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A "NICE" MAP COLOUR THEOREM

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ABSTRACT. A closed orientable triangulated surface is "nice" if its vertices can be assigned 4 colours in such a way that all 4 colours are used in the closed star of each edge. The 4-colouring can be interpreted as a simplicial map from the surface to the 4-vertex 2-sphere. If the surface has genus $(n-1)^2$, then the degree of this map is at least n^2. Conversely we show that, if n is not divisible by 2 and 3, then there are "nice" surfaces of genus $(n-1)^2$ for which the degree of the above map is exactly n^2. Complex analytically "nice" surfaces can be viewed as minimally triangulated meromorphic functions of a Riemann surface.

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By a *meromorphic function* of a (closed and connected) Riemann surface M we will mean a non-constant holomorphic function f from M to the complex sphere $S^2 = \mathbb{C} \cup \{\infty\}$. If $f'(z) = 0$, then $z \in M$ is called a *singular point* of f, and the value taken by f at a singular point z is called a *singular value* of f; all other points of M and S^2 are called *regular*. The cardinality of the inverse image $f^{-1}(w)$ is the same for any regular point $w \in S^2$ and is called the *degree* d of f. One can always find a simplicial complex L whose vertex set contains all the singular values of f, which triangulates S^2, and which pulls back under f to a simplicial complex $K = f^{-1}(L)$ which triangulates M. Note that L has at least 4 vertices. We will call f a *minimal meromorphic function* iff one can find such an $L = S^2_4$ with just 4 vertices (so a minimal f has ≤ 4 singular values). Our main result can be stated as follows.

THEOREM 1. *A minimal meromorphic function f of a Riemann surface M of genus g has degree $d \geq (\sqrt{g}+1)^2$. Furthermore, this bound is the best possible, in the sense that, if 2 and 3 do not divide n, then one does have a Riemann surface of genus $g = (n-1)^2$ which admits a minimal meromorphic function f having degree $d = n^2$.*

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The *branching number* of f at any $v \in M$ is the integer $k \geq 1$ such that, in suitably chosen local charts near v and $f(v)$, the holomorphic map f becomes $f(z) = z^k$. So a point $v \in M$ is singular iff $k \geq 2$ and the branched covering map $f : M \to S^2$ pulls back the (open) simplices of a triangulation L of S^2 to open simplices if the singular values of f are amongst the vertices of L. In general, this *simplicial subdivision* $K = f^{-1}(L)$ of M is not a *simplicial complex* (as we want) on account of two edges having the same pair of vertices; e.g., a meromorphic function f may well have ≤ 4 singular values, without being minimal. Also note that, even though all 4-vertex triangulations of S^2 are, of course, combinatorially isomorphic, the combinatorics of K can depend on how the 4 chosen vertices are joined to each other on S^2.

The key fact for us will be that this simplicial complex $K = f^{-1}(S^2_4)$ is *"nicely" four colourable*, as per the terminology introduced in our joint paper with Madahar [7]. That is, the 4-colouring which f determines on the vertices of K—the *four colours* being the 4 vertices $\{A, B, C, D\}$ of S^2_4—is such that the 4 vertices contained in the closed star of each edge of K get 4 distinct colours. To see this note that, if a vertex v of K has branching number k, then its *valence* in K is $3k$. For if, say, v is a *D-vertex* (i.e., $f(v) = D$), then each of the three edges DA, DB and DC of S^2_4 pulls back under f to k edges of K incident to v. Furthermore, as we cyclically go around the link of a D-vertex, every third vertex has one of the remaining three colours A, B, C (likewise for links of A-vertices, etc.) which, obviously, is an equivalent way of saying that the 4-colouring of K is "nice".

In the figures below, the *ABC-triangles* (those having their 3 vertices coloured A, B and C) will be shaded, so we will also refer to these as the *"black" triangles* of K. Clearly there are exactly d such triangles (and the same number of BCD-triangles, etc.) and, since f is orientation preserving, all these d triangles get oriented positively by the cyclic order of these 3 colours prescribed by the requirement that the triangle ABC of S^2 be positively oriented.

The construction given above can be reversed. This time we start with a *"nice" surface* K of genus g, i.e., an oriented simplicial 2-manifold K, equipped with a "nice" vertex-colouring f by the 4 colours $\{A, B, C, D\}$, having d "black" triangles (positively oriented). The colouring f induces a simplicial map $K \xrightarrow{f} S^2_4$, which topologically is a branched covering of the 2-sphere S^2, its branch points being precisely those vertices of K whose valence is bigger than 3, the branching number being always one-third of this valence (which, by "niceness", has to be a multiple of 3). We can now (cf. Springer [10, Chap. 4, p. 93]) equip $M = |K|$ with a (not necessarily unique) complex structure with respect to which (and with respect to the unique complex structure of S^2) the branched covering map $f : M \to S^2$ is holomorphic. For example, we can equip the open star of each vertex v of

K, having valence $3k$, with the chart obtained by using all the k-th roots of a chosen complex chart, of the open star in S_4^2 of $f(v)$, which takes $f(v)$ to $0 \in \mathbb{C}$. This gives, as required, a Riemann surface M of genus g, equipped with a minimal meromorphic function f of degree d.

