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P-HARMONIC FUNCTIONS AND THE MINIMAL GRAPH EQUATION IN A RIEMANNIAN MANIFOLD

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Abstract. We study the minimal graph equation in a Riemannian manifold. After explaining the geometric meaning of the solutions and giving some entire solutions of the minimal graph equation in Nil space and in a hyperbolic space we find a link among p-harmonicity, horizontal homothety, and the minimality of the vertical graphs of a submersion. We also study the transformation of the minimal graph equation under the conformal change of metrics. We prove that the foliation by the level hypersurfaces of a p-harmonic submersion is a minimal foliation with respect to a conformally deformed metric. This implies, in particular, that the graph of any harmonic function from a Euclidean space is a minimal hypersurface in a complete conformally flat space, thus providing an effective way to construct (foliations by) minimal hypersurfaces.

1. Preliminaries

In this paper, all objects, manifolds, vector fields, and maps are assumed to be smooth unless otherwise stated. Let (M, g) be a Riemannian manifold, and let ∇, $|.|$, $\triangle$, and div (respectively, $\bar{\nabla}$, $|.|_{\bar{g}}$, $\bar{\triangle}$ and $\mathrm{div}_{\bar{g}}$) denote the gradient, the norm, the Laplacian, and the divergence taken with respect to g (respectively, $\bar{g}$). We use the convention on the Laplacian so that on $\mathbb{R}^m$, $\triangle u = \sum_{i=1}^{m} \partial^2 u/\partial {x_i}^2$.

For $p \in (1, \infty)$, a function $u : (M, g) \longrightarrow \mathbb{R}$ is called a *p-harmonic function* if it is a critical point of the p-energy functional

$$E_p\,(u, \Omega) = \frac{1}{p}\int_{\Omega} |\nabla u|^p \,\mathrm{d}\, v_g \tag{1.1}$$

over all compact domains $\Omega \subseteq M$. Using the first variation formula, we see that a function u is p-harmonic if and only if it is a solution of p-Laplace equation $\triangle_p(u) \equiv 0$, where

$$\triangle_p(u) = |\nabla u|^{p-2}\,\triangle u + (p-2)\,|\nabla u|^{p-3}\,\mathrm{d}\,u(\nabla\,|\nabla u|).$$

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When $|\nabla u| \neq 0$, we can write

$$\begin{aligned} \triangle_p(u) &= \operatorname{div}(|\nabla u|^{p-2}\,\nabla u) \\ &= |\nabla u|^{p-2}\,\{\triangle u + (p-2)\,\mathrm{d}\,u(\nabla(\ln|\nabla u|))\}. \end{aligned} \tag{1.2}$$

Following [13] we call a submersion $u : (M, g) \longrightarrow \mathbb{R}$ a 1-*harmonic* if it is a critical point of the 1-energy functional, i.e., functional (1.1) with $p = 1$ defined on all functions on M which are submersions. It follows from [13] that a submersion $u : (M, g) \longrightarrow \mathbb{R}$ is 1-harmonic if and only if

$$\triangle_1(u) := |\nabla u|^{-1}\,\{\triangle u - g(\nabla u, \nabla \ln|\nabla u|)\} = \operatorname{div}(|\nabla u|^{-1}\,\nabla u) = 0. \tag{1.3}$$

Thus, combining (1.2) and (1.3) we have a unified form of the formula of the p-tension field of a submersion $u : (M^m, g) \longrightarrow \mathbb{R}$ including the $p = 1$ case,

$$\triangle_p(u) = |\nabla u|^{p-2}\,\{\triangle u + (p-2)g(\nabla u, \nabla \ln|\nabla u|)\}. \tag{1.4}$$

2. The minimal graph equation in a Riemannian manifold

By *the minimal graph equation (MGE)* in a Riemannian manifold (M, g) we mean the PDE

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, \tag{2.1}$$

or, equivalently,

$$\triangle u - \frac{|\nabla u|}{1+|\nabla u|^2}\,g(\nabla u, \nabla\,|\nabla u|) = 0, \tag{2.2}$$

where $u : M \supseteq \Omega \longrightarrow \mathbb{R}$ is a function.

Remark 2.1. (i) When (M^m, g) is Euclidean space $\mathbb{R}^m$ with the standard metric δ_{ij}, then the minimal graph equation (2.1) gives the well-known minimal graph equation in a Euclidean domain, which often appears in the form

$$\sum_{i=1}^{m}\left(\delta_{ij} - \frac{u_i u_j}{1+|\nabla u|^2}\right) u_{ij} = 0. \tag{2.3}$$

The study of this equation is the main contribution to the progress of nonlinear elliptic PDE theory in the last century. Indeed, most early works on nonlinear elliptic problems focused on this equation. For the beautiful theorems on existence, uniqueness and regularity of the solutions of the MGE in Euclidean domain see, e.g., [8], [17] and the references therein.

(ii) When (M, g) is the 2-dimensional hyperbolic space (B^2, g^H) with the open disk model, then we obtain the minimal graph equation in (B^2, g^H) (see Example 3.1), which has been studied by Nelli and Rosenberg in their recent papers [12], [15].

(iii) It is well-known (see, e.g., [2]) that any function $u : (M, g) \longrightarrow \mathbb{R}$ is horizontally weakly conformal with dilation λ given by $\lambda^2 = |\nabla u|^2 = g^{ij} u_i u_j$. u is said to be *horizontally homothetic* if $\eta(\lambda^2) = 0$ for any local vector field η normal to the level hypersurfaces of u. It follows from [13] that a submersion $u : (M, g) \longrightarrow \mathbb{R}$ is horizontally homothetic if and only if $g(\nabla u, \nabla |\nabla u|) = 0$. This and equation (2.2) imply that any horizontally homothetic harmonic submersion $u : (M, g) \longrightarrow \mathbb{R}$ is an entire solution of the minimal graph equation in (M, g).

EXAMPLE 2.1. Let $(\mathbb{R}^3, g_{\mathrm{Nil}})$ denote Nil space, one of the eight three-dimensional geometries, where the metric with respect to the standard coordinates (x, y, z) in $\mathbb{R}^3$ can be written as $g_{\mathrm{Nil}} = \mathrm{d}\,x^2 + \mathrm{d}\,y^2 + (\mathrm{d}\,z - x\,\mathrm{d}\,y)^2$. Then it follows from Theorem 3.11 in [13] that the function $u : (\mathbb{R}^3, g_{\mathrm{Nil}}) \longrightarrow \mathbb{R}$ defined by $u(x, y, z) = Ax + By + C(z - xy/2)$ is a horizontally homothetic harmonic submersion, where A, B, C are constants and not all of them are zero. So, by (iii) of Remark 2.1, u produces a family of entire solution of the minimal graph equation in Nil space.

The following theorem gives the geometric meaning of the minimal graph equation in a Riemannian manifold.

