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LIPSCHITZ TRIANGULATIONS

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ABSTRACT. In this paper we introduce a new tool called "Lipschitz triangulations", which gives combinatorially all information about the metric type. We show the existence of such triangulations for semi-algebraic sets. As a consequence we obtain a bi-Lipschitz version of Hardt's theorem. Hardt's theorem states that, given a family definable in an o-minimal structure, there exists (generically) a trivialization which is definable in this o-minimal structure. We show that, for a polynomially bounded o-minimal structure, there exists such an isotopy which is bi-Lipschitz as well.

1. Introduction

In this paper we introduce a new notion of triangulation describing exactly the metric type of a set, and we prove that semi-algebraic sets are triangulable in this sense. As an application we obtain a bi-Lipschitz version of Hardt's theorem. This theorem, which appeared in [H2], states that, given a semi-algebraic family of subsets of R^n, where R is a real closed field, we can find a partition of the set of parameters such that over each element of this partition the fibers of this family are semi-algebraically homeomorphic. Such a result is proved in [S], [C], [BCR1], [BCR2], [H1], [H2]. Here we prove an analog for semi-algebraic bi-Lipschitz equivalence. Our proof also holds over any polynomially bounded o-minimal structure over $\mathbb{R}$. Indeed, it works over any o-minimal structure in which Theorem 4.1 holds. It is inspired by the proof of M. Coste [C] of the topological case. There are also bi-Lipschitz equisingularity results proved by A. Parusiński [P1] and T. Mostowski [M] involving integration of vector fields, but these do not provide definable isotopies.

Our notion of *Lipschitz triangulation* is well adapted to the study of the metric type of a singular set. It is a combinatorial tool, as can be triangulations for the topological point of view, involving the entire Lipschitzian type of the singularity. It also allows us to generalize an elegant proof of Hardt's

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Theorem (see [C]). We construct simultaneous triangulations by triangulating the generic fiber defined along an ultrafilter of definable sets.

The main difficulty of such a construction consists in finding regular projections. This means that we will need to find a direction which is transverse to the tangent space at regular points of a given set. Given a set, there is no regular projection in general, but for a semi-algebraic set we are able to find such a projection up to a semi-algebraic bi-Lipschitz homeomorphism. We construct this mapping in Section 3. We also give an "L-regular decomposition" theorem (see [P1], [K]).

The existence of Lipschitz triangulations is proved by induction. A key tool is the preparation theorem for semi-algebraic continuous functions (Theorem 4.1). This theorem was first introduced by A. Parusiński in [P3] to prove the existence of Lipschitz stratifications. Later J.-M. Lion and J.-P. Rolin [LR] gave a version for global subanalytic sets and log-exp sets. Recently L. van den Dries and P. Speissegger [vD-S] proved an o-minimal version of this result for polynomially bounded o-minimal structures over $\mathbb{R}$. Here we give a proof in the semi-algebraic case over an arbitrary real closed field. Our statement is weaker than that of [P3] and [LR], since it does not give a precise description of the unit. However, it is just what we need. As a consequence of this result, we can compare a definable function to a distance function over a definable partition. This is done in Section 4.

In Section 5 we introduce our concept of Lipschitz triangulations. For instance, if we consider a cusp $\{(x;y) \in R^2 \mid y^2 = x^3\}$, it is clearly impossible to find a bi-Lipschitz homeomorphism of this cusp onto a simplicial complex. However, we can find a homeomorphism onto two edges joined at the origin such that the length of any vertical segment passing through $(x;0)$ joining two points of these two edges is multiplied by $x^{3/2}$. For real algebraic curves such a construction had been given by L. Birbrair [B]. We will generalize this to higher dimensions. What is important is that two sets having the same semi-algebraic Lipschitz trivialization will be semi-algebraically Lipschitz homeomorphic. Finally, as in [C], by performing simultaneous triangulations we will obtain a bi-Lipschitz version of Hardt's theorem. This theorem provides a bound for the number of Lipschitzian types of sets defined by polynomials of bounded degree over any real closed field, and over any polynomially bounded o-minimal structure over $\mathbb{R}$ (where the existence of Lipschitz stratifications have not been proved yet). Definability of the isotopy provides more information about the behavior of the Lipschitz constants when we approach the instability locus (see Remarks 2.3). We believe that Lipschitz triangulations may be useful in defining metric invariants and in counting the number of Lipschitz types definable in a polynomially bounded o-minimal structure over $\mathbb{R}$ (see Remark 5.8).

Throughout this paper R is a real closed field and $\mathbb{Q}^+$ denotes the set of strictly positive rational numbers. In Section 3 we will work over an o-minimal structure over R. All the results of Sections 4 and 5 hold for semi-algebraic sets over R or for a polynomially bounded o-minimal structure provided $R = \mathbb{R}$. Indeed, all the results of these sections could be stated over any polynomially bounded structure in which Theorem 4.1 holds. The word definable will always mean definable in the considered o-minimal structure.

2. Statement of the main result

We assume either that $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$ is the o-minimal structure of semi-algebraic sets over R or that $R = \mathbb{R}$ and $\mathcal{S}$ is a polynomially bounded structure over $\mathbb{R}$.

Let A be a definable subset of $R^n \times R^p$. We will consider such a subset as a family of definable subsets of R^n parametrized by R^p. For $U \subseteq R^p$ we denote by A_U the subfamily $\{q = (x;t) \in R^n \times R^p \mid q \in A,\ t \in U\}$, and for $t \in R^p$ we denote by A_t the fiber of A at t, i.e., $\{x \in R^n \mid q = (x;t) \in A\}$.

Definition 2.1. Let A be a definable subset of $R^n \times R^p$. We say that A is *definably bi-Lipschitz trivial along* $U \subseteq R^p$ if there exist $t_0 \in U$ and a definable homeomorphism $h : A_{t_0} \times U \to A_U$ of the form $h(x;t) = (h_t(x);t)$ together with a definable continuous function $C : U \to R$ satisfying for any elements x and x' in A_{t_0}, and $t \in U$,

$$|h_t(x) - h_t(x')| \le C(t) \cdot |x - x'| \tag{2.1}$$

and, for any $(x;x') \in A_t \times A_t$, $t \in U$,

$$|h_t^{-1}(x) - h_t^{-1}(x')| \le C(t) \cdot |x - x'|. \tag{2.2}$$

In this paper we will prove the following theorem:

Theorem 2.2. *Let A be a definable subset of $R^n \times R^p$. Then there exists a definable partition of R^p, such that the family A is definably bi-Lipschitz trivial along each element of this partition.*

Remarks 2.3. (a) In fact, we will prove a stronger statement, since the isotopy will also be defined on the ambient space. We could also require the isotopy to preserve a finite number of given definable families.

(b) In Definition 2.1, the Lipschitz constants are functions of the parameters. As they are continuous functions, we have Lipschitz triviality over any closed bounded subset of each element of the partition. We cannot hope for more. Indeed, for any non-Lipschitz stable family of sets, the Lipschitz constants tend to infinity when the parameters approach the instability locus. In the case of isotopies given by vector fields [P3], [M], constants of vector fields tend to infinity inversely proportional to the distance to the boundary of the strata. So the Lipschitz constant of the flow is given by the Gronwall lemma as exponential of the inverse of the distance. In Theorem 2.2, the bound obtained is better, since the constant is a definable function which, by Łojasiewicz's inequality, can be bounded by a power of the inverse of the distance to the boundary.

(c) Trivialization theorems obtained by integration of vector fields assume the properness of the mapping (see [P1]–[P4], [M]). This is not the case in the above statement, as in Hardt's theorem. Thus, also at infinity, we obtain the finiteness of metric types of the fibers.

(d) We can also have the Lipschitz property with respect to the parameters for bounded sets (as stated in Proposition 5.7).

In the non-bounded case, it is impossible to have in general bi-Lipschitzianity with respect to parameters, as shown by the following example:

EXAMPLE 2.4. Consider the family of sets $A = \{(x; y; t) \in \mathbb{R}^2 \times \mathbb{R} \mid y = tx\}$. It is easy to check that along no interval of type $]0; \varepsilon[$, we can have the bi-Lipschitz property with respect the parameter t (even with a constant depending on t).

3. Regular directions

This section deals with an arbitrary o-minimal structure. In the following sections we will need regular directions. A regular direction of a definable subset A is a vector whose distance to lines included in a tangent space at a smooth point of A is bounded below. We will prove that for a definable set over an o-minimal structure there exist such directions, up to a bi-Lipschitz homeomorphism of the ambient space (Proposition 3.13).

3.1. Notations, definitions and basic results. In this section we give definitions and prove lemmas which will be useful in the proof of Proposition 3.10 in finding regular directions for a definable subset. We start by introducing some notations.

We denote by $|.|$ the euclidian norm of R^n. We consider the induced distance over the sphere S^{n-1}. We write $B(x; r)$ for the ball. We denote by $\mathbb{G}_{k,m}$ the Grassmannian of k dimensional vector subspaces of R^m and we put $\mathbb{G}_m = \bigcup_{k=1}^{m-1} \mathbb{G}_{k,m}$.

A Lipschitzian function is said to be $\mathbb{Q}-$Lipschitzian when the Lipschitz constant can be chosen in $\mathbb{Q}^+$ (recall that R is not necessarily archimedian).

We define a metric d on $\mathbb{G}_{k,m}$ by

$$d(P; Q) = \sup\{d(\lambda; S^{m-1} \cap Q); \lambda \text{ is a unit vector of } P\},$$

where $d(\lambda; Q)$ denotes the distance in R^m.

Given a definable set $A \subseteq R^m$, we define $\tau(A)$ to be the closure of the set of vector spaces which are tangent to A at a regular point, i.e.,

$$\tau(A) = \mathrm{cl}\,\{T_x A \in \mathbb{G}_m \mid x \in A_{\mathrm{reg}}\}.$$

Given $X \subseteq \mathbb{G}_m$, we denote by $\Lambda(X)$ the projective subspace of S^{m-1} consisting of all unit vectors included in an element of X.

Let $\lambda \in S^n$. We denote by $\pi_\lambda : R^{n+1} \to N_\lambda$ the orthogonal projection onto the normal space of the vector λ, and by q_λ the coordinate of q along λ.

Let A and A' be elements of $\mathcal{S}_n$, with $A' \subseteq N_\lambda$, and $\xi : A' \to R$ be a definable function. The set A is said to be the graph of the function ξ for λ if

$$A = \{q \in R^{n+1} \mid \pi_\lambda(q) \in A' \quad \text{and} \quad q_\lambda = \xi(\pi_\lambda(q))\}.$$

The projection π_λ also induces a mapping $\widetilde{\pi}_\lambda : S^n \setminus \{\pm\lambda\} \to S^{n-1}(N_\lambda)$ defined by $\widetilde{\pi}_\lambda(u) = \pi_\lambda(u)/|\pi_\lambda(u)|$.