The second part of Theorem 1 is thus equivalent to Theorem 6 of Section 6, which says that, if n is not divisible by 2 and 3, then one has a "nice" surface K of genus $(n-1)^2$ having n^2 "black" triangles. The first part of Theorem 1 is proved in Section 2 by showing that it follows from the $t = 4$ case of a (presumably known) lower bound on the number of vertices required to triangulate a surface if we want a vertex colouring by t colours which assigns distinct colours to adjacent vertices. So our results are "nice" analogues of those of Ringel-Youngs [9], Jungerman [5] and others. In Section 3 there is a (also perhaps known) "holomorphic Reeb theorem". The next section gives an analogue of Theorem 1, with the additional condition that the minimal meromorphic function has no more than 3 singular values. The main item of Section 5 is Figure 2 below, a "nice" surface of degree 9 and genus 2, whose discovery paved the way for the later constructions of Section 6. The concluding Section 7 points out a connection between "nice" 4-colourings and orthogonal latin squares.

2

We recall first a numerical characterization of "niceness" which played a major rôle in [7]. Let K be any oriented simplicial 2-manifold, and let f be any 4-colouring of its vertices by $\{A, B, C, D\}$. Then, f still induces, in the usual way, a simplicial map $f : K \to S_4^2$, and it still makes sense to speak of this map's *degree* d, provided we now use the definition $f_*[K] = d \cdot [S_4^2]$, where $[K]$ and $[S_4^2]$ denote the *fundamental* 2-*cycles* of K and S_4^2. In other words, now $d \in \mathbb{Z}$ is the *algebraical number* (i.e., counted with orientation) of ABC-triangles of K. (Note that the triangles of K having 2, or all 3, vertices of the same colour are not counted.) This shows that the actual number of ABC-triangles (and, likewise, of BCD-triangles, etc.) is at least $|d|$, and so *one always has* $\alpha_2 \geq 4\,|d|$, *with equality holding if and only if the colouring is "nice"*; here, and below, α_i denotes the number of i-simplices of K. This follows at once by noting that, if, say, two ABC-triangles share an edge, then they must have opposite orientations. Alternatively, using $3\alpha_2 = 2\alpha_1$ and Euler's formula $\alpha_0 - \alpha_1 + \alpha_2 = 2 - 2g$, we see that *one always has* $\alpha_0 \geq 2 + 2\,|d| - 2g$, *with equality holding if and only if the colouring f is "nice"*.

Using this equation $\alpha_0 = 2 + 2d - 2g$, which incidentally gives at once the very rough bound $d \geq g + 1$, we see that the first part of Theorem 1, i.e., the stronger assertion that $d \geq (\sqrt{g} + 1)^2$ *for any "nice" surface* K, is equivalent to the inequality $\alpha_0 \geq 4 + 4\sqrt{g}$, which we will now check for a much bigger class of 4-colourings.

THEOREM 2. *If K is an orientable simplicial 2-manifold of genus g, with t colours (here $3 \leq t \leq \alpha_0$) assigned to its α_0 vertices in such a way that adjacent vertices have distinct colours, then*

$$\alpha_0 \leq 2 - 2g + \frac{1}{3}\binom{t}{2}\left(\frac{\alpha_0}{t}\right)^2 .$$

For example, when $t = 4$, such a K must have at least $4 + 4\sqrt{g}$ vertices.

Proof. Let $\alpha_{0,j}$, $1 \leq j \leq t$, denote the number of vertices of K which have been assigned the j-th colour. Since $\alpha_1 \leq \sum_{j \neq k}(\alpha_{0,j}\alpha_{0,k})$, we see, on maximizing this quadratic function of t variables $\alpha_{0,j}$ subject to the constraint $\sum_j \alpha_{0,j} = \alpha_0$, that $\alpha_1 \leq {}^tC_2 \cdot (\alpha_0/t)^2$. Substituting this in $\alpha_0 = 2 - 2g + \alpha_1/3$, we obtain the stated inequality. (Since we only used $3\alpha_2 = 2\alpha_1$, this inequality is valid even when K is a *pseudomanifold* having Euler characteristic $2 - 2g$.) For the case $t = 4$ it becomes $(\alpha_0)^2 - 8 \cdot \alpha_0 + 16 - 16g \geq 0$, which holds iff $\alpha_0 \geq 4 + 4\sqrt{g}$. □

