $S_e = \pi(s_e)$, $P_v = \pi(p_v)$, $P_{v,H} = \pi(p_{v,H})$, where $p_{v,H}$ are the projections defined in (2). If*

H is also saturated, then

$$
\begin{aligned}
Q_v &:= P_v & &\text{if } v \in (E/H)^0 \setminus H_\infty^{\text{fin}} \\
Q_v &:= P_{v,H} & &\text{if } v \in H_\infty^{\text{fin}} \\
Q_{\beta(v)} &:= P_v - P_{v,H} & &\text{if } v \in H_\infty^{\text{fin}} \\
T_e &:= S_e & &\text{if } r(e) \in (E^0 \setminus H) \setminus H_\infty^{\text{fin}} \\
T_e &:= S_e P_{r(e),H} & &\text{if } r(e) \in (E^0 \setminus H) \cap H_\infty^{\text{fin}} \\
T_{\beta(e)} &:= S_e(P_{r(e)} - P_{r(e),H}) & &\text{if } r(e) \in (E^0 \setminus H) \cap H_\infty^{\text{fin}}
\end{aligned}
$$

(3)

is a Cuntz-Krieger (E/H)-family in $C^*(E)/I_H$, and the homomorphism $\pi_{T,Q}$ is an isomorphism of $C^*(E/H)$ onto $C^*(E)/I_H$.

Proof. The argument of [1, Theorem 4.1(c)] shows that there is a natural isomorphism of $C^*(H)$ onto the corner of I_H determined by the projection $p_H := \sum_{v \in H} p_v \in M(I_H)$, and that this projection is full.

It is tedious but straightforward to verify that $\{T_e, Q_v\}$ is a Cuntz-Krieger (E/H)-family, and hence there is a homomorphism $\pi_{T,Q} : C^*(E/H) \to C^*(E)/I_H$ carrying the generating family $\{t_e, q_v\}$ of $C^*(E/H)$ into $\{T_e, Q_v\}$. To see that $\pi_{T,Q}$ is surjective, note that we can recover $\{S_e, P_v\}$ from $\{T_e, Q_v\}$:

$$
\begin{aligned}
P_v &= \begin{cases} Q_v & \text{if } v \notin H \cup H_\infty^{\text{fin}} \\ Q_v + Q_{\beta(v)} & \text{if } v \in H_\infty^{\text{fin}} \\ 0 & \text{if } v \in H \end{cases} \\[2em]
S_e &= \begin{cases} T_e & \text{if } r(e) \notin H \cup H_\infty^{\text{fin}} \\ T_e + T_{\beta(e)} & \text{if } r(e) \in H_\infty^{\text{fin}} \\ 0 & \text{if } r(e) \in H \end{cases}
\end{aligned}
$$

(4)

The formulas (4) also show how to construct a Cuntz-Krieger E-family $\{S_e, P_v\}$ from a Cuntz-Krieger (E/H)-family $\{T_e, Q_v\}$ in such a way that the formulas (3) recover $\{T_e, Q_v\}$. Thus there are Cuntz-Krieger E-families $\{S_e, P_v\}$ with $P_v = 0$ for $v \in H$ such that the projections $Q_v, Q_{\beta(v)}$ defined in (3) are all non-zero, and in particular this must be true of those defined by the universal family $\{\pi(s_e), \pi(p_v)\}$. It therefore follows from gauge-invariant uniqueness (Theorem 2.1) that the homomorphism $\pi_{T,Q}$ is injective. $\qquad\square$

The sinks in a directed graph give rise to ideals: indeed, if $B \subset E^0$ consists of sinks, then the projections p_v associated to the sinks generate a family $\{I_v : v \in B\}$ of mutually orthogonal ideals. Since the vertices $\{\beta(v) : v \in H_\infty^{\text{fin}}\}$ are sinks in E/H, they give rise to ideals in $C^*(E/H) \cong C^*(E)/I_H$, and hence to ideals in $C^*(E)$. More formally, if H is saturated and hereditary, then for $B \subset H_\infty^{\text{fin}}$ we let $J_{H,B}$ denote the ideal of $C^*(E)$ generated by the projections

$\{p_v : v \in H\} \cup \{p_v - p_{v,H} : v \in B\}$. The usual arguments show that

$$J_{H,B} = \overline{\text{span}}\{s_\alpha p_v s_\beta^*,\ s_\mu(p_w - p_{w,H})s_\nu^* :$$
$$v \in H,\ \alpha, \beta \in r^{-1}(v),\ w \in B,\ \mu, \nu \in r^{-1}(w)\}$$

and that $J_{H,B}$ is gauge-invariant. Note also that $I_H = J_{H,\emptyset}$, and that $I_H \subset J_{H,B}$ for all B.

