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A NECESSARY CONDITION FOR ESTIMATES FOR THE $\overline{\partial}_b$-COMPLEX

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ABSTRACT. A necessary condition is given for certain L^2 estimates to hold for the boundary Cauchy Riemann operator and the associated Kohn Laplacian on abstract CR manifolds of arbitrary codimension. A circle of ideas is presented related to a priori estimates for linear partial differential operators.

1. Introduction and discussion of the results

Let M be an abstract CR manifold of CR-dimension n and codimension $k \geq 1$ (see, e.g., Shaw and Wang [30] and Section 3 below for terminology). We are interested in intermediate estimates between the subelliptic estimates and certain estimates which correspond to the local solvability in L^2 of the Kohn Laplacian. More precisely, we consider the following estimate:

(1.1) *For a given* $x_0 \in M$ *and every* $\delta > 0$ *there exists an open neighborhood* $\Omega_\delta \subset M$ *of* x_0 *such that* $\|u\|_0 \leq \delta \|\Box_b^{(q)} u\|_0, \quad \forall u \in \mathcal{D}_{(0,q)}(\Omega_\delta)$.

Here $\|\cdot\|_0$ denotes the L^2-norm, $\mathcal{D}_{(0,q)}(\Omega_\delta)$ is the space of smooth $(0,q)$-forms with compact support in Ω_δ and

$$\Box_b^{(q)} = \overline{\partial}_b^{(q-1)} \overline{\partial}_b^{(q-1)^*} + \overline{\partial}_b^{(q)^*} \overline{\partial}_b^{(q)}$$

is the Kohn Laplacian acting on $(0,q)$-forms.

By a classical argument from Functional Analysis (see, e.g., Lemma 4.1 in Hörmander [17]) it follows that the local solvability in L^2 for $\Box_b^{(q)}$ is equivalent to requiring that the estimate in (1.1) is just valid for some fixed δ, whereas the hypoellipticity of $\Box_b^{(q)}$ with a loss of derivatives less than 2 implies (1.1). We therefore give a necessary condition for (1.1) to hold, which reads as follows.

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Let $x_0 \in M$ *and suppose that at some characteristic point* ρ_0 *above* x_0 *the Levi form* $\mathcal{L}(\rho_0)$ *has signature* $(q, n-q)$ *or* $(n-q, q)$, $q \in \{0, \dots, n\}$. *Then* (1.1) *does not hold.*

This is a weak analog of the well known result of Andreotti, Fredricks and Nacinovich [1] concerning the local solvability of $\overline{\partial}_b$. However, it does not seem to us that the method of proof in [1] can be applied in our context, since M is not required to be embeddable here. The proof we present is instead based on some localization techniques used in the study of hypoelliptic operators with multiple characteristics (see [15], Chapter 22 of [18], and [3]).

A problem that remains open for us is whether, in the situation described above, estimate (1.1) fails even for a fixed $\delta > 0$ (this would imply non-solvability in L^2), and also whether a similar result still holds if one allows a loss greater than 2 derivatives (i.e., if the L^2-norm in the left hand side of the estimate in (1.1) is replaced by a Sobolev norm $\|\cdot\|_s$, with $s < 0$). However, as the reader will notice, these possible improvements cannot be obtained directly with our approach, which is based on the study of a class of second order systems by looking just at the terms of order 2 and 1, and which therefore does not detect L^2-bounded perturbations (we observe that (1.1) is in fact independent of such perturbations). On the other hand, it is known that, in general, 0-order terms can be decisive. This interesting phenomenon was first put in evidence by Stein in [31] for the Kohn Laplacian on the Heisenberg group. More precisely, the Heisenberg group $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ is a classical example of strictly pseudoconvex CR manifold of the hypersurface type, where the vector bundle of vectors of type (1,0) is generated by

$$Z_j = \frac{\partial}{\partial z_j} + i\overline{z}_j \frac{\partial}{\partial t}, \quad j = 1, \dots, n, \quad (z,t) \in \mathbb{H}_n.$$

It is well known that $\Box_b^{(0)}$ is neither locally solvable, nor hypoelliptic. The result obtained by Stein states that instead, for every $\mu \in \mathbb{C} \setminus \{0\}$, $\Box_b^{(0)} + \mu$ is locally solvable, C^∞-hypoelliptic and analytic hypoelliptic (in this connection see also Folland and Stein [12], Nagel, Ricci and Stein [23], Peloso and Ricci [26], [27] and Treves [32]).

As a consequence of the above condition for (1.1) to hold, we also obtain a necessary condition for an estimate for the $\overline{\partial}_b$-complex, which is weaker than any subelliptic or even semi-maximal estimate:

For a given $x_0 \in M$ *and every* $\delta > 0$ *there exists an open neighborhood* $\Omega_\delta \subset M$ *of* x_0 *such that*

$$\delta^{-1}\|u\|_0^2 \leq \|\overline{\partial}_b u\|_0^2 + \|\overline{\partial}_b^* u\|_0^2, \quad \forall u \in \mathcal{D}_{(0,q)}(\Omega_\delta). \tag{1.2}$$

Precisely, we have the following result.

Let $x_0 \in M$ and suppose that at some characteristic point ρ_0 above x_0 the Levi form $\mathcal{L}(\rho_0)$ has signature $(q, n-q)$ or $(n-q, q)$, $q \in \{0, \dots, n\}$. Then (1.2) *does not hold.*

This result implies at once a necessary condition for any ϵ-subelliptic estimate to hold. Recall that for a given $\epsilon > 0$ we say that an ϵ-subelliptic estimate is satisfied at $x_0 \in M$ in degree $q \in \{0, \dots, n\}$ if there exists an open neighborhood $\Omega \subset M$ of x_0 and a constant $c > 0$ such that

$$(I_\epsilon) \qquad c\|u\|_\epsilon^2 \leq \|\overline{\partial}_b u\|_0^2 + \|\overline{\partial}_b^* u\|_0^2 + \|u\|_0^2, \quad \forall u \in \mathcal{D}_{(0,q)}(\Omega).$$

It is known that (I_ϵ) cannot hold with $\epsilon > 1/2$, and that Kohn's condition $Y(q)$ is *equivalent* to $(I_{1/2})$ (see, e.g., Shaw and Wang [30]; we reobtain this fact in Theorem 3.8 below). However, in general $Y(q)$ is not necessary for (I_ϵ) to hold with $\epsilon < 1/2$. For example, Catlin [5] (see also Diaz [11] and Derridj [8]) proved necessary and sufficient conditions for the (closely related) subellipticity for the $\overline{\partial}$-Neumann problem, when the boundary of the domain is pseudoconvex. Roughly speaking, the main result of [5] states that some subelliptic estimate holds in degree q at the point x_0 if and only if the boundary is not too holomorphically flat at x_0 in degree q (in the sense of D'Angelo, see [7] and the references therein).

Now, since any estimate (I_ϵ), $\epsilon > 0$, implies (1.2), we obtain the following necessary condition, which is very rough if compared with the now mentioned one, but which applies to abstract CR manifolds of arbitrary codimension:

If (I_ϵ) holds at $x_0 \in M$ and in degree $q \in \{0, \dots, n\}$ for some $\epsilon > 0$ and the Levi form is non-degenerate at some characteristic point ρ_0 above x_0 then $Y(q)$ holds true at ρ_0.

