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CONTINUITY WITH RESPECT TO DISORDER OF THE INTEGRATED DENSITY OF STATES

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ABSTRACT. We prove that the integrated density of states (IDS) associated to a random Schrödinger operator is locally uniformly Hölder continuous as a function of the disorder parameter λ. In particular, we obtain convergence of the IDS, as $\lambda \to 0$, to the IDS for the unperturbed operator at all energies for which the IDS for the unperturbed operator is continuous in energy.

1. Introduction and results

In this article, we use the methods recently developed in [2], [4] to prove that the integrated density of states (IDS) $N_\lambda(E)$ for a random Schrödinger operator $H_\omega(\lambda) = H_0 + \lambda V_\omega$ is a uniformly Hölder continuous function of the disorder parameter λ at energies E for which the unperturbed operator H_0 has a continuous IDS $N_0(E)$, under fairly general conditions. Moreover, the uniformity in λ implies that $N_\lambda(E) - N_0(E)$ is (Hölder) continuous in λ, as $\lambda \to 0$, at points E of (Hölder) continuity of $N_0(E)$. This result applies to random Schrödinger operators on the lattice $\mathbb{Z}^d$ and on the continuum $\mathbb{R}^d$, given as perturbations of a deterministic, background operator $H_0 = (-i\nabla - A_0)^2 + V_0$. We assume that the background operator is self-adjoint with operator core $C_0^\infty(X)$ (smooth, compactly supported functions on X) for $X = \mathbb{Z}^d$ or $X = \mathbb{R}^d$. For simplicity, we assume that $H_0 \geq -M_0 > -\infty$, for some finite constant M_0. In addition, we require that H_0 is gauge invariant under translations by elements of $\mathbb{Z}^d$. Specifically, this means that for every $m \in \mathbb{Z}^d$, we have $V_0(x + m) = V_0(x)$ and $A_0(x + m) = A_0(x) + \nabla \phi_m(x)$ for some function ϕ_m. For $X = \mathbb{Z}^d$, the operator $(-i\nabla - A_0)^2$ represents a short-range, e.g., nearest neighbor, hopping matrix.

We consider Anderson-type random potentials V_ω constructed from a family of independent, identically distributed (iid) random variables $\{\omega_j \mid j \in \mathbb{Z}^d\}$.

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On the lattice $\mathbb{Z}^d$, the potential acts as

$$(V_\omega f)(m) = \omega_m f(m),\ m \in \mathbb{Z}^d,\ f \in \ell^2(\mathbb{Z}^d), \tag{1.1}$$

On $\mathbb{R}^d$, the potential $V_\omega(x)$ also depends on the single-site potential u, and is a multiplication operator given by

$$(V_\omega f)(x) = \sum_{j\in\mathbb{Z}^d} \omega_j u(x-j) f(x),\ f \in L^2(\mathbb{R}^d). \tag{1.2}$$

Precise hypotheses on the single-site potential u and the random variables $\{\omega_j \mid j \in \mathbb{Z}^d\}$ are given below.

The family of random Schrödinger operators is given by

$$H_\omega(\lambda) = H_0 + \lambda V_\omega. \tag{1.3}$$

The parameter $\lambda > 0$ is a measure of the disorder strength, and we consider the other parameters entering into the construction of V_ω, that is, the norm $\|u\|_\infty$ and the distribution of ω_0, as fixed. As we are interested in the explicit dependence on λ, we will write H_λ for $H_\omega(\lambda)$ and suppress ω in the notation. Due to the assumed gauge invariance under shifts of H_0, and the explicit form of the random potential given in (1.1) and (1.2), the random operator $H_\omega(\lambda)$, for fixed λ, is ergodic with respect to the gauge twisted shifts

$$\psi(x) \mapsto e^{i\phi_m(x)}\psi(x-m)\ , \quad \psi \in L^2(X). \tag{1.4}$$

We mention that the results of this note are easily modified to apply to the random operators describing acoustic and electromagnetic waves in disordered media, and we refer the reader to [6], [8], [9].

Our result follows the investigation initiated in [2] where a proof of the Hölder continuity in energy of the IDS is given that relies on the continuity of the IDS for the unperturbed, background operator H_0. As in the first part of [2], we require that the IDS $N_0(E)$ for the background operator H_0 exists and that it is Hölder continuous in the energy. The proof is local in the energy and applies at any energy E at which $N_0(E)$ is Hölder continuous. In particular, it applies to Landau Hamiltonians away from the Landau levels, where $N_0(E)$ is discontinuous.