To identify the quotient $C^*(E)/J_{H,B}$, note that the set $\beta(B)$ is saturated and hereditary in $(E/H)^0$. Since the quotient map $C^*(E) \to C^*(E/H) = C^*(t_e, q_v)$ takes $p_v - p_{v,H}$ into $q_{\beta(v)}$, it maps $J_{H,B}$ onto the ideal $I_{\beta(B)}$ of $C^*(E/H)$ generated by $\{q_{\beta(v)} : v \in B\}$. Since $\beta(B)_\infty^{\text{fin}} = \emptyset$ in E/H, we see that the second quotient $(E/H)/\beta(B)$ is just equal to $(E/H) \setminus \beta(B) = \big((E/H)^0 \setminus \beta(B), r^{-1}((E/H)^0 \setminus \beta(B)), r, s\big)$. Thus we have:

COROLLARY 3.5.   *If H is a saturated hereditary subset of E^0 and $B \subset H_\infty^{\text{fin}}$, then $C^*(E)/J_{H,B}$ is naturally isomorphic to $C^*((E/H) \setminus \beta(B))$.*

The following theorem gives a complete list of the gauge-invariant ideals of $C^*(E)$ for an arbitrary infinite graph E.

THEOREM 3.6.   *Let E be a directed graph. Then the ideals*

$$\big\{J_{H,B} : H \text{ is saturated and hereditary, } B \subset H_\infty^{\text{fin}}\big\}$$

are distinct gauge-invariant ideals in $C^*(E)$, and every gauge-invariant ideal is of this form. Indeed, if I is a gauge-invariant ideal in $C^*(E) = C^*(s_e, p_v)$, $H := \{v \in E^0 : p_v \in I\}$, and $B := \{v \in H_\infty^{\text{fin}} : p_v - p_{v,H} \in I\}$, then $I = J_{H,B}$.

We begin by showing that we can recover H and B from $J_{H,B}$; this immediately implies that the ideals are distinct.

LEMMA 3.7.   *Let E be a directed graph. Suppose that H is a saturated hereditary subset of E^0 and $B \subset H_\infty^{\text{fin}}$. Then $H = H_{J_{H,B}}$. If we use the isomorphism of Proposition 3.4 to view $J_{H,B}/I_H$ as an ideal in $C^*(E/H)$, then $\beta(B) = H_{J_{H,B}/I_H}$.*

Proof. We trivially have $H \subset H_{J_{H,B}}$. Suppose $v \notin H$. Then the image of p_v under the isomorphism of $C^*(E)/J_{H,B}$ onto $C^*((E/H)\setminus\beta(B)) = C^*(t_e, q_v)$ dominates the projection q_v associated to the vertex $v \in ((E/H) \setminus \beta(B))^0$, and hence is nonzero; thus $p_v \notin J_{H,B}$. This gives the first assertion. For the second, note that the image of $J_{H,B}$ under the quotient map $C^*(E) \to C^*(E/H) = C^*(u_f, r_w)$ is the ideal $I_{\beta(B)}$, and $\beta(B) = \{w \in (E/H)^0 : r_w \in I_{\beta(B)}\}$ by the first assertion. $\square$

Proof of Theorem 3.6. Lemma 3.7 implies that the ideals are distinct. Given I and H, B as in the theorem, we note that $J_{H,B} \subset I$, and consider the image of $I/J_{H,B}$ in $C^*((E/H) \setminus \beta(B)) = C^*(t_e, q_v)$. We shall show by

contradiction that there is no vertex w of $(E/H) \setminus \beta(B)$ such that the corresponding projection q_w lies in $I/J_{H,B}$. If $w \in E^0 \setminus H$ and $w \notin H_\infty^{\mathrm{fin}}$, then $q_w \in I/J_{H,B}$ implies $p_w \in I$, which contradicts $w \notin H$. If $w \in H_\infty^{\mathrm{fin}}$, then $q_w \in I/J_{H,B}$ implies $p_{w,H} \in I$; now we can choose $e \in E^1$ such that $s(e) = w$ and $r(e) \notin H$, and then $p_{w,H} \in I$ implies $p_{r(e)} = s_e s_e^* \in I$, which is incompatible with $r(e) \notin H$. If $w = \beta(v)$ for some $v \in H_\infty^{\mathrm{fin}}$, then $q_w \in I/J_{H,B}$ implies $p_v - p_{v,H} \in I$, and $w = \beta(v) \in \beta(B)$, which is impossible because w is a vertex of $(E/H) \setminus \beta(B)$. Thus for all $w \in ((E/H) \setminus \beta(B))^0$, q_w has non-zero image in $C^*((E/H) \setminus \beta(B))/(I/J_{H,B})$. Now gauge-invariant uniqueness (Theorem 2.1) implies that the quotient map of $C^*((E/H) \setminus \beta(B))$ onto $C^*((E/H) \setminus \beta(B))/(I/J_{H,B})$ is injective, which says that $I/J_{H,B} = 0$ and $I = J_{H,B}$. □

COROLLARY 3.8. *Suppose E is a directed graph satisfying Condition* (K). *Then every ideal of $C^*(E)$ is gauge-invariant, and hence Theorem 3.6 gives a complete description of the ideals of $C^*(E)$.*