The analogous result for subelliptic estimates for the $\overline{\partial}$-Neumann problem was already proved by Derridj in [8] by different techniques. Notice that this necessary condition cannot be obtained from Catlin's result, for if $M \subset \mathbb{C}^{n+1}$ is the boundary of a bounded pseudoconvex domain of $\mathbb{C}^{n+1}$ and the Levi form is non-degenerate at $x_0 \in M$, then the condition $Z(q)$ is automatically satisfied for any $q \in \{1, \dots, n\}$ (in comparing this with the $\overline{\partial}$-Neumann problem we have of course to use the one-sided version $Z(q)$ rather than $Y(q)$).

More generally, as a consequence of the result above we see that the condition $Y(q)$ must still be satisfied at any characteristic point *at which the Levi form is non-degenerate* if an estimate like

$$(1.3) \quad c \sum_{0 \leq j \leq \binom{n}{q}} \left(\|Z_j u_j\|_0^2 + \|\overline{Z}_j u_j\|_0^2\right) \leq \|\overline{\partial}_b u\|_0^2 + \|\overline{\partial}_b^* u\|_0^2 + \|u\|_0^2, \quad \forall u \in \mathcal{D}_{(0,q)}(\Omega)$$

is satisfied, for complex non-vanishing vector fields Z_j in a neighborhood Ω of x_0. It is worth mentioning that, in particular, the so-called maximal estimates, as well as the semi-maximal ones considered by Derridj [9] and by Derridj and Tartakoff [10] imply (1.3).

Results in the spirit of the ones described now will be first proved in Section 2 for a class of systems of linear PDO's considered by Boutet de Monvel and Treves [4], Popivanov [29] (whose principal symbol is a scalar multiple of the identity matrix) and then applied to the analysis of the Kohn Laplacian and the $\overline{\partial}_b$-complex in Section 3.

2. A class of systems with double characteristics

In this section we prove some results for a class of systems considered in [4], [29] that contains, as main example, the Kohn Laplacian associated with a CR structure. In the next section we will interpret these results in the particular case of the Kohn Laplacian in terms of known geometric invariants.

Since one of these auxiliary results may be of some interest in its own right, we describe it now briefly in the context of the general theory of local solvability of linear PDO's. (For simplicity we consider scalar operators.)

Precisely, let X be an open subset of $\mathbb{R}^n$ and $P = P^* \in \mathrm{OP}\, S^2(X)$ be a classical formally self-adjoint and properly supported pseudodifferential operator in X. Hence the symbol p of P has an asymptotic expansion $p(x,\xi) \sim \sum_{j\geq 0} p_{2-j}(x,\xi)$, where the functions $p_{2-j}(x,\xi)$ are positively homogeneous of degree $2-j$ with respect to ξ.

We are concerned with the following estimate:

For every $x_0 \in X$ and every $\delta > 0$ there exists an open neighborhood $\Omega_\delta \subset X$ of x_0 such that

$$\|u\|_0^2 \leq \delta(Pu,u), \quad \forall u \in C_0^\infty(\Omega_\delta), \tag{2.1}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\mathbb{R}^n)$.

Estimate (2.1) is of interest both in regularity theory and in local solvability theory. Indeed, it is *sufficient* for the local solvability of P in L^2 near any point $x_0 \in X$, as follows from the Cauchy-Schwarz inequality and the Hahn-Banach theorem. On the other hand, it is *necessary* for any subelliptic estimate of the type

$$c\|u\|_\epsilon^2 \leq (Pu,u) + \|u\|_0^2, \quad \forall u \in C_0^\infty(\Omega),$$

for some $c > 0$, $\epsilon > 0$, with Ω an open neighborhood of x_0, which corresponds to a loss of $2-2\epsilon$ derivatives.

When $P = L^*L$, for a first order principal type operator L, many results are known. In particular, when L is a *differential* operator, we know that estimate (2.1) is equivalent to Nirenberg-Treves' condition (P), which remains sufficient if L is pseudodifferential of principal type (Beals and Fefferman [2], Hörmander [16]). However, if the condition $(\overline{\Psi})$ is only assumed for a *pseudodifferential* operator L, then estimate (2.1) may fail, even for a fixed δ, as shown by Lerner's counterexample in [19]. At present, necessary and sufficient conditions are not known, even in this special situation where P has the form L^*L; we refer to the paper by Lerner [20] for a survey of the main results in this connection.

Returning to the general case, let $\Sigma = \{(x,\xi) \in T^*X \setminus 0 : p_2(x,\xi) = 0\}$ be the characteristic set of P. Let $p_1^s := p_1 + \frac{i}{2}\langle \partial_\xi, \partial_x \rangle p_2$ be the subprincipal symbol of P, and denote by F_ρ the fundamental matrix (or symplectic Hessian) associated with p_2, defined by

$$\sigma(v, F_\rho w) = \frac{1}{2}\langle \text{Hess } p_2(\rho)\, v, w\rangle, \quad \forall v, w \in T_\rho T^*X, \tag{2.2}$$

where $\sigma = \sum_{j=1}^n d\xi_j \wedge dx_j$ is the canonical symplectic 2-form on T^*X. The definitions of p_1^s and of F_ρ in (2.2) have an invariant meaning at points where p_2 vanishes to second order; see Section 21.5 of [18]. Moreover, if p_2 is nonnegative, then the spectrum of F_ρ consists of the eigenvalue 0 and the eigenvalues $\pm i\mu_j$, with $\mu_j > 0$. One then sets $\mathrm{Tr}^+ F_\rho = \sum_j \mu_j$.

As a consequence of Melin [22], if the lower bound (2.1) holds true, then the following conditions must be satisfied:

$$\begin{cases} p_2(x,\xi) \geq 0, \ \forall (x,\xi) \in T^*X \setminus 0; \\ p_1^s(\rho) + \mathrm{Tr}^+ F_\rho \geq 0, \ \forall \rho \in \Sigma. \end{cases} \tag{2.3}$$

Indeed, the conditions (2.3) are *equivalent* to the so-called Melin inequality:

For every $\delta > 0$ and any compact $K \subset X$ there exists $C_{\delta,K} > 0$ such that

$$(Pu, u) \geq -\delta \|u\|_{1/2}^2 - C_{\delta,K}\|u\|_0^2, \quad \forall u \in C_0^\infty(K). \tag{2.4}$$

This inequality is clearly weaker than (2.1). However, it follows from the results in Popivanov [28] and Nicola [24] that in presence of a good geometry of Σ the conditions (2.3) are in fact sufficient for (2.1) to hold, *provided the symplectic form σ is degenerate when restricted to Σ.*

More precisely, suppose that the characteristic set Σ is a smooth connected manifold with constant symplectic rank (i.e., $\Sigma \ni \rho \mapsto \dim(T_\rho\Sigma \cap T_\rho\Sigma^\sigma)$ is constant), that the canonical one form $\sum_{j=1}^n \xi_j dx_j$ does not vanish identically on $T_\rho\Sigma$ for every $\rho \in \Sigma$ and finally that P is transversally elliptic (i.e., $\mathrm{Ker}\, F_\rho = T_\rho\Sigma$, for every $\rho \in \Sigma$). Then, if $\sigma|_\Sigma$ is degenerate, the conditions (2.3) are sufficient for (2.1) to hold, at least up to a similarity by Fourier integral operators (see also [25] for other related situations where the symplectic form is allowed to degenerate).