THEOREM 2.1. *Let $u : (M^m, g) \longrightarrow \mathbb{R}$ be a function, and let*

$$\Gamma(u, c) = \{(x, u(x) + c) \in M \times \mathbb{R} : x \in M\}$$

be the vertical graph of u at the height c. Then:

(I) *The vertical graphs $\{\Gamma(u, c) : c \in \mathbb{R}\}$ form a foliation of $(M \times \mathbb{R}, g + dt^2)$ by minimal hypersurfaces if and only if u is a solution of the minimal graph equation in (M, g).*

(II) *When (M, g) is a complete and connected manifold and u is a non-constant function, then the vertical graphs $\{\Gamma(u, c) : c \in \mathbb{R}\}$ form a Riemannian foliation of $(M \times \mathbb{R}, g + dt^2)$ if and only if M is diffeomorphic to the normal bundle of a complete hypersurface $u^{-1}(c) \subset M$.*

(III) *When (M, g) is a complete and connected manifold and u is a non-constant function, then the vertical graphs $\{\Gamma(u, c) : c \in \mathbb{R}\}$ form a foliation of $(M \times \mathbb{R}, g + dt^2)$ by totally geodesic hypersurfaces if and only if $|\nabla u| = constant > 0$ and ∇u is a Killing vector field.*

Proof. Consider $f : (M \times \mathbb{R}, g + dt^2) \longrightarrow \mathbb{R}$ defined by $f(x, t) = u(x) - t$. Let $\bar{g} = g + dt^2$ denote the product metric. Then a direct computation yields

$$\bar{\nabla} f = \nabla u - \frac{\partial}{\partial t}, \qquad \left|\bar{\nabla} f\right|_{\bar{g}} = \sqrt{1 + |\nabla u|^2}, \qquad \bar{\triangle} f = \triangle u. \tag{2.4}$$

Note that f is a submersion and its level hypersurface $f^{-1}(-c) = \{(x, t) \in M \times \mathbb{R} : t = u(x) + c\}$ is the vertical graph $\Gamma(u, c)$ of u at the height c. It follows from [13] that the foliation $\{\Gamma(u, c) : c \in \mathbb{R}\}$ of $(M \times \mathbb{R}, \bar{g})$ by the level hypersurfaces of f is a minimal foliation if and only if f is a 1-harmonic submersion, i.e., f is a solution of

$$\mathrm{div}_{\bar{g}}\left(\left|\bar{\nabla} f\right|_{\bar{g}}^{-1}\bar{\nabla} f\right)=0. \tag{2.5}$$

Using (1.3), (2.4) and the fact that $\bar{g}=g+dt^2$ is the product metric on $M\times\mathbb{R}$ and $\ln\sqrt{1+|\nabla u|^2}$ does not depend on t we have

$$\begin{aligned}\mathrm{div}_{\bar{g}}\left(\left|\bar{\nabla} f\right|_{\bar{g}}^{-1}\bar{\nabla} f\right) &= \frac{\triangle u-g(\nabla u,\nabla\ln\sqrt{1+|\nabla u|^2}\ )}{\sqrt{1+|\nabla u|^2}} \\ &= \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right).\end{aligned} \tag{2.6}$$

It follows from (2.6) that f is a solution of equation (2.5) if and only if u is a solution of the minimal graph equation (2.1). Thus we obtain Statement (I). To prove Statement (II) we recall (see [19], Theorem 8.9) that the foliation $\{\Gamma(u,c): c\in\mathbb{R}\}$ by the level hypersurfaces of the submersion $f:(M\times\mathbb{R}, g+dt^2)\longrightarrow\mathbb{R}$ defined by $f(x,t)=u(x)-t$ is a Riemannian foliation if and only if

$$V\bar{g}(\bar{\nabla} f,\bar{\nabla} f)=0 \quad \text{for} \quad \text{any} \ \ V\in\Gamma L, \tag{2.7}$$

where ΓL denotes the sub-bundle tangent to the hypersurfaces. Let $p\in\Gamma(u,c)\subset M$ and $(U,(x_i))$ be a local coordinate neighborhood of p. Then $(U\times\mathbb{R},(x_i,t))$ is a local coordinate neighborhood of $(p,t)\in M\times\mathbb{R}$ and in this neighborhood $\Gamma(u,c)$ can be parametrized as $F(x)=(x,u(x)+c)$. In the sequel we will use the notations $\partial_i=\frac{\partial}{\partial x_i}$, $u_i=\frac{\partial u}{\partial x_i}$, and $\partial_t=\frac{\partial}{\partial t}$. It is not difficult to check that

$$V_i=\partial_i+u_i\partial_t, \quad i=1,\ldots,m \tag{2.8}$$

is a local base of ΓL on $(U\times\mathbb{R})$. So, locally, equation (2.7) is equivalent to

$$V_i\bar{g}(\bar{\nabla} f,\bar{\nabla} f)=0 \quad \text{for } i=1,2,\ldots,m. \tag{2.9}$$

Note that

$$V_i\bar{g}(\bar{\nabla} f,\bar{\nabla} f)=(\partial_i+u_i\partial_t)(1+|\nabla u|^2)=\partial_i(|\nabla u|^2),$$

since $\left|\bar{\nabla} f\right|=1+|\nabla u|^2$ does not depend on t. It follows that equation (2.9) is equivalent to $|\nabla u|=$ constant on U. Since p is an arbitrary point and M is connected, we conclude that $|\nabla u|$ is constant on M. Furthermore, the constant is >0, since u is a nonconstant function. Now we can apply Proposition 2.1 in [16] to complete the proof of Statement (II). To prove Statement (III) we know from [19], page 110, that the foliation $\{\Gamma(u,c):c\in\mathbb{R}\}$ by the level hypersurfaces of the submersion $f : (M \times \mathbb{R}, g + dt^2) \longrightarrow \mathbb{R}$ with $f(x,t) = u(x) - t$ is a totally geodesic foliation if and only if

$$\mathrm{Hess}_f(V, W) = 0 \quad \text{for} \quad \text{any } V, \ W \in \Gamma L. \tag{2.10}$$

Since the Hessian of a function is a tensor we conclude that, locally, equation (2.10) is equivalent to

$$\mathrm{Hess}_f(V_i, V_j) = 0 \tag{2.11}$$

for the local base element V_i of ΓL given by (2.8). A straightforward computation yields

$$\begin{aligned} \mathrm{Hess}_f(V_i, V_j) &= (\partial_i + u_i \partial t)(\partial_j + u_j \partial t)(u(x) - t) \\ -\bar{\nabla}_{\partial_i + u_i \partial t}(\partial_j + u_j \partial t)(u(x) - t) &= u_{ij} - \Gamma_{ij}^k u_k = \mathrm{Hess}_u(\partial_i, \partial_j). \end{aligned} \tag{2.12}$$

It follows that equation (2.10) is equivalent to $\mathrm{Hess}_u = 0$ everywhere, and Statement (III) then follows from Lemma 2.3 in [16]. □

Note that if $u : M \longrightarrow \mathbb{R}$ is a constant function then it solves the MGE in (M, g) trivially; in fact, the foliation by vertical graphs of u is the canonical foliation $\{M \times \{t\} : t \in \mathbb{R}\}$, which is well-known to be a totally geodesic foliation.

