Illinois Journal of Mathematics
Volume 45, Number 1, Spring 2001, Pages 25–39
S 0019-2082

LOCAL PROPERTIES OF POLYNOMIALS ON A BANACH SPACE

RICHARD M. ARON, YUN SUNG CHOI, SUNG GUEN KIM, AND MANUEL MAESTRE

ABSTRACT. We introduce the concept of a smooth point of order n of the closed unit ball of a Banach space E and characterize such points for $E = c_0$, $L_p(\mu)$ $(1 \leq p \leq \infty)$, and $C(K)$. We show that every locally uniformly rotund multilinear form and homogeneous polynomial on a Banach space E is generated by locally uniformly rotund linear functionals on E. We also classify such points for $E = c_0$, $L_p(\mu)$ $(1 \leq p \leq \infty)$, and $C(K)$.

1. Introduction

This paper deals with smoothness and local uniform rotundity for n-homogeneous polynomials on a Banach space. The concept of smoothness is a linear one and we extend this notion to *n-smoothness,* in the context of n-homogeneous polynomials. In addition, we will study *locally uniformly rotund* points of the closed unit ball of the Banach space of n-homogeneous polynomials which, in a sense, is a dual notion to n-smoothness.

We recall that a point x_0 in the unit sphere S_E of a Banach space E is said to be a *smooth* point of the closed unit ball B_E if there is a unique norming functional for x_0, that is, if there is a unique linear form $\phi_0 \in E^*$ such that $\|\phi_0\| = 1 = \phi_0(x_0)$. The set of smooth points of B_E is denoted by $\mathrm{sm}(E)$. A dual notion to smoothness is rotundity: a point $x_0 \in S_E$ is called a *locally uniformly rotund point* if the condition $\|x_n + x_0\| \to 2$, for a sequence (x_n) in the closed unit ball B_E, implies that $\|x_n - x_0\| \to 0$. The set of locally uniformly rotund points of B_E is denoted by $\mathrm{lur}\, B_E$.

Received June 7, 1999.
2000 *Mathematics Subject Classification.* 46B20, 46E15.
The second author wishes to acknowledge the financial support of the Korea Research Foundation made in the program of 1998-001-D00065. The third author wishes to acknowledge the financial support of the Korea Research Foundation made in the program KRF-2000-015-DP0012. The fourth author was supported by DGESIC pr. no. P.B.96-0758 and grant PR 1997-0186.

To describe the problems we will be investigating, we first need to recall some terminology and notation. Let $\mathcal{L}(^nE)$ denote the Banach space of scalar-valued continuous n-linear mappings on $E \times \cdots \times E$, endowed with the norm $\|A\| = \sup_{\|x_i\|\leq 1} |A(x_1, \ldots, x_n)|$, and let $\mathcal{P}(^nE)$ denote the Banach space of scalar-valued continuous n-homogeneous polynomials on E endowed with the norm $\|P\| = \sup_{\|x\|\leq 1} |P(x)|$. We refer the reader to [4, 11, 14] for more details about polynomials on a Banach space. Given an n-homogeneous polynomial P, we denote the unique symmetric n-linear form associated to P by $\overset{\vee}{P}$.

A point $x_0 \in S_E$ is called a *smooth point of order* n of B_E if there is a *unique* element $P \in \mathcal{P}(^nE)$ such that $\|P\| = 1 = P(x_0)$. We denote the set of all smooth points of order n of B_E by $\mathrm{sm}^{(n)}(E)$, noting that $\mathrm{sm}^{(1)}(E)$ coincides with $\mathrm{sm}(E)$; see [9, Chapters 1 and 2] and [10, Chapter 2]. On the other hand, $\mathcal{P}(^nE)$ is isometrically isomorphic with the dual space of $\tilde{\otimes}^{n,s}_{\pi_s} E$, which is the completed n-th symmetric tensor product of E with the projective s-tensor norm (see [12]). From this viewpoint we can say that *the point* $x_0 \in S_E$ *is a smooth point of order* n *of* B_E if the n-symmetric tensor $x_0 \otimes \cdots \otimes x_0$ is a smooth point of $B_{\tilde{\otimes}^{n,s}_{\pi_s} E}$. Since $\mathcal{P}(^nE) \equiv (\tilde{\otimes}^{n,s}_{\pi_s} E)^*$, the concept of a smooth point of order n of B_E is closely related to that of an LUR point of $B_{\mathcal{P}(^nE)}$ (see, e.g., Corollary 3.5 below).

Boyd and Ryan [3, Proposition 17] showed that if E is a real Banach space of dimension at least 2, then for $n \geq 2$ the spaces $\tilde{\otimes}^{n,s}_{\pi_s} E$ and $\mathcal{P}(^nE)$ are neither smooth nor rotund. Extreme points and smooth points of $B_{\mathcal{P}(^nE)}$ were studied in [5, 6, 7, 8]. Here, we will study smooth points of $B_{\tilde{\otimes}^{n,s}_{\pi_s} E}$ and LUR points of $B_{\mathcal{P}(^nE)}$ and $B_{\mathcal{L}(^nE)}$ for classical Banach spaces E.

For use in the sequel, we now collect some results concerning smooth and locally uniformly rotund points. It is well-known (see [9, Examples 1.6], and [13, 26.5 Examples]) that $\mathrm{sm}(L_1[0,1]) = \{f \in L_1[0,1] \, : \, \|f\|_1 = 1 \text{ and } f(x) \neq 0 \ a.e.\}$, while $\mathrm{sm}(L_p[0,1]) = S_{L_p[0,1]}$ for $1 < p < \infty$ and $\mathrm{sm}(L_\infty[0,1]) = \emptyset$. Also, $\mathrm{sm}(\ell_p) = S_{\ell_p}$ for $1 < p < \infty$, $\mathrm{sm}(\ell_1) = \{(a_i) \in S_{\ell_1} : a_i \neq 0 \text{ for all } i\}$, $\mathrm{sm}(\ell_\infty) = \{(\lambda_i) \in \ell_\infty : \text{there exists } i_0 \text{ such that } |\lambda_{i_0}| = 1 > \sup\{|\lambda_i| \text{ for } i \neq i_0\}\}$, and $\mathrm{sm}(c_0) = \{(\lambda_i) \in c_0 : \text{there exists } i_0 \text{ such that } |\lambda_{i_0}| = 1 \text{ and } |\lambda_i| < 1 \text{ for } i \neq i_0\}$. Further, if X is a locally compact Hausdorff space and we denote by $C_0(X)$ the Banach space of continuous functions on X vanishing at infinity, endowed with the supremum norm, then $\mathrm{sm}(C_0(X)) = \{f : \text{there exists a unique } x_0 \in X \text{ satisfying } |f(x_0)| = 1 = \|f\|_\infty\}$. Finally, $\mathrm{lur}\, B_{C(K)} = \emptyset$ provided the compact set K has at least two points. Also, $\mathrm{lur}\, B_{\ell_p} = S_{\ell_p}$ and $\mathrm{lur}\, B_{L_p[0,1]} = S_{L_p[0,1]}$ for $1 < p < \infty$, because ℓ_p and $L_p[0,1]$ are uniformly convex.

In Section 2 we examine the set of smooth points of order n for all these spaces. In Section 3 we show that every locally uniformly rotund multilinear form and homogeneous polynomial on E is generated by locally uniformly rotund linear functionals, and we characterize such points for these spaces.