Before stating our results, let us make precise the hypotheses on the random potential.

Hypothesis (H1). *The family of iid random variables $\{\omega_j \mid j \in \mathbb{Z}^d\}$ is distributed with a density $h \in L^\infty(\mathbb{R})$ with compact support.*

Hypothesis (H2). *The single-site potential $u \geq 0$ is bounded with compact support. There exists an open subset $\mathcal{O} \subset \operatorname{supp} u$, and a positive constant $\kappa > 0$, so that $u_{|\mathcal{O}} > \kappa > 0$.*

We first recall a result of [2], that Hölder continuity in energy of the IDS for H_0 implies continuity of the IDS for $H_\omega(\lambda)$, with a constant and Hölder exponent independent of λ.

THEOREM 1.1. *We assume that the Schrödinger operator H_0 admits an IDS $N_0(E)$ that is Hölder continuous on the interval $I \subset \mathbb{R}$ with Hölder exponent $0 < q_1 \leq 1$, that is*

$$|N_0(E) - N_0(E')| \leq C_0(q_1, I)|E - E'|^{q_1}, \tag{1.5}$$

for all $E, E' \in I$, and some finite constant $0 < C_0(q_1, I) < \infty$. We assume hypotheses (H1) and (H2) on the random potential V_ω. Then, for any constant $0 < q \leq q_1 q^*/(q_1 + 2)$, where $q^* = 1$ for $\ell^2(\mathbb{Z}^d)$ and $0 < q^* < 1$ for $L^2(\mathbb{R}^d)$ (see (2.10)*), there exists a finite positive constant $C(q, I)$, independent of λ, so that for any $\lambda \neq 0$, and any $E, E' \in I$, we have*

$$|N_\lambda(E) - N_\lambda(E')| \leq C(q, I)|E - E'|^q. \tag{1.6}$$

Note that the exponent q obtained by this method is roughly $1/3$ whereas it is believed that it should hold with $q = 1$ (see Section 4). A similar result was obtained recently by one of us [13], using a method quite different from that in [2].

We now present the main result of this note.

THEOREM 1.2. *Under the hypotheses of Theorem 1.1, for any bounded interval $J \subset \mathbb{R}$, there exists a finite, positive constant $C(q, I, J)$, such that if $\lambda, \lambda' \in J$, we have*

$$|N_\lambda(E) - N_{\lambda'}(E)| \leq C(q, I, J)|\lambda - \lambda'|^{q_2}, \tag{1.7}$$

for any $E \in I$ and $0 < q_2 \leq 2q/(q + 3)$, where $0 < q \leq q_1 q^*/(q_1 + 2)$ is the exponent in (1.6)*.*

Until recently, it was not known that the IDS remained bounded in the weak disorder limit $\lambda \to 0$. In particular, result (1.7) was known only for closed intervals J *disjoint from zero.* This result follows from the Helffer-Sjöstrand formula (see Section 3 and also [10]). However, the constant $C(q_1, I, J)$ obtained from that proof scales like $[\mathrm{dist}(J, 0)]^{-1}$.

Recall that control of the IDS comes from the Wegner estimate,

$$\mathbb{P}\{\mathrm{dist}(\sigma(H_\Lambda), E) \leq \eta\} \leq C_q(\lambda)|\Lambda|\eta^q, \tag{1.8}$$

for any $0 < q \leq 1$. Here H_Λ is the restriction, with suitable boundary conditions, of $H_\omega(\lambda)$ to a bounded open set Λ of volume $|\Lambda|$. In the usual proof of the Wegner estimate [4], [14], the constant $C_q(\lambda)$ diverges as $1/\lambda$ as $\lambda \to 0$. In [2], a different proof of the Wegner estimate is given for which the constant $C(q, I)$ is *uniformly bounded in* λ. The only deficit of this proof is that the Hölder exponent q for the IDS $N_\lambda(E)$ must be taken sufficiently small (as stated in Theorem 1.1) relative to the assumed Hölder exponent $0 < q_1 \leq 1$ of the IDS of $N_0(E)$ in (1.5). In particular, the bound gives no information about the density of states (DOS) $\rho_\lambda(E) \equiv dN_\lambda(E)/dE$ (see Section 4 for a further discussion of the DOS).

We have the following two corollaries of Theorems 1.1 and 1.2.