Definition 3.1. Let A be a definable set of R^{n+1}. An element λ of S^n is said to be *regular for* A if

$$d(\lambda; (\Lambda \circ \tau)(A)) \geq \alpha$$

with $\alpha \in \mathbb{Q}^+$. A subset of S^n is said to be regular for A if all its elements are regular.

The following observations are consequences of the definitions and basic results about Lipschitz functions.

Observations. Let $\lambda \in S^n$ and $r \in \mathbb{Q}^+$.

(0) Given a definable partition of R^n, it is possible to refine it in such a way that on each element of this partition two given points q and q' may be joined by a definable path of length equivalent (up to constants in $\mathbb{Q}$) to the distance between these two points. This is a consequence of the existence of L-regular decompositions. (See [K], [P4]; actually, in Section 3.3 we shall prove a theorem guaranteeing the existence of such a decomposition having an extra property.)

(1) If A is a union of graphs for λ of $\mathbb{Q}-$Lipschitzian functions, then there exists $r \in \mathbb{Q}^+$ such that $B(\lambda; r)$ is regular for A. Also, if $B(\lambda; r) \subseteq S^n$ is regular for the definable set $A \subseteq R^{n+1}$, then A is the union of the graphs for λ of some $\mathbb{Q}-$Lipschitzian functions. Moreover, if A is the graph for λ of a Lipschitzian function $\xi : N_\lambda \to R$, then ξ is C-Lipschitzian with $C \leq 1/d(\lambda; \Lambda(\tau(A)))$.

Proof. The first assertion is obvious. For the second we may assume that λ is e_{n+1}, the last vector of the canonical basis. By cylindrical decomposition, the set A can be included in the graphs of some definable functions $\xi_i : V_i \to R$. By the first observation it is enough to bound their derivatives (that exit generically). Now for a regular point $x \in V_i$ we have $d(\lambda; T_{(x;\xi_i(x))}A) = 1/(\sqrt{1 + |\partial_x \xi_i|^2})$, and we are done. The last assertion also follows, since $d(\lambda; T) \leq d(\lambda; \Lambda(T))$. □

(2) Every definable $\mathbb{Q}-$Lipschitzian function ξ defined over a subset A of R^n can be extended to a definable $\mathbb{Q}-$Lipschitzian function $\widehat{\xi}$ defined over the whole of R^n. (Actually, it is enough to set $\widehat{\xi}(q) = \inf\{\xi(q')+C|q-q'| \ \mid q' \in A\}$ if $\xi \geq 0$ is a C-Lipschitz function on A.) If ξ_1 and ξ_2 are two definable functions such that $\xi_1 \leq \xi \leq \xi_2$, then by considering $\max(\min(\widehat{\xi};\xi_2);\xi_1)$, if necessary, we can require that $\widehat{\xi}$ also fulfills $\xi_1 \leq \widehat{\xi} \leq \xi_2$.

(3) If A is the union of the graphs for a direction λ of definable functions $\theta_1, \dots, \theta_k$ over R^n, we can find an ordered family of definable functions $\xi_1 \leq \dots \leq \xi_k$ such that A is the union of the graphs of these functions for λ.

(4) Given a family of definable Lipschitzian functions $f_1, \dots, f_k$, defined over R^n, we can find some definable Lipschitzian functions $\xi_1, \dots, \xi_m$ such that over each cell $C \subseteq R^{n+1}$ delimited by the graphs of two consecutive functions ξ_i and ξ_{i+1} the functions $|q_{n+1} - f_i(x)|$ (where $q = (x; q_{n+1})$) are comparable to each other (for the relation $\leq$) and comparable to the functions $f_i \circ \pi$ (where π is the canonical projection). Indeed, it suffices to consider the graphs of the functions f_i, $f_i + f_j$ and $(f_i + f_j)/2$; then the family $\xi_1, \dots, \xi_m$ is given by the last point.

Proposition 3.2. *Let B be a connected subset of S^n, $\lambda_0 \in B$, and let $\xi : N_{\lambda_0} \to R$ be a continuous definable function. Let H be the graph of ξ for λ_0. Suppose that B is regular for H. Then, for any $\lambda \in B$, the set H is the graph of a function $\xi^\lambda : N_\lambda \to R$. Moreover, the set "under the graph", namely,*

$$E(H;\lambda) = \{q \in R^{n+1} \mid q_\lambda \leq \xi^\lambda(\pi_\lambda(q))\},$$

is independent of $\lambda \in B$.

Proof. Let

$$C = \{\lambda \in B \mid \forall x \in N_\lambda \quad \text{card } \pi_\lambda^{-1}(x) \cap H = 1\}.$$

We have to check that $C = B$. Let $\lambda \in C$. Let $r(\lambda) = d(\lambda; \Lambda(\tau(H)))$.

We claim that $B(\lambda; r(\lambda)/2) \subseteq C$. Consider $\lambda' \in B(\lambda; r(\lambda)/2)$ different from λ, $x \in N_\lambda$, and set $l' = \pi_\lambda(\lambda')$. It suffices to show that the line L generated by λ' and passing through x intersects H in exactly one point. The line L is the graph for λ of the function $\eta(x + tl') = \alpha \cdot t$ with $\alpha > 2/r(\lambda)$. On the other hand, since $\lambda \in C$, the set H is the graph for λ of a $2/r(\lambda)$-Lipschitzian function ξ^λ (see Observation (1) above). This implies that for t large enough we have $\eta(x + tl') \geq \xi(x + tl')$, and that $\eta(x - tl') \leq \xi(x - tl')$. Thus there is a point q such that $\xi(q) = \eta(q)$, which implies that the line L cuts H. The uniqueness of the intersection is clear from the fact that $\alpha > 2/r(\lambda)$ and ξ is $2/r(\lambda)$-Lipschitzian. This proves the claim.

Note that this implies that C is open in B. Indeed, this proves also the closeness. Consider $\lambda \in B$ and a continuous definable arc γ in C tending to λ. Since $r(\gamma(t))$ tends to $r(\lambda)$, which is nonzero, the ball $B(\gamma(t); r(\gamma(t)))$

contains λ for t small enough. So the closeness of C follows from the above claim.

It remains to check that $E(H;\lambda)$ is independent of $\lambda \in B$. It is the closure of one of the two connected components of the complement of the graph. So, it suffices to find at least one point which is in common to all the $E(H;\lambda)$ and which lies outside H. The set H is the zero locus of the function $f(q) = q_{\lambda_0} - \xi(\pi_{\lambda_0}(q))$. Locally, at a smooth point q of H it is clear that $E(H;\lambda)$ is given by the sign of $d_q f(\lambda)$. But as B is regular for H, this never vanishes and so is of constant sign over B. ☐

Remark 3.3. Consider e_{n+1}, the last vector of the canonical basis. Suppose $B \subseteq S^{n-1}$ is regular for a subset $A \subseteq R^n$. Then, for any $a \in \mathbb{Q}^+$ the set

$$\widetilde{\pi}_{e_{n+1}}^{-1}(B) \cap \{\lambda \in S^n \mid d(\lambda; \{\pm e_{n+1}\}) \geq a\}$$

is regular for $\pi_{e_{n+1}}^{-1}(A)$. Moreover, if A is the graph of a $\mathbb{Q}-$Lipschitzian function for $\lambda \in B$, and if B is connected, then $\pi_{e_{n+1}}^{-1}(A)$ is the graph of a $\mathbb{Q}-$Lipschitzian function for any λ' in

$$\widetilde{\pi}_{\lambda}^{-1}(B) \cap \{\lambda \in S^n \mid d(\lambda; \{\pm e_{n+1}\}) \geq a\},$$

for any $a \in \mathbb{Q}^+$ (by Proposition 3.2). Moreover, in this case the following holds:

$$E(\pi_{e_{n+1}}^{-1}(A); \lambda') = \pi_{e_{n+1}}^{-1}(E(A;\lambda)).$$

We will also need a lemma due to K. Kurdyka [K]:

Lemma 3.4. *Let $\eta \in \mathbb{Q}^+$ be less than $1/2$ and $l \in S^n$. Given $\nu, n \in \mathbb{N}$ there exist strictly positive constants $\varepsilon \in \mathbb{Q}^+$ and $t \in \mathbb{Q}^+$ such that for any $P_1, \ldots, P_\nu$ in $\mathbb{G}_{n,n+1}$ there exists a line $P \notin B(l;\eta)$ such that for any $Y \in \bigcup_{i=1}^{\nu} B(P_i;\varepsilon)$ we have*

$$d(P; S^n \cap Y)) \geq t.$$

This statement is a bit different from that given in [K] (see Lemma 3 of [K]), since here we need to be able to choose the line P in the complement of a sufficiently small given ball. Moreover, we work over an arbitrary real closed field. However, the same proof gives the above version of the lemma.

3.2. Regular families.

Definition 3.5. A *regular family of hypersurfaces* of R^{n+1} is a family $H = (H_k; \lambda_k)_{1 \leq k \leq b}$ with $b \in \mathbb{N}$ of definable subsets of R^{n+1} together with elements of S^n such that the following properties hold for each $k < b$:

(i) The sets H_k and H_{k+1} are respectively the graphs for λ_k of two global $\mathbb{Q}-$Lipschitzian functions ξ_k and ξ'_k such that $\xi_k \leq \xi'_k$.

(ii) We have

$$E(H_{k+1};\lambda_k) = E(H_{k+1};\lambda_{k+1}).$$

Let A be a definable subset of R^{n+1} of empty interior. We say that the family H is *compatible* with A, if $A \subseteq \bigcup_{k=1}^{b} H_k$. An *extension* of H is a regular family compatible with the set $\bigcup_{k=1}^{b} H_k$.

Given $k < b$, we denote by G_k the cylinder defined by the functions ξ_k and ξ'_k, i.e., $G_k = E(H_{k+1};\lambda_k) \setminus \mathrm{Int}(E(H_k;\lambda_k))$. Note that by Proposition 3.2, if B is a connected regular subset for $H_k \cup H_{k+1}$ containing λ_k, then the set G_k may be defined using any $\lambda \in B$.

We say that another regular family H' *coincides with* H *outside* G_k if for each j' either $H'_j \subseteq G_k$ or there exists j' such that $H'_j = H_{j'}$.

Remark 3.6. We may always assume that the G_k are of nonempty interior. Indeed, if $\mathrm{Int}(G_k) = \emptyset$, then $H_k = H_{k+1}$, and in this case we may remove $(H_{k+1};\lambda_{k+1})$ from the sequence.

The following lemma allows us to assume that the interiors of the sets G_k are connected.