Proof. Suppose I is an ideal in $C^*(E)$. Let $H := \{v \in E^0 : p_v \in I\}$, which is saturated and hereditary by Lemma 3.2, and let $B := \{v \in H_\infty^{\mathrm{fin}} : p_v - p_{v,H} \in I\}$. Note that $J_{H,B} \subset I$. Let $(E/H) \setminus \beta(B)$ denote the graph appearing in Corollary 3.5. As in the first paragraph of the proof of [1, Theorem 4.1], both quotients $C^*(E)/I$ and $C^*(E)/J_{H,B}$ are generated by Cuntz-Krieger $((E/H) \setminus \beta(B))$-families in which all the projections associated to vertices are nonzero.

We claim that all loops in $(E/H) \setminus \beta(B)$ have exits. Suppose α is a loop in $(E/H) \setminus \beta(B)$. Since all the new vertices added to $E^0 \setminus H$ to form $(E/H) \setminus \beta(B)$ are sinks, the loop α must come from a loop $\tilde\alpha$ in E. Because E satisfies (K), each vertex in $\tilde\alpha$ must lie on another loop. Since this loop cannot enter the hereditary set H, there must be an exit from $\tilde\alpha$ which lies in $r^{-1}(E^0 \setminus H)$, and hence gives an exit from α in $(E/H) \setminus \beta(B)$. This justifies the claim.

Now two applications of the Cuntz-Krieger uniqueness theorem [20, Theorem 1.5] show that both quotients $C^*(E)/I$ and $C^*(E)/J_{H,B}$ are canonically isomorphic to $C^*((E/H) \setminus \beta(B))$. Thus the quotient map of $C^*(E)/J_{H,B}$ onto $C^*(E)/I$ is an isomorphism, and $I = J_{H,B}$. The corollary now follows from Theorem 3.6. □

We need the following proposition in the proof of Lemma 4.1 below, and in the analysis of the hull-kernel topology on the primitive ideal space in our sequel.

PROPOSITION 3.9. *Suppose E is a directed graph, $\{H_i : i \in \Lambda\}$ is a family of saturated hereditary subsets of E^0, and $B_i \subset (H_i)_\infty^{\mathrm{fin}}$ for $i \in \Lambda$. Let $H = \bigcap_{i \in \Lambda} H_i$ and $B = \left(\bigcap_{i \in \Lambda} H_i \cup B_i \right) \cap H_\infty^{\mathrm{fin}}$. Then*

$$\bigcap_{i \in \Lambda} J_{H_i, B_i} = J_{H,B}.$$

Proof. Since the intersection of gauge-invariant ideals is gauge-invariant, Theorem 3.6 says that $\bigcap_{i\in\Lambda} J_{H_i,B_i} = J_{K,C}$ for

$$K = \big\{v : p_v \in \textstyle\bigcap_{i\in\Lambda} J_{H_i,B_i}\big\}, \quad C = \big\{w \in K_\infty^{\mathrm{fin}} : p_w - p_{w,K} \in \textstyle\bigcap_{i\in\Lambda} J_{H_i,B_i}\big\}.$$

By two applications of Lemma 3.7, we have

$$K = H_{J_{K,C}} = \textstyle\bigcap_{i\in\Lambda} H_{J_{H_i,B_i}} = \bigcap_{i\in\Lambda} H_i = H.$$

It remains to identify C with B. Let $w \in K_\infty^{\mathrm{fin}}$; we want to show that $w \in \bigcap_{i\in\Lambda} H_i \cup B_i$ if and only if $w \in C$.

Suppose $w \in C$ and $i \in \Lambda$ is fixed. Then $p_w - p_{w,K} \in J_{K,C} \subset J_{H_i,B_i}$. For each of the finitely many e with $s(e) = w$ and $r(e) \notin K$, $r(e) \in H_i$ implies $p_{r(e)} \in J_{H_i,B_i}$ and $s_e s_e^* \in J_{H_i,B_i}$. If $r(e) \in H_i$ for all such e, then $p_{w,K} \in J_{H_i,B_i}$, $p_w = (p_w - p_{w,K}) + p_{w,K} \in J_{H_i,B_i}$, and $w \in H_i$. If $r(e) \notin H_i$ for some such e, then

$$p_w - p_{w,H_i} = p_w - p_{w,K} + \sum_{s(e)=w,\ r(e)\in H_i\backslash K} s_e s_e^* \in J_{H_i,B_i},$$

and $w \in B_i$. Either way, $w \in H_i \cup B_i$.