Corollary 1.3. *Under the same assumptions as Theorem 1.1, let $J \subset \mathbb{R}$ be any closed, bounded interval containing* 0. *Then, there exists a finite, positive constant $C(q, I, J)$, so that we have for any $E \in I$ and $\lambda \in J$,*

$$|N_\lambda(E) - N_0(E)| \leq C(q_1, I, J)|\lambda|^{q_2}, \tag{1.9}$$

where $0 < q_2 \leq 2q/(q + 3)$, where $0 < q \leq q_1 q^*/(q_1 + 2)$ is the exponent in (1.6).

There is a version of Theorem 1.2 and Corollary 1.3 with the weaker hypothesis of continuity for $N_0(E)$ and with a correspondingly weaker result.

Corollary 1.4. *We assume that the Schrödinger operator H_0 admits an IDS $N_0(E)$ that is continuous at E. Then, under the same hypotheses* (H1) *and* (H2) *as in Theorem 1.1, we have for any λ that the IDS N_λ is also continuous at E and that*

$$\lim_{\lambda' \to \lambda} N_{\lambda'}(E) = N_\lambda(E). \tag{1.10}$$

In general, as the IDS $N_0(E)$ is a monotone increasing function, this result applies at all but a countable set of energies.

In Section 2, we recall the proof of Theorem 1.1. The proofs of Theorem 1.2, and Corollaries 1.3-1.4, are given in Section 3. We conclude with some comments about the behavior of the density of states in Section 4. While preparing this article, we learned that Germinet and Klein [10] have proved a version of (1.7) for intervals J away from zero. We thank F. Germinet (private communication) for showing us the use of (3.9) that improves our original estimates on q_2.

It is clear that there are various generalizations of our results. For example, hypothesis (H1) can be weakened to allow unbounded random variables with the first two moments bounded.

2. Sketch of the proof of Theorem 1.1

For completeness, let us sketch the proof of Theorem 1.1 that appears in [2]. We assume hypotheses (H1)–(H2) and condition (1.5) on the IDS $N_0(E)$ for the background operator H_0. Let $\Delta \subset I$ be a sufficiently small closed interval, and let $\tilde{\Delta} \supset \Delta$ be a bounded interval with $|\tilde{\Delta}| = \mathcal{O}(|\Delta|^\alpha)$, for some $\alpha \in (0, 1)$. First, one proves that (1.5) implies that for all Λ sufficiently large, depending on $\tilde{\Delta}$, there exists a finite constant $C_1(I,d) > 0$ so that

$$\operatorname{Tr} E_0^\Lambda(\tilde{\Delta}) \leq C_1(I,d)|\tilde{\Delta}|^{q_1}|\Lambda|. \tag{2.1}$$

Next, we consider the local spectral projector $E_\Lambda(\Delta)$ for H_Λ and write

$$\operatorname{Tr} E_\Lambda(\Delta) = \operatorname{Tr} E_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}) + \operatorname{Tr} E_\Lambda(\Delta)E_0^\Lambda(\tilde{\Lambda}^c), \tag{2.2}$$

where $\tilde{\Delta}^c \equiv \mathbb{R}\backslash\tilde{\Delta}$. The first term on the right in (2.2) is easily seen to be bounded by

$$\operatorname{Tr} E_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}) \leq \operatorname{Tr} E_0^\Lambda(\tilde{\Delta}) \leq C_1(I,d)|\Delta|^{\alpha q_1}|\Lambda|, \tag{2.3}$$

and is already of order $|\Delta|^q$, for any $q \leq \alpha q_1$.

The second term on the right of (2.2) is estimated in second-order perturbation theory. Let $E \in \Delta$ be the center of the interval Δ, and write

$$\begin{aligned} \operatorname{Tr} E_\Lambda(\Delta)E_0^\Lambda(\tilde{\Lambda}^c) &= \operatorname{Tr} E_\Lambda(\Delta)(H_\Lambda - E)E_0^\Lambda(\tilde{\Lambda}^c)(H_0^\Lambda - E)^{-1} \\ &\qquad - \lambda \operatorname{Tr} E_\Lambda(\Delta)V_\Lambda E_0^\Lambda(\tilde{\Lambda}^c)(H_0^\Lambda - E)^{-1} \\ &= \text{(i)} + \text{(ii)}. \end{aligned} \tag{2.4}$$