2. Smooth points of high order

An easy observation is that if $x \in S_E$ is a norming point of a non-finite type polynomial $P \in \mathcal{P}(^nE)$, i.e. $P \notin \otimes^{n,s} E^*$, then x is not a smooth point of order n. Moreover, we can see that $\mathrm{sm}^{(n)}(E) \subset \mathrm{sm}^{(k)}(E)$ for $1 \le k \le n$. In fact, if $x_0 \in S_E$ is not a smooth point of order k, then there is $Q \in \mathcal{P}(^kE)$, $Q \neq \phi_0^k$, such that $1 = Q(x) = \|Q\|$, where $\phi_0 \in E^*$ is such that $1 = \|\phi_0\| = \phi_0(x_0)$. Hence $\phi_0^n = \phi_0^{n-k}\phi_0^k \neq \phi_0^{n-k}Q$, and so x_0 is not a smooth point of order n.

For a real Banach space E of dimension at least 2, we get $\mathrm{sm}^{(2)}(E) = \emptyset$. Indeed, given $x_0 \in S_E$, let $\phi_0 \in E^*$ and $P_0 = \phi_0^2 \in \mathcal{P}(^2E)$ be as above. Next choose $\eta \in E^*$ so that $\|\eta\| = 1$ and $\eta(x_0) = 0$. Define $Q \in \mathcal{P}(^2E)$ by $Q(x) = (\phi_0(x))^2 - (\eta(x))^2$. Then $P_0 \neq Q$ and $\|P_0\| = \|Q\| = 1 = P_0(x_0) = Q(x_0)$. Hence x_0 is not a smooth point of order 2. Consequently, all Banach spaces in this section will be assumed to be over $\mathbb{C}$.

Let K be a compact Hausdorff space and let Z be a closed subset of K. We denote by $C_Z(K)$ the Banach space of all continuous functions on K that vanish on Z, endowed with the supremum norm. It is well-known that any closed ideal of $C(K)$ is of this form.

THEOREM 2.1. (i) Let K be a compact Hausdorff space and let Z be a closed subset of K. Then for all n, $\mathrm{sm}^{(n)}(C_Z(K)) = \{f :$ there exists a unique $x_0 \in K$ satisfying $|f(x_0)| = 1 = \|f\|_\infty\}$.

(ii) If X is a locally compact space, then for all n, $\mathrm{sm}^{(n)}(C_0(X)) = \{f :$ there exists a unique $x_0 \in X$ satisfying $|f(x_0)| = 1 = \|f\|_\infty\}$.

Proof. (i) Given $x \in K$, let δ_x be the evaluation functional at x. Let $f \in S_{C_Z(K)}$ and suppose that there exist distinct points x_1 and x_2 in K satisfying $|f(x_1)| = |f(x_2)| = 1$. For each positive integer n, $P_j = \dfrac{1}{f(x_j)^n}(\delta_{x_j})^n$ $(j = 1, 2)$ are distinct continuous n-homogeneous polynomials with norm one on $C_Z(K)$ such that $P_j(f) = 1$ for $j = 1, 2$. Hence f is not a smooth point of order n. Conversely, let $f_0 \in C_Z(K)$ be a function such that there exists a unique $x_0 \in K$ with $\|f_0\|_\infty = 1 = |f_0(x_0)|$. To complete the proof it is enough to show that if $f_0(x_0) = 1$ and if $P \in \mathcal{P}(^nC_Z(K))$ satisfies $\|P\| = 1 = P(f_0)$, then $P = (\delta_{x_0})^n$.

Let $\mathcal{E}$ be the family of open neighborhoods U of x_0 with $U \neq K$. For each $U \in \mathcal{E}$ there exists $s_U : K \to [0, 1]$, continuous on K, satisfying

$$s_U(x) = \begin{cases} 1 & \text{if } x = x_0, \\ 0 & \text{if } x \in K \setminus U. \end{cases}$$

Next we can find $t_U : K \to [0, 1]$, continuous on K, such that

$$t_U(x) = \begin{cases} 1 & \text{if } x \in s_U^{-1}([0, 1/4]), \\ 0 & \text{if } x \in s_U^{-1}([1/2, 1]). \end{cases}$$

The functions $g_{1U} = \dfrac{t_U}{s_U + t_U}$ and $g_{2U} = \dfrac{s_U}{s_U + t_U}$ are continuous on K and satisfy $g_{1U} + g_{2U} \equiv 1$. For each $U \in \mathcal{E}$, we define $G_U : C_Z(K) \to \mathbb{C}$ by $G_U(f) = P(fg_{1U} + f_0 g_{2U})$ for $f \in C_Z(K)$. Since the map $\phi : C_Z(K) \to C_Z(K)$ defined by $\phi(f) = fg_{1U} + f_0 g_{2U}$ is affine, $G_U = P \circ \phi$ is a continuous polynomial of degree n on $C_Z(K)$. Moreover,

$$\begin{aligned} G_U(f) &= \sum_{k=0}^{n} \binom{n}{k} \overset{\vee}{P} \left((fg_{1U})^{(k)}, (f_0 g_{2U})^{(n-k)} \right) \\ &= P(fg_{1U}) + \sum_{k=1}^{n-1} \binom{n}{k} \overset{\vee}{P} \left((fg_{1U})^{(k)}, (f_0 g_{2U})^{(n-k)} \right) + P(f_0 g_{2U}). \end{aligned}$$

If $\|f\|_\infty \leq 1$, then

$$|G_U(f)| \leq \|P\| \|fg_{1U} + f_0 g_{2U}\|_\infty^n \leq \|P\| \max\{\|f\|_\infty, \|f_0\|_\infty\}^n \leq 1.$$

Since $G_U(f_0) = P(f_0 g_{1U} + f_0 g_{2U}) = P(f_0) = 1$, we have $\max\{|G_U(f)| : \|f\|_\infty \leq 1\} = 1$. Now consider the continuous function $h_U : K \to [0,1]$, given by

$$h_U(x) = \begin{cases} 0 & \text{if } x \in s_U^{-1}([7/8, 1]), \\ 1 & \text{if } x \in s_U^{-1}([0, 3/4]). \end{cases}$$

If $g_{1U}(x) > 0$, then $s_U(x) < 1/2 < 3/4$, and so $h_U(x) = 1$. As a consequence, $h_U g_{1U} = g_{1U}$ and therefore $G_U(f_0 h_U) = P(f_0 h_U g_{1U} + f_0 g_{2U}) = P(f_0) = 1$. Note that $\|f_0 h_U\|_\infty < 1$. By the maximum modulus theorem G_U is a constant polynomial on $C_Z(K)$. Hence, for $1 \leq k \leq n$, all k-homogeneous polynomials in the above representation of the polynomial G_U must be 0. In particular,

$$P(fg_{1U} + f_0 g_{2U}) = P(f_0 g_{2U}) = G_U(0) = 1 \text{ and } P(fg_{1U}) = 0 \tag{1}$$

for all $f \in C_Z(K)$. For each $f \in C_Z(K)$ and $U \in \mathcal{E}$ define $f_U = fg_{1U} + f(x_0) f_0 g_{2U}$. Since $f_U(x) - f(x) = 0$ for all $x \in K \backslash U$ and all $U \in \mathcal{E}$, it is easy to check that the net $(f_U)_{U \in \mathcal{E}}$ is $\|\cdot\|_\infty$-convergent to f. If $f(x_0) = 0$, then (1) and the definition of f_U imply that $P(f) = 0 = (\delta_{x_0})^m(f)$. If $f(x_0) \neq 0$, then it follows from (1) that $P(f_U) = f(x_0)^m P(\frac{1}{f(x_0)} fg_{1U} + f_0 g_{2U}) = f(x_0)^m = (\delta_{x_0})^m(f)$. Hence $P(f) = \delta_{x_0}^m(f)$ for all $f \in C_Z(K)$.