The next theorem gives a link among p-harmonicity, horizontal homothety, and the minimality of the vertical graphs of a function.

THEOREM 2.2. *Let $u : M \longrightarrow \mathbb{R}$ be a submersion. Then any two of the following statements imply the other one:*

(a) *u is a p-harmonic function for some $p \geq 1$,*
(b) *u is a solution of the MGE in (M, g), i.e., the vertical graph $\Gamma(u, c)$ is minimal in $(M \times \mathbb{R}, g + dt^2)$,*
(c) *u is horizontally homothetic.*

Proof. $(a) + (b) \Rightarrow (c)$: Since u is a p-harmonic submersion we use (1.4) to have

$$\triangle u + (p - 2) g(\nabla u, \nabla \ln |\nabla u|) = 0.$$

Combining this equation and the MGE (2.2) we obtain

$$\left(\frac{|\nabla u|^2}{1 + |\nabla u|^2} + p - 2 \right) g(\nabla u, \nabla \ln |\nabla u|) = 0. \tag{2.13}$$

If $p = 1$, then (2.13) implies that $g(\nabla u, \nabla \ln |\nabla u|) = 0$ identically since the first factor on the left hand side of (2.13) is never zero. It follows that u is horizontally homothetic. If $p \neq 1$, we claim that $g(\nabla u, \nabla \ln |\nabla u|) = 0$ identically and hence u is also horizontally homothetic in this case. Suppose otherwise. Then there exists a point $x \in M$ such that

$$g_x(\nabla u, \nabla \ln |\nabla u|) \neq 0. \tag{2.14}$$

By continuity, there exists a neighborhood W of x on which the inequality (2.14) holds. It follows from (2.13) that

$$\frac{|\nabla u|^2}{1+|\nabla u|^2}+p-2=0$$

for any point in W, which implies that $|\nabla u|=\pm\sqrt{(p-2)/(1-p)}$ is a constant on W. It follows that $\nabla \ln|\nabla u|=0$ and hence $g(\nabla u, \nabla \ln|\nabla u|)=0$ on W, a contradiction.

$(a)+(c)\Rightarrow(b)$: Since u is a horizontally homothetic p-harmonic submersion, by Corollary 3.6 in [13], it is q-harmonic for any $q\geq 1$. In particular, it is a horizontally homothetic harmonic submersion. Thus, as in Remark 2.1(iii), u is a solution of the MGE in (M,g) and the vertical graphs of u are minimal.

$(b)+(c)\Rightarrow(a)$: Since u is horizontally homothetic, $g(\nabla u,\nabla|\nabla u|)=0$. This, together with the MGE (2.2), gives $\triangle u=0$. Thus, u is a horizontally homothetic harmonic submersion and, by Corollary 3.6 in [13] again, it is a p-harmonic submersion for any $p\geq 1$. □

Corollary 2.3. *A submersion $u: M\longrightarrow\mathbb{R}$ has minimal level hypersurfaces and minimal vertical graphs if and only if it is a horizontally homothetic p-harmonic function for some $p\geq 1$.*

Proof. If a submersion u has minimal level hypersurfaces, then it is 1-harmonic by [13]. If the vertical graphs of u are minimal, then Theorem 2.2 implies that u is horizontally homothetic. By [13], a horizontally homothetic 1-harmonic submersion is p-harmonic for any $p\geq 1$. Conversely, if u is a horizontally homothetic p-harmonic submersion for some $p\geq 1$, then, by [13], u is also a 1-harmonic submersion and hence the level hypersurfaces are minimal. On the other hand, Theorem 2.2 implies that the vertical graphs of a horizontally homothetic p-harmonic submersion are minimal. This completes the proof of the corollary. □

3. The conformal transformations of the minimal graph equation

Now we prove the following proposition which gives the minimality of vertical graphs with respect to two kinds of conformally deformed metrics.

Proposition 3.1.

(A) *The MGE in $(M^m, F^{-2}g)$ (called the conformal minimal graph equation in (M,g)) is given by*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+F^2|\nabla u|^2}}\right)-\frac{(m-2)g(\nabla u,\nabla F)}{F\sqrt{1+F^2|\nabla u|^2}}=0. \tag{3.1}$$

(B) *The vertical graphs* $\{\Gamma(u,c) : c \in \mathbb{R}\}$ *of* $u : (M^m, g) \longrightarrow \mathbb{R}$ *form a foliation of* $(M \times \mathbb{R}, F^{-2}(g + dt^2))$ *by minimal hypersurfaces if and only if* u *is a solution of the PDE*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) - \frac{m}{F\sqrt{1+|\nabla u|^2}}[g(\nabla u, \nabla F) - F_t] = 0 \tag{3.2}$$

Proof. Let $\bar{g} = F^{-2}g$. Then a direct computation gives

$$\begin{cases} \bar{\nabla} u = F^2 \nabla u, \quad \left|\bar{\nabla} u\right|_{\bar{g}} = F\left|\nabla u\right|, \\ \operatorname{div}_{\bar{g}}(X) = \operatorname{div}(X) - mF^{-1}X(F), \end{cases} \tag{3.3}$$

from which we have

$$\operatorname{div}_{\bar{g}}\left(\frac{\bar{\nabla} u}{\sqrt{1+\left|\bar{\nabla} u\right|_{\bar{g}}^2}}\right) = F^2 \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+F^2\left|\nabla u\right|^2}}\right) - \frac{(m-2)Fg(\nabla u, \nabla F)}{\sqrt{1+F^2\left|\nabla u\right|^2}}.$$

This and the MGE in $(M, \bar{g})$ give the conformal minimal graph equation (3.1) and hence Statement (A). To prove Statement (B), we know from [13] that the foliation $\{\Gamma(u,c) : c \in \mathbb{R}\}$ of $(M \times \mathbb{R}, F^{-2}\bar{g})$ is a minimal foliation if and only if $f(x,t)$ is a 1-harmonic submersion with respect to $\tilde{g} = F^{-2}\bar{g}$, i.e., f is a solution of

$$\operatorname{div}_{\tilde{g}}\left(\left|\tilde{\nabla} f\right|_{\tilde{g}}^{-1} \tilde{\nabla} f\right) = 0. \tag{3.4}$$

A straightforward computation using the conformal transformation law (3.3) with $(m+1) = \dim(M \times \mathbb{R})$ in place of m yields

$$\operatorname{div}_{\tilde{g}}\left(\left|\tilde{\nabla} f\right|_{\tilde{g}}^{-1} \tilde{\nabla} f\right) = F \operatorname{div}_{\bar{g}}\left(\left|\bar{\nabla} f\right|_{\bar{g}}^{-1} \bar{\nabla} f\right) - \frac{m\bar{g}(\bar{\nabla} f, \bar{\nabla} F)}{\left|\bar{\nabla} f\right|_{\bar{g}}}. \tag{3.5}$$

Using (2.4) and the fact that $f_t = -1$ and $\bar{g}$ is the product metric we have

$$\bar{g}(\bar{\nabla} f, \bar{\nabla} F) = g(\nabla u, \nabla F) - F_t. \tag{3.6}$$

It follows from equations (2.6), (3.5) and (3.6) that f is a solution of (3.4) if and only if u is a solution of (3.2). Thus we obtain Statement (B) and complete the proof of the proposition. □

In Examples 3.1, 3.2, 3.3 and 3.4 below, we will use ∇, $|.|$, and div to denote the gradient, the norm, and the divergence taken with respect to the standard Euclidean metric δ_{ij}.