Since the distance from $\tilde{\Delta}^c$ to E is of order $|\Delta|^\alpha$, we easily see that term (i) of (2.4) is bounded as

$$|\text{(i)}| \leq |\Delta|^{1-\alpha} \operatorname{Tr} E_\Lambda(\Delta), \tag{2.5}$$

so that as $0 < \alpha < 1$ and $|\Delta| < 1$, we can move this term to the left in (2.2). Continuing with (ii), we repeat the calculation in (2.4), now to the left of $E_\Lambda(\Delta)$, and obtain

$$\begin{aligned} \text{(ii)} &= -\lambda \operatorname{Tr}(H_\Lambda - E)E_\Lambda(\Delta)V_\Lambda E_0^\Lambda(\tilde{\Lambda}^c)(H_0^\Lambda - E)^{-2} \\ &\qquad + \lambda^2 \operatorname{Tr} V_\Lambda E_\Lambda(\Delta)V_\Lambda E_0^\Lambda(\tilde{\Lambda}^c)(H_0^\Lambda - E)^{-2} \\ &= \text{(iii)} + \text{(iv)}. \end{aligned} \tag{2.6}$$

Term (iii) is estimated as in (2.5) and we obtain

$$|\text{(iii)}| \leq \lambda|\Delta|^{1-2\alpha}\|\tilde{V}_\Lambda\| \operatorname{Tr} E_\Lambda(\Delta), \tag{2.7}$$

where $\tilde{V}_\lambda$ is the potential obtained by replacing ω_j by the maximal value of $|\omega_j|$. Term (iv) in (2.6) can be bounded above by

$$|\text{(iv)}| \leq \lambda^2|\Delta|^{-2\alpha} \operatorname{Tr} V_\Lambda E_\Lambda(\Delta)V_\Lambda. \tag{2.8}$$

Taking the expectation and replacing V_λ^2 by the upper bound $\tilde{V}_\Lambda^2$, we find that we must estimate

$$\mathbb{E}\{\operatorname{Tr}(\tilde{V}_\Lambda^2 E_\Lambda(\Delta))\}. \tag{2.9}$$

This is done using estimates on the spectral shift function comparing the two local Hamiltonians with one random variable fixed respectively at its maximum and minimum values. For the lattice case, this is a rank one perturbation, so the corresponding spectral shift is bounded by one, the rank of the perturbation. For the continuous case, the perturbation is no longer of finite rank, but we may use the local L^p-estimate on the spectral shift function proved in [4]. In either case we obtain

$$\mathbb{E}\{\mathrm{Tr}(\tilde{V}_\Lambda^2 E_\Lambda(\Delta))\} \leq C_4(I, q^*, u)\lambda^{-1}|\Delta|^{q^*}|\Lambda|, \tag{2.10}$$

where the exponent q^* in (2.10) is (i) $q^* = 1$ in the lattice case, (ii) $0 < q^* < 1$ in the continuum.

As a consequence, term (iv) in (2.6) can be bounded by

$$\mathbb{E}\{|(\mathrm{iv})|\} \leq \lambda|\Delta|^{q^*-2\alpha} C_1(I, d) C_4(I, q^*, u)|\Lambda|. \tag{2.11}$$

Putting together (2.3), (2.5), (2.7), and (2.11), we obtain

$$\begin{aligned} &\{1 - |\Delta|^{1-\alpha} - \lambda|\Delta|^{1-2\alpha}\|\tilde{V}_\Lambda\|\}\mathbb{E}\{\mathrm{Tr}\, E_\Lambda(\Delta)\} \\ &\qquad \leq (\lambda|\Delta|^{q^*-2\alpha} C_1(u, d) C_4(q^*, u) + C_2|\Delta|^{\alpha q_1})|\Lambda|. \end{aligned} \tag{2.12}$$

By choosing the optimal $\alpha < 1/2$, it is clear from this expression that Theorem 1.1 holds with $0 < q \leq q_1 q^*/(q_1 + 2)$. □

3. Proof of Theorem 1.2

The almost-sure existence of the IDS for random Hamiltonians of the type considered here is well-known and we refer the reader to [1], [11], [12]. The IDS $N_\lambda(E)$ is given in terms of the spectral projector $P_\lambda(E)$ associated with $H_\omega(\lambda)$ and the interval $(-\infty, E] \subset \mathbb{R}$. For the lattice case, with Hilbert space $\ell^2(\mathbb{Z}^d)$, the IDS $N_\lambda(E)$ is given by

$$\begin{aligned} N_\lambda(E) &= \mathbb{E}\{\mathrm{Tr}\, \Pi_0 P_\lambda(E) \Pi_0\} \\ &= \mathbb{E}\{\langle 0|P_\lambda(E)|0\rangle\}, \end{aligned} \tag{3.1}$$

where Π_0 is the projection onto the site at 0 and $|x\rangle$ is the state at site $x \in \mathbb{Z}^d$. For the continuous case on $\mathbb{R}^d$, the IDS $N_0(E)$ is given by

$$N_\lambda(E) \equiv \mathbb{E}\{\mathrm{Tr}\, \chi_0 P_\lambda(E) \chi_0\}, \tag{3.2}$$

with χ_0 the characteristic function on the unit cube in $\mathbb{R}^d$. *To unify the notation, we will write* χ_0 *for the characteristic function on the unit cube as in* (3.2) *in the continuous case, or for the projector* Π_0 *as in* (3.1) *in the lattice case.*