Lemma 3.7. *Let H be a regular system of hypersurfaces. There exists an extension $\widehat{H}$ of H such that all the sets* $\mathrm{Int}(\widehat{G}_k)$ *are connected.*

Proof. Let $1 \leq m \leq b-1$. Suppose that G_m is not connected. Let $A_1, \dots, A_\nu$ be the connected components of G_m. Set $A'_i = \pi_{\lambda_m}(A_i)$. Each A_i is of the form

$$\{(x_{\lambda_m};q_{\lambda_m}) \in A'_i \oplus N_{\lambda_m} \mid \xi_m(x_{\lambda_m}) \leq q_{\lambda_m} \leq \xi'_m(x_{\lambda_m})\}.$$

Clearly $\xi_m = \xi'_m$ over the boundary of A'_i. Thus we can define Lipschitzian functions η_i, $1 \leq i \leq \nu - 1$, as follows. We set over A'_j, $\eta_i = \xi'_j$ whenever $1 \leq j \leq i$, and $\eta_i = \xi_j$ when $j > i$. Therefore we have that $\eta_1 \leq \cdots \leq \eta_{\nu-1}$. Now it suffices to

- let $\widehat{H}_k = H_k$ and $\widehat{\lambda}_k = \lambda_k$ if $k \leq m$,
- let $\widehat{H}_k$ be the graph of η_{i-m} for λ_m and $\widehat{\lambda}_k = \lambda_m$ for $m+1 \leq k \leq m+\nu-1$,
- let $\widehat{H}_k = H_k$ and $\widehat{\lambda}_k = \lambda_k$ if $m+1 \leq k \leq b+\nu-1$.

This is clearly a regular system of hypersurfaces. Note that the sets $\mathrm{Int}(\widehat{G}_k)$ are the connected components of $\mathrm{Int}(G_m)$. □

Given a regular family of hypersurfaces H, it will be convenient to extend the notations in the following way. Define $H_0 = \{-\infty\}$ and $H_{b+1} = \{+\infty\}$. Suppose all the elements of S^n are regular for these two sets. Suppose also that they are the respective graphs of two functions which take $-\infty$ and $+\infty$ as constant values (for any λ). Define also $\lambda_0 = \lambda_1$, $\lambda_{b+1} = \lambda_b$, $E(H_0;\lambda_0) = \emptyset$,

$G_0 = E(H_1; \lambda_1)$, $G_b = R^{n+1} \setminus \mathrm{Int}(E(H_b; \lambda_b))$. Observe that now $R^{n+1} = \bigcup_{k=0}^{b} G_k$.

We want to prove that every definable set with empty interior admits a regular family compatible with it (Proposition 3.10). To this end, we first prove the following lemma, which is a consequence of Observations (1) and (2) of the previous section.

LEMMA 3.8. *Let $H = (H_k; \lambda_k)_{1 \le k \le b}$ be a regular family of hypersurfaces of R^{n+1} and let $m \in \{0, \dots, b\}$. Let A be a definable subset of G_m of empty interior such that λ_m is regular for A. Then H can be extended to a regular family of hypersurfaces $\widetilde{H}$ compatible with A, which coincides with H outside G_m.*

Proof. By property (i) of Definition 3.5 the sets H_m and H_{m+1} are respectively the graphs for λ_m of two $\mathbb{Q}$−Lipschitzian functions ξ_m and ξ'_m. By Observation (1) the set A can be included in a finite number of graphs for λ_m of $\mathbb{Q}$−Lipschitzian functions. By Observation (2) these functions, say $\theta_1, \dots, \theta_\nu$, can be assumed to be ordered and satisfy $\xi_m \le \theta_i \le \xi'_m$. So, it suffices to

- let $H'_k = H_k$ and $\lambda'_k = \lambda_k$ whenever $1 \le k \le m$,
- let H'_k be the graph of θ_{k-m} for λ_m and $\lambda'_k = \lambda_m$ for $m < k \le m + \nu$,
- let $H'_k = H_{k-\nu}$ and $\lambda'_k = \lambda_{k-\nu}$, whenever $m + 1 + \nu \le k \le b + \nu$.

Properties (i) and (ii) clearly hold by construction. □

We will need a second lemma, which concerns the extension of a regular family by a second one.

LEMMA 3.9. *Let $H = (H_k; \lambda_k)_{1 \le k \le b}$ be a regular family of hypersurfaces of R^{n+1}, $m \le b$ an integer and $r \in \mathbb{Q}^+$ such that $B(\lambda_m; r)$ is regular for $H_m \cup H_{m+1}$. Let $\widehat{H} = (\widehat{H}_k; \widehat{\lambda}_k)_{k \le \widehat{b}}$ be a second regular system of hypersurfaces of R^{n+1}, such that for each k, $\widehat{\lambda}_k \in B(\lambda_m; r)$. Then we can find an extension $\widetilde{H} = (\widetilde{H}_k; \widetilde{\lambda}_k)_{k \le \widetilde{b}}$ of H, which coincides with H outside G_m, such that we have $G_m = \bigcup_{j=m}^{m+\widehat{b}} \widetilde{G}_j$. Moreover, $\widetilde{\lambda}_m = \widehat{\lambda}_1$ and for each $1 \le k \le \widehat{b}$ we may require $\widetilde{\lambda}_{k+m} = \widehat{\lambda}_k$ and $\widetilde{G}_{k+m} \subseteq (\widehat{G}_k \cap G_m) \cup H_m \cup H_{m+1}$.*

Proof. Let $k \le \widehat{b}$ be an integer. Since $\widehat{\lambda}_k \in B(\lambda_m; r)$, by Proposition 3.2, the sets H_m and H_{m+1} are respectively the graphs for $\widehat{\lambda}_k$ of two $\mathbb{Q}$−Lipschitzian functions μ_k and μ'_k. Moreover, the set $\widehat{H_k}$ is also the graph for $\widehat{\lambda}_k$ of a $\mathbb{Q}$−Lipschitzian function $\widehat{\xi}_k$. Define

$$\eta_k = \min(\max(\mu_k; \widehat{\xi}_k); \mu'_k),$$

in order to get a function whose graph is included in G_m. Now we define the desired regular family $(\widetilde{H}_k; \widetilde{\lambda}_k)_{1 \le k \le \widetilde{b}}$ as follows.

- Let $\widetilde{H}_k = H_k$ and $\widetilde{\lambda}_k = \lambda_k$ if $k < m$.
- Let $\widetilde{H}_m = H_m$ and $\widetilde{\lambda}_m = \widehat{\lambda}_1$.
- Let $\widetilde{H}_k$ be the graph of η_{k-m} for $\widehat{\lambda}_{k-m}$, and let $\widetilde{\lambda}_k = \widehat{\lambda}_{k-m}$, whenever $m+1 \leq k \leq m+\widehat{b}$.
- Finally, let $\widetilde{H}_k = H_{k-\widehat{b}}$ and $\widetilde{\lambda}_k = \lambda_{k-\widehat{b}}$ if $m+\widehat{b}+1 \leq k \leq b+\widehat{b}$.

Now we check that the properties (i) and (ii) hold for the family $\widetilde{H}$.

For $k < m-1$, or $k \geq m+\widehat{b}+1$, the result is clear since the family $\widetilde{H}$ is, in fact, the family H. For $k = m-1$ property (i) follows from (i) for H and property (ii) is a consequence of Proposition 3.2, since we assumed $\lambda_1 \in B(\lambda_m; r)$, which is regular for H_m. For $k = m$ property (i) also follows from Proposition 3.2 for the same reason and property (ii) is trivial.

So assume now $m < k \leq m+\widehat{b}$. As $\widetilde{\lambda}_k = \widehat{\lambda}_{k-m}$, the sets H_m and H_{m+1} are the graphs for $\widetilde{\lambda}_k$ of μ_{k-m} and μ'_{k-m}. Moreover, by (i) for $\widehat{H}$, the set $\widehat{H}_{k+1-m}$ is the graph for $\widetilde{\lambda}_k$ of a $\mathbb{Q}$−Lipschitzian function $\widehat{\xi}'_{k-m}$ such that $\widehat{\xi}_{k-m} \leq \widehat{\xi}'_{k-m}$.

Define

$$\eta'_k = \min(\max(\mu_{k-m}; \widehat{\xi}'_{k-m}); \mu'_{k-m}).$$

We claim that the graph of η'_k for $\widetilde{\lambda}_k$ is that of η_{k+1-m} for $\widetilde{\lambda}_{k+1}$. To see this, note that the graph of η'_k (resp. η_{k+1-m}) coincides

- with H_m over $E(H_m; \widetilde{\lambda}_k)$ (resp. $E(H_m; \widetilde{\lambda}_{k+1})$),
- with $\widehat{H}_{k+1-m}$ over $E(H_{m+1}; \widetilde{\lambda}_k) \setminus E(H_m; \widetilde{\lambda}_k)$ (resp. $E(H_{m+1}; \widetilde{\lambda}_{k+1}) \setminus E(H_m; \widetilde{\lambda}_{k+1})$),
- with H_{m+1} over $R^{n+1} \setminus E(H_{m+1}; \widetilde{\lambda}_k)$ (resp. $R^{n+1} \setminus E(H_{m+1}; \widetilde{\lambda}_{k+1})$).

But, by Proposition 3.2, as the ball $B(\lambda_m; r)$ is connected and regular for $H_m \cup H_{m+1}$, the sets $E(H_m; l)$ and $E(H_{m+1}; l)$ do not depend on $l \in B(\lambda_m; r)$. As $\widetilde{\lambda}_k$ and $\widetilde{\lambda}_{k+1}$ both belong to $B(\lambda_m; r)$, the claim is clear.

This claim proves that $\widetilde{H}_{k+1}$ is the graph of η'_k for $\widetilde{\lambda}_k$. Now, to check (i), we just have to prove that $\eta_{k-m} \leq \eta'_k$. But this is clear, as $\widehat{\xi}_{k-m} \leq \widehat{\xi}'_{k-m}$ by the definition of η'_k and η_{k-m}.

It remains to check property (ii). If $k = m+\widehat{b}-1$, this is a consequence of Proposition 3.2, since we have assumed that $\widetilde{\lambda}_k$ belongs to $B(\lambda_m; r)$.

Let k be such that $m \leq k \leq m+\widehat{b}-1$. First note that by (ii) for $\widehat{H}$ we have

$$E(\widehat{H}_{k+1}; \widehat{\lambda}_k) = E(\widehat{H}_{k+1}; \widehat{\lambda}_{k+1}).$$

But, as above, $E(\widetilde{H}_{k+1}; \widetilde{\lambda}_k)$ (resp. $\widetilde{\lambda}_{k+1}$) is equal

- to $E(H_m; \widetilde{\lambda}_k)$ over $E(H_m; \widetilde{\lambda}_k)$ (resp. $\widetilde{\lambda}_{k+1}$),
- to $E(\widehat{H}_{k+1-m}; \widetilde{\lambda}_k)$ over $E(H_{m+1}; \widetilde{\lambda}_k) \setminus E(H_m; \widetilde{\lambda}_k)$ (resp. $\widetilde{\lambda}_{k+1}$),
- to $E(H_{m+1}; \widetilde{\lambda}_k)$ over $R^{n+1} \setminus E(H_{m+1}; \widetilde{\lambda}_k)$ (resp. $\widetilde{\lambda}_{k+1}$).