Available evidence suggests that, for any t, the above inequality is probably the best possible for infinitely many, and perhaps even almost all, values of the genus g. For example, for $t = \alpha_0$, i.e., when there is effectively no restriction on K, it becomes

$$\alpha_0 \geq \frac{7 + \sqrt{1 + 48g}}{2} ,$$

and the celebrated *map colour theorem* (see Ringel [8]) essentially just says that this bound is the best possible for all g. Again, for the other extreme case $t = 3$, it reads $(\alpha_0)^2 - 9 \cdot \alpha_0 + 18 - 18g \geq 0$, and is best possible for all g's of the type $t(t-1)/2$, t a natural number. This is so because $(3n)^2 - 9(3n) + 18 - 18 \cdot {}^{n-1}C_{n-3} = 0$ and Ringel and Youngs [9] showed that the genus ${}^{n-1}C_{n-3}$ orientable surface has a triangulation T whose 1-skeleton is the complete 3-partite graph $K_{n,n,n}$. Likewise others, notably Jungerman [5], have constructed orientable simplicial surfaces K with $g = (n-1)^2 \neq 4$, whose 1-skeleton is $K_{n,n,n,n}$. This shows that, for the case $t = 4$ which interests us most, the inequality $\alpha_0 \geq 4 + 4\sqrt{g}$ of Theorem 2 is known to be the best possible for all g's which are perfect squares $\neq 4$. However, none of these authors—besides [5] see also Garman [3] and White [11]—required their K's to be "nice", and indeed, we were unable to locate any "nice" K's amongst these known examples. (For instance, if $n \equiv 1 \bmod 4$, the circuit given by the current graph, Fig. 6, of Jungerman [5, p. 184], contains two light edges separated by just one other edge, which means in our terminology that, in the link of a D-vertex, one of the other three colours does not have periodicity 3, as required by "niceness".) So Theorem 4 below can be viewed as a "nice"

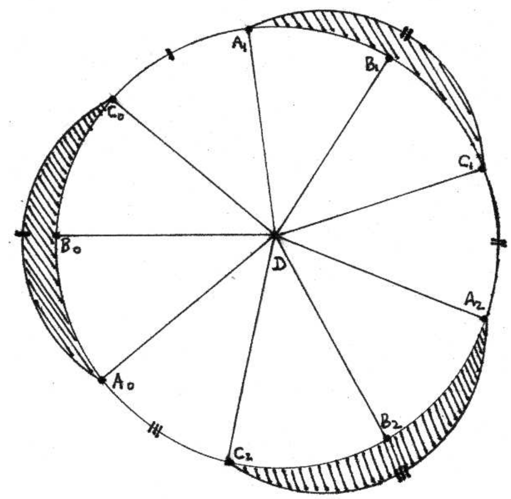

FIGURE 1

analogue of the map colour theorem, for it gives a new "nice" series of K's, with g a perfect square, and having $K_{n,n,n,n}$ as 1-skeleton. Before giving these and other constructions, we note that "nice" 4-colourability is obviously much stronger than requiring that adjacent vertices have distinct colours; e.g., the four colour theorem assures us that any triangulation K of S^2 is of the second type, but most (e.g., all those having a vertex whose valence is not divisible by 3) are not of the first type.

3

Nothing much of combinatorial interest happens if the number of singular values (colours) is at most 2; e.g., we will see in the course of the proof of the following "holomorphic Reeb theorem" that $K = S^2_4$ is the only such "nice" surface.

THEOREM 3. *A (closed and connected) Riemann surface M admits no (non-constant) meromorphic function having just one singular value, and if it admits a meromorphic function f having either none or precisely two singular values, then M must be the complex 2-sphere.*

Proof. To see this consider the star of a D-vertex of $K = f^{-1}(S^2_4)$ whose $3k$ incident edges have in cyclic order, the other vertex A_i, B_i, C_i, $i \in \mathbb{Z}/k$; see Figure 1, which shows the case $k = 3$. Now, if B is a regular value of f, i.e., if all the B-vertices of K have valence 3, then K must contain the k

"black" triangles $A_iB_iC_i$. If all the C-vertices of K are also of valence 3, then we must have the identification of edge pairs $A_iC_i \equiv A_{i+1}C_i$, $i \in \mathbb{Z}/k$, which show that K has no triangles other than the indicated ones, and that $|K|$ is a 2-sphere. Note further that all the vertices A_i are same, and that this vertex has also valence $3k$. If $k = 1$ (no singular value), then $K = S^2_4$, while for $k \geq 2$ (two singular values) the simplicial subdivision K is *not* a simplicial complex, i.e., f is *not minimal*, and has just two singular points, both with the same branching number k, this being equal to the degree d of the map f. □

4

In contrast to the above, one obtains lots of "nice" surfaces $K \neq S^2_4$ as soon as one allows 3 singular colours.