(ii) Let X be a locally compact set. If we denote its Alexandroff compactification by $\tilde{X}$, then $C_0(X) = C_{\{\infty\}}(\tilde{X})$, and the conclusion follows from (1). □

Corollary 2.2. *For every positive integer n,*

$$\begin{aligned} \mathrm{sm}^{(n)}(c_0) = \{(\lambda_i) \in c_0 : & \text{ there exists } i_0 \text{ such that} \\ & |\lambda_{i_0}| = 1 \text{ and } |\lambda_i| < 1 \text{ for } i \neq i_0\}, \\ \mathrm{sm}^{(n)}(\ell_\infty) = \{(\lambda_i) \in \ell_\infty : & \text{ there exists } i_0 \text{ such that} \\ & |\lambda_{i_0}| = 1 > \sup\{|\lambda_i| \text{ for } i \neq i_0\}\} \end{aligned}$$

Proof. Since $c_0 = C_0(\mathbb{N})$, where $\mathbb{N}$ is endowed with the discrete topology, and $\ell_\infty = C(\beta\mathbb{N})$, where $\beta\mathbb{N}$ denotes the Stone-Čech compactification of $\mathbb{N}$, the result is immediate from Theorem 2.1. Note also that the results are very well-known for the case $n = 1$ (see, e.g., [9, Example 1.6.b)]), and Theorem 2.1 implies that the sets of smooth points of any order coincide for these spaces. $\square$

We now turn our attention to $\mathrm{sm}^{(n)}(\ell_p)$ for $1 \leq p < \infty$.

Remark. In general, if x_0 is a norm one element of ℓ_p such that some coordinate of x_0 equals 0, then x_0 is not contained in $\mathrm{sm}^{(n)}(\ell_p)$ for $n \geq p$. Indeed, suppose that x_0 is 0 in the j^{th} coordinate and let $\phi \in (\ell_p)^*$ be a norm one functional such that $\phi(x_0) = 1$. Note that ϕ itself must be 0 in the j^{th} coordinate. Then for any $n \geq p$, the n-homogeneous polynomial $P(x) = [\phi(x)]^n + (x_j)^n$ is such that $P(x_0) = 1 = ||P|| = \phi^n(x_0)$. Therefore, since P and ϕ^n are different, x_0 is not a smooth point of order n of B_{ℓ_p}.

Theorem 2.3. *Let $1 \leq p < \infty$ and $n \geq 2$ be a positive integer. Then*

$$\mathrm{sm}^{(n)}(\ell_p) = \begin{cases} \{\lambda e_j : |\lambda| = 1, j \in \mathbb{N}\} & \text{if } 2 \leq n < p, \\ \emptyset & \text{if } p \leq n, \end{cases}$$

and if $k \geq 2$ is a positive integer, then

$$\mathrm{sm}^{(n)}(\ell_p^k) = \begin{cases} \{\lambda e_j : |\lambda| = 1, 1 \leq j \leq k\} & \text{if } 2 \leq n < p, \\ \emptyset & \text{if } p \leq n. \end{cases}$$

In our proof (which will be given only for ℓ_p), we will examine the three cases $p = 1$, $1 < p \leq 2$ and $2 < p < \infty$ separately.

Step 1: $p = 1$. We need to prove that $\mathrm{sm}^{(2)}(\ell_1) = \emptyset$. Let $a,\ b,\ c \in \mathbb{R}$, $|a| < 1$, $|b| < 1$ and $2 < |c| \leq 4$ and $P(x, y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2\ell_1^2)$. We make use of a result in [8, Theorem 2.4] that

$$\|P\| = 1 \text{ if and only if } 4|c| - c^2 = 4(|a + b| - ab).$$

Since $\mathrm{sm}(\ell_1) = \{(a_i) \in S_{\ell_1} : a_i \neq 0 \text{ for all } i\}$, it is enough to show that $\mathrm{sm}^{(2)}(\ell_1)$ does not contain any point $a = (a_i) \in S_{\ell_1}$ such that $a_i \neq 0$ for all i. Choose a positive integer i_0 so that $0 < |a_{i_0}| < 1/2$. Set $c = 2/(1 - |a_{i_0}|)$ and define

$$P(x) = (c - \frac{c^2}{4}) \left(\sum_{i \neq i_0} \mathrm{sgn}(a_i) x_i \right)^2 + c x_{i_0} \left(\sum_{i \neq i_0} \mathrm{sgn}(a_i a_{i_0}) x_i \right)$$

for $x = (x_i) \in \ell_1$, where $\mathrm{sgn}(d) = |d|/d$ if $d \neq 0$ and 1 if $d = 0$. Since

$$\begin{aligned} \|P\| &\leq \sup_{\|x\|\leq 1} (c - \frac{c^2}{4}) \left(\sum_{i\neq i_0} |x_i|\right)^2 + c|x_{i_0}| \left(\sum_{i\neq i_0} |x_i|\right) \\ &= \sup_{|\alpha|+|\beta|\leq 1} (c - \frac{c^2}{4})|\alpha|^2 + c|\alpha||\beta| = 1 \end{aligned}$$

and $P(a) = 1$, we have $\|P\| = 1$.

On the other hand, the functional $\phi_0 \in (\ell_1)^*$ with $\|\phi_0\| = 1 = \phi_0(a)$ has the property that all its coordinates have modulus 1. Since every coefficient of the monomial expansion of $Q = \phi_0^2$ has modulus 1 or 2, P and Q are distinct polynomials. Hence x_0 is not a smooth point of order 2. □

Step 2: $1 < p \leq 2$. We need the following lemma.

LEMMA 2.4. *Let $1 < p \leq 2$. Given $(z_0, w_0) \in \ell_p^2$ with $\|(z_0, w_0)\|_p = 1$ and $2^{-1/p} \leq |z_0| < 1$, there exist $b > 0$ and $c \geq 0$ (with $c = 0$ if and only if $|z_0| = 2^{-1/p}$) such that*

$$P(z, w) = b\left(\left(\frac{|z_0|}{z_0}\right)^2 cz^2 + \frac{|z_0|}{z_0}\frac{|w_0|}{w_0} zw\right)$$

satisfies $\|P\| = 1 = P(z_0, w_0)$.

Proof. Given $c \geq 0$, consider the polynomial $Q_c(z, w) = cz^2 + zw$. Letting $f_c(x) = cx^2 + x(1-x^p)^{1/p}$, it is immediate that $\|Q_c\| = \max\{f_c(x) \ : \ 2^{-1/p} \leq x \leq 1\}$. We denote $(|z_0|, |w_0|) = (x_0, y_0)$. For the case where $x_0 = 2^{-1/p}$ we set $c = 0$ and easily check that $\|Q_0\| = Q_0(2^{-1/p}, 2^{-1/p})$. Taking $b = 1/\|Q_0\|$, we have $\|P\| = 1 = P(z_0, w_0)$.