Hence, assuming (2.3), the main obstruction to the validity of (2.1) is represented by the points $\rho \in \Sigma$ at which the characteristic set is symplectic (i.e., $T_\rho\Sigma \cap T_\rho\Sigma^\sigma = \{0\}$); this is already clear from the paper by Hörmander [14] (see also Lewy [21]).

We then present a necessary condition for (2.1) to hold, which concerns just those points (more generally, for an operator P which is not transversally elliptic, the points $\rho \in \Sigma$ at which F_ρ is not semi-simple), namely

$$\forall \rho \in \Sigma,\ \operatorname{Ker} F_\rho \cap \operatorname{Im} F_\rho = \{0\} \Longrightarrow p_1^s(\rho) + \operatorname{Tr}^+ F_\rho > 0. \tag{2.5}$$

We observe that in (2.5) no assumption on the characteristic set Σ or on the vanishing order of p_2 at Σ is made.

It is worth mentioning that, when $P = L^*L$ for a first order pseudodifferential operator L with principal symbol $\sigma_1(L)$, (2.5) yields exactly the well-known Hörmander condition, that is, $d\sigma_1(L) \neq 0$ and $\{\operatorname{Re}\sigma_1(L), \operatorname{Im}\sigma_1(L)\} \geq 0$, where $\sigma_1(L) = 0$ (see Theorem 2.3 below). This is essentially a weak version of the already mentioned condition $(\overline{\Psi})$. It was introduced in [14] and allowed a geometric explanation of the nonsolvability of Lewy's operator.

We can now establish our results. We begin with a necessary condition for an estimate like (2.1) when the inner product on the right hand side is replaced by $\|Pu\|^2$. (Since we are only dealing here with the space $L^2(\mathbb{R}^n;\mathbb{C}^N)$, we denote the corresponding norm by $\|\cdot\|$, instead of $\|\cdot\|_0$, and the inner product by $(\cdot,\cdot)$.)

Theorem 2.1. *Let $P = P^* \in \operatorname{OP} S^2(X;\mathbb{C}^N)$ be a $N \times N$ formally self-adjoint system of classical properly supported pseudodifferential operators in X, and assume that its principal symbol has the form $p_2(x,\xi)I$ ($I := \mathrm{Id}_{\mathrm{Mat}_{N\times N}(\mathbb{C})}$). Let*

$$\Sigma = \{(x,\xi) \in T^*X \setminus 0 : p_2(x,\xi) = 0\}$$

be the characteristic set of P, p_1^s its subprincipal symbol and F_ρ, $\rho \in \Sigma$, the fundamental matrix associated with p_2. Let $x_0 \in X$ and suppose that there exist an open neighborhood $\Omega \subset X$ of x_0 and a constant $C > 0$ such that

$$(Pu, u) \geq -C\|u\|^2, \quad \forall u \in C_0^\infty(\Omega;\mathbb{C}^N), \tag{2.6}$$

and that for every $\delta > 0$ there exists an open neighborhood $\Omega_\delta \subset X$ of x_0 such that

$$\|u\| \leq \delta\|Pu\|, \quad \forall u \in C_0^\infty(\Omega_\delta;\mathbb{C}^N).$$

If for some $\rho_0 = (x_0,\xi_0) \in \Sigma$ above x_0 we have

$$\operatorname{Ker} F_{\rho_0} \cap \operatorname{Im} F_{\rho_0} = \{0\}, \tag{2.7}$$

then

$$p_1^s(\rho_0) + \operatorname{Tr}^+ F_{\rho_0} I \quad \textit{is invertible.}$$

Proof. The proof is by contradiction. We begin by observing that from (2.6) and Melin's result [22] (see also Popivanov [29]) it follows that $p_2(x,\xi) \geq 0$ for every $(x,\xi) \in T^*X \setminus 0$, with $x \in \Omega$, and that $p_1^s(\rho_0) + \operatorname{Tr}^+ F_{\rho_0} I \geq 0$, as Hermitian matrix. Hence we may suppose

$$\operatorname{Ker}\left(p_1^s(\rho_0) + \operatorname{Tr}^+ F_{\rho_0} I\right) \neq \{0\}. \tag{2.8}$$

For simplicity we consider the case of an operator with scalar-valued symbol ($N = 1$), and we describe the changes needed to treat the general case at the end of the proof.

We can assume $|\xi_0| = 1$ and also that the symbol p of P is compactly supported with respect to x.

Consider now a real-valued function $\chi \in C_0^\infty(\Omega_\delta)$, with $\chi = 1$ in a neighborhood of x_0. We have

$$\|\chi u\| \leq \delta \|P\chi u\|, \quad \forall u \in \mathcal{S}(\mathbb{R}^n). \tag{2.9}$$

We will test (2.9) on wave packets which are localized in a small conic neighborhood of (x_0, ξ_0) of the type

$$u_t(x) = e^{it^2 x\xi_0}\left(v_1(t(x-x_0)) + t^{-1}v_2(t(x-x_0))\right), \quad t \geq 1, \tag{2.10}$$

where $v_1, v_2 \in \mathcal{S}(\mathbb{R}^n)$ will be chosen later on.

Now, by dominated convergence we have

$$\|\chi u_t\| = t^{-n/2}\left(\|v_1\| + o(1)\right), \quad \text{as } t \to +\infty. \tag{2.11}$$

Moreover, we have

$$P\chi u_t(x) = e^{it^2 x\xi_0}\phi_t(t(x-x_0)), \tag{2.12}$$

where, upon setting p' for the symbol of $P\chi$,

$$\phi_t(y) = (2\pi)^{-n}\int e^{iy\eta}p'(x_0 + y/t, t^2\xi_0 + t\eta)\left(\widehat{v}_1(\eta) + t^{-1}\widehat{v}_2(\eta)\right)\, d\eta. \tag{2.13}$$

In an open neighborhood $\Omega' \subset \Omega$ of x_0 we have $p - p' \in S^{-\infty}(\Omega' \times \mathbb{R}^n)$, so that a Taylor expansion at $(x_0, t^2\xi_0)$ gives

$$p'(x_0 + y/t, t^2\xi_0 + t\eta) = \sum_{r=0}^{2} t^{2-r}p^{(2+r)}(\rho_0; y, \eta) + g_t(y, \eta), \tag{2.14}$$

where

$$p^{(h)}(\rho_0; y, \eta) := \sum_{|\alpha|+|\beta|+2j=h} \frac{1}{\alpha!}\frac{1}{\beta!}\partial_x^\alpha \partial_\xi^\beta p_{2-j}(\rho_0) y^\alpha \eta^\beta, \quad h = 2, 3, 4,$$

and

$$|g_t(y, \eta)| \leq Ct^{-1}\langle y\rangle^5 \langle \eta\rangle^{11}, \ \forall (y, \eta) \in \mathbb{R}^{2n}, \forall t \geq 1. \tag{2.15}$$