For the converse, suppose $w \in H_i \cup B_i$ for all i, and fix i; we want to show $p_w - p_{w,K} \in J_{H_i,B_i}$. If $w \in H_i$, then $p_w \in J_{H_i,B_i}$, so this is trivially true. If $w \in B_i$, then $p_w - p_{w,H_i} \in J_{H_i,B_i}$, and

$$p_w - p_{w,K} = p_w - p_{w,H_i} + \sum_{s(e)=w,\ r(e)\in H_i\backslash K} s_e s_e^* \in J_{H_i,B_i},$$

as required. $\qquad\square$

COROLLARY 3.10.   *If E is a directed graph, H_1 and H_2 are saturated hereditary subsets of E^0, and $B_i \subset (H_i)_\infty^{\mathrm{fin}}$ for $i = 1,2$, then $J_{H_1,B_1} \subset J_{H_2,B_2}$ if and only if $H_1 \subset H_2$ and $B_1 \subset H_2 \cup B_2$.*

4. Gauge-invariant primitive ideals

The primitive ideal spaces of the C^*-algebras of row-finite graphs satisfying Condition (K) were described in [1, §6]. In particular, [1, Corollary 6.5] gives a bijection between the primitive ideals and certain subsets of the vertex set, called maximal tails. The concept of a maximal tail also plays a crucial role in our analysis of primitive gauge-invariant ideals in $C^*(E)$. However, we need to adjust the definition to accommodate non-row-finite graphs.

LEMMA 4.1.   *Suppose I is an ideal in $C^*(E)$. Then $M := E^0 \backslash H_I$ satisfies:*
  (a) *if $v \in E^0$, $w \in M$, and $v \geq w$ in E, then $v \in M$, and*
  (b) *if $v \in M$ and $0 < |s^{-1}(v)| < \infty$, then there exists $e \in E^1$ with $s(e) = v$ and $r(e) \in M$.*

If I is a primitive ideal, then in addition:

(c) *for every $v, w \in M$ there exists $y \in M$ such that $v \geq y$ and $w \geq y$.*

Proof. Conditions (a) and (b) say that H_I is hereditary and saturated, so the first part follows from Lemma 3.2.

Now suppose I is primitive, $v, w \in M$, and there is no y as in (c). The sets $H_v = \{x \in E^0 : v \geq x\}$ and $H_w = \{x \in E^0 : w \geq x\}$ are hereditary and satisfy $H_v \cap H_w \subset H_I$. The set $H_I \cup (E^0 \setminus H_w)$ is saturated. Indeed, let $x \in E^0$ be such that $0 < |s^{-1}(x)| < \infty$ and $r(e) \in H_I \cup (E^0 \setminus H_w)$ for each edge e with $s(e) = x$. Then either $x \in H_I$ or there is at least one e with $s(e) = x$ and $r(e) \notin H_w$, in which case $x \in E^0 \setminus H_w$ because H_w is hereditary. Thus $\Sigma H(v) = \Sigma(H_v) \subset H_I \cup (E^0 \setminus H_w)$. The same argument shows $\Sigma H(w) \subset H_I \cup (E^0 \setminus H_v)$, and the hypothesis $H_v \cap H_w \subset H_I$ forces $\Sigma H(v) \cap \Sigma H(w) \subset H_I$. It follows from Lemma 3.9 that

$$I_{\Sigma H(v)} \cap I_{\Sigma H(w)} = I_{\Sigma H(v) \cap \Sigma H(w)} \subset I,$$

which is impossible because neither $I_{\Sigma H(v)}$ nor $I_{\Sigma H(w)}$ is contained in I. Thus there must exist $y \in M$ such that $v \geq y$ and $w \geq y$. $\square$

We now define a *maximal tail* in E to be a nonempty subset M of E^0 satisfying conditions (a), (b) and (c) of Lemma 4.1. If M is a maximal tail in E then we say that *every loop in M has an exit* if every loop with vertices in M has an exit $e \in E^1$ with $r(e) \in M$.

PROPOSITION 4.2.   *Let E be a directed graph. Then $C^*(E)$ is primitive if and only if every loop in E has an exit and for every $v, w \in E^0$ there exists $y \in E^0$ such that $v \geq y$ and $w \geq y$.*

Proof. Suppose that $C^*(E)$ is primitive. Then E^0 is a maximal tail by Lemma 4.1. If there is a loop L in E without exits and L^0 is the set of vertices on L, then the ideal I_{L^0} is Morita equivalent to $C(\mathbb{T})$ (cf. [15, §2] and formula (1) above), contradicting primitivity of $C^*(E)$.

Conversely, suppose the two conditions of the proposition are satisfied. It suffices to show that $\{0\}$ is a prime ideal of $C^*(E)$. Indeed, let I_1, I_2 be two non-zero ideals of $C^*(E)$. Since all loops in E have exits, [9, Theorem 2] implies that there exist $v, w \in E^0$ such that $p_v \in I_1$ and $p_w \in I_2$. If $y \in E^0$ satisfies $v \geq y$ and $w \geq y$, then we have $p_y \in I_1 \cap I_2$, so $I_1 \cap I_2 \neq \{0\}$, and $\{0\}$ is prime. $\square$

REMARK 4.3.   The conditions of Proposition 4.2 are equivalent to:

(1) for every $v, w \in E^0$ we have $\Sigma H(v) \cap \Sigma H(w) \neq \emptyset$, and
(2) every loop in E has an exit.