Example 3.1 (The minimal graph equation in hyperbolic space). Let (B^m, g^H) be the m-dimensional hyperbolic space with the open-ball model, where $B^m = \{x \in \mathbb{R}^m : |x| < 1\}$ and $g^H = F^{-2}\delta_{ij}$ with $F = 2^{-1}(1 - |x|^2)$. Then the minimal graph equation in the hyperbolic space (B^m, g^H) is the conformal minimal graph equation in the Euclidean space $(\mathbb{R}^m, \delta_{ij})$, which, by (A) of Proposition 3.1, can be written as

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+F^2\left|\nabla u\right|^2}}\right)+\frac{(m-2)\langle x,\nabla u\rangle}{F\sqrt{1+F^2\left|\nabla u\right|^2}}=0,\quad x\in\mathbb{R}^m. \tag{3.7}$$

When $m=2$, the minimal graph equation in B^2 becomes

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+F^2\left|\nabla u\right|^2}}\right)=0. \tag{3.8}$$

A solution of this equation is a function $u : B^2 \supseteq \Omega \longrightarrow \mathbb{R}$ whose vertical graph is a minimal surface in $B^2 \times \mathbb{R}$.

Note that recently Nelli and Rosenberg [12], [15] derived the MGE (3.8) by computing the mean curvature of the graph of u using a special adapted orthonormal frame along the graph of u. They also proved that for any rectifiable Jordan curve at infinity of $B^2\times\mathbb{R}$ there is a minimal graph over B^2 whose asymptotic boundary is the given curve. In the next theorem we give a class of entire solutions of the MGE in an m-dimensional hyperbolic space.

Theorem 3.2. *In hyperbolic space (B^m, g^H) of the open-ball model as in Example 3.1, the function $u(x)=(a_1x_1+\cdots+a_{m-1}x_{m-1})(1+|x|^2-2x_m)^{-1}$ is an entire solution of the MGE, where $a_1,\ldots,a_{m-1}$ are constant. Furthermore, the vertical graphs of u produce a foliation by minimal hypersurfaces none of which is totally geodesic.*

Proof. Consider the hyperbolic space in the upper-half space model $(\mathbb{R}^m_+, g^H_+)$, where $\mathbb{R}^m_+=\{(y_1,\ldots,y_m)\in\mathbb{R}^m : y_m>0\}$ and $g^H_+=F^{-2}\delta_{ij}$ with $F(y)=y_m$. Let $f:(\mathbb{R}^m_+, g^H_+)\longrightarrow\mathbb{R}$ be a function defined by $f(y)=a_1y_1+\cdots+a_{m-1}y_{m-1}=\langle A,y\rangle$, where $A=(a_1,\ldots,a_{m-1},0), y=(y_1,\ldots,y_m)\in\mathbb{R}^m_+$. Let $\bar g=F^{-2}\delta_{ij}$. Then a straightforward computation using (4.2) shows that

$$\bar\triangle f=F^2\triangle f+(2-m)F\langle\nabla F,\nabla f\rangle=0,$$
$$\bar g(\bar\nabla f,\bar\nabla\ln\left|\bar\nabla f\right|_{\bar g})=F^2\langle\nabla f,\nabla\ln|\nabla f|\rangle+F\langle\nabla F,\nabla f\rangle=0.$$

It follows that f is a horizontally homothetic harmonic submersion in hyperbolic space $(\mathbb{R}^m_+, g^H_+)$ and hence, by (iii) of Remark 2.1, f is an entire solution of the minimal graph equation in hyperbolic space $(\mathbb{R}^m_+, g^H_+)$. The foliation of the vertical graphs of f is not a totally geodesic foliation, for otherwise, by (iii) of Theorem 2.1, we would have $\left|\bar\nabla f\right|_{\bar g}=y_m\,|A|=$ constant, which is impossible. It is well-known that the hyperbolic spaces (B^m, g^H) and $(\mathbb{R}^m_+, g^H_+)$ are isometric to each other. In fact, it is easily checked that the map $\varphi: B^m\longrightarrow\mathbb{R}^m_+$ with $\varphi(x)=(1+|x|^2-2x_m)^{-1}(2x_1,\ldots,2x_{m-1},1-|x|^2)$

is an isometry. Since the MGE is invariant under isometries we conclude that $u = f \circ \varphi : (B^m, g^H) \longrightarrow \mathbb{R}$ with $u(x) = (2a_1x_1 + \cdots + 2a_{m-1}x_{m-1})(1 + |x|^2 - 2x_m)^{-1}$ is an entire solution of the MGE in (B^m, g^H), which gives the theorem. $\square$

Example 3.2 (Minimal graphs in hyperbolic space). Let (R^{m+1}_+, g^H_+) be the $(m+1)$-dimensional hyperbolic space with the upper half space model, where $g_H = F^{-2}\delta_{ij}$ with $F(x,t) = t$. Since $R^{m+1}_+ \cong \mathbb{R}^m \times \mathbb{R}^+$, a function $u : \mathbb{R}^m \supseteq \Omega \longrightarrow \mathbb{R}^+$ will have vertical graphs in R^{m+1}_+. Then, by equation (3.2) and a direct computation, the vertical graph of u is a minimal hypersurface in (R^{m+1}_+, g^H_+) if and only if u is a solution of the PDE

$$\sum_{i=1}^{m} \left(\delta_{ij} - \frac{u_i u_j}{1 + |\nabla u|^2} \right) u_{ij} + \frac{m}{u(x)} = 0. \tag{3.9}$$

Note that the minimal graph equation (3.9) was first derived by Anderson in [1]. The existence, uniqueness and stability of solutions of this equation was studied by Lin [10] and recently by Sa Earp and Toubiana [4].

Example 3.3 (The minimal graph equation in sphere). Let (S^m, g_{can}) be the m-dimensional sphere with the standard metric. It is well-known that we can identify $(S^m \setminus \{N\}, g_{\mathrm{can}})$ with $(\mathbb{R}^m, F^{-2}\delta_{ij})$, where $F = 2^{-1}(1 + |x|^2)$. Then, the minimal graph equation in the sphere S^m is the conformal minimal graph equation in the Euclidean space $(\mathbb{R}^m, \delta_{ij})$, which, by (A) of Proposition 3.1, can be written as

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + F^2 |\nabla u|^2}} \right) - \frac{(m-2)\langle x, \nabla u\rangle}{F\sqrt{1 + F^2 |\nabla u|^2}} = 0, \quad x \in \mathbb{R}^m. \tag{3.10}$$

When $m = 2$, the minimal graph equation in S^2 becomes

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + F^2 |\nabla u|^2}} \right) = 0. \tag{3.11}$$

A solution of this equation is a function $u : S^2 \supset \Omega \longrightarrow \mathbb{R}$ whose vertical graph is a minimal surface in $S^2 \times \mathbb{R}$.