Let us verify (2.15). We have

$$
\begin{aligned}
&|g_t(y,\eta)| \\
&\le C_1 \sup_{|\alpha|+|\beta|+2j=5} \sup_{\tau\in[0,1]} t^{|\beta|-|\alpha|}|y|^{|\alpha|}|\eta|^{|\beta|}\left|\left(\partial_x^\alpha\partial_\xi^\beta p'_{2-j}\right)(x_0+\tau y/t, t^2\xi_0+\tau t\eta)\right| \\
&\qquad\qquad + |(p'-p'_2-p'_1-p'_0)(x_0+y/t,t^2\xi_0+t\eta)|) \\
&\le C_2 t^{-1} \sup_{|\alpha|+|\beta|+2j=5} \underbrace{\sup_{\tau\in[0,1]} |y|^{|\alpha|}|\eta|^{|\beta|}\left(\frac{1}{t^2}+\left|\xi_0+\tau\frac{\eta}{t}\right|\right)^{2-j-|\beta|}}_{A_{\alpha\beta}(t,y,\eta)} + C'_2 t^{-1}\langle\eta\rangle.
\end{aligned}
$$

We see that for $|\eta|<t/2$ we have

$$|A_{\alpha\beta}(t,y,\eta)| \le C_3|y|^{|\alpha|}|\eta|^{|\beta|}.$$

For $|\eta|\ge t/2$ we consider two cases: if $2-j-|\beta|\ge 0$, then

$$|A_{\alpha\beta}(t,y,\eta)| \le C_4|y|^{|\alpha|}\langle\eta\rangle^{2-j},$$

whereas if $2-j-|\beta|<0$, then

$$|A_{\alpha\beta}(t,y,\eta)| \le C_5|y|^{|\alpha|}|\eta|^{|\beta|}t^{-4+2j+2|\beta|} \le C_6|y|^{|\alpha|}|\eta|^{2|\beta|-|\alpha|+1}.$$

Hence (2.15) is verified.

Now,

$$\|P\chi u_t\| = t^{-n/2}\|\phi_t\|, \tag{2.16}$$

and by (2.13), (2.14), (2.15) and similar estimates for $\partial_\eta^\beta g_t(y,\eta)$ we can write

$$
\begin{aligned}
\phi_t(y) = t^2 p^{(2)}(\rho_0;y,D)v_1 &+ t\left(p^{(2)}(\rho_0;y,D)v_2 + p^{(3)}(\rho_0;y,D)v_1\right) \\
&+ p^{(4)}(\rho_0;y,D)v_1 + p^{(3)}(\rho_0;y,D)v_2 + r_t(y),
\end{aligned}
\tag{2.17}
$$

where $|r_t(y)|\le C_N t^{-1}\langle y\rangle^{-N}$, for every $y\in\mathbb{R}^n$, $t\ge 1$, $N\in\mathbb{Z}_+$.

We are now going to choose v_1 and v_2 in such a way that the first two terms in (2.17) vanish.

We begin by observing that

$$p^{(2)}(\rho_0;y,D) = Q^w_{\rho_0}(y,D) + p_1^s(\rho_0),$$

where

$$Q_{\rho_0}(y,\eta) = \sigma\left(\begin{bmatrix} y\\ \eta\end{bmatrix}, F_\rho\begin{bmatrix} y\\ \eta\end{bmatrix}\right)$$

denotes the Hessian of $p_2/2$ at ρ_0, and $Q^w_{\rho_0}(y,D)$ its Weyl quantization. Now, by Hörmander's theorem on the symplectic classification of semidefinite quadratic forms (Theorem 21.5.3 of [18]), there exists a linear symplectic map

$\chi : \mathbb{R}^{2n}_{y,\eta} \to \mathbb{R}^{2n}_{z,\zeta}$ such that

$$\left(Q_{\rho_0} \circ \chi^{-1}\right)(z,\zeta) = \sum_{j=1}^{\nu} \mu_j(\rho_0)(z_j^2 + \zeta_j^2) + \sum_{j=\nu+1}^{\nu+l} \zeta_j^2, \tag{2.18}$$

with $l = \dim(\operatorname{Ker} F_{\rho_0} \cap \operatorname{Im} F_{\rho_0})$, $\mu_j(\rho_0) > 0$ and $\sum_{j=1}^{\nu} \mu_j(\rho_0) = \operatorname{Tr}^+ F_{\rho_0}$. By the hypothesis (2.7) we have $l = 0$ (hence the second sum in (2.18) does not appear) and therefore by (2.8)

$$\operatorname{Ker}\left(\left(Q_{\rho_0} \circ \chi^{-1}\right)^w (z,D) + p_1^s(\rho_0)\right) \cap \mathcal{S}(\mathbb{R}^n) = \operatorname{span}\left\{e^{-\sum_{j=1}^{\nu} z_j^2/2}\right\} \otimes \mathcal{S}(\mathbb{R}^{n-\nu})$$

is non-empty. Now, it follows from the Segal Theorem (Theorem 18.5.9 of [18]) that there exists a metaplectic transformation U_{ρ_0} in $L^2(\mathbb{R}^\nu_z)$ (which is therefore, in particular, a unitary transformation in $L^2(\mathbb{R}^\nu)$ and an automorphism of $\mathcal{S}(\mathbb{R}^\nu)$ and of $\mathcal{S}'(\mathbb{R}^\nu)$) such that $Q^w_{\rho_0} = U^*_{\rho_0}(Q_{\rho_0} \circ \chi^{-1})^w U_{\rho_0}$. We observe that, since χ is linear (and not merely affine), the transformation U_{ρ_0} preserves the parity of functions in $L^2(\mathbb{R}^\nu)$ (i.e., it commutes with the involution $L^2(\mathbb{R}^\nu) \ni f(y) \mapsto f(-y) \in L^2(\mathbb{R}^\nu)$), as follows by an inspection of the explit expression of U_{ρ_0} (see, e.g., the proof of Theorem 18.5.9 of [18]).