The above two conditions should be compared with the criterion of simplicity for graph algebras [23, Theorem 12]:

(1) for every $v \in E^0$ we have $\Sigma H(v) = E^0$, and

(2) every loop in E has an exit.

COROLLARY 4.4.   *Let E be a directed graph and let H be a saturated hereditary subset of E^0. Then $C^*(E \setminus H)$ is primitive if and only if $M := E^0 \setminus H$ is a maximal tail such that all loops in M have exits.*

By Theorem 3.6, we can determine all the gauge-invariant primitive ideals of $C^*(E)$ by deciding which of the ideals $J_{H,B}$ are primitive. To this end, we use Corollary 4.4 to see which quotient algebras $C^*(E)/J_{H,B} \cong C^*((E/H) \setminus \beta(B))$ are primitive. If $H_\infty^{\mathrm{fin}} \setminus B$ contains distinct vertices v, w, there are at least two sinks $\beta(v), \beta(w)$ in $(E/H) \setminus \beta(B)$ and $C^*((E/H) \setminus \beta(B))$ cannot be primitive by Proposition 4.2. So we only need to consider two possibilities: $B = H_\infty^{\mathrm{fin}}$ and $B = H_\infty^{\mathrm{fin}} \setminus \{v\}$ for some $v \in H_\infty^{\mathrm{fin}}$.

LEMMA 4.5.   *Let E be a directed graph and let H be a saturated and hereditary subset of E^0. Then $J_{H,H_\infty^{\mathrm{fin}}}$ is primitive if and only if $M := E^0 \setminus H$ is a maximal tail such that all loops in M have exits.*

Proof. Since $C^*(E)/J_{H,H_\infty^{\mathrm{fin}}}$ is isomorphic to $C^*((E/H) \setminus \beta(H_\infty^{\mathrm{fin}})) = C^*(E \setminus H)$, this follows from Corollary 4.4.   □

For any non-empty subset X of E^0 we denote by $\Omega(X)$ the collection of vertices $w \in E^0 \setminus X$ such that there is no path from w to any vertex in X. That is,
$$\Omega(X) := \left\{ w \in E^0 \setminus X : w \not\geq v \text{ for all } v \in X \right\}.$$
Note that we have $\Omega(M) = E^0 \setminus M$ for any maximal tail M. The following lemma shows that it is important to look at the sets $\Omega(v)$ corresponding to certain vertices v.

LEMMA 4.6.   *Let E be a directed graph, let H be a saturated hereditary subset of E^0, and let $v \in H_\infty^{\mathrm{fin}}$. Then $J_{H,H_\infty^{\mathrm{fin}} \setminus \{v\}}$ is primitive if and only if $H = \Omega(v)$.*

Proof. The ideal $J_{H,H_\infty^{\mathrm{fin}} \setminus \{v\}}$ is primitive if and only if the corresponding quotient algebra $C^*((E/H) \setminus \beta(H_\infty^{\mathrm{fin}} \setminus \{v\}))$ is primitive. Since the graph $(E/H) \setminus \beta(H_\infty^{\mathrm{fin}} \setminus \{v\})$ contains a sink $\beta(v)$, Corollary 4.4 implies that $J_{H,H_\infty^{\mathrm{fin}} \setminus \{v\}}$ is primitive if and only if for any vertex $w \in E^0 \setminus H$ there exists a path from w to v. This, however, is equivalent to $H = \Omega(v)$.   □

We call a vertex $v \in E^0$ with the property described in Lemma 4.6 a *breaking vertex*, and write $v \in BV(E)$. More formally, we define
$$BV(E) := \left\{ v \in E^0 : |s^{-1}(v)| = \infty \text{ and } 0 < |s^{-1}(v) \setminus r^{-1}(\Omega(v))| < \infty \right\}.$$
Note that if a vertex v emits infinitely many edges then $\Omega(v)$ is automatically saturated and hereditary, and hence v is a breaking vertex if and only if $v \in \Omega(v)_\infty^{\mathrm{fin}}$. If the graph E is row-finite, there are no breaking vertices.

THEOREM 4.7. *Let E be a directed graph. Then the gauge-invariant primitive ideals in $C^*(E)$ are the ideals $J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_\infty}$ associated to the maximal tails M in which all loops have exits, and the ideals $J_{\Omega(v),\Omega(v)^{\mathrm{fin}}_\infty\setminus\{v\}}$ associated to breaking vertices $v \in BV(E)$. These ideals are distinct.*