For the case where $2^{-1/p} < x_0 < 1$ we first claim that for any $c > 0$ there exists a unique $u_c \in (2^{-1/p}, 1)$ such that

$$f_c'(x) \begin{cases} > 0 & \text{if } 2^{-1/p} \leq x < u_c, \\ = 0 & \text{if } x = u_c, \\ < 0 & \text{if } u_c < x < 1. \end{cases}$$

To show this, we observe that on the interval $(2^{-1/p}, 1)$, the functions x^{p-1}, $(1-x^p)^{(1/p)-2}$, $(2x^p - 1)$, and $(1 - x^p)^{(1/p)-1}$ are positive and strictly increasing. Consequently, a computation shows that f_c'' is continuous and strictly decreasing to $-\infty$ on $[2^{-1/p}, 1)$. Now, if $f_c''(2^{-1/p}) \leq 0$, then f' is strictly decreasing on the interval. Since $f_c'(2^{-1/p}) > 0$ and $\lim_{x\to 1} f_c'(x) = -\infty$, the claim follows. On the other hand, if $f_c''(2^{-1/p}) > 0$, then there is a unique $v_c \in (2^{-1/p}, 1)$ such that $f_c''(x) > 0$ for $2^{-1/p} \leq x < v_c$, $f_c''(v_c) = 0$, and $f_c''(x) < 0$ for $v_c < x < 1$. As above, the claim follows.

Taking $c_0 = (1-x_0^p)^{(1/p)-1}(2x_0^p - 1)/(2x_0)$, we have that $f'_{c_0}(x_0) = 0$ and $c_0 > 0$, and so $x_0 = u_{c_0}$ by the claim. This implies that $\|Q_{c_0}\| = f_{c_0}(x_0) = Q_{c_0}(x_0, y_0) > 0$. Letting $b = 1/\|Q_{c_0}\|$ and $c = c_0$, we have $\|P\| = 1 = P(z_0, w_0)$, as required. $\square$

Lemma 2.4 allows us to prove that if $1 < p \leq 2$ then $\mathrm{sm}^{(2)}(\ell_p) = \emptyset$. Indeed, given $a = (a_n)_{n=1}^\infty \in \ell_p$ with $\|(a_n)\| = 1$, we will show that a is not contained in $\mathrm{sm}^{(2)}(\ell_p)$. By the Remark preceding the Theorem, we may assume that $a_n \neq 0$ for all n. Let $Q(x) = \phi(x)^2$, where $\phi \in (\ell_p)^*$ and $\|\phi\| = 1 = \phi(a)$. Given $x = (x_1, x_2, \cdots) \in \ell_p$, we put $x = x_1 e_1 + x'$ with $x' = (0, x_2, x_3, \cdots)$. Let $z_0 = a_1$ and $w_0 = \|a'\|_p$. Clearly $|z_0|^p + w_0^p = \|a\|_p^p = 1$. Assume first that $2^{-1/p} \leq |z_0| < 1$. By Lemma 2.4 there exist real numbers $b > 0$ and $c \geq 0$ such that the polynomial R defined by

$$R(z,w) = b\left(\left(\frac{|z_0|}{z_0}\right)^2 cz^2 + \frac{|z_0|}{z_0} zw \right)$$

satisfies $\|R\| = 1 = R(z_0, w_0)$. Let $\theta \in (\ell_p)^*$ be of norm one such that $\theta(a') = \|a'\|_p = w_0$. Define $P \in \mathcal{P}(^2\ell_p)$ by $P(x) = R(x_1, \theta(x'))$. Clearly $\|P\| = 1 = P(a)$, but $P \neq Q$ because $P(e_2) = 0 \neq Q(e_2)$. Therefore a is not a smooth point of order 2 of B_{ℓ_p}. If $2^{-1/p} \leq |w_0| < 1$, we can apply a similar argument to (w_0, z_0). $\square$

Step 3: $2 < p < \infty$. Given $n \in \mathbb{N}$, let $r_1(t)$ and $r_2(t)$ be distinct generalized Rademacher functions associated with n-th roots of unity, as described in [1]. Given two vectors v_1, v_2 of a complex Banach space E and $P \in \mathcal{P}(^mE)$, we have

$$P(v_1) + P(v_2) = \int_0^1 P(r_1(u)v_1 + r_2(u)v_2)du.$$

Moreover, taking $\beta_1, \beta_2 \in \mathbb{C}$ such that $|\beta_1| = |\beta_2| = 1$, $\beta_1^m P(v_1) = |P(v_1)|$ and $\beta_2^m P(v_2) = |P(v_2)|$, we have

$$|P(v_1)| + |P(v_2)| = \int_0^1 P(\beta_1 r_1(u)v_1 + \beta_2 r_2(u)v_2)du.$$

(For details see [11, Lemma 1.57].) Again we need a lemma.

Lemma 2.5. *Let n be a positive integer with $n < p$.*

(a) Let $P : \ell_p^2 \to \mathbb{C}$ be an n-homogeneous polynomial of the form $P(z,w) = z^n + bw^n$ with $b \geq 0$. Then $\|P\| = 1$ if and only if $b = 0$.

(b) Let $P(z,w)$ be an n-homogeneous polynomial on ℓ_p^2. If $\|P\| = 1 = P(1,0)$, then $P(z,w) = z^n$.

Proof. (a) Applying Lagrange multipliers to find the maximum of $P(s,t)$ on the compact set $K = \{(s,t) \in \mathbb{R}^2 : s \geq 0, t \geq 0, s^p + t^p \leq 1\}$, we get $\|P\| = (1 + b^{\frac{p}{p-n}})^{\frac{p-n}{p}}$. Hence $\|P\| = 1$ if and only if $b = 0$.

(b) We write $P(z,w) = a_0 z^n + a_1 z^{n-1} w + \ldots + a_{n-1} z w^{n-1} + a_n w^n$. Clearly $a_0 = 1$. First we prove that $a_n = 0$. To do this we define the polynomial R by $R(z,w) = z^n + |a_n| w^n$. By the remark preceding this lemma, given $(z,w) \in \mathbb{C}^2$, there exist β_1 and β_2 in $\mathbb{C}$ such that $|\beta_1| = |\beta_2| = 1$ and

$$\begin{aligned} |R(z,w)| &\leq |z|^n + |a_n||w|^n = |P(z,0)| + |P(0,w)| \\ &= \int_0^1 P(\beta_1 r_1(u)(z,0) + \beta_2 r_2(u)(0,w))du \\ &= \int_0^1 P((\beta_1 r_1(u)z, \beta_2 r_2(u)w))du. \end{aligned}$$

Since $\|(\beta_1 r_1(u)z, \beta_2 r_2(u)w)\|_p = \|(z,w)\|_p$ for all (z,w), we get $\|R\| \leq \|P\| = 1$. Since $R(1,0) = 1$, $a_n = 0$ by part (a).