Example 3.4 (Minimal graphs in sphere). Identify $(S^{m+1} \setminus \{N\}, g_{\mathrm{can}})$ with $(\mathbb{R}^{m+1}, F^{-2}\delta_{ij})$, where $F = 2^{-1}(1 + \sum_{i=1}^{m+1} x_i^2)$. Then, by equation (3.2) and a direct computation, a function $u : \mathbb{R}^m \supseteq \Omega \longrightarrow \mathbb{R}$ whose vertical graph is a minimal hypersurface in the sphere $(\mathbb{R}^{m+1}, F^{-2}\delta_{ij}) \equiv (S^{m+1} \setminus \{N\}, g_{\mathrm{can}})$ if and only if u is a solution of the PDE

$$\sum_{i=1}^{m} \left(\delta_{ij} - \frac{u_i u_j}{1 + |\nabla u|^2} \right) u_{ij} - \frac{2m[\langle x, \nabla u\rangle - u(x)]}{1 + |x|^2 + u^2(x)} = 0, \tag{3.12}$$

where $x \in \Omega \subseteq \mathbb{R}^m$.

4. p-harmonic functions and foliations by minimal hypersurfaces

One fundamental question raised by Harvey and Lawson [7] in the study of foliations is the following: Given a foliation $\mathcal{F}$ of a manifold M, when can one find a Riemannian metric g on M so that all the leaves of $\mathcal{F}$ are minimal submanifolds of (M, g)? In this section, we prove that for the foliation defined by the level hypersurfaces of a p-harmonic submersion $u : (M^m, g) \longrightarrow \mathbb{R}$, one can always find such a metric (and in some cases, one can find more than one such metric). In particular, we give a method to construct a metric with respect to which the foliation by the parallel graphs of a harmonic function is a minimal foliation.

THEOREM 4.1. *Let $p \in (1, \infty)$, and let $u : (M^m, g) \longrightarrow \mathbb{R}$ be a p-harmonic submersion. Then $u : (M^m, \bar{g}) \longrightarrow \mathbb{R}$ is a 1-harmonic submersion and the foliation by the level hypersurfaces $\Gamma = \{u^{-1}(c) : c \in \mathbb{R}\}$ of u is a minimal foliation of $(M, \bar{g})$, where $\bar{g} = |\nabla u|^{2(p-1)/(m-1)} g$ is a Riemannian metric conformal to g.*

Proof. By Theorem 3.4 in [13], we only need to prove that $u : (M^m, \bar{g}) \longrightarrow \mathbb{R}$ is a 1-harmonic submersion. This is equivalent to showing that

$$\tau_1(u, \bar{g}) = \left|\bar{\nabla} u\right|_{\bar{g}}^{-1} \{\bar{\triangle} u - \bar{g}(\bar{\nabla} u, \bar{\nabla} \ln \left|\bar{\nabla} u\right|_{\bar{g}})\} \equiv 0. \tag{4.1}$$

Let $F = |\nabla u|^{(1-p)/(m-1)}$. Then $\bar{g} = F^{-2} g$. A direct computation gives

$$\begin{cases} \left|\bar{\nabla} u\right|_{\bar{g}} = F\,|\nabla u|, \quad \sqrt{\det(\bar{g}_{ij})} = F^{-m}\sqrt{\det(g_{ij})}, \\ \bar{\triangle} u = F^2 \triangle u + (2-m) F g(\nabla F, \nabla u), \\ \bar{g}(\bar{\nabla} u, \bar{\nabla} \ln \left|\bar{\nabla} u\right|_{\bar{g}}) = F^2 g(\nabla u, \nabla \ln |\nabla u|) + F g(\nabla F, \nabla u). \end{cases} \tag{4.2}$$

Using (4.2) and the first equality in (4.1) we have

$$\begin{aligned} &\tau_1(u, \bar{g}) = \\ &\quad (F\,|\nabla u|)^{-1}\{F^2 \triangle u + (1-m) F g(\nabla F, \nabla u) - F^2 |\nabla u|^{-1} g(\nabla u, \nabla\,|\nabla u|)\}. \end{aligned} \tag{4.3}$$

Substituting $F = |\nabla u|^{(1-p)/(m-1)}$ into (4.3) we obtain

$$\tau_1(u, \bar{g}) = |\nabla u|^{m(1-p)/(m-1)} \cdot |\nabla u|^{p-2} \{\triangle u + (p-2) g(\nabla u, \nabla \ln |\nabla u|)\},$$

from which, together with (1.4), we obtain

$$\tau_1(u, \bar{g}) = |\nabla u|^{m(1-p)/(m-1)}\, \tau_p(u, g). \tag{4.4}$$

It follows from (4.4) that $u : (M^m, g) \longrightarrow \mathbb{R}$ is a p-harmonic submersion if and only if $u : (M^m, \bar{g}) \longrightarrow \mathbb{R}$ if a 1-harmonic submersion. This completes the proof of the theorem. $\square$

In [3], Proposition 1, Chruściel proved that a function $u : (M^3, g) \longrightarrow \mathbb{R}$ is a 3-harmonic submersion if and only if there exists a conformal metric $\lambda^2 g$ such that $u : (M^m, \lambda^2 g) \longrightarrow \mathbb{R}$ is a harmonic Riemannian submersion, in which case $\Gamma = \{u^{-1}(c) : c \in \mathbb{R}\}$ is a minimal foliation of $(M, \lambda^2 g)$ by the level hypersurfaces of u. Now we give the following generalization.

PROPOSITION 4.2. *A function $u : (M^m, g) \longrightarrow \mathbb{R}$ is an m-harmonic submersion if and only if $u : (M^m, |\nabla u|^2 g) \longrightarrow \mathbb{R}$ is a harmonic Riemannian submersion, in which case $\Gamma = \{u^{-1}(c) : c \in \mathbb{R}\}$ is a minimal foliation of $(M, |\nabla u|^2 g)$ by the level hypersurfaces of u.*

Proof. Applying Theorem 4.1 to u with $p = m$ we conclude that the function $u : (M^m, |\nabla u|^2 g) \longrightarrow \mathbb{R}$ is a 1-harmonic submersion and $\Gamma = \{u^{-1}(c) : c \in \mathbb{R}\}$ is a minimal foliation of $(M, |\nabla u|^2 g)$ by level hypersurfaces of u. On the other hand, it is easily checked that $u : (M^m, |\nabla u|^2 g) \longrightarrow \mathbb{R}$ is a Riemannian submersion and an m-harmonic function since m-harmonicity is invariant under the conformal change of the metric on the domain manifold. It follows from [13] that $u : (M^m, |\nabla u|^2 g) \longrightarrow \mathbb{R}$ is horizontally homothetic since it is p-harmonic for two different p values. In this case, it is p-harmonic for any $p \in [1, \infty)$. In particular, $u : (M^m, |\nabla u|^2 g) \longrightarrow \mathbb{R}$ is a harmonic Riemannian submersion. Conversely, if $u : (M^m, |\nabla u|^2 g) \longrightarrow \mathbb{R}$ is a harmonic Riemannian submersion, then, by [13], it is p-harmonic for any $p \in [1, \infty)$, and in particular, it is m-harmonic. Therefore, $u : (M^m, g) \longrightarrow \mathbb{R}$ is also an m-harmonic submersion since g is conformal to $|\nabla u|^2 g$ and m-harmonicity is invariant under the conformal change of metric on the domain manifold. □

As an application of Theorem 4.1, we obtain a convenient formula for the mean curvature of the level hypersurfaces of a p-harmonic submersion.