Proof. By Theorem 3.6, all gauge-invariant ideals in $C^*(E)$ have the form $J_{H,B}$, with H a saturated hereditary subset of E^0 and B a subset of H^{fin}_∞, and these ideals are distinct. So we only need to decide which of these ideals are primitive. If $H^{\mathrm{fin}}_\infty \setminus B$ has two or more vertices then $J_{H,B}$ is not primitive, since $C^*(E)/J_{H,B} \cong C^*((E/H) \setminus \beta(B))$ and the graph $(E/H) \setminus \beta(B)$ contains at least two sinks, contradicting the conditions of Proposition 4.2. Thus we may assume that either $B = H^{\mathrm{fin}}_\infty$ or $B = H^{\mathrm{fin}}_\infty \setminus \{v\}$ for some $v \in H^{\mathrm{fin}}_\infty$. If $B = H^{\mathrm{fin}}_\infty$, the ideal $J_{H,B}$ is primitive if and only if $H = \Omega(M)$ for some maximal tail M in which all loops have exits by Lemma 4.5. If $B = H^{\mathrm{fin}}_\infty \setminus \{v\}$, the ideal $J_{H,B}$ is primitive if and only if $H = \Omega(v)$ for some breaking vertex $v \in BV(E)$ by Lemma 4.6.   □

The following corollary now follows from Corollary 3.8 and Theorem 4.7.

COROLLARY 4.8. *If E is a directed graph satisfying Condition (K) then Theorem 4.7 gives a complete description of the primitive ideals of $C^*(E)$.*

5. Examples

We illustrate our results with four examples which are not covered by the existing literature.

EXAMPLE 5.1. The following directed graph satisfies Condition (K) but is not row-finite.
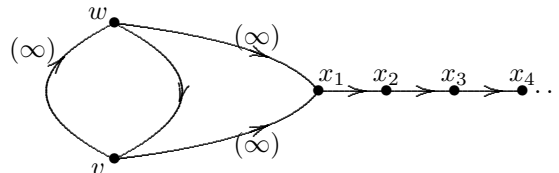

There are only two maximal tails, $M_1 = E^0$ and $M_2 = \{v, w\}$, and in both every loop has an exit. The primitive ideal corresponding to M_1 is $\{0\}$, and hence $C^*(E)$ is primitive. The primitive ideal corresponding to M_2 is $J_{X,\{w\}}$, where $X = \{x_1, x_2, x_3, \ldots\}$. The only breaking vertex is w, and the corresponding primitive ideal is I_X.

Since I_X is infinite-dimensional and Morita equivalent to $C^*(X)$ [15, §2], and $C^*(X)$ is isomorphic to the C^*-algebra $\mathcal{K}$ of compact operators on a separable infinite-dimensional Hilbert space, we have $I_X \cong \mathcal{K}$ also. By Proposition 3.4, the quotient graph E/X contains one sink $\beta(w)$, and $\beta(w)$ is the range of

infinitely many paths. Thus $J_{X,\{w\}}/I_X \cong \mathcal{K}$. The ideal $J_{X,\{w\}}$ is an extension of $\mathcal{K}$ by $\mathcal{K}$, and is the unique essential extension by [26, Lemma 1.1]. Another application of Proposition 3.4 shows that $C^*(E)/J_{X,\{w\}} \cong C^*(E \setminus X) \cong M_2(\mathbb{C}) \otimes \mathcal{O}_\infty$.

EXAMPLE 5.2.   Let E be the following graph:
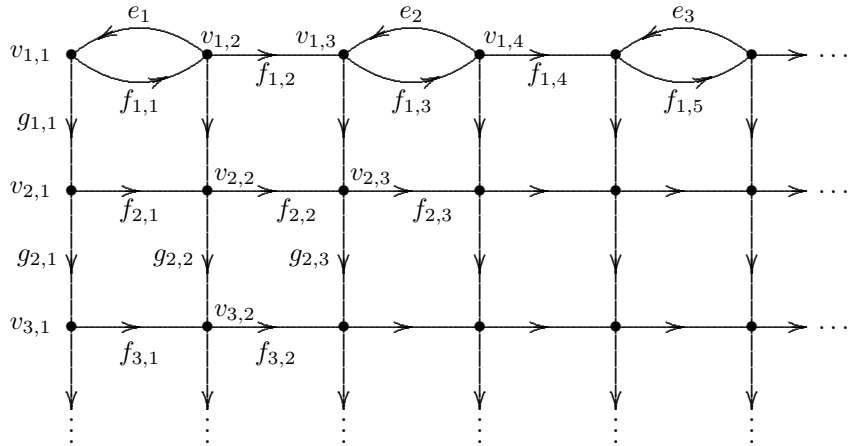
This is an infinite row-finite graph which does not satisfy Condition (K). There are four families of maximal tails indexed by the integers $n \geq 1$:

$$M_n = \{v_{i,j} : 1 \leq i \leq n, : 1 \leq j < \infty\},$$
$$M^{2n-1} = \{v_{i,j} : 1 \leq i < \infty, : 1 \leq j \leq 2n - 1\} \cup \{v_{1,2n}\},$$
$$M^{2n} = \{v_{i,j} : 1 \leq i < \infty, : 1 \leq j \leq 2n\},$$
$$R_n = \{v_{1,j} : 1 \leq j \leq 2n\}.$$

In addition, E^0 is a maximal tail. Each maximal tail R_n contains a loop without exits. On the other hand, all loops in M_n and M^n have exits. Since E is row-finite there are no breaking vertices. Thus the gauge-invariant primitive ideals in $C^*(E)$ are $I_{\Omega(M_n)}$, $I_{\Omega(M^n)}$ and $\{0\}$.