As $\widetilde{\lambda}_{k+1}$ and $\widetilde{\lambda}_k$ both belong to the ball $B(\lambda_m; r)$, this proves (ii).

The second part of the statement of the lemma follows from the construction. □

PROPOSITION 3.10. *For each definable set A of R^{n+1}, of empty interior, there exists a regular family of hypersurfaces of R^{n+1} compatible with A.*

Proof. Actually, we will prove by induction on n that there exists a regular family of hypersurfaces of R^{n+1} compatible with a given definable subset such that all the λ_k can be chosen in a given ball $B(\lambda; \eta)$ in S^n, for $\eta \in \mathbb{Q}^+$.

For $n = 0$ the result is clear. So assume it is true for $n - 1$. Let A be a definable subset and $B(\lambda; \eta) \subseteq S^n$, $\eta \in \mathbb{Q}^+$.

Take $e \notin B(\lambda; \eta)$ such that $-e \notin B(\lambda; \eta)$. We can choose e in such a way that π_e restricted to A is finite to one. Let ν be the maximal number of points of a fiber of π_e restricted to A. For any $\varepsilon \in \mathbb{Q}^+$, there exists a partition $(A_i)_{i \in I}$ of N_e such that, for each $i \in I$, the set $\tau(\pi_e^{-1}(\mathrm{cl}(A_i)) \cap A)$ is included in the union of ν balls of radius ε. Consider such a partition for the ε given by Lemma 3.4. Choose $\eta' \in \mathbb{Q}^+$ such that

$$B(\widetilde{\pi}_e(\lambda); \eta') \subseteq \widetilde{\pi}_e(B(\lambda; \eta/2)). \tag{3.1}$$

Apply the induction hypothesis (identify N_e with R^n) to the set consisting of the union of the boundaries of the sets A_i to get a regular family of R^n, $H = (H_k; \lambda_k)_{1 \le k \le b}$, such that all the λ_k belong to $B(\widetilde{\pi}_e(\lambda); \eta')$. By Lemma 3.7 we may assume that each set $\mathrm{Int}(G_k)$ is connected. We may also assume it to be of nonempty interior (see Remark 3.6). Now this implies that each G_k is included in $\mathrm{cl}(A_j)$ for some j as follows. The set $A_i \cap \mathrm{Int}(G''_k)$ is an open set of empty boundary of $\mathrm{Int}(G''_k)$. Hence it is a connected component. But as $\mathrm{Int}(G''_k)$ is connected, it must be either the empty set or $\mathrm{Int}(G''_k)$ itself.

Then define

$$H'_k = \pi_e^{-1}(H_k).$$

By (3.1), $\lambda_k \in \widetilde{\pi}_e(B(\lambda; \eta/2))$. So, choose some $\lambda'_k \in \widetilde{\pi}_e^{-1}(\lambda_k) \cap B(\lambda; \eta/2)$.

As $\lambda'_k \in B(\lambda; \eta/2)$ and e does not belong to the ball $B(\lambda; \eta)$, we have $d(\lambda'_k; e) \ge \eta/2$. So, by Remark 3.3 (identify again N_e with R^n), the set H'_k is the graph of a $\mathbb{Q}-$Lipschitzian function. And, as H satisfies (ii), again by Remark 3.3, condition (ii) is clearly fulfilled by H'.

Fix m less than b. By Lemma 3.4 and the choice of ε, over each G_m the set A is the union of the graphs of some definable functions having a common regular ball $B(P; t)$ with t in $\mathbb{Q}^+$ (since we have seen that each G_m is included in some $\mathrm{cl}(A_j)$).

We would like to extend H' to a regular family containing these graphs, applying Lemma 3.8. The problem is that all the tangent spaces of these graphs may cover all the regular directions of H'_m near λ'_m. So we project a second time, now along P, and construct another family that we can extend to a family compatible with these graphs.

We may assume that π_P restricted to A is finite to one. Lemma 3.4 also states that we may require that P and $-P$ do not belong to the ball $B(\lambda'_m; r)$.

Note that as λ'_m is regular for $H'_m \cup H'_{m+1}$, by Observation (1), there exists $r \in \mathbb{Q}^+$ such that $B(\lambda'_m; r)$ is regular for $H'_m \cup H'_{m+1}$. Moreover, as $\lambda'_m \in B(\lambda; \eta/2)$, we may choose r in such a way that we have $B(\lambda'_m; r) \subseteq B(\lambda; \eta)$. Note also that there exists a constant $r' \in \mathbb{Q}^+$ such that

$$B(\widetilde{\pi}_P(\lambda'_m); r') \subseteq \widetilde{\pi}_P(B(\lambda'_m; r/2)), \tag{3.2}$$

where the first ball is taken in S^{n-1}.

To complete the proof we need a lemma.

Lemma 3.11. *Let $l \in S^n$, $r \in \mathbb{Q}^+$ and $n_1 \in \mathbb{N}$. Let C be a closed subset of $\mathbb{G}_{n+1}$ and P an element of S^n such that*

$$d(P; \Lambda(C)) \geq t, \tag{3.3}$$

where $t \in \mathbb{Q}^+$. Then there exists $\alpha \in \mathbb{Q}^+$ such that for any $P_1, \dots, P_{n_1}$ in C and any $y \in \widetilde{\pi}_P(B(l; r/2))$ there exists $\widehat{\lambda} \in B(l; r/2) \cap \widetilde{\pi}_P^{-1}(y)$ such that

$$d\left(\widehat{\lambda}; \Lambda\left(\bigcup_{i=1}^{n_1} P_i\right)\right) \geq \alpha.$$

The proof of this lemma will be given later; we first show that it is enough to finish the proof of the proposition. Let n_0 be the maximal value of the cardinality of fibers of $\pi_{P_{|A}}$. Applying this lemma with $n_1 = 2n_0$, $C = \tau(A \cap G'_m)$ and $l = \lambda'_m$, we get a constant α.

Consider $(V_\sigma)_{\sigma \in \Sigma}$, a finite partition of N_P into definable sets, such that for any V_σ

$$\tau(A \cap \pi_P^{-1}(V_\sigma)) \subseteq \bigcup_{i=0}^{n_0} B(P_i; \alpha/2),$$

for some $P_1, \dots, P_{n_0}$ in $\tau(A \cap G'_m)$.

Note that, as a consequence of Lemma 3.11, this implies that for any elements σ and σ' in Σ and for any $y \in \widetilde{\pi}_P(B(\lambda'_m; r/2))$, there exists $\widehat{\lambda} \in B(\lambda'_m; r/2) \cap \widetilde{\pi}_P^{-1}(y)$ such that

$$d\left(\widehat{\lambda}\,;\, \Lambda(\tau(\pi_P^{-1}(V_\sigma \cup V_{\sigma'}) \cap G'_m \cap A))\right) \geq \frac{\alpha}{2}. \tag{3.4}$$

Apply the induction hypothesis to get a regular family of hypersurfaces H'' of R^n (identify N_P with R^n) compatible with the boundaries of the sets V_σ. Do it in such a way that all the associated lines λ''_k are elements of $B(\widetilde{\pi}_P(\lambda'_m); r')$.

Define now

$$\widehat{H}_k = \pi_P^{-1}(H''_k). \tag{3.5}$$

The compatibility with the boundaries of the sets V_σ implies that every G''_k is included in a subset $\mathrm{cl}(V_\sigma)$ (by the argument we used for G_k and the partition $(A_i)_{i\in I}$). Thus, according to (3.4) for $y = \lambda''_k$, for each $k > 1$ there exists $\widehat{\lambda}_k \in B(\lambda_m; r/2) \cap \widetilde{\pi}_P^{-1}(\lambda''_k)$ such that

$$d\left(\widehat{\lambda}_k\,;\,\Lambda(\tau(\pi_P^{-1}(G''_k) \cap G'_m \cap A))\right) \geq \frac{\alpha}{2}. \tag{3.6}$$

Also, as a consequence of (3.4) and the compatibility of the G''_k with the sets, we can find $\widehat{\lambda}_1 \in B(\lambda_m; r/2) \cap \widetilde{\pi}_P^{-1}(\lambda''_1)$ such that

$$d\left(\widehat{\lambda}_1\,;\,\Lambda(\tau(\pi_P^{-1}(G''_0 \cup G''_1) \cap G'_m \cap A))\right) \geq \frac{\alpha}{2}. \tag{3.7}$$

Note that, as $P \notin B(\lambda'_m; r)$, we have for each k

$$d(\widehat{\lambda}_k; P) \geq \frac{r}{2}.$$

Hence, by construction and Remark 3.3, as $\widehat{\lambda}_k \in \widetilde{\pi}_P^{-1}(\lambda''_k)$, this implies that the family $\widehat{H}$ is a regular system of hypersurfaces. Moreover, as $B(\lambda'_m; r/2) \subseteq B(\lambda; \eta)$, all the $\widehat{\lambda}_k$ belong to $B(\lambda; \eta)$. Note also that, as $B(\lambda'_m; r)$ is regular for $H'_m \cup H'_{m+1}$ and $\widehat{\lambda}_k \in B(\lambda_m; r/2)$, the ball $B(\widehat{\lambda}_k; r/2)$ is regular for $H'_m \cup H'_{m+1}$.

So we can apply Lemma 3.9 to H' and $\widehat{H}$ to obtain a regular family of hypersurfaces $\widetilde{H}$ which is compatible with the sets $G'_m \cap \widehat{H}_k$.

But, by (3.5) we have

$$\pi_P^{-1}(G''_k) = \widehat{G}_k.$$

Together with (3.6) this implies for $k > 1$

$$d\left(\widehat{\lambda}_k\,;\,\Lambda(\tau(\widehat{G}_k \cap G'_m \cap A))\right) \geq \frac{\alpha}{2}. \tag{3.8}$$

Similarly, from (3.7) we obtain

$$d\left(\widehat{\lambda}_1\,;\,\Lambda(\tau((\widehat{G}_0 \cup \widehat{G}_1) \cap G'_m \cap A))\right) \geq \frac{\alpha}{2}. \tag{3.9}$$

Note that as for each $1 \leq k \leq \widehat{b}$ we have $\lambda_k \in B(\lambda'_k; r)$, λ_k is regular for $H'_m \cup H'_{m+1}$. Recall that Lemma 3.9 also states that $\widetilde{\lambda}_m = \widehat{\lambda}_1$, $\widetilde{\lambda}_{k+m} = \widehat{\lambda}_k$ and

$$\widetilde{G}_{k+m} \subseteq H'_m \cup H'_{m+1} \cup (\widehat{G}_k \cap G'_m) \tag{3.10}$$

for each $1 \leq k \leq \widehat{b}$.