Now, to apply induction, take $0 \leq k \leq n-2$ and assume that $a_n = a_{n-1} = \cdots = a_{n-k} = 0$. We define $\tilde{P}(z,w) = z^{n-k-1} + a_1 z^{n-k-2} w + \cdots + a_{n-k-1} w^{n-k-1}$ and $\tilde{R}(z,w) = z^{n-k-1} + |a_{n-k-1}| w^{n-k-1}$. Given $(z,w) \in \mathbb{C}^2$ with $|z|^p + |w|^p = 1$, there exist $\gamma_1, \gamma_2 \in \mathbb{C}$ with $|\gamma_1| = |\gamma_2| = 1$ such that

$$\begin{aligned} |z|^{n-k-1} + |a_{n-k-1}||w|^{n-k-1} &= |\tilde{P}(z,0)| + |\tilde{P}(0,w)| \\ &= \int_0^1 \tilde{P}((\gamma_1 r_1(u)z, \gamma_2 r_2(u)w))du. \end{aligned}$$

Since

$$\begin{aligned} \left| z^n + |a_{n-k-1}| w^{n-k-1} z^{k+1} \right| &\leq \int_0^1 |z|^{k+1} \tilde{P}((\gamma_1 r_1(u)z, \gamma_2 r_2(u)w))du \\ &\leq \int_0^1 \left| (\gamma_1 r_1(u)z)^{k+1} \tilde{P}((\gamma_1 r_1(u)z, \gamma_2 r_2(u)w)) \right| du \\ &= \int_0^1 |P((\gamma_1 r_1(u)z, \gamma_2 r_2(u)w))| \, du \leq \|P\| = 1, \end{aligned}$$

the polynomial $Q(z,w) = z^n + |a_{n-k-1}| w^{n-k-1} z^{k+1}$ has norm one. If $|a_{n-k-1}| \geq 1$, then

$$1 = \|Q\| \geq Q(2^{-1/p}, 2^{-1/p}) \geq 2^{1-n/p} > 1,$$

which is a contradiction. Hence $|a_{n-k-1}| < 1$. Taking

$$z_0 = \frac{1}{(1 + |a_{n-k-1}|^{p/(p-n)})^{1/p}}, \quad w_0 = \frac{|a_{n-k-1}|^{1/(p-n)}}{(1 + |a_{n-k-1}|^{p/(p-n)})^{1/p}},$$

we have

$$1 = \|Q\| \geq Q(z_0, w_0) \geq (1 + |a_{n-k-1}|^{p/(p-n)})^{(p-n)/p}.$$

Hence $a_{n-k-1} = 0$, which completes the proof of the lemma. $\square$

We can now complete the proof of Theorem 2.3. We need to check that if $2 < p < \infty$, then

$$\mathrm{sm}^{(n)}(\ell_p) = \begin{cases} \{\lambda e_j : |\lambda| = 1, j \in \mathbb{N}\} & \text{if } 2 \le n < p, \\ \emptyset & \text{if } p \le n. \end{cases}$$

First, we discuss the subcase $2 \le n < p$. Given $a = (a_i) \in S_{\ell_p}$, define

$$P(x) = \sum_i (\mathrm{sgn}(a_i))^p a_i^{p-n} x_i^n$$

for $x = (x_i) \in \ell_p$, where the values of $(\mathrm{sgn}(a_i))^p$ and a_i^{p-n} are taken for the principal branch of $\log z$. Then $P(a) = \|a\|_p^p = 1$ and by Hölder's inequality we get

$$\|P\| \le \sup_{\|x\|_p \le 1} \left(\sum_i |a_i|^p \right)^{(p-n)/p} \|x\|_p^n \le 1;$$

hence $\|P\| = 1 = P(a)$.

Let us consider the case where there are $i \ne j$ such that $a_i \ne 0 \ne a_j$. As usual, let $\phi_0 \in (\ell_p)^*$ be the norm one functional which attains its norm at x_0. Since ϕ_0 has nonzero i^{th} and j^{th} coordinates, all coefficients of $x_i^k x_j^{n-k}$, $k = 1, \cdots, n$, in the monomial expansion of ϕ_0^n must be nonzero. Hence P and ϕ_0^2 are distinct polynomials and a is not a smooth point of order n.

Suppose that there is only one i such that $a_i \ne 0$. Without loss of generality, we may assume $a = e_1$, so that the corresponding norm-attaining polynomial is given by $P(x) = x_1^n$. Suppose that $Q \in \mathcal{P}({}^n\ell_p)$ satisfies $\|Q\| = 1 = Q(e_1)$. Using the same notation as in the proof of Step 2 of Theorem 2.3, we have

$$Q(x) = Q(x_1 e_1 + x') = x_1^n + \sum_{k=1}^{n} \binom{n}{k} x_1^{n-k} \overset{\vee}{Q}(e_1^{(n-k)}, x'^{(k)}),$$

where $\overset{\vee}{Q}$ is the symmetric multilinear form associated to Q. For each $k = 1, \cdots, n$, let $Q_k(x) = \overset{\vee}{Q}(e_1^{(n-k)}, x'^{(k)}) \in \mathcal{P}({}^k\ell_p)$. We claim that $Q_k = 0$ for each k, $k = 1, \cdots, n$. Otherwise, there is y', $\|y'\|_p = 1$, with its first coordinate zero such that $Q_j(y') \ne 0$, for some j, $1 \le j \le n$. Define $R(z, w) = Q(ze_1 + wy')$ for all $(z, w) \in \mathbb{C}^2$. Since $\|ze_1 + wy'\|_p = \|(z, w)\|_p$, we have $\|R\| \le \|Q\| = 1$ and $R(1, 0) = 1$. By Lemma 2.5(b), $R(z, w) = z^n$. Hence $\overset{\vee}{Q}(e_1^{(n-k)}, y'^{(k)}) = Q_k(y') = 0$ for all k, $1 \le k \le n$, which is a contradiction. Therefore, for $2 \le n < p$, $\mathrm{sm}^{(n)}(\ell_p) = \{\lambda e_j : |\lambda| = 1, j \in \mathbb{N}\}$.

It remains to show that $\mathrm{sm}^{(n)}(\ell_p) = \emptyset$ for $2 < p \le n$. To this end, it is enough to prove that e_1 is not a smooth point of order n, because $\mathrm{sm}^{(2)}(\ell_p) = \{\lambda e_j : |\lambda| = 1, j \in \mathbb{N}\}$. This follows immediately from the remark preceding the statement of the theorem. $\square$

THEOREM 2.6. *We have* $\mathrm{sm}^{(n)}(L_p[0, 1]) = \emptyset$ *for* $1 \le p \le \infty$ *and* $n \ge 2$.

Proof. It is enough to show that no function $f \in L_p[0,1]$ of norm one is contained in $\mathrm{sm}^{(2)}(L_p[0,1])$. Note that there is nothing to prove if $p = \infty$, since $\mathrm{sm}(L_\infty[0,1]) = \emptyset$.

We begin by proving the case where $p = 1$. Choose a measurable subset D of $[0,1]$ so that

$$0 < \int_D |f(x)|dx < 1/2.$$

Let $c = 2/(1 - \int_D |f(x)|dx)$. Clearly $2 < c < 4$. Choose $\varphi \in L_1[0,1]^* = L_\infty[0,1]$ so that $\|\varphi\| = 1 = \varphi(f)$. Define $P,\ Q \in \mathcal{P}(^2L_1[0,1])$ by

$$P(h) = (c - \frac{c^2}{4})\big[\varphi(h \cdot \chi_D)\big]^2 + c\varphi(h \cdot \chi_D)\varphi(h \cdot \chi_{[0,1]\setminus D}),$$

and $Q(h) = \big[\varphi(h)\big]^2$. As in Step 1 of the proof of Theorem 2.3 we have that $\|P\| = 1 = P(f)$. Clearly $\|Q\| = 1 = Q(f)$ and $P \neq Q$, because $P(f \cdot \chi_{[0,1]\setminus D}) = 0 \neq Q(f \cdot \chi_{[0,1]\setminus D})$. Hence $\mathrm{sm}^{(2)}(L_1[0,1]) = \emptyset$.