We therefore define the operators

$$\tilde{p}^{(h)}(\rho_0; z, D) := U_{\rho_0} p^{(h)}(\rho_0; y, D) U^*_{\rho_0}, \quad h = 2, 3,$$

on $\mathcal{S}(\mathbb{R}^n)$, so that in particular

$$\tilde{p}^{(2)}(\rho_0; z, D) = \sum_{j=1}^{\nu} \mu_j(\rho_0)(z_j^2 + D_j^2) + p_1^s(\rho_0).$$

Consider now any *even* function $\tilde{v}_1 \in \mathcal{S}(\mathbb{R}^n) \cap \operatorname{Ker} \tilde{p}^{(2)}(\rho_0; y, D)$, $\tilde{v}_1 \not\equiv 0$. Then in (2.10) we choose $v_1 := U^* \tilde{v}_1$, so that

$$p^{(2)}(\rho_0; y, D) v_1 = U^*_{\rho_0} \tilde{p}^{(2)}(\rho_0; z, D) \tilde{v}_1 = 0. \tag{2.19}$$

It remains to choose v_2. To this end, we first show that

$$\left(p^{(3)}(\rho_0; y, D) w, w\right) \geq 0, \quad \forall w \in \mathcal{S}(\mathbb{R}^n) \cap \operatorname{Ker} p^{(2)}(\rho_0; y, D). \tag{2.20}$$

Indeed, upon setting $w_t := e^{it^2 x \xi_0} w(t(x - x_0))$, it follows as above from (2.6) that

$$\begin{aligned} t^{-n}\|w\|^2 &\leq (\chi P \chi w_t, w_t) \\ &= t^{2-n}\left(p^{(2)}(\rho_0; y, D) w, w\right) \\ &\quad + t^{1-n}\left(p^{(3)}(\rho_0; y, D) w, w\right) + o(t^{1-n}), \text{ as } t \to +\infty. \end{aligned} \tag{2.21}$$

Since $p^{(2)}(\rho_0; y, D) w = 0$, we deduce at once (2.20) from (2.21) after dividing by t^{1-n} and letting $t \to +\infty$. (Similarly, from the fact that P is formally self-adjoint one also sees that the operator $p^{(3)}(\rho_0; y, D)$ is symmetric when restricted to the vector space $\mathcal{S}(\mathbb{R}^n) \cap \operatorname{Ker} p^{(2)}(\rho_0; y, D)$.)

Write $z = (z', z'') \in \mathbb{R}^\nu \times \mathbb{R}^{n-\nu}$ and set $h_0(z') := \pi^{-\nu/4} e^{-|z'|^2/2}$. We now claim that, with our choice of $\tilde{v}_1$, we have

$$\left(\tilde{p}^{(3)}(\rho_0; z, D)\tilde{v}_1, h_0 k\right) = 0, \quad \forall k \in \mathcal{S}(\mathbb{R}^{n-\nu}_{z''}). \tag{2.22}$$

Indeed, since $\tilde{v}_1$, $h_0 k \in \mathcal{S}(\mathbb{R}^n) \cap \operatorname{Ker} \tilde{p}^{(2)}(\rho_0; z, D)$, from (2.20) it follows that

$$0 \leq \left(\tilde{p}^{(3)}(\rho_0; z, D)(\tilde{v}_1 + \lambda h_0 k), \tilde{v}_1 + \lambda h_0 k\right), \quad \forall \lambda \in \mathbb{R}.$$

Then it suffices to expand this inner product and, after dividing by λ, let $\lambda \to 0$, taking into account that $(\tilde{p}^{(3)}(\rho_0; z, D)\tilde{v}_1, \tilde{v}_1) = 0$, since $\tilde{v}_1$ is an even function, whereas $\tilde{p}^{(3)}(\rho_0; z, D)\tilde{v}_1$ is odd (for $p^{(3)}(\rho_0; y, D)$ changes the parity).

By the arbitrariness of k in (2.22) we deduce that

$$\left(\tilde{p}^{(3)}(\rho_0; z, D)\tilde{v}_1, h_0\right)_{L^2(\mathbb{R}^\nu_{z'})} = 0, \quad \forall z'' \in \mathbb{R}^{n-\nu}. \tag{2.23}$$

As a consequence, by performing a partial Fourier expansion of the function $\tilde{p}^{(3)}(\rho_0; z, D)\tilde{v}_1 \in \mathcal{S}(\mathbb{R}^n_z)$ with respect to the Hilbert basis of Hermite functions in $L^2(\mathbb{R}^\nu_{z'})$, we see that the equation

$$\tilde{p}^{(2)}(\rho_0; z, D)\tilde{v}_2 = -\tilde{p}^{(3)}(\rho_0; z, D)\tilde{v}_1$$

has a solution $\tilde{v}_2 \in \mathcal{S}(\mathbb{R}^n)$. We can now choose $v_2 := U^* \tilde{v}_2$ in (2.10), so that

$$p^{(2)}(\rho_0; y, D)v_2 + p^{(3)}(\rho_0; y, D)v_1 = 0. \tag{2.24}$$

From (2.9), (2.11), (2.16), (2.17), (2.19) and (2.24), upon letting $t \to +\infty$ it follows by dominated convergence that

$$\|v_1\| \leq \delta \|p^{(4)}(\rho_0; y, D)v_1 + p^{(3)}(\rho_0; y, D)v_2\|. \tag{2.25}$$

On the other hand, if δ is small enough, it is clear that (2.25) fails.

This concludes the proof in the scalar case. For general $N \geq 1$, the proof is the same (with the obvious changes of notation) by choosing $\tilde{v}_1 \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N) \cap \operatorname{Ker} \tilde{p}_2(\rho_0; y, D) \setminus \{0\}$ even and $v_2 \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N)$ which solves (2.24). □

In the next corollary we prove the necessary condition mentioned above for (2.1) to hold. We present the result for the class of systems considered in Theorem 2.1.

Corollary 2.2. *Let $P = P^* \in \mathrm{OP}\, S^2(X; \mathbb{C}^N)$ be a $N \times N$ formally self-adjoint system of classical properly supported pseudodifferential operators in X, and assume that its principal symbol has the form $p_2(x, \xi)I$. Let us suppose that for a given $x_0 \in X$ and every $\delta > 0$ there exists an open neighborhood Ω_δ of x_0 such that*

$$\|u\|^2 \leq \delta (Pu, u), \quad \forall u \in C_0^\infty(\Omega_\delta; \mathbb{C}^N). \tag{2.26}$$

Then for every $\rho_0 = (x_0, \xi_0) \in \Sigma$ *above* x_0 *we have*

$$p_1^s(\rho_0) + \mathrm{Tr}^+ F_{\rho_0} I \geq 0, \tag{2.27}$$

and

$$\operatorname{Ker} F_{\rho_0} \cap \operatorname{Im} F_{\rho_0} = \{0\} \Longrightarrow p_1^s(\rho_0) + \mathrm{Tr}^+ F_{\rho_0} I \text{ is invertible.} \tag{2.28}$$

Proof. As we observed in the proof of Theorem 2.1, (2.27) follows at once from the results in [22], [29]. As regards (2.28), it suffices to apply the Cauchy-Schwarz inequality in (2.26) and Theorem 2.1. □

Let us show that from Corollary 2.2 we can easily deduce the following well-known result of Hörmander [14].