THEOREM 4. *Any closed oriented surface M of genus $g \geq 0$ can be triangulated by a "nicely" four coloured simplicial complex K, all of whose D-vertices are of valence* 3 *(and so M can be made into a Riemann surface admitting a minimal meromorphic function having precisely 3 singular values). Such a K has degree $d \geq 2g + (1/2)\,(5 + 3\sqrt{1+8g})$, and this bound is the best possible if g is of the type $t(t-1)/2$.*

Proof. Start with any triangulation T whose vertices can be assigned 3 colours $\{A, B, C\}$ in such a way that adjacent vertices have distinct colours (e.g., take T to be the barycentric derived of any triangulation). Under the cyclic order of colours, ABC, exactly half the triangles of T are oriented positively. Derive the remaining triangles by taking one new vertex in each of them, to which is assigned the fourth colour D. This gives a "nicely" 4-coloured K of the required kind having degree $d = (1/2)\,\alpha_2(T)$.

If we start with a barycentric derived as our T, the degree d of K is pretty high, but it can be lowered by making a more prudent choice of T, e.g., as in [7]. Indeed, this construction gives lowest degree "nice" surfaces of genus $g = {}^{n-1}C_{n-3}$ having only three singular colours, if we start with the triangulation T of Ringel-Youngs [9] mentioned in Section 2.

This follows because any such "nice" K is obtainable in the above way from the 3-coloured T obtained by erasing the D-vertices of K and their incident edges. So $d = (1/2)\,\alpha_2(T) = \alpha_0(T) - 2 + 2g$, which gives, on using the case $t = 3$ of Theorem 2, $d \geq 2g + (1/2)\,(5 + 3\sqrt{1+8g})$. (Note that this is much bigger than $(\sqrt{g}+1)^2$.) For a Ringel-Youngs triangulation T, we have $d = (1/2)\,\alpha_2(T) = n^2$ and $g = {}^{n-1}C_{n-3}$, and then equality holds. □

One can increase the number of triangles of T by any even number ≥ 4 without losing 3-colourability: Put a new edge within an edge to increase this number by 4, and put a new triangle within a triangle to increase this number by 6. So one also has "nice" K's, with all D-vertices of valence 3, and of genus ${}^{n-1}C_{n-3}$ and degree $n^2 + t$ for any $t \geq 2$.

5

From now on we will allow all the colours to be singular. The "nice" degree 3 sphere, and the "nice" degree 6 torus, of [7] turn out to be the first two instances ($g = 0, 1$) of a pretty sequence.

THEOREM 5. *For each $g \geq 0$, there is a "nice" surface K of genus g having degree $d = 3(g+1)$. (This is less than the bound $2g + (1/2)\,(5 + 3\sqrt{1+8g})$ attained in Section 4 iff $g \leq 17$.)*

Proof. Our surface K has $g+2$ vertices of each colour. There is a D-vertex having valence $3(g+1)$, which we denote by D_*, and we denote by $\dots A_i B_i C_i A_{i+1} \dots$, $i \in \mathbb{Z}/g+1$, the vertices which occur, in cyclic order, in its link. The three other vertices of these three colours are denoted A_*, B_* and C_*, and there is precisely one of these in each "black" triangle. So these $3(g+1)$ triangles are $A_i C_* B_i$, $B_i A_* C_i$ and $C_i B_* A_{i+1}$, $i \in \mathbb{Z}/g+1$. The remaining vertices are denoted by D_i, $i \in \mathbb{Z}/g+1$; each of them has valence 6, the link of D_i being $B_i C_* A_{i+1} B_* C_{i+1} A_*$. We equip the D-stars and the "black" triangles with the orientations which induce the indicated cyclic order of their boundaries. So for $g = 2$ the D-stars and the "black" triangles form a plane polygon as shown in Figure 2 below (where edges incident to the D-vertices have been omitted to reduce clutter) and a pairwise identification of its boundary edges gives K.