But, as for any k the vector $\widehat{\lambda}_k$ is regular for $H'_m \cup H'_{m+1}$ by (3.8), (3.9) and (3.10), this implies that for each $m \leq j \leq m + \widehat{b}$, the vector $\widetilde{\lambda}_j$ is regular for $\widetilde{G}_j \cap A$. So, by Lemma 3.8, we can extend $\widetilde{H}$ to a family compatible with the set $\bigcup_{j=m}^{m+\widehat{b}} \widetilde{G}_j \cap A$. But Lemma 3.9 also states that $\bigcup_{j=m}^{m+\widehat{b}} \widetilde{G}_j = G'_m$. Therefore the family obtained is indeed compatible with $G'_m \cap A$. Since all the extensions do not modify the family outside G'_m we can carry out the construction over all the G'_m successively. This provides the desired family. □

It remains to prove Lemma 3.11. For this it will be convenient to work up to a coordinate system. We will provide S^n with projective coordinates in the following sense. Let U_i^+ (resp. U_i^-) denote

$$\{x \in S^n \mid x_i \geq \epsilon\}$$

(resp. $x_i \leq -\epsilon$) with $\epsilon \in \mathbb{Q}^+$ small enough. Then define $h_i : U_i \to R^n$ by $h_i(x_1; \ldots; x_n) = (x_1/x_i; \ldots; x_n/x_i)$. Note that h_i is a $\mathbb{Q}$-Lipschitzian isomorphism.

Now through such a chart the set $S^n \cap N_P$ is a vector subspace and $\widetilde{\pi}_P$ becomes an orthogonal projection. The following lemma is fairly elementary, but describes a useful property of $\widetilde{\pi}_P$.

LEMMA 3.12. *Let $\lambda \in S^n$, $T \in \mathbb{G}_{n+1}$ and $x \in T \cap \widetilde{\pi}_P^{-1}(\lambda)$. Let v be a unit vector tangent at x to $\widetilde{\pi}_P^{-1}(\lambda)$. Then*

$$d(P;T) \leq d(v; S^n \cap T).$$

Proof. Let w be the vector of $S^n \cap T$ which realizes $d(v; S^n \cap T)$. Observe that the vectors x, P, and v are in the same two dimensional vector space. Moreover, $(x;v)$ is an orthonormal basis of this plane. Let $P = \alpha x + \beta v$ with $\alpha^2 + \beta^2 = 1$. Then $|P - (\alpha x + \beta w)| = \beta |v - w| \leq |v - w|$. □

Proof of Lemma 3.11. Up to a choice of the coordinate system of S^n (identify P with the last vector of the canonical basis) we can identify each $\Lambda(\{T\})$, for $T \in C$, with an element of $\mathbb{G}_{k,n}$, $k \leq n$, and the mapping $\widetilde{\pi}_P$ with the orthogonal projection onto R^{n-1}. Denote by Q the direction of this projection. By Lemma 3.12, hypothesis (3.3) implies that there exists $u \in \mathbb{Q}^+$ such that

$$d\left(Q; \Lambda(\{T\})\right) \geq u$$

for any $T \in C$.

This implies that Q is transverse to all the elements of the set C and that for any $x \in Q$ and any $P_1, \ldots, P_{n_0}$ in C

$$d\left(x\,; \bigcup_{i=1}^{n_0} \Lambda(P_i)\right) \leq \frac{1}{u} \cdot d\left(x; \bigcup_{i=1}^{n_0} \Lambda(P_i) \cap Q\right). \tag{3.11}$$

For any $y \in \widetilde{\pi}_P(B(l;r/2))$ the length of the line segment $\widetilde{\pi}_P^{-1}(y) \cap B(l;r)$ is bounded below by a strictly positive rational number α_0.

Let α be the rational number $\alpha_0 u/(2n_0)$. Then, using (3.11) one can easily see that if the conclusion of the lemma failed for some $y \in \widetilde{\pi}_P(B(l;r/2))$, we could cover the segment $\widetilde{\pi}_P^{-1}(y) \cap B(l;r)$ by n_0 segments of length less than $\alpha_0/(2n_0)$. This contradicts the fact that the length of this segment is not less than α_0. □

PROPOSITION 3.13. *Let A be a definable subset of R^n of empty interior. Then there exists a definable bi-Lipschitz homeomorphism $h : R^n \to R^n$ such that $h(A)$ has a regular projection.*

Proof. By the above proposition there exists a regular system of hypersurfaces $H = (H_k; \lambda_k)_{1 \le k \le b}$ compatible with A. We define h over $E(H_k; \lambda_k)$, by induction on k, in such a way that $h(E(H_k; \lambda_k)) = E(F_k; e_n)$ (and so $h(H_k) = F_k$), where F_k is the graph of a function η_k for e_n, the last vector of the canonical basis of R^n.

For $k = 1$ choose an orthonormal basis of N_{λ_1} and set $h(q) = (x_{\lambda_1}; q_{\lambda_1})$, where x_{λ_1} are the coordinates of $\pi_{\lambda_1}(q)$ in this basis. Let $k \ge 1$. By (i) the sets H_k and H_{k+1} are the graphs for λ_k of two Lipschitzian functions ξ_k and ξ'_k. For $q \in E(H_{k+1}; \lambda_k) \setminus E(H_k; \lambda_k)$ define $h(q)$ to be the element

$$h(\pi_{\lambda_k}(q); \xi_k \circ \pi_{\lambda_k}(q)) + (q_{\lambda_k} - \xi_k \circ \pi_{\lambda_k}(q))e_n.$$

Thanks to the property (ii) of Definition 3.5 we have $E(H_{k+1}; \lambda_{k+1}) = E(H_{k+1}; \lambda_k)$. Hence h is, in fact, defined over $E(H_{k+1}; \lambda_{k+1})$. Since ξ_k is Lipschitzian, this is a bi-Lipschitz homeomorphism. Note also that the image is $E(F_{k+1}; e_n)$, where F_{k+1} is the graph of the Lipschitzian function

$$\eta_{k+1}(q) = \eta_k \circ \pi_{e_n}(q) + (\xi'_k - \xi_k) \circ \pi_{\lambda_k} \circ h^{-1}(q; \eta_k \circ \pi_{e_n}(q)).$$

This gives h over $E(H_b; \lambda_b)$, and we can extend h to all of R^n as in the case $k = 1$. Now it is easy to check that h defines a bi-Lipschitz homeomorphism. □

3.3. About L-regular cell decompositions. $L-$regular decompositions were introduced by A. Parusiński to prove the existence of Lipschitz stratification of subanalytic sets ([P2], [P4], [K]). In Section 4 we will need a specific version of such a decomposition. We want to be able to choose the directions involved in this decomposition in a given finite subset S^n depending only on n. This is stated in Proposition 3.16. The proof is inspired from the proof of existence of $L-$regular decompositions given in [K].

We recall a definition from [K]:

DEFINITION 3.14. Let $\alpha \in \mathbb{Q}^+$ and let $A \in \mathcal{S}_n$. We say that A is $\alpha-$*flat* if there exists $P \in \mathbb{G}_{n-1,n}$ such that

$$\Lambda(\tau(A)) \subseteq \Lambda(B(P; \alpha)).$$

LEMMA 3.15. *Let $s \in \mathbb{N}$. There exist $\lambda_1, \dots, \lambda_N$ in S^n and $\alpha \in \mathbb{Q}^+$ such that for any $P_1, \dots, P_s$ in $\mathbb{G}_{n,n+1}$ we can find $i \le N$ such that*

$$\lambda_i \notin \Lambda\left(\bigcup_{i=1}^{s} B(P_i; \alpha)\right).$$

Proof. Let α be the rational number given by Lemma 3.4. Let $\lambda_1, \dots, \lambda_N$ in S^n, such that $\bigcup_{i=1}^N B(\lambda_i; \alpha/2) = S^n$. Let $P_1, \dots, P_s$ in $\mathbb{G}_{n,n+1}$ and suppose that for any $i \in \{1, \dots N\}$ we have $\lambda_i \in \Lambda(\bigcup_{i=1}^s B(P_i; \alpha/2))$. This implies that $\Lambda(\bigcup_{i=1}^s B(P_i; \alpha)) \supseteq S^n$. This contradicts Lemma 3.4. □

Using this lemma we can now prove the existence of the required $L-$regular decomposition:

Proposition 3.16. *There exists $\{\lambda_1, \dots, \lambda_N\} \subseteq S^n$ such that for any elements $A_1, \dots, A_m$ of $\mathcal{S}_{n+1}$, there exists a cylindrical decomposition $(C_i)_{i \in I}$ of R^{n+1} adapted to all the sets A_k, $1 \leq k \leq m$, such that for each open cell C_i, we can find $\lambda_{j(i)}$, $1 \leq j(i) \leq N$, regular for the boundary of C_i.*

Proof. According to Lemma 3.15 it is sufficient to prove by induction on n the following assertions: given $\alpha \in \mathbb{Q}^+$ and $A_1, \dots, A_m$ in $\mathcal{S}_n$, there exists a cylindrical decomposition adapted to $A_1, \dots, A_m$, such that the boundary of each open cell is the union of $2n$ definable subsets which are $\alpha-$flat.

For $n = 0$ the result is clear. Let $n \in \mathbb{N}^*$, $\alpha \in \mathbb{Q}^+$ and let $A_1, \dots, A_m$ be elements of $\mathcal{S}_{n+1}$. Denote by $\pi : R^{n+1} \to R^n$ the canonical projection. Choose a cylindrical definable cell decomposition $V = (V_j)_{j \in J'}$ adapted to $A_1, \dots, A_m$. Consider a partition $(W_j)_{j \in J}$ of R^n such that if $V_{j'}$ is a cell of V of empty interior in R^n, then $V_{j'} \cap \pi^{-1}(W_j)$ is $\alpha-$flat. Apply the induction hypothesis to the sets W_j and to the elements of the cylindrical decomposition of R^n defined by $(V_j)_{j \in J'}$ to get a cylindrical decomposition $(C'_i)_{i \in I'}$. Define

$$C_{i,j} = \pi^{-1}(C'_i) \cap V_j.$$

The boundary of an open cell $C_{i,j}$ is included in the union of $\pi^{-1}(\partial C'_i)$ and two subsets of two $\alpha-$flat graphs. This completes the induction step. □

4. On the preparation theorem

In this section we prove the preparation theorem. As we mentioned in the introduction, our statement is less precise than that in [P3] or [LR], since it does not give a precise description of the unit. However, it is exactly what we need for the construction of definable bi-Lipschitz trivializations. We will prove the preparation theorem for semi-algebraic subsets over any real closed field. As a consequence of this theorem, we will then prove a proposition concerning positive definable functions, which can be stated over any o-minimal structure satisfying it.