Proof of Theorem 1.2. Fix $\lambda, \lambda' \in J$ and $E \in I$. Choose $g \in C^4(\mathbb{R})$, depending on E, λ, and λ', with $0 \leq g \leq 1$ and

$$g(s) = \begin{cases} 1 & s \leq E, \\ 0 & s \geq E + |\lambda - \lambda'|^\alpha, \end{cases} \tag{3.3}$$

where $0 < \alpha \leq 1$ will be determined. The choice of $g \in C^4$ obeying (3.3) is basically arbitrary; however we require that

$$\|g^{(j)}\|_\infty \leq C|\lambda - \lambda'|^{-j\alpha} \quad (j = 1, 2, 4)\ , \tag{3.4}$$

with some constant independent of E, λ, and λ' (this can be done).

We have

$$N_\lambda(E) - N_{\lambda'}(E) = N_\lambda(E) - \mathbb{E}\left\{\operatorname{Tr}\chi_0[g(H_\lambda)]^2\chi_0\right\} + \mathbb{E}\left\{\operatorname{Tr}\chi_0[g(H_{\lambda'})]^2\chi_0\right\} - N_{\lambda'}(E) + \mathbb{E}\left\{\operatorname{Tr}\chi_0([g(H_\lambda)]^2 - [g(H_{\lambda'})]^2)\chi_0\right\}. \tag{3.5}$$

The monotonicity of $N_\lambda(E)$ with respect to energy and the properties of g imply that

$$\mathbb{E}\left\{\operatorname{Tr}\chi_0[g(H_\lambda)]^2\chi_0\right\}\chi_0 \le N_\lambda(E + |\lambda - \lambda'|^\alpha). \tag{3.6}$$

It follows from Theorem 1.1 that

$$\mathbb{E}\left\{\operatorname{Tr}\chi_0[g(H_\lambda)]^2\chi_0\right\} - N_\lambda(E) \le N_\lambda(E + |\lambda - \lambda'|^\alpha) - N_\lambda(E) \le C(q_1, I, J)|\lambda - \lambda'|^{\alpha q}, \tag{3.7}$$

for any $0 < q \le q_1 q^*/(q_1 + 2)$, and an identical estimate holds for the second term on the right in (3.5).

It remains to estimate the last term on the right in (3.5). Using the identity $2(A^2 - B^2) = \{A(A - B) + (A - B)A\} + \{B(A - B) + (A - B)B\}$, we can write the last term in (3.5) as

$$\begin{aligned}\mathbb{E}\left\{\operatorname{Tr}\chi_0([g(H_\lambda)]^2 - [g(H_{\lambda'})]^2)\chi_0\right\} &= \mathbb{E}\left\{\operatorname{Tr}\chi_0 g(H_\lambda)(g(H_\lambda) - g(H_{\lambda'}))\chi_0\right\} \\ &\quad + \mathbb{E}\left\{\operatorname{Tr}\chi_0 g(H_{\lambda'})(g(H_\lambda) - g(H_{\lambda'}))\chi_0\right\},\end{aligned} \tag{3.8}$$

where, to reduce the number of terms, we have made use of the following identity: If A_ω and B_ω are $\mathbb{Z}^d$-ergodic operators such that $\chi_0 A_\omega B_\omega \chi_0$ is trace class, then we have

$$\mathbb{E}\left\{\operatorname{Tr}\chi_0 A_\omega B_\omega \chi_0\right\} = \mathbb{E}\left\{\operatorname{Tr}\chi_0 B_\omega A_\omega \chi_0\right\}. \tag{3.9}$$

(We use this identity in a more crucial way below.) We note that the trace norm $\|\chi_0 g(H_\lambda)\|_1$ is bounded uniformly in $\lambda \in J$ as well as in the random couplings ω_j. In the continuum, we use the fact that H_λ is bounded from below.