The D-links are all edge-disjoint, and each of the $9(g+1)$ edges constituting them occurs exactly once, with opposite orientation, as an edge of one of the $3(g+1)$ "black" triangles. So K is an oriented pseudomanifold with $\alpha_0 - (1/2)\,\alpha_2 = (4g+8) - (1/2)\,(12g+12) = 2 - 2g$ as desired. To see that $|K|$ is a manifold we verify finally that the other vertex links are also (single) polygons. We have $\text{link}(A_i) = C_* B_i D_* C_{i-1} B_* D_{i-1}$, $\text{link}(B_i) = C_* D_i A_* C_i D_* A_i$ and $\text{link}(C_i) = A_* D_{i-1} C_* A_{i+1} D$, all hexagons; while $\text{link}(A_*) = \dots C_i B_i D_i C_{i+1} \dots$, $\text{link}(B_*) = \dots C_i D_{i-1} A_i C_{i-1} \dots$ and $\text{link}(C_*) = \dots D_i C_i A_i D_{i-1} \dots$, all $3(g+1)$-gons. □

The only smaller degree "nice" sphere is S^2_4. If there was one with $d = 2$, it would have $\alpha_0 = 6$, so a colour which is assigned to just one vertex, which must have valence $3d = 6$, which needs 7 vertices already.

There is also no smaller degree "nice" torus. For example, if there were one with $d = 4$ (similar arguments apply to other values of $d \leq 5$), it would have $\alpha_0 = 8$, so there is a colour, say D, occurring at most twice. The sum of the valences of these D-vertices being $3d = 12$, obviously there cannot be just one D-vertex, and there must be two D-vertices, both of valence 6, with the same 6 vertices in their links. Consider now the 6 edges of the first link. Each is incident to one of the 4 "black" triangles, so two of them, necessarily adjacent, must be incident to the same "black" triangle. The common vertex

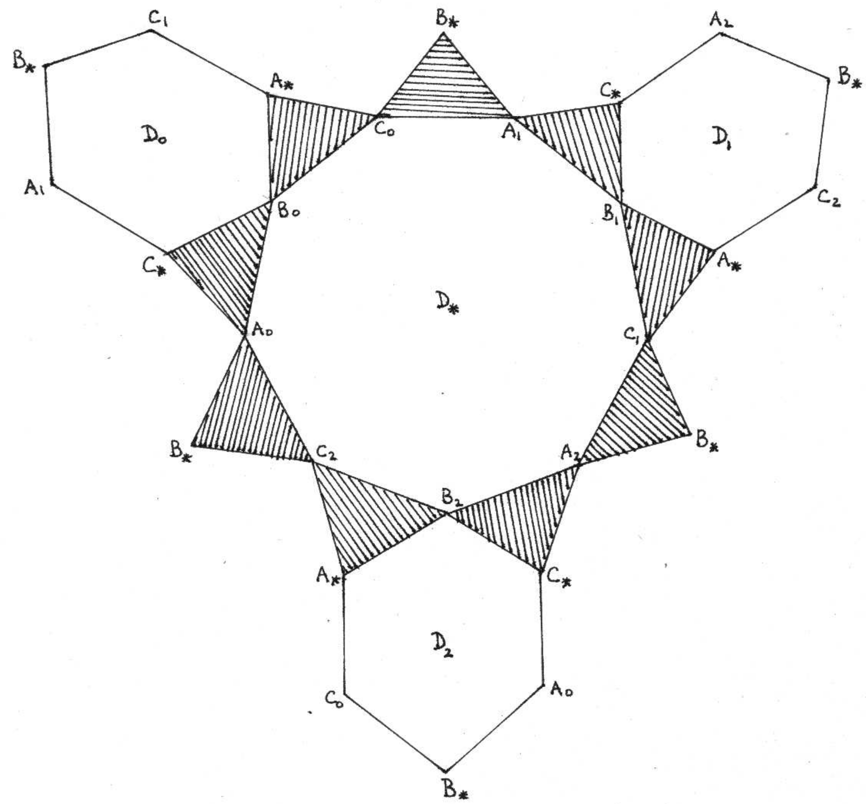

FIGURE 2

of these edges then has valence 3 and does not occur in the second link, which is a contradiction.

The same arguments also show easily that *there is no "nice" genus* 2 *surface with* $d \leq 7$*, and if there is one with* $d = 8$*, then the colour assigned least frequently in it must occur* 3 *times with valences* 9*,* 9 *and* 6. But, vexingly enough, we were unable to rule this out, and so still do not know if degree 9 is least possible for $g = 2$. (It was the complexity of this argument which first suggested to us that, for g large, $3(g + 1)$ was nowhere near the least degree possible; soon thereafter, in June 1999, we became aware of Ringel-Youngs [9]. Using [9] we were able to lower this value easily, for $g \geq 18$, by the construction of Section 4.)

We feel that a computer program, based on the methods of this paper, will probably give quickly least degree "nice" surfaces for many low values of g. However, as such, these triangulations are rare; this was made clear by the fact that a long computer search by Frank Lutz, using the already extant bistellar flips program of Björner-Lutz [1], failed to find any degree 9 "nice" genus two surface. The example of Figure 2 was found shortly after this, by hand, in May 1999.