Now we consider the case where $1 < p \leq 2$. Let $D \subset [0,1]$ be a measurable set with $2^{-1/p} \leq (\int_D |f(x)|^p dx)^{1/p} < 1$, and let $z_0 = (\int_D |f(x)|^p dx)^{1/p}$, $w_0 = (\int_{[0,1]\setminus D} |f(x)|^p dx)^{1/p}$. Clearly $2^{-1/p} \leq z_0 < 1$, and $\|(z_0, w_0)\|_p$=1. By Lemma 2.4, there exist real numbers $b > 0$ and $c \geq 0$ such that $R(z,w) = b(cz^2 + zw) \in \mathcal{P}(^2\ell_p^2)$ satisfies $\|R\| = R(z_0, w_0) = 1$. Consider $\varphi \in L_p(D)^*$ and $\phi \in L_p([0,1] \setminus D)^*$ with $\|\varphi\| = \|\phi\| = 1$ and

$$\varphi(f \cdot \chi_D) = \|f \cdot \chi_D\|_p \text{ and } \phi(f \cdot \chi_{[0,1]\setminus D}) = \|f \cdot \chi_{[0,1]\setminus D}\|_p.$$

Define $P \in \mathcal{P}(^2L_p[0,1])$ by $P(h) = R(\varphi(h\chi_D), \phi(h\chi_{[0,1]\setminus D}))$, for all $h \in L_p([0,1])$. Clearly $\|P\| = 1 = P(f)$. Let $\eta \in L_p[0,1]^*$ with $\|\eta\| = 1 = \eta(f)$. Define $Q(h) = \big[\eta(h)\big]^2$ for all $h \in L_p[0,1]$. Then $\|Q\| = 1 = Q(f)$. In order to show that $Q \neq P$, we consider the two dimensional subspace $Y = \{zf\chi_D + wf\chi_{[0,1]\setminus D} \ :\ z, w \in \mathbb{C}\}$. If $P = Q$ on Y, then b must be zero, which is a contradiction. Therefore $\mathrm{sm}^{(2)}(L_p[0,1]) = \emptyset$.

Finally we prove the result when $2 < p < \infty$. Let $g(x) = \mathrm{sgn}(f(x))$. Define $P \in \mathcal{P}(^2L_p[0,1])$ by $P(h) = \int_{[0,1]} (g(x))^p (f(x))^{p-2} (h(x))^2 dx$. Then

$$\begin{aligned} \|P\| &\leq \sup_{\|h\|_p = 1} \int |f(x)|^{p-2} |h(x)|^2 dx \\ &\leq \sup_{\|h\|_p = 1} \left(\int |f(x)|^p dx \right)^{\frac{p-2}{p}} \left(\int |h(x)|^p dx \right)^{\frac{2}{p}} = 1, \end{aligned}$$

and $P(f) = 1$; hence $\|P\| = 1$. Let $\varphi \in L_p[0,1]^*$ such that $\|\varphi\| = 1 = \varphi(f)$. Define $Q \in \mathcal{P}(^2L_p[0,1])$ by $Q(h) = \big[\varphi(f)\big]^2$. In fact, we have $Q(h) = \big(\int (g(x))^p (f(x))^{p-1} h(x) dx\big)^2$. We can see that $\|Q\| = 1 = Q(f)$ and $P \neq Q$.

Indeed, let D be a measurable subset of $[0,1]$ and $\|f \cdot \chi_D\|_p = 1/2$. Define

$$h(x) = \begin{cases} f(x) & \text{if } x \in D, \\ -f(x) & \text{if } x \in [0,1] \setminus D. \end{cases}$$

Then $P(h) = 1$, but $Q(h) = 0$. Hence $\operatorname{sm}^{(2)}(L_p[0,1]) = \emptyset$. $\square$

3. LUR polynomials and multilinear forms.

In this section, the Banach space E may be assumed to be *either* real *or* complex. For $n, m \in \mathbb{N}$, $A \in \mathcal{L}(^nE)$ and $B \in \mathcal{L}(^mE)$ define

$$A \cdot B(x_1, \dots, x_{n+m}) = A(x_1, \dots, x_n)B(x_{n+1}, \dots, x_{n+m}),$$

where $x_1, \dots, x_{n+m} \in E$. It is obvious that $A \cdot B \in \mathcal{L}(^{n+m}E)$ and that $\|A \cdot B\| = \|A\|\|B\|$. We first show that the multilinear forms and homogeneous polynomials on E which are locally uniformly rotund are generated by locally uniformly rotund linear functionals. To do so we need the following lemma, which is easy to check.

Lemma 3.1. *Let m and n be positive integers, $A \in S_{\mathcal{L}(^nE)}$ and $B \in S_{\mathcal{L}(^mE)}$. If $A \cdot B \in \operatorname{lur} B_{\mathcal{L}(^{n+m}E)}$, then $A \in \operatorname{lur} B_{\mathcal{L}(^nE)}$ and $B \in \operatorname{lur} B_{\mathcal{L}(^mE)}$.*

Proposition 3.2. *If $A \in \operatorname{lur} B_{\mathcal{L}(^mE)}$ for a positive integer m, then $A = \prod_{k=1}^m f_k$ for some $f_k \in \operatorname{lur} B_{E^*}$.*

Proof. Let $\{(x_{1j}, \dots, x_{mj})\}_{j=1}^\infty$ in E^m be such that $\|x_{1j}\| = \cdots = \|x_{mj}\| = 1$ for all j and $\lim_j A(x_{1j}, \dots, x_{mj}) = 1$. Let $\{f_{kj}\}$ be a sequence in B_{E^*} such that $\|f_{kj}\| = f_{kj}(x_{kj}) = 1$ for $j \in \mathbb{N}$ and $k = 1, \dots, m$. Clearly $\lim_j \|A + \prod_{k=1}^m f_{kj}\| = 2$. Since $A \in \operatorname{lur} B_{\mathcal{L}(^mE)}$, we get $A(x_1, \dots, x_m) = \lim_j \prod_{k=1}^m f_{kj}(x_k)$ for any $x_1, \dots, x_m \in E$. Since the set $\{f_{kj} : j = 1, 2, \cdots\}$ is relatively weak-star compact for each k, $k = 1, \cdots, m$, there are $f_1, \dots, f_m$ in B_{E^*} and a subnet $\{(f_{1j_\beta}, \dots, f_{mj_\beta})\}$, such that $f_k(x) = \lim_{j_\beta} f_{kj_\beta}(x)$ for each k, $k = 1, \dots, m$, and each $x \in E$. Hence $A(x_1, \dots, x_m) = \prod_{k=1}^m f_k(x_k)$ for any $x_1, \dots, x_m \in E$. Since $A \in \operatorname{lur} B_{\mathcal{L}(^mE)}$, it follows from Lemma 3.1 that each $f_k \in \operatorname{lur} B_{E^*}$. $\square$

Lemma 3.3. *Let m and n be positive integers and let $P \in \mathcal{P}(^mE)$. If $P^{n+1} \in \operatorname{lur} B_{\mathcal{P}(^{(n+1)m}E)}$, then $P^n \in \operatorname{lur} B_{\mathcal{P}(^{nm}E)}$.*