COROLLARY 4.3. *Let $u : (M^m, g) \longrightarrow \mathbb{R}$ be a p-harmonic submersion with $p > 1$. Let $\eta = \frac{\nabla u}{|\nabla u|}$ be the unit normal vector field of the level hypersurfaces. Then the mean curvature $H(\eta)$ of the level hypersurfaces with respect to η is given by*

$$\begin{aligned} H(\eta) &= (p-1)(m-1)^{-1}\eta(\ln|\nabla u|) \\ &= (p-1)[(m-1)\,|\nabla u|\,]^{-1} g(\nabla u, \nabla \ln|\nabla u|). \end{aligned} \tag{4.5}$$

Proof. Let $\bar{g} = |\nabla u|^{2(p-1)/(m-1)}\, g$, and let $H(\bar{g})$ denote the mean curvature of the level hypersurfaces of u with respect to the conformal metric $\bar{g}$. Then, by Lemma 2.1 in [11], we have

$$H(\bar{g}) = \frac{H(\eta)}{\lambda} - \frac{1}{\lambda^2}\eta(\lambda), \tag{4.6}$$

where $\lambda = |\nabla u|^{(p-1)/(m-1)}$, and $\eta(\lambda)$ denotes the derivative of λ in the direction of η. By Theorem 4.1, the level hypersurfaces of u are minimal in $(M^m, \bar{g})$, hence it follows from (4.6) that

$$\begin{aligned} H(\eta) &= \frac{1}{\lambda}\eta(\lambda) = |\nabla u|^{-(p-1)/(m-1)}\,\eta(|\nabla u|^{(p-1)/(m-1)}) \\ &= (p-1)[(m-1)\,|\nabla u|\,]^{-1}\nabla u(\ln|\nabla u|) \\ &= (p-1)[(m-1)\,|\nabla u|\,]^{-1}g(\nabla u, \nabla \ln|\nabla u|). \end{aligned}$$

Note that in obtaining the last equality we have used the fact that $(\nabla u)(f) = (\operatorname{grad} u)(f) = g(\nabla u, \nabla f)$ for any functions u and f. $\square$

It was proved in [13] that the level hypersurfaces of a p-harmonic submersion $u : (M^m, g) \longrightarrow \mathbb{R}$ produce a minimal foliation $\mathcal{F} = \{u^{-1}(c) : c \in \mathbb{R}\}$ if and only if (i) $p = 1$, or (ii) $p \neq 1$, and u is horizontally homothetic, i.e., $g(\nabla u, \nabla \ln|\nabla u|) \equiv 0$. It follows, in particular, that the level hypersurfaces of a horizontally homothetic harmonic submersion always produce a minimal foliation. This is equivalent to saying that a solution of

$$\triangle u = 0, \quad g(\nabla u, \nabla \ln|\nabla u|) = 0 \tag{4.7}$$

is always a solution of *the* 1*-harmonic submersion equation*

$$\triangle u - g(\nabla u, \nabla \ln|\nabla u|) = 0. \tag{4.8}$$

Call a solution u of equation (4.7) a *trivial* 1-harmonic submersion, or a *trivial solution* of the 1-harmonic submersion equation (4.8). Then a natural question to ask is the following: Can there be any nontrivial 1-harmonic submersion, i.e., a 1-harmonic submersion which is not a horizontally homothetic harmonic submersion? The following proposition shows that there are many such 1-harmonic submersions, i.e., there are many nontrivial solutions to the 1-harmonic submersion equation (4.8).

PROPOSITION 4.4. *Let $m \geq 3$, and let $u : (M^m, g) \longrightarrow \mathbb{R}$ be a submersive harmonic function whose level hypersurfaces are not minimal in (M^m, g). Let $\bar{g} = |\nabla u|^{2/(m-1)}\, g$. Then $u : (M^m, \bar{g}) \longrightarrow \mathbb{R}$ is a nontrivial* 1*-harmonic submersion whose level hypersurfaces are minimal in $(M^m, \bar{g})$.*

Proof. Applying Theorem 4.1 to u with $p = 2$ we see that, with respect to the metric $\bar{g}$, u is a 1-harmonic submersion. By Corollary 3.6 in [13], $u : (M^m, \bar{g}) \longrightarrow \mathbb{R}$ is a trivial 1-harmonic submersion if and only if it is horizontally homothetic. Thus, to see that $u : (M^m, \bar{g}) \longrightarrow \mathbb{R}$ is a nontrivial 1-harmonic submersion it is enough to show that it is not horizontally homothetic, i.e.,

$$\bar{g}(\bar{\nabla} u, \bar{\nabla} \ln \left|\bar{\nabla} u\right|_{\bar{g}}) \neq 0 \quad \text{for some point.} \tag{4.9}$$

Since u is assumed to be a harmonic submersion whose level hypersurfaces are not minimal in (M, g), we see from (4.5) that

$$g(\nabla u, \nabla \ln |\nabla u|) \neq 0 \quad \text{for some point.} \tag{4.10}$$

Substituting $F = |\nabla u|^{-1/(m-1)}$ into the last equation of (4.2) and performing a further calculation yields

$$\bar{g}(\bar{\nabla} u, \bar{\nabla} \ln \left|\bar{\nabla} u\right|_{\bar{g}}) = (m-2)(m-1)^{-1} \, |\nabla u|^{\frac{-2}{m-1}} \, g(\nabla u, \nabla \ln |\nabla u|).$$

This, together with the assumptions $m \geq 3$, $|\nabla u| \neq 0$ and (4.10), shows that (4.9) holds. Hence $u : (M^m, \bar{g}) \longrightarrow \mathbb{R}$ is a nontrivial 1-harmonic submersion. Thus, we complete the proof of the proposition. □

Example 4.1. It is easily checked that $u : \mathbb{R}^5 \longrightarrow \mathbb{R}$, $u(x_1, \ldots, x_5) = x_5 - x_1 x_2 - x_3 x_4$ is a harmonic submersion with $|\nabla u|^2 = 1 + x_1^2 + \ldots + x_4^2$. A direct computation using (4.5) shows that the mean curvature of the level hypersurfaces of u is given by

$$H(\eta) = \frac{1}{4\,|\nabla u|} \langle \nabla u, \nabla \ln |\nabla u| \rangle = -\frac{1}{2}(x_1 x_2 + x_3 x_4)\, |\nabla u|^{-3} \neq 0,$$

where $\langle , \rangle$ denotes the standard Euclidean metric on $\mathbb{R}^5$. Thus, the level hypersurfaces of the harmonic submersion u are not minimal in Euclidean space $\mathbb{R}^5$. It follows from Proposition 4.4 that $u : (\mathbb{R}^5, (1+x_1^2+\ldots+x_4^2)^{1/4}\delta_{ij}) \longrightarrow \mathbb{R}$ is a nontrivial 1-harmonic submersion whose level hypersurfaces are minimal in the complete (see Theorem 4.5 below) conformally flat space $(\mathbb{R}^5, (1+x_1^2+ \ldots + x_4^2)^{1/4}\delta_{ij})$.