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The "nice" surfaces of Sections 4–5 still have degree d far bigger than the lower bound $(\sqrt{g}+1)^2$ of Section 2. We were able to close this gap completely, in June 1999, for infinitely many values of g, by means of the construction given below. Note that this is not much different from that of Theorem 5; we now use many analogous plane polygons, and identify their boundary edges in pairs.

THEOREM 6. *If n is not divisible by* 2 *and* 3*, then there exists a "nice" surface of genus* $(n-1)^2$ *and degree* n^2.

Proof. We want an oriented simplicial 2-manifold K, with $4n$ vertices and $6n^2$ edges, with each of its vertices assigned one of 4 colours $\{A, B, C, D\}$ so that each colour is given to precisely n vertices, with any two vertices adjacent if and only if their colours are distinct, and the 3 colours $\{A, B, C\}$ are given to the 3 vertices of precisely n^2 ("black") triangles, which all get oriented positively under the cyclic ordering ACB of these 3 colours.

Definition of K. From now on all suffixes are integers mod n. We will describe K by giving the boundaries of the n 2-chains formed by the sums of the oriented triangles of the stars of its D vertices, and by its "black" triangles. We want $\partial(\mathrm{Star}\,(D_j)) = \sum_i \{A_i B_{i+j}\} + \{B_{i+j} C_{i+2j}\} + \{C_{i+2j} A_{i+1}\}$, which we will write more briefly as

$$\partial(D_j) = \dots A_i\, B_{i+j}\, C_{i+2j}\, A_{i+1} \dots ,$$

and the n^2 oriented "black" triangles of K will be

$$\Delta_{j,i} = A_i\, C_{i-3+2j}\, B_{i+j} \,.$$

Less formally, the vertices occurring in the link of D_0 are, in cyclic order, $A_0 B_0 C_0 A_1 B_1 C_1 \dots A_{n-1} B_{n-1} C_{n-1}$. Now, keeping suffixes of the A vertices the same, increase those of the B and C vertices by 1 and 2, respectively, to obtain the link of D_1. Repeating this gives the link of D_2, and so on. Figure 3 below displays $\mathrm{Star}(D_0)$ for the least non-trivial case $n = 5$; it shows also the "black" triangles $\Delta_{0,i}$ incident to the AB edges of this link. Note that the triangles of the figure are equipped with the clockwise orientation, and that all the simplices shown are distinct, except that each of the C vertices occurs twice, and these pairs of vertices will of course be identified in K. The remaining simplices of K are obtained from those shown in the figure by

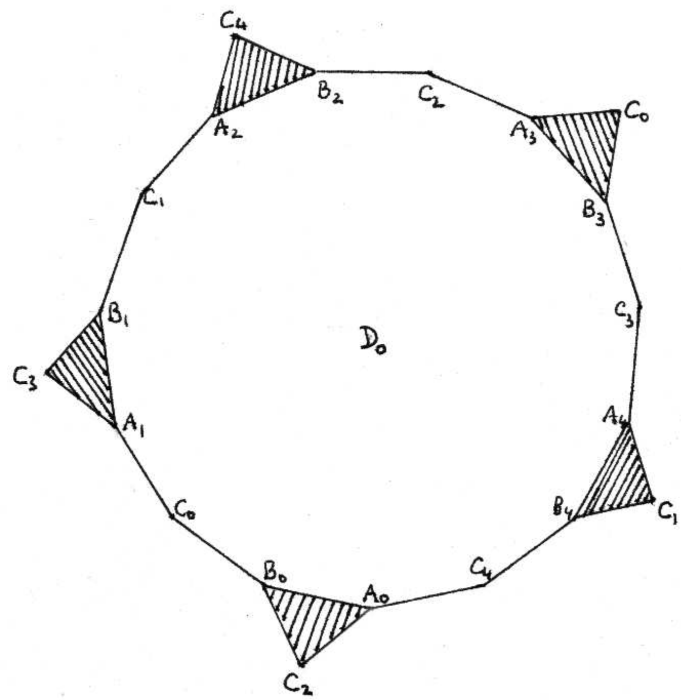

FIGURE 3

repeatedly applying an order n (colour and orientation preserving) simplicial *automorphism* λ of K, viz., that defined by

$$\lambda(A_i) = A_i\ ,\ \lambda(B_i) = B_{i+1}\ ,\ \lambda(C_i) = C_{i+2}\ ,\ \lambda(D_i) = D_{i+1}\ .$$

So, for $n = 5$, one has 5 copies of the displayed polygonal region. Each white/black boundary edge, of any of these, repeats exactly once as a black/white boundary edge of some other polygon, and K is obtained from the 5 polygonal regions by identifying these pairs of edges. Since the case $n = 1$ is trivial—now $K = S_4^2$—we will assume below that $n > 1$.