Proof. Let $\{Q_j\}$ be a sequence in $B_{\mathcal{P}(^{nm}E)}$ such that $\lim_j \|Q_j + P^n\| = 2$. It is easy to see that see that $\lim_j \|PQ_j + P^{n+1}\| = 2$. Therefore $\|P(Q_j - P^n)\| = \|PQ_j - P^{n+1}\| \to 0$. The proof follows by an application of the fact ([2, Theorem 3 and Proposition 9]) that given $r, s \in \mathbb{N}$ there exists an $M_{r,s} > 0$ such that for all $P_1 \in \mathcal{P}(^rE)$ and $P_2 \in \mathcal{P}(^sE)$, we have $\|P_1P_2\| \geq M_{r,s}\|P_1\|\|P_2\|$. $\square$

Proposition 3.4. *If $P \in \operatorname{lur} B_{\mathcal{P}(^m E)}$ for a positive integer m, then $P = f^m$ for some $f \in \operatorname{lur} B_{E^*}$.*

Proof. Let $\{x_j\}$ be a sequence in B_E such that $\|x_j\| = 1$ and $\lim_j P(x_j) = 1$. Let $\{f_j\}$ be a sequence in B_{E^*} such that $\|f_j\| = f_j(x_j) = 1$. Clearly $\lim_j \|P + f_j^m\| = 2$ so that $P(x) = \lim_j f_j^m(x)$ for all $x \in E$, since P is locally uniformly rotund. Since $\{f_j\}_{j=1}^\infty$ is relatively weak-star compact, there are $f \in E^*$ and a subnet $\{f_{j_\beta}\}$ such that $f(x) = \lim_\beta f_{j_\beta}(x)$ for all $x \in E$. Clearly $P = f^m$. Since $P \in \operatorname{lur} B_{\mathcal{P}(^m E)}$, Lemma 3.3 implies that $f \in \operatorname{lur} B_{E^*}$. □

Corollary 3.5. *Let E be a reflexive Banach space and let m be a positive integer. If $P \in \operatorname{lur} B_{\mathcal{P}(^m E)}$, then P is norm attaining at some $x_0 \in \operatorname{sm}^{(m)}(E)$.*

Proof. If $P \in \operatorname{lur} B_{\mathcal{P}(^m E)}$, then $P = f^m$ for some $f \in \operatorname{lur} B_{E^*}$ by Proposition 3.4. Since E is reflexive, f and P are norm attaining at some $x_0 \in S_E$. If x_0 were not a smooth point of order m of B_E, then there would exist $Q \in \mathcal{P}(^m E)$ such that $P \neq Q$ and $\|Q\| = 1 = Q(x_0)$. This would imply that P is not in $\operatorname{lur} B_{\mathcal{P}(^m E)}$, which is a contradiction. □

We observe that none of the converses of 3.2, 3.4 and 3.5 hold in general.

Corollary 3.6. *(1) Let E be any of the following spaces: c_0, ℓ_1, ℓ_∞, $L_p[0,1]$, $1 \le p \le \infty$; ℓ_∞^k, ℓ_1^k for $k \ge 2$; $C(K)$ for any compact set K with at least two points. Then $\operatorname{lur} B_{\mathcal{L}(^m E)} = \emptyset$ and $\operatorname{lur} B_{\mathcal{P}(^m E)} = \emptyset$ for each positive integer $m \ge 2$.*

(2) If E is a real reflexive Banach space of dimension greater than or equal to 2, then $\operatorname{lur} B_{\mathcal{P}(^m E)} = \emptyset$ for each positive integer $m \ge 2$.

Proof. Almost everything follows from 3.2, 3.4, 3.5, and the remarks in the Introduction. The case $L_p[0,1]$, $1 < p < \infty$, follows from 3.5 and 2.6. For the sake of completeness we prove that $\operatorname{lur} B_{C(K)^*} = \emptyset$. First we note that $\lambda\delta_t$ is not a locally uniformly rotund point of $B_{C(K)^*}$ for $t \in K$ and $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Indeed, choose $t' \in K$ distinct from t and apply the Tietze extension theorem to conclude $\|\lambda\delta_t + \delta_{t'}\| = 2$, but $\lambda\delta_t \neq \delta_{t'}$. Suppose that $\varphi \in S_{C(K)^*}$ is not of the form $\lambda\delta_t$ for any $t \in K$ and $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Choose a sequence (f_j), $f_j \in S_{C(K)}$, such that $\varphi(f_j) \to 1$. For each j choose $t_j \in K$ and $\lambda_j \in \mathbb{C}$, $|\lambda_j| = 1$, so that $\lambda_j f_j(t_j) = 1$. Clearly $\|\varphi + \lambda_j\delta_{t_j}\| \to 2$ as $j \to \infty$, but $(\lambda_j\delta_{t_j})$ does not converge to φ. Otherwise, letting $\lambda \in \mathbb{C}$ and $t \in K$ be limit points of (λ_j) and (t_j) respectively, it would follow that $\varphi = \lambda\delta_t$, which is a contradiction. □

Theorem 3.7. *Let $1 < p < \infty$ and $m \ge 2$ and $k \ge 2$ be positive integers. For the complex Banach spaces ℓ_p and ℓ_p^k,*

$$\operatorname{lur} B_{\mathcal{P}(^m \ell_p)} = \begin{cases} \{\lambda x_j^m : |\lambda| = 1, j \in \mathbb{N}\} & \text{if } 2 \le m < p, \\ \emptyset & \text{if } p \le m, \end{cases}$$

and

$$\operatorname{lur} B_{\mathcal{P}(^m\ell_p^k)} = \begin{cases} \{\lambda x_j^m : |\lambda| = 1, 1 \le j \le k\} & \text{if } 2 \le m < p, \\ \emptyset & \text{if } p \le m. \end{cases}$$

Proof. We give the proof only for ℓ_p. From Theorem 2.3 and Corollary 3.5 the only part to prove is that for $2 \le m < p$, $\operatorname{lur} B_{\mathcal{P}(^m\ell_p)} = \{\lambda x_j^m : |\lambda| = 1, j \in \mathbb{N}\}$. Applying Theorem 2.3 and Corollary 3.5 again, it is enough to show that $x_1^m \in \operatorname{lur} B_{\mathcal{P}(^m\ell_p)}$. Assume that there exists a sequence $(P_h)_{h=1}^\infty \in \mathcal{P}(^m\ell_p)$, $\|P_h\| = 1$, such that $\lim_{h\to\infty} \|x_1^m + P_h\| = 2$.