EXAMPLE 5.3.   The following infinite graph E is row-finite but does not satisfy Condition (K).
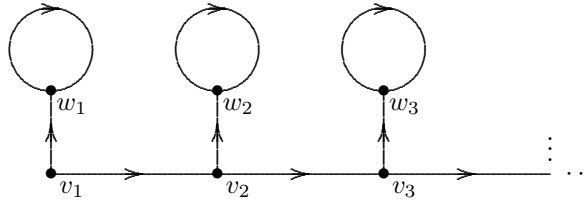
The maximal tails are $M = \{v_i : i \geq 1\}$ and $M_n = \{w_n, v_1, \ldots, v_n\}$ for all $n \geq 1$. Each M_n contains a loop without exits, but M does not contain any loops. Thus $I_{\Omega(M)}$ is the only gauge-invariant primitive ideal in $C^*(E)$.

EXAMPLE 5.4. It was suggested in [10, Remark 3.11] that describing the ideals of the C^*-algebra of the following graph would be an interesting test question:
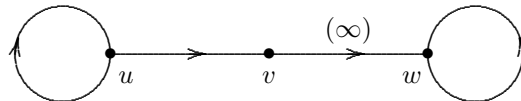
This graph is not row-finite and does not satisfy Condition (K). There are no breaking vertices, and there are three maximal tails: $M_1 = \{u\}$, $M_2 = \{u, v\}$ and $M_3 = \{u, v, w\}$. All loops in M_2 have exits, but M_1 and M_3 contain loops without exits. Thus there is exactly one gauge-invariant primitive ideal I_w, which corresponds to M_2, .

We have $p_w C^*(E) p_w \cong C(\mathbb{T})$ and there are infinitely many paths ending at w, so $I_w \cong C(\mathbb{T}) \otimes \mathcal{K}$. By Proposition 3.4, the quotient $C^*(E)/I_w \cong C^*(E \setminus \{w\})$ is isomorphic to the Toeplitz algebra.

6. An application to K-theory

One of our original motivations for analysing ideals in graph algebras was to extend [20, Theorem 3.2] to arbitrary graphs, and we now do this; the only difference between Theorem 6.1 below and [20, Theorem 3.2] is the definition of W. Recall that $\hat{\gamma}$ is the dual action of $\mathbb{Z} = \hat{\mathbb{T}}$ on the crossed product $C^*(E) \rtimes_\gamma \mathbb{T}$ by the gauge action γ, and that applying the integrated form of the canonical embedding $u : \mathbb{T} \to M(C^*(E) \rtimes_\gamma \mathbb{T})$ to the function $z \in L^1(\mathbb{T})$ yields a projection $\chi_1 = \int z u_z \, dz \in C^*(E) \rtimes_\gamma \mathbb{T}$.

THEOREM 6.1. *Let E be a directed graph, let W be the set of those vertices $w \in E^0$ that $s^{-1}(w)$ is either empty or infinite, and let $V = E^0 \setminus W$. With respect to the decomposition $E^0 = V \cup W$, the $E^0 \times E^0$ vertex matrix*

$$M(v, w) := \#\{e \in E^1 : s(e) = v \ \text{ and } \ r(e) = w\},$$

takes the block form

$$M = \begin{pmatrix} B & C \\ * & * \end{pmatrix}$$

where B and C have nonnegative integer entries. We define $K : \mathbb{Z}^V \to \mathbb{Z}^V \oplus \mathbb{Z}^W$ by $K(x) = \big((1-B^t)x, -C^t x\big)$, and $\phi : \mathbb{Z}^V \oplus \mathbb{Z}^W \to K_0(C^*(E) \rtimes_\gamma \mathbb{T})$ in terms of the usual basis by $\phi(v) = [p_v \chi_1]$. Then ϕ restricts to an isomorphism $\phi|$ of $\ker K$ onto $K_1(C^*(E))$, and induces an isomorphism $\overline{\phi}$ of $\operatorname{coker} K$ onto $K_0(C^*(E))$ such that the following diagram commutes:

$$\begin{CD} \ker K @>>> \mathbb{Z}^V @>K>> \mathbb{Z}^V \oplus \mathbb{Z}^W @>>> \operatorname{coker} K \\ @V\phi|VV @V\phi VV @V\phi VV @V\overline{\phi}VV \\ K_1(C^*(E)) @>>> K_0(C^*(E) \rtimes_\gamma \mathbb{T}) @>1-\hat{\gamma}_*^{-1}>> K_0(C^*(E) \rtimes_\gamma \mathbb{T}) @>>> K_0(C^*(E)). \end{CD}$$