Given two definable functions, $f, g : A \to R$, we write $f \sim_K g$, for $K \subseteq R$, if there exist two positive constants C_1 and C_2 in K such that $C_1 f \leq g \leq C_2 f$.

Theorem 4.1 (Preparation Theorem). *Let $\xi : R^{n+1} \to R$ be a semi-algebraic function. Then there exists a semi-algebraic partition $(V_i)_{i \in I}$ of*

R^{n+1}, such that for any element V_i there exist semi-algebraic continuous functions a, $\theta : V_i \to R$, and $r \in \mathbb{Q}$ such that

$$\xi(q) = (q_{n+1} - \theta(x))^r \, a(x)U(q), \tag{4.1}$$

for $q = (x; q_{n+1}) \in V_i$, with U a semi-algebraic function over V_i bounded below and above by rational numbers.

Proof. We first prove the result in the particular case of a polynomial function. Let $Q : R^n \times R \to R$ be a polynomial function. Consider a partition of R^n such that, over any element of this partition, $Q(x; y)$ can be factorized in the following form:

$$A(x)(y - \theta_1(x)) \dots (y - \theta_k(x))((y - a_1(x))^2 + \delta_1(x)) \dots ((y - a_l(x))^2 + \delta_l(x))$$

with a_i, δ_i and θ_i semi-algebraic functions over R^n and with δ_i positive on the considered element of the partition. Clearly, up to a subpartition, we may suppose that over any element and for any i either $((y - a_i(x))^2 + \delta_i(x)) \sim_{\mathbb{Q}} (y - a_i(x))^2$ or $((y - a_i)^2 + \delta_i) \sim_{\mathbb{Q}} \delta_i$. Also we may require that for any i and j, either $|y - \theta_i(x)| \sim_{\mathbb{Q}} |y - \theta_j(x)|$, or $|y - \theta_i(x)| \sim_{\mathbb{Q}} |\theta_j(x) - \theta_i(x)|$, or $|y - \theta_j(x)| \sim_{\mathbb{Q}} |\theta_j(x) - \theta_i(x)|$. Thus we obtain a simultaneous preparation of all the terms of the factorization of Q. This completes the proof in the polynomial case.

We now consider the general case. There exists a partition of R^{n+1} such that ξ is algebraic over each element of this partition. So write $P(x; y; z) = \sum_{i=0}^{d} q_i(x; y)z^i$, where $P(x; y; \xi(x; y)) \equiv 0$ with q_i polynomial functions. Up to a subpartition, we may suppose that the terms $q_i(x; y)\xi(x; y)^i$ have constant signs and are comparable (for relation $\leq$) with each other. Let I be the set of indices for which the term is negative, and J the set of those for which it is positive. We may also suppose that the functions q_i, for $i \in \{1, \dots, d\}$, and ξ either are zero or do not vanish over an element of the partition. Thus we get

$$\sum_{i \in I} q_i(x; y)\xi(x; y)^i = -\sum_{i \in J} q_i(x; y)\xi(x; y)^i.$$

As all the terms are comparable with each other, both sums are equivalent to their maximal term. Hence, we can find $i_0 \in I$ and $j_0 \in J$ such that

$$q_{i_0}\xi^{i_0} \sim_{\mathbb{Q}} -q_{j_0}\xi^{j_0}.$$

Therefore

$$|\xi| \sim_{\mathbb{Q}} \left| \frac{q_{i_0}}{q_{j_0}} \right|^{1/(j_0 - i_0)}.$$

But the q_i are polynomials and so can be prepared. Then, as in the polynomial case, we can find a simultaneous preparation of q_{i_0} and q_{j_0} to get the desired preparation. □

Remark 4.2. Let α be an element of $\widetilde{R^p}$ (the Stone space of the Boolean algebra of definable sets; see [BCR1], [BCR2], [C], or Section 5.2 of this paper for precise definitions). If an o-minimal structure satisfies the preparation theorem, then the o-minimal extension over $k(\alpha)$ also satisfies the preparation theorem. Actually, given $\alpha \in \widetilde{R^p}$, if we have a preparation of $\xi : R^p \times R^n \to R$ by a and θ, the functions a_α and θ_α obviously provide a preparation of ξ_α. This follows from the definitions.

For the last result of this section we fix an o-minimal structure which is either the semi-algebraic sets over R or a polynomially bounded structure over $\mathbb{R}$ (and in this case $R = \mathbb{R}$).

The preparation theorem allows us to compare a semi-algebraic function to a product of powers of distance functions to semi-algebraic subsets, as described in the following proposition.

Proposition 4.3. *Let $\xi : R^n \to R$ be a positive definable function. Then there exists a partition of R^n such that over each element of this partition the function ξ is $\sim_R$ to a product of powers of distances to definable subsets of R^n.*

Proof. We proceed by induction on n. For $n = 1$ the result follows from

Theorem 4.1 or from Theorem 2.1 of [vD-S] (in the o-minimal case). Assume that it is true for some $n \geq 0$. Let $\lambda_1, \dots, \lambda_N$ be the elements of S^n given by Proposition 3.16. Applying Theorem 4.1 (or Theorem 2.1 of [vD-S] in the o-minimal case) to $\xi \circ A_i$, where A_i is an orthogonal linear mapping of R^n sending the vector λ_i onto the last vector of the canonical basis for $i \in \{1, \dots, N\}$, and taking the intersections of all the obtained partitions, we get a definable partition $(V_j)_{j \in J}$ of R^{n+1}. Therefore over each V_j and for each i we can find continuous functions $a, \theta : \pi_{\lambda_i}(V_j) \to R$ and $r \in \mathbb{R}$ such that

$$\xi(q) = (q_{\lambda_i} - \theta(x_{\lambda_i}))^r \, a(x_{\lambda_i}) U(q), \tag{4.2}$$

for $q = (x_{\lambda_i}; q_{\lambda_i}) \in V_j$, with U a continuous semi-algebraic function over V_j bounded below and above by rational numbers.

Apply Proposition 3.16 to the family consisting of all the sets of the partition $(V_j)_{j \in J}$ and the zero locus of ξ. This gives a partition $(V'_j)_{j \in J'}$ such that each element V'_j which is open is of the form

$$\{q = (x_{\lambda_k}; q_{\lambda_k}) \in \pi_{\lambda_k}(V'_j) \; \oplus \langle \lambda_k \rangle \;\; | \;\; \xi_1(x_{\lambda_k}) \leq q_{\lambda_k} \leq \xi_2(x_{\lambda_k})\},$$

for some $k \in \{1, \dots, N\}$, where $\xi_\nu : \pi_{\lambda_k}(V'_j) \to R$, $\nu = 1, 2$, are $\mathbb{Q}-$Lipschitzian functions, and such that the function ξ is of the form (4.2) for each vector λ_i. Let V'_j be an open element of the partition V'. By the induction hypothesis (identify N_{λ_k} with R^n) it is sufficient to prove the result for the function $|q_{\lambda_k} - \theta(x_{\lambda_k})|$. By the compatibility of the partition with the zero locus of ξ, we have either $\theta \leq \xi_1$ or $\theta \geq \xi_2$. Up to a subpartition we may assume that only one case occurs over V_i, for instance, $\theta \leq \xi_1$. So we see that $|q_{\lambda_k} - \theta(x_{\lambda_k})|$ is $\sim_{\mathbb{Q}}$ either to $|q_{\lambda_k} - \xi_1(x_{\lambda_k})|$ or to $|\xi_1(x_{\lambda_k}) - \theta(x_{\lambda_k})|$. In the first case, as ξ_1 is Lipschitzian, we deduce that $|q_{\lambda_k} - \theta(x_{\lambda_k})|$ is $\sim_{\mathbb{Q}}$ to the distance to the graph of ξ_1 for λ_k. In the second case this is a consequence of the induction hypothesis. For sets V'_j of strictly positive codimension one can also deduce the result from the induction hypothesis. □

5. Lipschitz triangulations

Throughout this section we fix an o-minimal structure $\mathcal{S}$ over R in which Theorem 4.1 is valid. So, in particular, the results are true for the semi-algebraic sets over an arbitrary real closed field R or for a polynomially bounded structure over $\mathbb{R}$.

5.1. Lipschitz triangulations and proof of the main theorem. As mentioned in the introduction, we will define a triangulation adapted to the study of the Lipschitzian type. This will be a homeomorphism onto a simplicial complex, so it will be a triangulation in the usual sense. Clearly this cannot be a bi-Lipschitz homeomorphism. We shall require that over each simplex the distances are preserved up to "some contractions" explicitly described along the directions of specific coordinate systems of R^n. All these contractions will be defined by sums of products of powers of distance to faces. Namely, given a simplex σ, we will consider functions $\phi_{\sigma,i}$ over σ in a complex K which are a finite sum of functions of type

$$d(q;\sigma_1)^{\alpha_1}. \dots .d(q;\sigma_k)^{\alpha_k}, \tag{5.1}$$

where $\sigma_1, \dots, \sigma_k$ are simplices of K and $\alpha_1, \dots, \alpha_k$ are real numbers. A finite sum of such function will be called a *standard simplicial function.* Indeed, Definition 5.2 will involve standard simplicial functions over $\sigma \times \sigma$, that is, such sums of distances involving q or another point q'.

What is important is that two sets having the same triangulation will be Lipschitz homeomorphic (with same coordinates systems and equivalent contraction functions). Let us point out that distances in the given set will not be equivalent to distances in the simplicial complex.

The reader can refer to [C] or [S] for basic definitions about triangulations. Given a point $q \in R^n$, we write $q_1, \dots, q_n$ for the coordinates of q in the canonical basis and $\pi_i : R^n \to R^i$ for the canonical projection.

First we introduce the concept of a tame system of coordinates on R^n, which will describe the directions of contractions:

DEFINITION 5.1. A *tame system of coordinates on R^n* is a family of functions $(\psi_1; \dots; \psi_n)$ of the form

$$\psi_i(q) = \frac{q_i - \theta_i(\pi_{i-1}(q))}{|\theta_i(\pi_{i-1}(q)) - \theta'_i(\pi_{i-1}(q))|} \tag{5.2}$$

(and 0 whenever $\theta_i \circ \pi_{i-1}(q) = \theta'_i \circ \pi_{i-1}(q)$), where θ_i and θ'_i are piecewise linear functions on R^{i-1}.

This gives the following definition:

DEFINITION 5.2. A *Lipschitz triangulation of* R^n is the data of a finite simplicial complex K together with a homeomorphism $h : |L| \to R^n$, where L is a union of open simplices of K, such that for every $\sigma \in L$ there exist $\varphi_{\sigma,1}, \dots, \varphi_{\sigma,k}$, standard simplicial functions over $\sigma \times \sigma$, satisfying for any q and q' in σ

$$|h(q) - h(q')| \sim_R \sum_{i=1}^{n} \varphi_{\sigma,i}(q;q') \cdot |q_{i,\sigma} - q'_{i,\sigma}|, \tag{5.3}$$

where $(q_{1,\sigma}, \dots, q_{n,\sigma})$ is a tame system of coordinates of R^n. Let $A_1, \dots, A_k$ be subsets of R^n. A Lipschitz triangulation of $A_1, \dots, A_k$ is a Lipschitz triangulation of R^n such that each $h^{-1}(A_i)$ is a union of open simplices.