We express the difference $(g(H_\lambda) - g(H_{\lambda'}))$ in terms of the resolvents using the Helffer-Sjöstrand formula, which we now recall (see, for example, [7] for details). Given $f \in C_0^{k+1}(\mathbb{R})$, we denote by $\tilde{f}_k$ an almost analytic extension of f of order k, which is a function $\tilde{f}_k$ defined in a complex neighborhood of the support of f having the property that $\tilde{f}_k(x + i0) = f(x)$ and that

$$|\partial_{\bar{z}}\tilde{f}_k(x + iy)| \sim |f^{(k+1)}(x)||y|^k, \text{ as } |y| \to 0, \tag{3.10}$$

where $\partial_{\bar{z}} = \partial_x + i\partial_y$. For the construction of such a function, which is not unique, we refer to [7]. Let $R_\lambda(z) = (H_\lambda - z)^{-1}$ denote the resolvent of H_λ.

For functions g as in (3.3), the functional calculus gives

$$g(H_\lambda) - g(H_{\lambda'}) = \frac{(\lambda - \lambda')}{\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{g}(z) R_\lambda(z) V_\omega R_{\lambda'}(z) d^2 z, \tag{3.11}$$

with $\tilde{g}$ an extension of order 3 (recall that $g \in C^4$).

Let us estimate the first term on the right in (3.8). The estimate for the second term is similar. We substitute the Helffer-Sjöstrand formula (3.11) and find

$$\begin{aligned} &\operatorname{Tr} \chi_0 g(H_\lambda)(g(H_\lambda) - g(H_{\lambda'}))\chi_0 \\ &\qquad = \frac{(\lambda - \lambda')}{\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{g}(z) \operatorname{Tr} \chi_0 g(H_\lambda) R_\lambda(z) V_\omega R_{\lambda'}(z) \chi_0 d^2 z. \end{aligned} \tag{3.12}$$

Using the second resolvent identity, we rewrite the operator involving resolvents as

$$\begin{aligned} R_\lambda(z) V_\omega R_{\lambda'}(z) &= R_\lambda(z) V_\omega (R_{\lambda'}(z) - R_\lambda(z)) + R_\lambda(z) V_\omega R_\lambda(z) \\ &= (\lambda' - \lambda) R_\lambda(z) V_\omega R_{\lambda'}(z) V_\omega R_\lambda(z) \\ &\qquad + R_\lambda(z) V_\omega R_\lambda(z) \\ &= \text{(i)} + \text{(ii)}. \end{aligned} \tag{3.13}$$

The integral in (3.12) involving the first term (i) in (3.13) is estimated as follows. The resolvents are bounded by $|\Im z|^{-1}$ as $|\Im z| \to 0$, but this divergence is canceled by the estimate (3.10) for $\partial_{\bar{z}} \tilde{g}$ (since we take an extension of order 3). Recalling the estimate (3.4) on the derivatives of g and noting $|\operatorname{supp} g'| \sim \delta_1^{-1}$, with $\delta_1 = |\lambda - \lambda'|^\alpha$, we find that

$$\int_{\mathbb{C}} |\partial_{\bar{z}} \tilde{g}(z)| |\Im z|^{-3} d^2 z \leq C |\lambda - \lambda'|^{-3\alpha}, \tag{3.14}$$

since we obtain a factor of δ_1^{-4} from the derivatives and a factor of δ_1 from the size of the domain of integration. Consequently, we find

$$\begin{aligned} &\mathbb{E} \left\{ \left| \frac{(\lambda - \lambda')^2}{\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{g}(z) \operatorname{Tr} \chi_0 g(H_\lambda) R_\lambda(z) V_\omega R_{\lambda'}(z) V_\omega R_\lambda(z) \right| \right\} \\ &\qquad \leq C \mathbb{E} \left\{ \|\chi_0 g(H_\lambda)\|_1 \|V_\omega\|_\infty^2 \right\} |\lambda - \lambda'|^{2-3\alpha} \leq C_0 |\lambda - \lambda'|^{2-3\alpha}. \end{aligned} \tag{3.15}$$

To evaluate the integral involving the second term (ii) of (3.13), we apply (3.9) to the operator integrand in (ii) of (3.13). Inserting this into (3.12), we obtain for the integrand

$$\mathbb{E} \left\{ \operatorname{Tr} \chi_0 g(H_\lambda)(R_\lambda(z) V_\omega R_\lambda(z) \chi_0) \right\} = \mathbb{E} \left\{ \operatorname{Tr} \chi_0 g(H_\lambda) R_\lambda(z)^2 V_\omega \chi_0 \right\}. \tag{3.16}$$