THEOREM 2.3. *Let* $L \in \mathrm{OP}\, S^1(X)$ *be a classical and properly supported pseudodifferential operator, with principal symbol* l_1. *Let us suppose that for any given* $x_0 \in X$ *and every* δ *there exists a neighborhood* $\Omega_\delta \subset X$ *of* x_0 *such that*

$$\|u\|^2 \leq \delta \|Lu\|^2, \quad \forall u \in C_0^\infty(\Omega_\delta). \tag{2.29}$$

Then

$$dl_1 \neq 0 \text{ and } \{\operatorname{Re} l_1, \operatorname{Im} l_1\} \geq 0 \text{ where } l_1 = 0. \tag{2.30}$$

Proof. It suffices to apply Corollary 2.2 to the operator $P = L^* L$ ($N = 1$). We have $p_2 = |l_1|^2$, $\Sigma = \{(x, \xi) \in T^*X \setminus 0 : l_1(x, \xi) = 0\}$. Moreover,

$$F_\rho w = \frac{1}{2}\left(\sigma(w, H_{l_1}) H_{\bar{l}_1} + \sigma(w, H_{\bar{l}_1}) H_{l_1}\right), \quad w \in T_\rho T^*X, \ \rho \in \Sigma,$$

where H_{l_1} (resp. $H_{\bar{l}_1}$) denotes the Hamilton field associated with l_1 (resp. $\bar{l}_1$). Hence it is easily seen that we have

$$\operatorname{Ker} F_\rho \cap \operatorname{Im} F_\rho = \{0\} \Longleftrightarrow dl_1(\rho) = 0 \text{ or } \{\operatorname{Re} l_1, \operatorname{Im} l_1\}(\rho) \neq 0,$$

and

$$\begin{cases} p_1^s(\rho) = \{\operatorname{Re} l_1, \operatorname{Im} l_1\}(\rho), \\ \mathrm{Tr}^+ F_\rho = |\{\operatorname{Re} l_1, \operatorname{Im} l_1\}(\rho)|. \end{cases}$$

Therefore it follows from (2.5) that (2.29) implies (2.30). □

REMARK 2.4. Actually, the second condition in (2.30) is still necessary for (2.29) to hold with a fixed constant δ and, in fact, even for the local solvability of L^* with a loss of more than 1 derivative. (Moreover, this condition has been refined to the so-called condition $(\overline{\Psi})$; see Hörmander [18], Section 26.4, and Lerner [20].) Instead, the first condition in (2.30) is no longer necessary for such weaker estimates. More generally, (2.28) *is not necessary for* (2.26) *to hold with a fixed constant* $\delta > 0$, even for transversally elliptic operators.

Indeed, consider for example the operator $P = P^*$ in $\mathbb{R}^2$ given by

$$P = D_1^2 + x_1^2 D_2^2 - D_2 + 1.$$

We have $P = MM^* + 1$, where $M = D_1 + ix_1 D_2$ is the Mizohata operator, so that $(Pu, u) \geq \|u\|^2$ for every $u \in \mathcal{S}(\mathbb{R}^n)$. However, for every $\rho = (x, \xi) \in \Sigma = \{x_1 = \xi_1 = 0, \xi_2 \neq 0\}$ we have $\operatorname{Ker} F_\rho \cap \operatorname{Im} F_\rho = \{0\}$, whereas $p_1^s(\rho) + \operatorname{Tr}^+ F_\rho = -\xi_2 + |\xi_2|$ vanishes when $\xi_2 > 0$.

We also observe that P is locally solvable with a loss of 2 derivatives, whereas the operator $P - 1 = MM^*$ it not locally solvable, for otherwise M would be (which is not true, see, e.g., [18]). Hence the local solvability depends here on the terms of order 0, as in Stein's example [31] discussed in the introduction.

3. Estimates for the $\overline{\partial}_b$-complex

We briefly recall the definition of the $\overline{\partial}_b$-complex on a CR manifold M (for details see, e.g., Chen and Shaw [6] and Shaw and Wang [30]).

A CR manifold M of CR-dimension n and real codimension k is a real smooth manifold of dimension $2n + k$ together with a subbundle $T^{1,0}(M) \subset \mathbb{C}TM$ satisfying the following properties:

(a) $\operatorname{rank}_{\mathbb{C}} T^{1,0}(M) = n$.
(b) $T^{1,0}(M) \cap T^{0,1}(M) = 0$ (the zero section), where $T^{0,1}(M) = \overline{T^{1,0}(M)}$.
(c) (Integrability condition) For any given $X_1, X_2 \in \Gamma(U, T^{1,0}(M))$, we have $[X_1, X_2] \in \Gamma(U, T^{1,0}(M))$, for every open $U \subset M$.

We define the *characteristic bundle* $N^*(M) \subset T^*M$ as the space of the real covectors which are conormal to $T^{1,0}M \oplus T^{0,1}M$.

We fix a Hermitian metric on $\mathbb{C}TM$ so that $T^{0,1}(M)$ is orthogonal to $T^{1,0}(M)$ and we denote by $T^{*0,1}(M) \subset \mathbb{C}T^*M$ the complex vector bundle whose sections are forms which annihilate $T^{0,1}(M)^\perp$. We set $\Lambda^{0,q}(M) = \Lambda^q T^{*0,1}(M)$ and we denote by $\mathcal{E}^q(M)$ (resp. $\mathcal{E}^{0,q}(M)$) the space of smooth sections of $\Lambda^q \mathbb{C}T^*M$ (resp. $\Lambda^{0,q}(M)$). Let $\pi_q : \mathcal{E}^q(M) \to \mathcal{E}^{0,q}(M)$ be the projection and define $\overline{\partial}_b^{(q)} := \pi_{q+1} \circ d : \mathcal{E}^{0,q}(M) \to \mathcal{E}^{0,q+1}(M)$. We also consider its adjoint $\overline{\partial}_b^{(q)*} : \mathcal{E}^{0,q+1}(M) \to \mathcal{E}^{0,q}(M)$.

We observe that the estimate (I_ϵ) in the introduction can be written as

$$c\|u\|_\epsilon^2 \leq (\Box_b^{(q)} u, u) + \|u\|_0^2, \quad \forall u \in \mathcal{D}_{(0,q)}(\Omega), \tag{3.1}$$

where

$$\Box_b^{(q)} := \overline{\partial}_b^{(q-1)} \overline{\partial}_b^{(q-1)^*} + \overline{\partial}_b^{(q)^*} \overline{\partial}_b^{(q)} : \mathcal{E}^{0,q}(M) \to \mathcal{E}^{0,q}(M)$$

(for $q = 0$, $\Box_b^{(0)} = \overline{\partial}_b^{(0)^*} \overline{\partial}_b^{(0)}$) is the Kohn Laplacian associated with the $\overline{\partial}_b$-complex acting to $(0, q)$-forms.

We now consider a local basis $\overline{L}_1, \ldots, \overline{L}_n$ of sections of $T^{0,1}(M)$ and real vector fields $T_1, \ldots, T_k$ such that $L_1, \ldots, L_n, \overline{L}_1, \ldots, \overline{L}_n, T_1, \ldots, T_k$ is a local orthonormal basis of $\mathbb{C}TM$. Let $\omega^1,\dots,\omega^n,\overline{\omega}^1,\dots,\overline{\omega}^n,\tau_1,\dots,\tau_k$, be the dual basis to $L_1,\dots,L_n,\overline{L}_1,\dots,\overline{L}_n$, $T_1,\dots,T_k$, and let $u=\sum_I \phi_I\overline{\omega}^I$ be a $(0,q)$-form, with $I=(i_1,\dots,i_q)$, $1\le i_1<\dots<i_q\le n$, and $\overline{\omega}^I=\overline{\omega}^{i_1}\wedge\dots\wedge\overline{\omega}^{i_q}$. We now see that, with respect to such a basis, the Kohn Laplacian takes a relatively simple form.