With this definition two definable subsets admitting the same simplicial complex as definable triangulation, with $\sim_R$ functions φ_σ and the same tame systems of coordinates, are definably bi-Lipschitz homeomorphic. So simultaneous Lipschitz triangulations of fibers of a family will provide definable trivializations. We first prove the existence of such triangulations.

THEOREM 5.3. *Let $A_1, \dots, A_k$ be definable subsets of R^n. Then there exists a definable Lipschitz triangulation of $A_1, \dots, A_k$.*

Proof. In fact, we will prove the following stronger statement: given a finite family of definable functions $(\eta_l)_{l \in L}$, we can construct a Lipschitz triangulation of $A_1, \dots, A_k$ such that over each simplex each function $\eta_l \circ h$ is $\sim_R$ to a standard simplicial function.

We prove this by induction on n. We denote by $\pi : R^{n+1} \to R^n$ the canonical projection. For $n = 1$ the result is clear. Assume that it is true for some $n \geq 1$. We may assume that the subsets $A_1, \dots A_k$ are of empty interior, since it is enough to construct a triangulation of their boundaries.

Apply Proposition 4.3 to each function η_l to obtain a partition $(V_i)_{i \in I}$ of R^{n+1} such that, over each V_i, each function η_l is equivalent to a product of powers of distances to definable subsets $(W_j)_{j \in J}$ of R^{n+1}. By Proposition 3.13 we may assume that all the boundaries of the elements of the families $(V_i)_{i \in I}$, $(A_i)_{1 \leq i \leq k}$ and $(W_i)_{i \in J}$ are included in the union of a finite number of graphs of some Lipschitzian functions $\theta_1 \leq \dots \leq \theta_\mu$. By Observation (4) applied to the functions θ_i and to the functions $d(x; \pi(\partial W_i))$, there exist a finite number of functions $\xi_1, \dots, \xi_m$ such that over each cell delimited by the graphs of two consecutive functions ξ_i and ξ_{i+1} all the functions $|q_{n+1} - \theta_i|$ are comparable to each other and comparable to the functions $d(x; \pi(\partial W_i))$.

Consider a cylindrical definable cell decomposition of R^{n+1} adapted to the graphs of the functions ξ_i and the sets W_i. This provides a partition $X_1, \dots, X_s$ of R^n. Refining it we may assume that the functions $d(x; \pi(\partial W_i))$ are comparable with each other. Apply the induction hypothesis to get a Lipschitz triangulation $(h; K)$ of the sets X_i, $1 \le i \le s$.

Also by the induction hypothesis, we can do this in such a way that over each simplex, each function $|\xi_j - \theta_i| \circ h$ and all the functions $q \to d(h(x); \pi(\partial W_j \cap \Gamma_{\theta_i}))$ are $\sim_R$ to standard simplicial functions, where Γ_{θ_i} is the graph of θ_i.

The simplicial complex is constructed as for topological triangulations (see [C], [S]). Let $\zeta_1 \le \cdots \le \zeta_m$ be piecewise linear functions over $|K|$ such that $\zeta_i = \zeta_{i+1}$ whenever $\xi_i \circ h = \xi_{i+1} \circ h$. Let also $\zeta_0 = \zeta_1 - 1$ and $\zeta_{m+1} = \zeta_m + 1$. Let

$$N = \{(x; q_{n+1}) \in R^n \times R \;\mid\; \zeta_0(x) \le q_{n+1} \le \zeta_{m+1}(x)\}.$$

We obtain a polyhedral decomposition of N by taking the inverse image by $\pi_{|N}$ of the simplices of K of dimension $n-1$ on the one hand, and by taking all the images of $|K|$ by the mappings $x \to (x; \zeta_i(x))$ on the other hand. After a barycentric subdivision of this polyhedral we obtain a simplicial complex $\widetilde{K}$.

The union of the graphs gives a simplicial complex. Let $\widetilde{L}$ be the union of open simplices σ lying over $|K|$ and included in

$$\{(x; q_{n+1}) \in R^n \times R \mid \zeta_0(x) < q_{n+1} < \zeta_{m+1}(x)\}.$$

Define now over $\widetilde{L}$ the desired homeomorphism $\widetilde{h}$ by

$$\widetilde{h}(x; t\,\zeta_i(x) + (1-t)\zeta_{i+1}(x)) = (h(x); t\,\xi_i(h(x)) + (1-t)\xi_{i+1}(h(x)))$$

for $1 \le i \le m$, $x \in R^n$ and $t \in [0;1]$. Define also

$$\widetilde{h}(x; t\,\zeta_0(x) + (1-t)\,\zeta_1(x)) = \left(h(x); \xi_1(h(x)) - \frac{t}{1-t}\right)$$

and

$$\widetilde{h}(x; t\,\zeta_{m+1}(x) + (1-t)\,\zeta_m(x)) = \left(h(x); \xi_m(h(x)) + \frac{t}{1-t}\right)$$

for $t \in [0;1)$. This defines a homeomorphism $\widetilde{h} : |\widetilde{L}| \to R^{n+1}$.

It remains to check that over each simplex σ the mapping $\widetilde{h}$ fulfills an inequality of type (5.3). Let σ be a simplex of $\widetilde{K}$, q and q' two points of σ. Set now $q = (x; t\zeta_i(x) + (1-t)\zeta_{i+1}(x))$ and $q' = (x'; t'\zeta_i(x') + (1-t')\zeta_{i+1}(x'))$ with $0 \le i \le m$ for t and t' in $[0;1]$. Then define $q'' = (x; t'\zeta_i(x) + (1-t')\zeta_{i+1}(x))$.

We begin with the case when $1 \le i \le m-1$. Let $p = \widetilde{h}(q)$, $p' = \widetilde{h}(q')$ and $p'' = \widetilde{h}(q'')$. We can consider x, x', p, p' and p'' as functions of q and q'. As ξ_i and ξ_{i+1} are Lipschitzian functions, we have over $\sigma \times \sigma$

$$|p - p'| \sim_R |p - p''| + |h(x) - h(x')|. \tag{5.4}$$

Let σ' be the simplex of K containing $\pi(\sigma)$. Since h is a Lipschitz triangulation, we can find functions $\varphi_{\sigma',1},\dots,\varphi_{\sigma',n}$ and a tame system of coordinates $(x_{1,\sigma};\dots;x_{n,\sigma})$ such that for any x and x' in σ'

$$|h(x)-h(x')| \sim_R \sum_{l=1}^{n} \varphi_{\sigma',l}(x;x')|x_{l,\sigma}-x'_{l,\sigma}|. \tag{5.5}$$

On the other hand, as $\pi(q)=\pi(q'')$, by construction we have

$$|p_{n+1}-p''_{n+1}| \sim_R |q_{n+1}-q''_{n+1}| \cdot \frac{\xi_{i+1}(h(x))-\xi_i(h(x))}{\zeta_{i+1}(x)-\zeta_i(x)}.$$

Recall that we have constructed the triangulation K in such a way that $(\xi_{i+1}-\xi_i)\circ h$ are respectively $\sim_R$ to standard functions ψ_i of K. Note that the composite $\psi_i\circ\pi$ gives a standard simplicial function of $\widetilde{K}$. Moreover, ζ_i and ζ_{i+1} defines a tame coordinate on R^{n+1}. Denote it by $q_{n+1,\sigma}$. Thus we have

$$|p-p''| \sim_R |q_{n+1,\sigma}-q'_{n+1,\sigma}| \cdot \varphi_{\sigma,n+1}(q;q') \tag{5.6}$$

for a standard simplicial function $\varphi_{\sigma,n+1}$ (which here actually depends only on q). Define $\varphi_{\sigma,l}(q;q')=\varphi_{\sigma',l}(\pi(q);\pi(q'))$. Then by (5.6), (5.5) and (5.4) we get the desired equivalence.

Now consider the case $i=0$. By construction we have over $\sigma\times\sigma$,

$$|p_{n+1}-p''_{n+1}| \sim_R \frac{1}{(q_{n+1}-\zeta_0(x))(q'_{n+1}-\zeta_0(x'))} \cdot |q_{n+1}-q''_{n+1}|. \tag{5.7}$$

Note that in this case $|q_{n+1}-q''_{n+1}|$ is $\sim_R$ to the linear coordinate defined by ζ_0 and ζ_1, and $1/(q_{n+1}-\zeta_0(x))$ is $\sim_R$ to distance to the graph of ζ_0. Therefore we can apply the same argument as in the above case to get the desired equivalence.

The same argument works for the case $i=m$. This proves the Lipschitz property of the triangulation $\widetilde{h}$.

It remains to check that the given functions η_l are $\sim_R$ to standard simplicial functions over any simplex σ. Let $\sigma\in\widetilde{K}$; the set $h(\sigma)$ is included in the cell delimited by ξ_i and ξ_{i+1}, for some $1\le i\le m-1$. As the boundaries of the V_i are subsets of the union of the graphs of the functions $\xi_1,\dots,\xi_m$, over $h(\sigma)$ the functions η_l are $\sim_R$ to a product of powers of distances to some W_j, so it suffices to show the result for functions of type $q\to d(h(q);W_j)$. As $\widetilde{K}$ is also a triangulation of the sets W_j, for each j we have that either $h(\sigma)$ is included in W_j or the distance to W_j is $\sim_R$ to the distance to its boundary. In the first case the result is obvious, since the function is zero over σ. By construction, the boundary ∂W_j is included in the union of the graphs of the functions θ_ν.

Moreover, clearly we have for any $\nu\in\{1,\dots,\mu\}$

$$d(q;\partial W_i\cap\Gamma_{\theta_\nu}) \sim_R |q_{n+1}-\theta_\nu(x)| + d(x;\pi(\partial W_i\cap\Gamma_{\theta_\nu})), \tag{5.8}$$

where $q=(x;q_{n+1})$ in $R^n\times R$.

As both terms of the right-hand-side are positive, the sum is $\sim_R$ to the maximum of these two terms, and hence $\sim_R$ to one of them, since they are comparable over $h(\sigma)$. Note that clearly

$$d(q;\partial W_i) = \min_{1\leq\nu\leq\mu} d(q;\partial W_i \cap \Gamma_{\theta_\nu}).$$

But as $g_\nu = d(\pi(q);\pi(\partial W_i \cap \Gamma_{\theta_\nu}))$ are comparable functions and are comparable with all the functions $|q_{n+1}-\theta_\nu(x)|$, the function $d(q;\partial W_i)$ is equivalent over $h(\sigma)$ to one of the functions g_ν or to some function $|q_{n+1}-\theta_\nu(x)|$.