Any edge, with neither of its two vertices coloured D, occurs once and only once in the n links of the D vertices. This follows from the above formula for $\partial(D_j)$ because there is a unique pair $\{i, j\}$ of integers mod n which satisfies each of the following pairs of equations: $\{x = i\ ,\ y = i + j\}$, $\{x = i + j\ ,\ y = i+2j\}$, $\{x = i+2j\ ,\ y = i+1\}$. For the last pair of equations we used here the hypothesis $2 \nmid n$ which gives the unique solution $\{i = y-1\ ,\ j = (x-y+1)/2\}$.

Likewise, each of these $3n^2$ edges occurs once and only once amongst the edges of the "black" triangles. This follows from the formula for $\Delta_{j,i}$ because there is a unique pair $\{i, j\}$ of integers mod n which satisfies each of the pairs of equations $\{x = i\,,\ y = i{+}j\}$, $\{x = i{+}j,\,,\ y = i{-}3{+}2j\}$, $\{x = i\,,\ y = i{-}3{+}2j\}$, using again, for the last pair, the hypothesis $2 \nmid n$.

So each edge of K is incident to precisely two triangles. Also note that the two occurrences—in the boundary of the star of a D vertex, and in the boundary of a "black" triangle—of each of the aforementioned $3n^2$ edges, are with opposite incidence numbers. So the 2-chain formed by the sum of all the oriented triangles of K has zero boundary; i.e., for any odd n, the K defined above is an oriented *pseudo* 2*-manifold* of the correct genus and degree. So it remains only to check that the A, B and C vertices are all nonsingular.

The links of the A, B and C vertices are all $3n$-gons, and are given by the following formulae, where we have used the same notation as used previously for the case of the D vertices:

$$\begin{aligned}
\partial(A_j) &= \ldots D_i\, C_{2i+j-1}\, B_{i+j+1}\, D_{i+1} \ldots\,,\\
\partial(B_j) &= \ldots A_i\, C_{2j-i-3}\, D_{j-i-3}\, A_{i+3} \ldots\,,\\
\partial(C_j) &= \ldots A_i\, D_{(j-i+1)/2}\, B_{(i+j-1)/2}\, A_{i-4} \ldots\,.
\end{aligned}$$

The proofs of all three formulae are similar. For example, for the second formula (which is the only point where we use $3 \nmid n$) note that $A_iC_{2j-i-3}\,B_j = \Delta_{j-i,i}$, so the edge A_iC_{2j-i-3} is in the link of B_j, then that $A_{i+3}B_jC_{2j-i-3}$ occurs as a subsequence in the formula for $\partial(D_{j-i-3})$, so the edges $C_{2j-i-3}D_{j-i-3}$ and $D_{j-i-3}A_{i+3}$ are also in the link of B_j. (Likewise for the first formula note that $A_jC_{2i+j-1}B_{i+j+1} = \Delta_{i+1,j}$ and $C_{2i+j+1}A_jB_{i+j+1}$ is a subsequence of ∂D_{i+1}, and for the third formula that $A_{i-4}C_jB_{(i+j-1)/2} = \Delta_{(j-i+7)/2,i-4}$ and $B_{(i+j-1)/2}C_jA_i$ is a subsequence of $\partial D_{(j-i+1)/2}$.) As we continue in this manner, the suffixes of the successive A vertices occurring in the link of B_j increase by 3, and of the successive C and D vertices decrease by 3. Since $3 \nmid n$, the initial vertex A_i recurs only after we have gone through all the $3n$ vertices not coloured D. So the link of B_j is a single polygon with $3n$ sides, i.e., B_j is nonsingular. □

Using Theorem 6 as a starting point, and the (for $K \neq S^2_4$) local constructions of [7, Figs. 4b and 4c], it follows that *if $2 \nmid n$ and $3 \nmid n$, then one also has "nice" surfaces of genus $(n-1)^2$ and degree n^2+t for all $t \geq 2$.* However, for $t = 1$, the construction of [7, Fig. 4a] cannot be used now because the link of every edge consists of the two vertices of some other edge. It seems likely nevertheless that if $2 \nmid n$, $3 \nmid n$ and $n > 1$, then one also has a "nice" surface of genus $(n-1)^2$ and degree n^2+1.

7

An $n \times n$ matrix over $\mathbb{Z}/n$ (or any other cardinality n set) is called an *Euler square* if each element of $\mathbb{Z}/n$ occurs exactly once in each row and in each column. Superimposing two such squares, one gets an $n \times n$ matrix over $\mathbb{Z}/n \times \mathbb{Z}/n$, and if all entries of this matrix are distinct, the two Euler squares are deemed *orthogonal*, and this superimposition is often called a Graeco-Latin square.