The next theorem shows that we can turn harmonic graphs into minimal hypersurfaces by a suitable choice of a metric.

Theorem 4.5. *Let $u : (M^m, g) \longrightarrow \mathbb{R}$ be a harmonic function from a complete Riemannian manifold. Then the vertical graphs $\mathcal{G} = \{(x, u(x)+c) \in M \times \mathbb{R} : c \in \mathbb{R}\}$ produce a foliation of the complete manifold $(M^m \times \mathbb{R}, (1 + |\nabla u|^2)^{1/m}(g + \mathrm{d}\,t^2))$ by minimal hypersurfaces.*

Proof. Consider the function $f : (M^m \times \mathbb{R}, g + \mathrm{d}\,t^2) \longrightarrow \mathbb{R}$ given by $f(x, t) = t - u(x)$. The level hypersurface of f is $f^{-1}(c) = \{(x, u(x) + c) \in M \times \mathbb{R} : c \in \mathbb{R}\}$. Thus, the foliation by the vertical graphs of $u(x)$ is the foliation by the level hypersurfaces of $f(x, t)$. Let $\bar{g}$ denote the product metric $g + \mathrm{d}\,t^2$. Then, by (2.4), we have $\left|\bar{\nabla} f\right|_{\bar{g}}^{2} = 1 + |\nabla u|^2$, and $\bar{\triangle} f = \triangle u \equiv 0$ since u is assumed to be harmonic. Thus $f : (M^m \times \mathbb{R}, g + \mathrm{d}\,t^2) \longrightarrow \mathbb{R}$ is a harmonic submersion. Applying Theorem 4.1 to f with $p = 2$, we obtain the Theorem except for the completeness of the metric $(1 + |\nabla u|^2)^{1/m}(g + \mathrm{d}\,t^2)$. Note that the product metric $\bar{g} = g + \mathrm{d}\,t^2$ is complete since g is assumed to be complete.

It follows from Theorem 4.2 in [6] that the pointwise conformally deformed metric $(1+|\nabla u|^2)^{1/m}(g+\mathrm{d}\,t^2)$ is complete since $(1+|\nabla u|^2)^{1/m} \geq 1 > 0$. $\square$

The following corollary shows that there are many minimal hypersurfaces in complete conformally flat spaces.

COROLLARY 4.6. *For any harmonic function $u : \mathbb{R}^m \longrightarrow \mathbb{R}$ with $m \geq 2$, the foliation $\mathcal{F} = \{(x, u(x) + c) \in \mathbb{R}^m \times \mathbb{R} : c \in \mathbb{R}\}$ of $\mathbb{R}^{m+1}$ by the parallel graphs of u is a minimal foliation with respect to the complete conformally flat metric $(1 + |\nabla u|^2)^{1/m}\,\delta_{ij}$ on $\mathbb{R}^{m+1}$. Furthermore, each graph is a homologically area-minimizing hypersurface.*

Proof. The first part of the corollary follows immediately form Theorem 4.5. That each graph is a homologically area-minimizing hypersurface follows from the fact that $\mathbb{R}^{m+1}$ is orientable and Corollary 3.9 in [13]. $\square$

By the classification theorem of Fischer-Colbrie and Schoen [5] (see also the recent paper of Li and Wang [9]), a complete, oriented, stable, minimal surface in a complete 3-manifold of nonnegative scalar curvature must be either conformally a plane $\mathbb{R}^2$ or conformally a cylinder $\mathbb{R} \times S^1$. The following proposition gives some examples of the same type of minimal surfaces in a complete conformally flat space of strictly negative scalar curvature.

PROPOSITION 4.7. *With respect to the complete conformally flat metric $\frac{1}{2}\sqrt{4 + x^2 + y^2}\,\delta_{ij}$ of strictly negative scalar curvature on $\mathbb{R}^3$, we have:*

(i) *The foliation of $\mathbb{R}^3$ by the parallel hyperbolic paraboloids $\mathcal{F} = \{z = \frac{1}{2}xy + c : c \in \mathbb{R}\}$ is a non-totally geodesic minimal foliation with each leaf a complete, orientable, stable, minimal surface.*

(i) *The foliation by the parallel planes $\mathcal{F}_3 = \{(x, y, z) \in \mathbb{R}^3 : z = c, c \in \mathbb{R}\}$ is a totally geodesic foliation with each leaf a complete, orientable, stable, minimal surface which is conformally a Euclidean plane $\mathbb{R}^2$.*

Proof. To prove Statement (i), we apply Corollary 4.6 to the harmonic function $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $u(x, y) = \frac{1}{2}xy$ to conclude that the foliation $\mathcal{F}$ of $\mathbb{R}^3$ by parallel hyperbolic paraboloids is a minimal foliation of the complete conformally flat space $(\mathbb{R}^3, \frac{1}{2}\sqrt{4 + x^2 + y^2}\,\delta_{ij})$. Clearly, each leaf is an orientable surface in $\mathbb{R}^3$. It is complete since it is a close subset in a complete manifold. By Corollary 4.6 again, each leaf is homologically area-minimizing, hence stable. By the transformation law of conformal change of metrics on an m-dimensional manifold, we know that the scalar curvatures $\bar{S}$ of $\bar{g} = e^{2\psi}g$ and S of g are related by

$$\bar{S} = e^{-2\psi}[S - 2(m-1)\triangle\psi - (m-1)(m-2)\,|\nabla\psi|^2]. \tag{4.11}$$

A direct computation using (4.11) shows that the scalar curvature $\bar{S}$ of the conformally flat metric $\frac{1}{2}\sqrt{4 + x^2 + y^2}\,\delta_{ij}$ is given by $\bar{S} = -(32+x^2+y^2)/(4+$