The integral becomes

$$\int_{\mathbb{C}} \partial_{\bar{z}} \tilde{g}(z) \operatorname{Tr} \chi_0 g(H_\lambda) R_\lambda(z)^2 V_\omega \chi_0 d^2 z = -\pi \operatorname{Tr} \chi_0 g(H_\lambda) g'(H_\lambda) V_\omega \chi_0. \tag{3.17}$$

As a result, we obtain the following estimate for the term involving (ii):

$$|\mathbb{E}\{\operatorname{Tr}\chi_0 g(H_\lambda)g'(H_\lambda)V_\omega\chi_0\}| \leq C_1|\lambda-\lambda'|^{-\alpha}. \tag{3.18}$$

Combining the estimates (3.7), (3.15), and (3.18), we obtain the upper bound for the right side of (3.5),

$$\begin{aligned}|N_\lambda(E) - N_{\lambda'}(E)| \\ \leq 2C(q_1,I)|\lambda-\lambda'|^{\alpha q} + C_0|\lambda-\lambda'|^{2-3\alpha} + C_1|\lambda-\lambda'|^{1-\alpha}.\end{aligned} \tag{3.19}$$

Comparing the exponents of $|\lambda-\lambda'|$ in (3.19), we can take $0<\alpha<1$ so that $\alpha q = 2-3\alpha$, giving the exponent $2q/(q+3)$. □

The proof of Corollary 1.3 follows simply by taking $\lambda' = 0$. The continuity result of Corollary 1.4 is proved as follows.

Proof of Corollary 1.4. It suffices to note that the proof of Theorem 1.1 in [2] can be extended to prove that if $N_0(E)$ continuous at E, then so is $N_\lambda(E)$. To see this, fix $E\in\mathbb{R}$ at which N_0 is continuous. Following the argument of [2], we see that the finite-volume estimate (2.1) becomes the following. For any $\epsilon>0$, there exists $\eta>0$ such that for $\tilde{\Delta} = [E-\eta, E+\eta]$, and all Λ sufficiently large, one has

$$\operatorname{Tr} E_0^\Lambda(\tilde{\Delta}) \leq \epsilon|\Lambda|. \tag{3.20}$$

Without loss, we assume that $\eta<\epsilon$, since the left side of (3.20) is non-increasing in η. Choose a closed subinterval $\Delta = [E-\eta^p, E+\eta^p]$, with $p>1$. Following the argument in Section 2 with this choice of $\tilde{\Delta}$ and Δ, the estimates (2.5), (2.7), and (2.11) now have the form:

$$|(\mathrm{i})| \leq \eta^{p-1}\operatorname{Tr} E_\Lambda(\Delta), \tag{3.21}$$

$$|(\mathrm{iii})| \leq C_0\lambda\eta^{p-2}\operatorname{Tr} E_\Lambda(\Delta), \tag{3.22}$$

$$\mathbb{E}\{|(\mathrm{iv})|\} \leq C_1\lambda\eta^{pq^*-2}|\Lambda|, \tag{3.23}$$

where the constants C_0 and C_1 are independent of Λ, η, and ϵ and the exponent q^* appears in (2.10). These imply that (2.12) has the form

$$\{1-\eta^{p-1} - C_0\lambda\eta^{p-2}\}\mathbb{E}\{\operatorname{Tr} E_\Lambda(\Delta)\} \leq C(\lambda\eta^{pq^*-2}+\epsilon)|\Lambda|. \tag{3.24}$$

If we pick $p>3/q^*$, then for sufficiently small ϵ, we get for all Λ sufficiently large

$$\mathbb{E}\{\operatorname{Tr} E_\Lambda(\Delta)\} \leq \epsilon C|\Lambda|, \tag{3.25}$$

for some finite constant $C>0$ since $\eta<\epsilon$. This shows that the IDS N_λ is continuous at E.