First we introduce the following notation: given two multi-indices K and L such that $|K|=|L|=q$ and $|\{K\cap L\}|=q-1$, we set $\epsilon(K,L)=(-1)^m$, where m is the number of elements in $K\cap L$ between the unique element $k\in K\setminus L$ and the unique element $l\in L\setminus K$.

All summations below are performed on *strictly increasing* multi-indices.

Proposition 3.1. *Given a $(0,q)$-form $u=\sum\limits_K \phi_K\overline{\omega}^K$, we have*

$$\Box_b^{(q)}\left(\sum_K \phi_K\overline{\omega}^K\right)=\sum_L\left(\sum_K \Box_{LK}\phi_K\right)\overline{\omega}^L+\mathcal{E}(L,\overline{L},1)u. \tag{3.2}$$

Here $\mathcal{E}(L,\overline{L},1)$ denotes a sum of terms in which L_j, $\overline{L}_j$, $j=1,\dots,n$, and the constant function 1 are multiplied by smooth matrices, and

$$\Box_{LK}=-\delta_{LK}\sum_{k=1}^n L_j\overline{L}_j+M_{LK},$$

with

$$M_{LK}=\begin{cases}\sum\limits_{k\in K}[L_k,\overline{L}_k] & \text{if } K=L,\\ \epsilon(K,L)[L_k,\overline{L}_l] & \text{if } |\{K\cap L\}|=q-1,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. The computations are the same as those in the proof of Proposition 2.1 of Peloso and Ricci [27]. Hence we refer the reader directly to that paper. □

As a consequence we see that the system $\Box_b^{(q)}$ has a principal symbol which is a scalar multiple of the $N\times N$ identity matrix, with $N=\binom{n}{q}$, i.e., of the form p_2I. Indeed, if we set $v_j:=\sigma_1(L_j)$ for the principal symbols of the vector fields L_j, $j=1,\dots,n$, we have $p_2=\sum_{j=1}^n|v_j|^2$. Hence its characteristic set is given by

$$\Sigma=\{\rho\in T^*M\setminus 0:\ v_j(\rho)=0,\ j=1,\dots,n\},$$

which is exactly $N^*(M)$ with the 0-section removed.

We now recall the definition of the Levi form and that of the conditions $Z(q)$ and $Y(q)$ at a given point $\rho\in N^*(M)\setminus 0$. By definition, the Levi form $\mathcal{L}(\rho)$ at $\rho\in N^*(M)$ is the Hermitian matrix whose entries are

$$\mathcal{L}(\rho)_{jk}:=\sigma_1([L_j,\overline{L}_k])(\rho)=i\langle\rho,[L_j,\overline{L}_k]\rangle=i\{v_j,\overline{v}_k\}(\rho),$$

$j,k=1,\dots,n$, $\rho\in N^*(M)$. We observe that changing an orthornormal basis L_j, $j=1,\dots,n$, results in unitary intertwinings of $\mathcal{L}(\rho)$.

For $\rho\in\Sigma$, let $n^+(\rho)$ (resp. $n^-(\rho)$) be the number of strictly positive (resp. strictly negative) eigenvalues of $\mathcal{L}(\rho)$.

Definition 3.2. We say that condition $Z(q)$, $q\in\{0,\dots,n\}$, holds at a given $\rho\in N^*(M)\setminus 0$ if

$$Z(q) \qquad n^+(\rho)>n-q \quad \text{or} \quad n^-(\rho)>q.$$

We say that condition $Y(q)$ holds at ρ if both $Z(q)$ and $Z(n-q)$ hold at ρ (equivalently, if $Z(q)$ holds at $\rho=(x,\xi)$ and at $-\rho:=(x,-\xi)$).

Remark 3.3. If $\mathcal{L}(\rho)$ is non-degenerate, then condition $Z(q)$ at ρ is equivalent to $(n^+(\rho),n^-(\rho))\neq(n-q,q)$.

We can now state the main result of this section.

Theorem 3.4. *Let $x_0\in M$ and suppose that at some characteristic point $\rho_0\in N^*(M)\setminus 0$ above x_0 the Levi form $\mathcal{L}(\rho_0)$ has signature $(q,n-q)$ or $(n-q,q)$, $q\in\{0,\dots,n\}$. Then* (1.1) *does not hold.*

Corollary 3.5. *Under the assumptions of Theorem 3.4,* (1.2) *does not hold.*

Corollary 3.5 is an immediate consequence of Theorem 3.4 and the Cauchy-Schwarz inequality. Theorem 3.4 will follow from Theorem 2.1 applied to $P=\Box_b^{(q)}$ and the following two propositions, which express the hypotheses of Theorem 2.1 in terms of the spectrum of the Levi form. In what follows, we denote by p_1^s and F_ρ the subprincipal symbol and the fundamental matrix of the system $\Box_b^{(q)}$.

Proposition 3.6. *For any $\rho\in\Sigma$ we have*

$$\operatorname{Ker}F_\rho\cap\operatorname{Im}F_\rho=\{0\}\Longleftrightarrow \mathcal{L}(\rho)\ \text{is non-degenerate.} \tag{3.3}$$

Proof. The equivalence (3.3) easily follows from the fact that the map

$$\operatorname{Ker}\mathcal{L}(\rho)\ni\alpha\mapsto u=\sum_{j=1}^n \overline{\alpha}_j H_{v_j}+\alpha_j H_{\overline{v}_j}\in\operatorname{Ker}F_\rho\cap\operatorname{Im}F_\rho$$

is an isomorphism of vector spaces on $\mathbb{R}$, as one easily sees using the explicit expression

$$F_\rho w=\frac{1}{2}\sum_{j=1}^n\sigma(w,H_{\overline{v}_j})H_{v_j}+\sigma(w,H_{v_j})H_{\overline{v}_j},\quad w\in T_\rho T^*M,\ \rho\in\Sigma,$$

and the fact that the Hamilton fields H_{v_j}, $H_{\overline{v}_j}$ are linearly independent in view of the condition (b) at the beginning of this section. $\Box$

PROPOSITION 3.7. *For any $\rho \in \Sigma$ the spectrum of Melin's invariant $p_1^s(\rho)+\operatorname{Tr}^+ F_\rho I$ for the system $\Box_b^{(q)}$ acting on $(0,q)$-forms is given by*

$$\begin{aligned}(3.4)\quad &\operatorname{Spec}\left(p_1^s(\rho)+\operatorname{Tr}^+ F_\rho I\right)\\ &=\left\{\frac{1}{2}\left(\sum_{j\in K}(|\lambda_j(\rho)|+\lambda_j(\rho))+\sum_{j\notin K}(|\lambda_j(\rho)|-\lambda_j(\rho))\right);\ |K|=q\right\},\end{aligned}$$

where $\lambda_j(\rho)$, $j=1,\dots,n$, are the eigenvalues of $\mathcal{L}(\rho)$, counted with multiplicity. In particular,

$$(3.5)\qquad p_1^s(\rho)+\operatorname{Tr}^+ F_\rho I \text{ is invertible} \Longleftrightarrow Z(q) \text{ holds at } \rho.$$

Proof. We have to compute Melin's invariant $p_1^s(\rho)+\operatorname{Tr}^+ F_\rho I$ from the expression in (3.2).