$x^2 + y^2)^{5/2}$, which is strictly negative. By [19], page 110, the foliation $\mathcal{F}$ of (M, g) by the level hypersurfaces of a submersion $u : (M^m, g) \longrightarrow \mathbb{R}$ is totally geodesic if and only if $\mathrm{Hess}_f(X, Y) = 0$ for all vector fields X, Y tangent to the leaves of $\mathcal{F}$. Thus, to see that $\mathcal{F}$ is not a totally geodesic foliation it is enough to show that $\mathrm{Hess}_f(X, Y) \neq 0$ for some vector fields X, Y tangent to the leaves of $\mathcal{F}$. For $f(x, y, z) = z - \frac{1}{2}xy$, we can easily check that $X = \partial_1 + \frac{1}{2}y\ \partial_3$ and $Y = \partial_2 + \frac{1}{2}x\ \partial_3$ are a base of vector fields tangent to the leaves, where $\partial_1 = \frac{\partial}{\partial x}$, $\partial_2 = \frac{\partial}{\partial y}$ and $\partial_3 = \frac{\partial}{\partial z}$. A straightforward computation gives

$$\begin{aligned}
\mathrm{Hess}_f(X, Y) &= XYf - (\nabla_X Y)f = -(\nabla_{\partial_1 + \frac{1}{2}y\partial_3}(\partial_2 + \frac{1}{2}x\partial_3))f \\
&= -\frac{1}{2}\ \partial_3 f - (\Gamma^1_{12} + \frac{1}{2}x\ \Gamma^1_{13} + \frac{1}{2}y\ \Gamma^1_{23} + \frac{1}{4}xy\ \Gamma^1_{33})\partial_1 f \\
&\quad - (\Gamma^2_{12} + \frac{1}{2}x\ \Gamma^2_{13} + \frac{1}{2}y\ \Gamma^2_{23} + \frac{1}{4}xy\ \Gamma^2_{33})\partial_2 f \\
&\quad - (\Gamma^3_{12} + \frac{1}{2}x\ \Gamma^3_{13} + \frac{1}{2}y\ \Gamma^3_{23} + \frac{1}{4}xy\ \Gamma^3_{33})\partial_3 f,
\end{aligned} \tag{4.12}$$

where Γ^k_{ij} are connection coefficients of the conformally flat metric $\frac{1}{2}\sqrt{4 + x^2 + y^2}\ \delta_{ij}$. Define $\lambda = -\frac{1}{4}\ \ln[1 + (x^2 + y^2)/4]$. Then, by [18], we have

$$\begin{aligned}
&\Gamma^i_{ii} = -\lambda_i = -\Gamma^i_{jj},\ \ \Gamma^i_{ij} = \Gamma^i_{ji} = -\lambda_j\ \ (i \neq j); \\
&\text{all other } \Gamma^i_{jk} = 0.
\end{aligned} \tag{4.13}$$

A further calculation using (4.13) and (4.12) shows that

$$\mathrm{Hess}_f(X, Y) = -\frac{1}{2}\ - \frac{x^2 y^2}{8(4 + x^2 + y^2)} \neq 0.$$

Therefore, the foliation $\mathcal{F}$ is not totally geodesic. This completes the proof of Statement (i). For Statement (ii), we note from (4.13) that $\Gamma^3_{ij} = 0$ for any $i, j \neq 3$. Thus, by Corollary 4.1 in [13], we conclude that the coordinate plane foliation $\mathcal{F}_3$ is a totally geodesic foliation. By reasons similar to those given in the proof of (i), each plane is a complete, orientable, stable, minimal surface. Since the induced metric of $\frac{1}{2}\sqrt{4 + x^2 + y^2}\ \delta_{ij}$ on the plane $z = c$ is given by $ds^2 = \frac{1}{2}\sqrt{4 + x^2 + y^2}(dx^2 + dy^2)$, we conclude that each plane is conformally equivalent to the Euclidean plane $\mathbb{R}^2$. This completes the proof of (ii). □

Remark 4.1. It was proved in [13] that the foliation of $\mathbb{R}^3$ by the parallel hyperbolic paraboloids $\mathcal{F} = \{z = \frac{1}{2}xy + c : c \in \mathbb{R}\}$ is also a non-totally geodesic minimal foliation with respect to Nil metric $g_{\mathrm{Nil}} = dx^2 + dy^2 + (dz - xdy)^2$ on $\mathbb{R}^3$. Therefore, we have an example of a foliation of $\mathbb{R}^3$ which is a non-totally geodesic minimal foliation with respect to two different Riemannian metrics.

Now we give an example showing that the minimal foliations obtained in Theorem 4.1 need not be non-totally geodesic in general.

Example 4.2. For $m \geq 3$, the foliation of the conformally flat space $(\mathbb{R}^m \setminus \{0\}, (2-m)^{2/(1-m)} |x|^{-2} \ \delta_{ij})$ by the concentric spheres $\mathcal{F} = \{x \in \mathbb{R}^m \setminus \{0\} : |x|^2 = c, c \in \mathbb{R}^+\}$ is a totally geodesic foliation.

In fact, it is not difficult to check that $u : \mathbb{R}^m \setminus \{0\} \longrightarrow \mathbb{R}$ with $u(x) = |x|^{2-m}$ is a harmonic submersion with $|\nabla u|^2 = (2-m)^2 |x|^{2(1-m)}$. Thus, by Theorem 4.1, $u : (\mathbb{R}^m \setminus \{0\}, (m-2)^{2/(m-1)} |x|^{-2} \ \delta_{ij}) \longrightarrow \mathbb{R}$ is a 1-harmonic submersion and hence the level hypersurfaces determine a minimal foliation of $(\mathbb{R}^m \setminus \{0\}, (m-2)^{2/(m-1)} |x|^{-2} \ \delta_{ij})$. Noting that the level hypersurfaces $u^{-1}(c) = \{x \in \mathbb{R}^m \setminus \{0\} : |x|^{2-m} = c\}$ of u are nothing but the concentric spheres we conclude that the foliation $\mathcal{F}$ is a minimal foliation. We claim that this minimal foliation, unlike the one obtained in Proposition 4.7 (i), is totally geodesic. To see this, we write, by using polar coordinates, the Euclidean subspace $(\mathbb{R}^m \setminus \{0\}, \delta_{ij})$ as a warped product $(S^{m-1} \times \mathbb{R}^+, r^2 g_{m-1} + \mathrm{d}\, r^2)$ of (S^{m-1}, g_{m-1}) and $(\mathbb{R}^+, dt^2)$, where g_{m-1} denotes the standard metric on the sphere S^{m-1}. One can easily check that, with respect to the polar coordinates, the conformally flat space $(\mathbb{R}^m \setminus \{0\}, (m-2)^{2/(1-m)} |x|^{-2} \ \delta_{ij})$ is isometric, up to a scalar, to $(S^{m-1} \times \mathbb{R}^+, g_{m-1} + [(m-2)^{1/(m-1)} r]^{-2} \,\mathrm{d}\, r^2)$, a twisted product of (S^{m-1}, g_{m-1}) and $(\mathbb{R}^+, dt^2)$ with the twisting function $[(m-2)^{1/(m-1)} r]^{-2}$. It follows from [14] that the canonical foliation (i.e., the foliation by concentric spheres in our case) of a twisted product is a totally geodesic foliation.

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