Almost the entire proof of [20, Theorem 3.2] applies in this more general situation, and indeed only one point needs a different argument. Recall that for integers $m \le n$ we denote by $E \times_1 [m,n]$ the subgraph of $E \times_1 \mathbb{Z}$ with vertices $\{(v,k) : m \le k \le n, \ v \in E^0\}$ and edges $\{(e,k) : m < k \le n, \ e \in E^1\}$. It is essential for the proof of the theorem to know the K-theory of the corresponding algebra $C^*(E \times_1 [m,n])$. We claim that $K_0(C^*(E \times_1 [m,n]))$ is a free abelian group with free generators

$$\{[p_{(v,n)}] : v \in V\} \cup \{[p_{(v,k)}] : v \in W, \ m \le k \le n\}.$$

In the row-finite case this algebra is a direct sum of copies of the compacts (on Hilbert spaces of varying dimensions), and the claim is quite obvious. In general it is an AF-algebra with a more complicated structure. However, since any path in $E \times_1 [m,n]$ has length at most $n - m$ the following lemma applies.

LEMMA 6.2. *Let E be a directed graph such that the length of any path $\alpha \in E^*$ does not exceed a fixed number d. Then $K_1(C^*(E)) = 0$ and $K_0(C^*(E))$ is the free abelian group generated by*

$$\{[p_v] : s^{-1}(v) \text{ is either empty or infinite}\}.$$

Proof. We proceed by induction on d. If $d = 0$ then $C^*(E)$ is a direct sum of copies of $\mathbb{C}$, and the claim is clear. Suppose the lemma holds for all graphs with the maximum path length at most d, and let E be a graph with the maximum path length at most $d + 1$. We denote by F the set of sinks in E. By Lemma 3.4 $C^*(E)/I_{\Sigma(F)} \cong C^*(E \setminus \Sigma(F))$, and thus there is an exact sequence

(5) $0 \longrightarrow I_{\Sigma(F)} \longrightarrow C^*(E) \longrightarrow C^*(E \setminus \Sigma(F)) \longrightarrow 0.$

In the graph $E \setminus \Sigma(F)$, the maximum path length is at most d. Thus the inductive hypothesis implies that $K_1(C^*(E \setminus \Sigma(F))) = 0$ and $K_0(C^*(E \setminus \Sigma(F)))$ is free abelian with free generators corresponding to vertices which emit infinitely many edges and sinks in $E \setminus \Sigma(F)$. Moreover, the sinks in this quotient graph are the vertices $v \in E$ which emit infinitely many edges but for which there is no path in E from v to another vertex which emits infinitely many edges. If $v \in F$ then the ideal of $C^*(E)$ generated by p_v is isomorphic to $\mathcal{K}$, with $\{s_\mu s_\nu^* : \mu, \nu \in E^*, r(\mu) = r(\nu) = v\}$ as a system of matrix units. Moreover, any two such ideals corresponding to different sinks have trivial

intersection. Thus $I_{\Sigma(F)}$ is the direct sum of these ideals, $K_1(I_{\Sigma(F)}) = 0$, and $K_0(I_{\Sigma(F)})$ is the free abelian group generated by $\{[p_v] : v \in F\}$. Thus the six-term exact sequence of K-theory associated to (5) yields $K_1(C^*(E)) = 0$ and the short exact sequence

$$(6) \qquad 0 \longrightarrow K_0(I_{\Sigma(F)}) \longrightarrow K_0(C^*(E)) \longrightarrow K_0(C^*(E \setminus \Sigma(F))) \longrightarrow 0.$$

Since $K_0(C^*(E \setminus \Sigma(F)))$ is free abelian, the sequence (6) splits. Furthermore, such a splitting $K_0(C^*(E \setminus \Sigma(F))) \to K_0(C^*(E))$ may be determined by lifting free generators; we choose to lift $[p_v]$ to $[p_v]$. This completes the proof of the inductive step and the lemma. $\qquad\square$

The rest of the proof of Theorem 6.1 is exactly the same as in [20, Theorem 3.2].

REMARK 6.3.   Of course this computation of K-theory is not entirely new, though we believe the approach taken in [20] has much to commend it. Cuntz's original calculation of K-theory for Cuntz-Krieger algebras applies as it stands to finite graphs without sinks or sources in which every loop has an exit [2, Proposition 3.1]. This was extended to locally finite graphs in [17] and [16], and to arbitrary row-finite graphs in [20, Theorem 3.2]; for non-row-finite graphs without sinks or sources, we can apply the computations of K-theory for the Cuntz-Krieger algebras of infinite matrices ([7, Theorem 4.5], [22, §6], [20, Theorem 4.1]). Infinite graphs with finitely many vertices are covered by [24, Proposition 2], and arbitrary graphs by [5, Theorem 3.1], which was proved by reducing to the row-finite case and applying [20, Theorem 3.2].

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