Recall that we have required the triangulation $(h;K)$ to be such that $d(h(x);\pi(\partial W_j \cap \Gamma_{\theta_\nu}))$ is $\sim_R$ to a standard simplicial function of K. So it suffices to prove that the function $|p_{n+1}-\theta_\nu \circ h|$ is $\sim_R$ over σ to a standard simplicial function of $\widetilde{K}$. Assume that σ is between the graphs of ζ_i and ζ_{i+1}. We can write

$$|p_{n+1}-\theta_\nu \circ h| = p_{n+1} - \xi_i \circ h + (\xi_i \circ h - \theta_\nu \circ h)$$

if $\nu \leq i$ and

$$|p_{n+1}-\theta_\nu \circ h| = \xi_{i+1} \circ h - p_{n+1} + (\theta_\nu \circ h - \xi_{i+1} \circ h)$$

if $\nu \geq i+1$. But, by (5.6) and (5.7), we have over σ

$$p_{n+1} - \xi_i(h(x)) \sim_R |q_{n+1} - \zeta_i(x)| \,.\, \varphi_{\sigma,n+1}(q;q').$$

Clearly, the function $|q_{n+1}-\zeta_i(x)|$ is $\sim_R$ to a standard simplicial function. As all the $|\xi_i \circ h - \theta_j \circ h|$ have been assumed to be equivalent to standard simplicial functions, the theorem is proved. □

REMARK 5.4. If σ is a simplex such that $h(\sigma)$ is bounded, it is not necessary to involve in the definition of Lipschitz triangulation standard simplicial functions depending on q and q'. More precisely, instead of (5.3), it is enough to require

$$|h(q)-h(q')| \sim_R \sum_{i=1}^{n} \varphi_{\sigma,i}(q)\cdot|q_{i,\sigma}-q'_{i,\sigma}|$$

with $\varphi_{\sigma,i}(q)$ standard simplicial functions of q. Moreover, in this case, by construction the simplices involved in the expression of the functions $\varphi_{\sigma,i}$ are of dimension $i-2$, so that for $i=1$, the simplex is reduced to the empty set and $\varphi_{\sigma,1} \equiv 1$.

5.2. Proof of the main theorem. As an application of the above theorem we will now generalize the argument of [C] to bi-Lipschitz triviality. That is, we will realize simultaneous triangulations by triangulating the generic fibers.

We shall need some basic facts about the Stone space of the Boolean algebra of definable sets. We recall some results about ultrafilters from [C] or [BCR1]–[BCR2].

A definable ultrafilter α of R^n is a collection of definable subsets satisfying

(1) $\emptyset \notin \alpha$,
(2) A and B belong to α iff $A \cap B \in \alpha$,
(3) $A \in \alpha$ iff $R^n \setminus A \notin \alpha$.

We denote by $\widetilde{R^n}$ the set of definable ultrafilters together with the topology that makes $\widetilde{U} = \{\alpha \mid U \in \alpha\}$, for $U \subseteq R^n$ definable, a basis of closed-open sets. This topology is quasi-compact. Let $\mathcal{D}(\widetilde{U})$ denote the set of definable functions over U. We obtain a sheaf over $\widetilde{R^n}$. The residue field of the local ring at α is denoted by $k(\alpha)$, which is a real closed extension of R. Now for $U \subseteq R^p \times R^n$ definable we set

$$U_\alpha = \{g \in k(\alpha)^p \mid \exists V \in \alpha, \forall x \in V, (g(x); x) \in U\}.$$

This defines an o-minimal structure over $k(\alpha)$. A definable function $f : U \to R$ induces $f_\alpha : U_\alpha \to k(\alpha)$ defined by $\Gamma_{f_\alpha} = (\Gamma_f)_\alpha$ (where Γ_f denotes the graph).

In model theory the set $\widetilde{R^n}$ can be identified with the sets of complete n types with the Stone topology [Ma]. Then $k(\alpha)$ is an elementary extension which realizes the type α.

These notions provide powerful methods to prove Theorem 2.2, giving an elegant application of the notion of Lipschitz triangulations. Roughly speaking, we will obtain a "parameterized version" by working in a field of functions (of the parameters). A parameterized version of the triangulation theorem immediately gives a trivialization theorem.

Proof of Theorem 2.2. Let $\alpha \in \widetilde{R^p}$ and let $(h; K)$ be a Lipschitz triangulation of A_α. We may suppose that this triangulation has its vertices in $\mathbb{Q}^n$. This implies (see [C]) that we can find $U_\alpha \in \alpha$, a simplicial complex $K' \subseteq R^n$ with $|K'|_{k(\alpha)} = |K|$, and a mapping $H : L' \times U_\alpha \to R^n \times U_\alpha$, with L' the union of some open simplices of K', such that each $(H_t; K')$ is a Lipschitz triangulation of A_t. Let $\psi_t = H_t \circ H_{t_0}^{-1}$ for some $t_0 \in U_\alpha$. Then, by the definition of Lipschitz triangulations, each ψ_t is a bi-Lipschitz homeomorphism. This implies that ψ_α is a bi-Lipschitz homeomorphism. Therefore there exists $U'_\alpha \in \alpha$ such that the mapping $\psi : A_{t_0} \times U'_\alpha \to A_{U'_\alpha}$ defined by $\psi(x; t) = (\psi_t(x); t)$ is a family of bi-Lipschitz homeomorphisms. The sets $\widetilde{U'_\alpha}$ constitute an open covering of $\widetilde{R^p}$. By the compactness of this set we have the desired covering. □

5.3. Lipschitzianity with respect to parameters. Isotopies constructed with vector fields are also Lipschitzian with respect to the parameters. We will prove this in the case of a family with bounded fibers. However, as shown by Example 2.4, we cannot expect such a result to hold in general for non-bounded sets. Isotopies constructed using Lipschitz stratifications are always given by integration of vector fields and involve bounded sets, since integration theorems require a properness assumption (see [P1], [P2], or [M]).

We first prove a proposition to help us in this direction, which can be regarded as a Lipschitzian version of Theorem 5.19 of [C]. We recall that the dimension of $\alpha \in \widetilde{R^n}$ is the minimal $p \in \mathbb{N}$ such that there exists $U \in \alpha$ of dimension p (as a definable set).

PROPOSITION 5.5. *Let $\alpha \in \widetilde{R^p}$, $\dim \alpha = p$, and let $f : R^n \times R^p \to R$ be a definable function such that f_α is Lipschitzian. Then the generic fiber of the differential $(d_x f)_\alpha$ is bounded in $k(\alpha)$ (partial derivatives exist generically) over any bounded subset of $k(\alpha)^n$.*

Proof. The problem comes from derivatives with respect to parameters. Suppose that the conclusion fails. This implies that we can find a bounded curve $\gamma :]0;\varepsilon[\to (\Gamma_f)_\alpha$ (where Γ_f denotes the graph of f), such that $e_{n+1} \in \tau = \lim_{t\to 0} T_{\gamma(t)}\, \Gamma_{f_{k(\alpha)}}$ (where e_{n+1} is the $(n+1)$st vector of the canonical basis). As γ is bounded, we can extend it at zero. As f_α is Lipschitzian, e_{n+1} is regular for $(\Gamma_f)_\alpha$. This implies that τ and $k(\alpha)^{n+1} \times \{0\}$ are not transverse in $k(\alpha)^{n+1} \times k(\alpha)^p$. Let $\mu : [0;\varepsilon[\times U' \to R^{n+1} \times R^p$ be such that $\mu_\alpha = \gamma$. If ε is chosen small enough, the wing μ is a smooth manifold with boundary. Note that the image of $\mu'(x) = \mu(x;0)$ is transverse to $R^n \times \{0\}$ in $R^n \times R^p$. But, as the image of μ is a C^1 manifold with boundary, at any point $x \in U'$ the limit $P(x) = \lim_{t\to 0} T_{\mu(x;t)}\, \mu([0;\varepsilon[\times U')$ contains $T_{(x;\mu'(x))}\, \Gamma_{\mu'}$. So, as obviously $P_\alpha \subseteq \tau$, we get a contradiction. □

REMARK 5.6. As shown by Example 2.4, the above proposition is not true over a non-bounded set.

Let $\alpha \in \widetilde{R^p}$, $\dim \alpha = p$, and A be a definable family of sets of $R^n \times R^p$ such that each A_t is a bounded set. Consider a family of functions $f : R^n \times R^p \to R \times R^p$ satisfying over A for each $1 \le i \le n+p$

$$\left| \left(\frac{\partial f}{\partial x_i} \right)_\alpha \right| \le M,$$

for $M \in k(\alpha)$. Then we can find $U \in \alpha$ and a continuous function $C : U \to R$ such that for $(x;t) \in A_U$

$$|df(x;t)| \le C(t).$$

This implies that we can find a continuous function $C : U \times U \to R$, such that for any $(q;q') \in A_U \times A_U$

$$|f(x;t) - f(x';t')| \le C(t;t') \cdot |q - q'|. \tag{5.9}$$

Thus this argument leads us to the following proposition, which is an improvement of Theorem 2.2 in the bounded case:

Proposition 5.7. *Let $A \in \mathcal{S}_{n+p}$ be such that each A_t is bounded, for $t \in R^p$. Then the isotopy given by Theorem 2.2 satisfies*

$$|h(x;t) - h(x';t')| \leq C(t;t') \cdot |q - q'|$$

and

$$|h^{-1}(x;t) - h^{-1}(x';t')| \leq C(t;t') \cdot |q - q'|,$$

where $q = (x;t)$ and $q' = (x';t')$, for any $(q;q') \in A_U \times A_U$.

Proof. Let $\alpha \in \widetilde{R^p}$. To trivialize over an element of α we may assume that α is of dimension p, since we can work up to a coordinate system of the support of the ultrafilter. We then apply inequality (5.9) to the components of the fibers ψ_α and ψ_α^{-1} in the proof of Theorem 2.2. □

Lipschitz triangulations can have another point of interest:

Remark 5.8. In [SS] L. Siebenmann and D. Sullivan asked whether the number of Lipschitz types of analytic germs is countable. It is well known that this is true if we replace the Lipschitz type by the topological type. Actually, this is a consequence of the existence of triangulations (in the topological sense), since it is clear that the number of topological types of finite simplicial complexes is countable. Therefore we see that another application of the existence of Lipschitz triangulations for polynomially bounded o-minimal structure is a positive answer to the conjecture by L. Siebenmann and D. Sullivan (considering the o-minimal structure of global subanalytic sets). In fact, we can show that the number of Lipschitz types in a polynomially bounded o-minimal structure over $\mathbb{R}$ is the cardinality of the set of exponents Λ involved in the preparation theorem of L. van den Dries and P. Speissegger [vD-S]. The details will appear in the forthcoming paper [V].

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