Now, the non-zero spectrum of F_ρ coincides with that of the restriction $F_\rho|_{\operatorname{Im} F_\rho} : \operatorname{Im} F_\rho \to \operatorname{Im} F_\rho$, which, in the basis given by the H_{v_j}, $H_{\overline{v}_j}$, $j=1,\dots,n$, is expressed by the block matrix

$$-\frac{i}{2}\begin{pmatrix}\mathcal{L}(\rho) & 0\\ 0 & -\mathcal{L}(\rho)\end{pmatrix}.$$

Hence

$$\operatorname{Tr}^+ F_\rho=\frac{1}{2}\sum_{j=1}^{n}|\lambda_j(\rho)|.$$

The subprincipal symbol of the term $-\sum_{j=1}^n L_j\overline{L}_j\otimes I$ in (3.2), evaluated at $\rho\in\Sigma$, is given by

$$-\frac{i}{2}\sum_{j=1}^{n}\{v_j,\overline{v}_j\}(\rho)I=-\frac{1}{2}\operatorname{Tr}\mathcal{L}(\rho)I=-\frac{1}{2}\sum_{j=1}^{n}\lambda_j(\rho)I.$$

Finally, the principal symbol of the first order system (M_{LK}) in (3.2) evaluated at $\rho=(x,\xi)\in\Sigma$ is the endomorphism $\mathcal{L}^{(q)}(\rho):\Lambda^q T_x^{*0,1}(M)\to\Lambda^q T_x^{*0,1}(M)$, given by

$$\mathcal{L}^{(q)}(\rho)(\alpha)=\sum_{j,k=1}^{n}\mathcal{L}(\rho)_{kj}\overline{\omega}_j\wedge(\overline{\omega}_k\rfloor\alpha).$$

We recall that, for any given $(0,q)$-form $\alpha=\sum_{|J|=q}\alpha_J\overline{\omega}^J$ the $(0,q-1)$-form $\overline{\omega}_k\rfloor\alpha$ is defined by

$$\overline{\omega}_k\rfloor\alpha=\sum_{|J|=q}\sum_{|L|=q-1}\epsilon^J_{kL}\alpha_J\overline{\omega}^L,$$

where $\epsilon^J_{kL}=0$ if $J\neq\{k\}\cup L$ as sets, and it equals the parity of the permutation that rearranges (k,I) in increasing order if $J=\{k\}\cup L$ (of course, if $q=0$ it is understood that $\overline{\omega}_k\rfloor\alpha=0$).

Altogether, we have obtained

$$p_1^s(\rho)+\mathrm{Tr}^+ F_\rho\, I=\mathcal{L}^{(q)}(\rho)+\frac{1}{2}\left(\sum_{j=1}^n|\lambda_j(\rho)|-\lambda_j(\rho)\right)I.$$

It is clear that (3.5) is therefore proved once we show that the eigenvalues of $\mathcal{L}^{(q)}(\rho)$ consist exactly of the sums $\lambda_{j_1}(\rho)+\ldots+\lambda_{j_q}(\rho)$, $1\leq j_1<\ldots<j_q\leq n$, the corresponding eigenvectors being $\overline{\omega}'_{j_1}\wedge\ldots\wedge\overline{\omega}'_{j_q}$, where $\overline{\omega}'_j$, $j=1,\ldots,n$, are the eigenvectors of $\mathcal{L}^{(1)}(\rho)$, that is, just the transpose of $\mathcal{L}(\rho)$. On the other hand, this is easily seen by the property $\mathcal{L}^{(q_1+q_2)}(\alpha_1\wedge\alpha_2)=\mathcal{L}^{(q_1)}(\alpha_1)\wedge\alpha_2+\alpha_1\wedge\mathcal{L}^{(q_2)}(\alpha_1)$, for any $(0,q_1)$-form α_1 and $(0,q_2)$-form α_2. (To verify this it is useful to recall the formula $\overline{\omega}_k\rfloor(\alpha_1\wedge\alpha_2)=(\overline{\omega}_k\rfloor\alpha_1)\wedge\alpha_2+(-1)^{q_1}\alpha_1\rfloor(\overline{\omega}_k\wedge\alpha_2)$.) □

Proof of Theorem 3.4. The theorem is an immediate consequence of Theorem 2.1, (3.3), (3.5) and Remark 3.3. □

As regards the subelliptic estimates (I_ϵ) for the $\overline{\partial}_b$ we recalled in the introduction we have therefore the following result.

THEOREM 3.8.

(i) *If (I_ϵ) holds, then $\epsilon\leq 1/2$.*

(ii) *$(I_{1/2})$ holds at x_0 in degree q if and only if there exists an open neighborhood $U\subset M$ of x_0 such that $Y(q)$ holds at every characteristic point $(x,\xi)\in N^*(M)$, with $x\in U$, $\xi\neq 0$.*

(iii) *If (I_ϵ) holds at x_0 in degree q and the Levi form $\mathcal{L}(\rho_0)$ is non-degenerate at some characteristic point $\rho_0\in N^*(M)\setminus 0$ above x_0, then the condition $Y(q)$ holds true at ρ_0.*

Proof. The points (i) and (ii) are well-known. They are also a consequence of the results by Melin [22] once one characterizes the invertibility of Melin's invariant for the $\Box_b^{(q)}$, as it is done in (3.5) (see also Grigis [13] for the case of codimension 1). The point (iii) is a consequence of Corollary 3.5 and Remark 3.3, since the estimate (I_ϵ) implies of course (1.2). □

Similarly we deduce the following result.

THEOREM 3.9. *Let us suppose that estimate* (1.3) *holds in an open neighborhood Ω of $x_0\in M$ in degree q, and that the Levi form $\mathcal{L}(\rho_0)$ is non-degenerate at some characteristic point $\rho_0\in N^*(M)\setminus 0$ above x_0. Then the condition $Y(q)$ holds true at ρ_0.*

Proof. Indeed, if X is a non-vanishing real vector field, then for every $\delta>0$ we have $\|v\|_0\leq\delta\|Xv\|_0$ if $v\in C_0^\infty(\Omega_\delta)$ for a sufficiently small neighborhood

Ω_δ of x_0. (We may see this by straightening X or also from [14].) On the other hand, if $Z = X + iY$ is a complex vector field, with X and Y real, then

$$\|Zv\|_0^2 + \|\overline{Z}v\|_0^2 \geq (2-\delta')\left(\|Xv\|_0^2 + \|Yv\|_0^2\right) - C_{\delta'}\|v\|_0^2,$$

for every $v \in C_0^\infty(\Omega)$ and $\delta' > 0$. Hence we see that (1.3) implies (1.2), and therefore it suffices to apply Theorem 3.4 and Remark 3.3. □

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