A notable feature of the construction of Section 6 *was the use of many Euler squares*; e.g., in the definition of the D-links we used $[i+j, i+2j]$ which is a Graeco-Latin square because $2 \nmid n$. More generally any Graeco-Latin square $[g_{ij}, \ell_{ij}]$ will give n edge-disjoint $3n$-gons if we similarly define $\partial(D_j) = \dots A_i\, B_{g_{ij}}\, C_{\ell_{ij}}\, A_{i+1} \dots$. Likewise, in the definition of the "black triangles", we used $[i-3+2j, i+j]$, which too is Graeco-Latin because $2 \nmid n$. This property ensured edge-disjointness of the boundaries of the "black triangles", and so gave a "nice" pseudomanifold of the correct degree and genus. This much of Theorem 6 will remain true if now $[\ell_{i-1,j-1}, g_{ij}]$ is also Graeco-Latin. There are even sized Graeco-Latin square having this property, e.g.,

$$\begin{bmatrix} 1,1 & 2,2 & 3,3 & 0,0 \\ 2,3 & 1,0 & 0,1 & 3,2 \\ 3,0 & 0,3 & 1,2 & 2,1 \\ 0,2 & 3,1 & 2,0 & 1,3 \end{bmatrix}$$

is one, thus giving a "nice" pseudomanifold K of degree 16 and genus 9. (All the A- and B-vertices are nonsingular, but the links of the C-vertices turn out to be two disjoint hexagons.)

In fact, *any Graeco-Latin square of size* n *determines a rather trivial "nice" pseudomanifold* K *of degree* n^2 *and genus* $(n-1)^2$. There is a standard way (see, e.g., Hall [4, p. 190]) in which one can consider the pair of orthogonal Euler squares as an *orthogonal array* $OA(n,4)$ over $\mathbb{Z}/n$ having 4 rows and n^2 columns. Think of an entry i of $OA(n,4)$ as the vertex A_i, B_i, C_i or D_i depending on whether it occurs in the first, second, third or fourth row. As the $4n^2$ triangles of K we take all those which have all their three vertices in the same column of $OA(n,4)$. Note that each column of $OA(n,4)$ thus contributes a 4-vertex sphere, and K is simply the union of these edge-disjoint spheres (and so K desingularizes—see below—to give a union of disjoint 2-spheres).

One can *desingularize* a simplicial pseudomanifold in an obvious way, by replacing each vertex, whose link consists of the union of t polygons, by t distinct vertices. We remark that, if $3 \,|\, n$ but $2 \nmid n$, then desingularizing the pseudomanifolds given by the construction of Section 6 one gets "nice" connected surfaces with degrees very close to the lower bound. However, though the divisibility conditions of Theorem 6 are probably redundant for large n, we have so far no construction which actually attains the lower bound when these conditions fail.

The K's of Section 6 have the property that *the two vertices of each edge occur exactly once as the link of some other edge.* Starting from any such K of degree n^2 and genus $(n-1)^2$ one gets an $OA(n,4)$, whose r-th column consists of the four indices of the vertices of the r-th ABC-triangle and the adjacent ABD-triangle. Equivalently, any such K determines a Graeco-Latin square of size n, which implies that *there is no such "nice" K of degree* 36 *and genus* 25. This follows because there is no Graeco-Latin square of size six: 36 officers of 6 different ranks from 6 different regiments cannot be arranged in a 6×6 array so that no rank or regiment repeats in any row or column.

As is well-known, Euler (1782) had in fact expected more generally that there is no Graeco-Latin square of size $n \equiv 2 \bmod 4$. An interesting topological misproof of Euler's conjecture was given in the seminal paper of MacNeish [6]. He gave a construction (quite different from those of this paper) which associates to any Graeco-Latin square an oriented 2-pseudomanifold, and showed that *Euler's conjecture is true whenever this MacNeish complex has first Betti number even.* He erred in assuming this to be always the case; to see that it is not so, consider, e.g., S^2 with two of its points identified when $b_1 = 1$. Later, of course, Euler's conjecture was completely demolished by Bose, Parker and Shrikhande in 1959 (see, e.g., [2] and [4]).

The minimality of the meromorphic function, e.g., the fact that it has at most 4 singular values, is necessary for the lower bound of Theorem 1. We remark that if one lifts this condition, one can, by using instead of S^2_4 bigger triangulations of the 2-sphere, construct by similar combinatorial methods *examples of meromorphic functions on genus g surfaces with degree less than εg where $\varepsilon > 0$ is preassigned*. Also, as we shall show elsewhere, there are natural and interesting analogues of "nice" surfaces in dimensions ≥ 3.

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