To complete the proof of Corollary 1.4, we return to equations (3.5) and (3.7). We use the continuity of $N_\lambda(E)$ to control the first and the last terms on the right in (3.5). For example, we need to estimate

$$N_\lambda(E) - \mathbb{E}\left\{\operatorname{Tr} \chi_0 g(H_\lambda)\chi_0\right\}. \tag{3.26}$$

The monotonicity of $N_\lambda(E)$ with respect to energy, and the properties of g, imply that

$$\mathbb{E}\left\{\operatorname{Tr} \chi_0 g(H_\lambda)\chi_0\right\} \leq N_\lambda(E + |\lambda - \lambda'|^\alpha). \tag{3.27}$$

It follows from the continuity and monotonicity in E of $N_\lambda(E)$ that

$$\begin{aligned} 0 &\leq \mathbb{E}\left\{\operatorname{Tr} \chi_0 g(H_\lambda)\chi_0\right\} - N_\lambda(E) \\ &\leq N_\lambda(E + |\lambda - \lambda'|^\alpha) - N_\lambda(E) \xrightarrow{\lambda' \to \lambda} 0. \end{aligned} \tag{3.28}$$

The estimate for the middle term of (3.5) remains the same. Consequently, we have that

$$\lim_{\lambda' \to \lambda} N_{\lambda'}(E) = N_\lambda(E), \tag{3.29}$$

at any point E of continuity of N_0, proving Corollary 1.4. □

Remark. This proof shows that in general one can control the modulus of continuity for the IDS $N_\lambda(E)$ of the random model using that of the free model.

4. Additional comments and conjectures

In certain situations, we are able to obtain more information about the density of states (DOS) $\rho_\lambda(E)$ and its behavior as $\lambda \to 0$. The DOS is the derivative of the IDS $N_\lambda(E)$ with respect to energy. Since the spectral shift function is pointwise bounded for the lattice model, it follows from [2] that the DOS is bounded except at possibly a countable set of energies. In this case, the DOS is given by

$$\rho_\lambda(E) \equiv \frac{dN_\lambda}{dE}(E) = \lim_{\epsilon \to 0} \mathbb{E}\{\Im\langle 0|(H_\lambda - E - i\epsilon)^{-1}|0\rangle\}. \tag{4.1}$$

Let us suppose that the random variables ω_j are Gaussian with mean zero. In this case, the almost-sure spectrum of $H_\omega(\lambda)$ is $\mathbb{R}$, for $\lambda \neq 0$, and the spectrum of $H_0 = \Delta$ is $[-2d, 2d]$. If $E \in \mathbb{R} \backslash [-2d, 2d]$, the resolvent can be expanded in a Neumann series,

$$R_\lambda(E + i\epsilon) = \sum_{k=0}^{\infty} R_0(E + i\epsilon)\left[-\lambda V_\omega R_0(E + i\epsilon)\right]^k. \tag{4.2}$$

The matrix elements of the free resolvent decay exponentially by the Combes-Thomas argument. Let $d_0(E)$ be the distance from the spectrum of H_0 to E.

We then have the bound,

$$|\langle x|R_0(E)|y\rangle| \leq \frac{C_0}{d_0(E)} e^{-d_0(E)|x-y|/2}. \tag{4.3}$$

We take the expectation of the zero-zero matrix element in (4.2). We expand the potentials V_ω and use the estimate (4.3) to control the sum over sites. We easily see that the power series converges absolutely provided

$$\frac{|\lambda| \mathbb{E}\{|\omega_0|\} C_1(d)}{d_0(E)^{d+1}} < 1, \tag{4.4}$$

where the constant $C_1(d)$ depends on C_0 in (4.3) and the dimension. For example, for all $\lambda < 1$, we have the convergent expansion

$$\rho_\lambda(E) = \lambda^2 \rho^{(2)}(E) + \sum_{k=3}^{\infty} \lambda^k \rho^{(k)}(E), \tag{4.5}$$

for all $|E| > [C_1(d)\mathbb{E}\{|\omega_0|\}]^{1/(d+1)} + 2d$.

This result, and the results on the IDS in this article, are steps towards proving the general conjecture concerning the regularity of the DOS. In particular, under the hypotheses (H1)–(H2), we expect that the IDS is Lipschitz continuous, that is, we have $q = 1$ in (1.6), with a constant independent of λ. Furthermore, if the unperturbed operator H_0 has a Lipschitz continuous IDS, then we expect that

$$|\rho_\lambda(E) - \rho_0(E)| \leq C_q |\lambda|^q, \tag{4.6}$$

for some constant $0 < C_q < \infty$, independent of λ, and some $0 < q \leq 1$. Finally, if the distribution function for the random variable ω_0 is sufficiently regular, we expect that the IDS is also regular.

Note added in proof. Recent results [3] show that we can take $q^* = 1$ in Theorem 1.1 under (H1), and that we can extend Theorem 1.1 to include Hölder continuous probability measures and take q^* to be the Hölder exponent.

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