We claim that the sequence (P_h) converges to x_1^m in $\mathcal{P}(^m\ell_p)$. To show this we will use the same notation as in the proof of Theorem 2.3, obtaining for a polynomial $P \in \mathcal{P}(^m\ell_p)$ the representation

$$P(x) = P(x_1e_1 + x') = \sum_{k=0}^m \binom{m}{k} x_1^{m-k} \overset{\vee}{P}(e_1^{(m-k)}, x'^{(k)}).$$

Passing to a subsequence, we can choose a sequence $(c_he_1 + d_h) \in S_{\ell_p}$ so that $c_h \in \mathbb{C}$, the first coordinate of $d_h \in \ell_p$ is zero and $|c_h^m + P_h(c_he_1 + d_h)| > 2 - 1/h$. Now we consider the sequence $(R_h) \subset \mathcal{P}(^2\ell_p^2)$ defined by

$$R_h(s,t) = \begin{cases} P_h(se_1 + t\frac{d_h}{\|d_h\|_p}) = \sum_{k=0}^m \binom{m}{k} \ s^{m-k}t^k \overset{\vee}{P}\left(e_1^{(m-k)}, \frac{d_h}{\|d_h\|_p}^{(k)}\right) & \\ & \text{if } d_h \neq 0, \\ P_h(se_1) = s^m P_h(e_1) & \text{if } d_h = 0. \end{cases}$$

Since $\|se_1 + t\frac{d_h}{\|d_h\|_p}\|_p = \|(s,t)\|_p$ for all $(s,t) \in \mathbb{C}^2$ and $d_h \neq 0$, we can see that $\|R_h\| \le \|P_h\| = 1$. Thus a subsequence of (R_h), again denoted by (R_h), converges to R in the finite dimensional Banach space $\mathcal{P}(^m\ell_p^2)$. Since

$$\begin{aligned} 2 - \frac{1}{h} &< |c_h^m + P_h(c_he_1 + d_h)| = |c_h^m + R_h(c_h, \|d_h\|_p)| \\ &\le \|s^m + R_h\| \le \|s^m\| + \|R_h\| \le 2 \end{aligned}$$

for all h, we have $\|R\| \le 1$ and $\|s^m + R\| = 2$. Hence $\|R\| = 1 = R(1,0)$. By Lemma 2.5 (b), $R(s,t) = s^m$. Since (R_h) converges to R in $\mathcal{P}(^m\ell_p^2)$, we get $\lim_{h\to\infty} P_h(e_1) = 1$. Fix k, $1 \le k \le m$, and define $Q_h \in \mathcal{P}(^k\ell_p)$ by

$$Q_h(x) = \overset{\vee}{P}_h(e_1^{(m-k)}, x'^{(k)}).$$

Then the sequence (Q_h) of k-homogeneous polynomials on ℓ_p converges to 0 in $\mathcal{P}(^k\ell_p)$, which implies that the sequence (P_h) converges to x_1^m in $\mathcal{P}(^m\ell_p)$. Indeed, if it did not converge to 0, then there would be a number $\delta > 0$ and, passing to a subsequence, a sequence (y_h'), $\|y_h'\| = 1$, with its first coordinate zero, such that $Q_h(y_h') > \|Q_h\|/2 > \delta/2$. Define $\tilde{R}_h(s,t) = P_h(se_1 + ty_h')$ for all $(s,t) \in \mathbb{C}^2$. Clearly $\|\tilde{R}_h\| \le \|P_h\| = 1$. Thus a subsequence $(\tilde{R}_h)$ converges to $\tilde{R}$ in $\mathcal{P}(^k\ell_p^2)$, which implies that $\tilde{R}(1,0) = \lim_{h\to\infty} P_h(e_1) = 1$.

By Lemma 2.5 (b), $\tilde{R}(s,t) = s^m$. Since the coefficient of $s^{m-k}t^k$ in the monomial expansion of $\tilde{R}_h(s,t)$ is $Q_h(y'_h)$, the sequence $(Q_h(y'_h))$ converges to 0, which is a contradiction. $\square$

Let $\mathcal{P}(E)$ denote the normed space of scalar-valued continuous polynomials on E endowed with norm $\|P\| = \sup_{\|x\|\leq 1} |P(x)|$. The space $\mathcal{P}(E)$ is not a Banach space, but it is worth observing that the balls of both it and its completion $\mathcal{A}(B_E)$ (the algebra of uniformly continuous holomorphic functions on the interior of B_E) contain no locally uniformly rotund points. In fact, let P be a continuous polynomial of degree k and $\|P\| = 1$. Then there is a sequence (x_n) in S_E such that $|P(x_n)| \to 1$ as $n \to \infty$. By the Hahn-Banach theorem there is a sequence (φ_n) in S_{E^*} such that $\varphi_n(x_n) = 1$ for all n. Let m be a positive integer greater than k. It is clear that $\|P + \frac{P(x_n)}{|P(x_n)|}(\varphi_n)^m\| \to 2$. However, $\|P - \frac{P(x_n)}{|P(x_n)|}\varphi_n^m\|$ does not converge to 0. Otherwise, $P(x) = \lim_n \frac{P(x_n)}{|P(x_n)|}(\varphi_n)^m(x)$ for each $x \in E$. By the Banach-Steinhaus theorem for polynomials (see [4]) P must be an m-homogeneous polynomial, which is a contradiction. The proof for $\mathcal{A}(B_E)$ follows easily.

Acknowledgements. This paper was completed while the second and the fourth authors were Visiting Professors in the Department of Mathematical Sciences at Kent State University during the 1998-99 academic year. Both authors express their thanks to this Department for its hospitality.

References

[1] R.M. Aron and J. Globevnik, *Analytic functions on c_0*, Rev. Mat. (Madrid) **2** (1989), 27-34.

[2] C. Benítez, Y. Sarantopoulos and A. Tonge, *Lower bounds for norms of products of polynomials*, Math. Proc. Cambridge Philos. Soc. **124** (1998), 395-408.

[3] C. Boyd and R. Ryan, *Geometric theory of spaces of integral polynomials and symmetric tensor products*, *J. Funct. Anal.* **179** (2001), 18–42.

[4] S.B. Chae, *Holomorphy and Calculus in Normed Spaces*, Marcel Dekker, New York, 1985.

[5] Y.S. Choi and S.G. Kim, *The unit ball of $\mathcal{P}(^2\ell_2^2)$*, Arch. Math. **71** (1998), 472–480.

[6] ———, *Smooth points of the unit ball of the space $\mathcal{P}(^2\ell_1)$*, Results Math. **36** (1999), 26–33.

[7] ———, *Extreme polynomials on c_0*, Indian J. Pure Appl. Math. **29** (1998), 983–989.

[8] Y.S. Choi, H. Ki and S.G. Kim, *Extreme polynomials and multilinear forms on ℓ_1*, J. Math. Anal. Appl. **228** (1998), 467–482.

[9] R. Deville, G. Godefroy and V. Zizler, *Smoothness and renorming in Banach spaces*, Pitman Monogr. Survey Pure Appl. Math., vol. 64, Longman, Harlow, 1993.

[10] J. Diestel, *Geometry of Banach Spaces – Selected Topics*, Lecture Notes in Math., vol. 485, Springer-Verlag, New York, 1975.

[11] S. Dineen, *Complex analysis on infinite-dimensional spaces*, Springer Monogr. in Math., Springer-Verlag, London, 1999.

[12] K. Floret, *Natural norms on symmetric tensor products of normed spaces*, Note Mat. **17** (1997), 153–188.

[13] G. Köthe, *Topological Vector Spaces I,* Springer-Verlag, New York, 1969.

[14] J. Mujica, *Complex Analysis in Banach Spaces*, vol. 120, North-Holland Math. Stud., Amsterdam, 1986.

Richard M. Aron, Department of Mathematical Sciences, Kent State University, Kent, OH 44240

E-mail address: aron@mcs.kent.edu

Yun Sung Choi, Department of Mathematics, POSTECH, Pohang 790-784, Korea

E-mail address: mathchoi@euclid.postech.ac.kr

Sung Guen Kim, Department of Mathematics, Kyungpook National University, Taegu 702-701, Korea

E-mail address: sgk317@bh.knu.ac.kr

Manuel Maestre, Departamento de Análisis Matemático, Facultad de Matemáticas, Universitat de Valencia, Doctor Moliner 50, 46100 Valencia, Spain

E-mail address: maestre@uv.es