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A UNIFYING RADON-NIKODÝM THEOREM THROUGH NONSTANDARD HULLS

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ABSTRACT. We present a Radon-Nikodým theorem for vector measures of bounded variation that are absolutely continuous with respect to the Lebesgue measure on the unit interval. Traditional Radon-Nikodým derivatives are Banach space-valued Bochner integrable functions defined on the unit interval or some other measure space. The derivatives we construct are functions from $^*[0,1]$, the nonstandard extension of the unit interval into a nonstandard hull of the Banach space E. For these generalized derivatives we have an integral that resembles the Bochner integral. Furthermore, we can standardize the generalized derivatives to produce the weak*-measurable E''-valued derivatives that Ionescu-Tulcea, Dinculeanu and others obtained in [8] and [5].

1. Introduction and notation

For notions and notations not defined here we refer to the book by Albeverio, Fenstad, Hoegh-Krøhn and Lindstrom [1] and the survey on nonstandard hulls by Henson and Moore [6]. An introduction to nonstandard analysis can be found in the book [11] by Loeb and Wolff, or in the book [7] by Hurd and Loeb. We think of the nonstandard model in terms of superstructures $V(X)$ and $V(^*X)$ connected by the monomorphism $* : V(X) \to V(^*X)$ and call an element $b \in V(^*X)$ internal if it is an element of a standard entity, i.e., if there is an $a \in V(X)$ with $b \in {}^*a$. The most important example of a set which is not internal is the set of natural numbers $\mathbb{N} \subseteq {}^*\mathbb{N}$. For elements in $^*\mathbb{R}$ which are bounded by a standard real number, we can define the standard part map $^\circ : \mathrm{fin}(^*\mathbb{R}) \to \mathbb{R}$. Sometimes this map is also denoted by $\mathrm{st} : \mathrm{fin}(^*\mathbb{R}) \to \mathbb{R}$. We assume that the nonstandard model is at least $\aleph$-saturated, where $\aleph$ is an uncountable cardinal number. In standardizing to bidual-valued derivatives we need to assume that the nonstandard extension is κ-saturated, where $\mathrm{card}(E'') < \kappa$ for the standard Banach space E and where the cardinality of the Lebesgue measurable subsets of $[0,1]$ is less than κ.

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Our underlying measure space is $([0,1], \mathcal{C}, \lambda)$, the unit interval with Lebesgue measure. Let $({}^*[0,1], L_\lambda({}^*\mathcal{C}), \widehat{\lambda})$ denote the Loeb space constructed from the nonstandard extension of the Lebesgue unit interval. This measure space is obtained by extending the measure ${}^\circ {}^*\lambda$ from ${}^*\mathcal{C}$ to the σ-algebra generated by ${}^*\mathcal{C}$. The completion of this σ-algebra is denoted by $L_\lambda({}^*\mathcal{C})$. The Loeb measure $\widehat{\lambda}$ is a standard countably additive measure. Most of what we do in this paper also works for a more general measure space. Only for the existence of the lifting choice function $S : [0,1] \to {}^*[0,1]$, which we need in standardizing to a function defined on $[0,1]$ instead of a derivative defined on ${}^*[0,1]$, we would need additional assumptions on a more general measure space than $[0,1]$.

The functions we work with take their values in a Banach space E or its nonstandard hull $\widehat{E}$, which is defined as $\widehat{E} = \mathrm{fin}({}^*E)/\approx$, the quotient of the elements of bounded norm by elements of infinitesimal norm. The nonstandard hull is a standard Banach space and contains E as a subspace. We denote the quotient map from $\mathrm{fin}({}^*E)$ onto $\widehat{E}$ by π. By a result of Loeb in [9], there exists an internal *-finite partition of ${}^*[0,1]$ consisting of sets $A_1, \ldots, A_H \in {}^*\mathcal{C}$ ($H \in {}^*\mathbb{N}$) such that the partition is finer than the image under $*$ of any finite partition of $[0,1]$ into sets in $\mathcal{C}$. The proof uses a concurrent relation argument. From this fine partition we discard all partition sets of ${}^*\lambda$-measure zero. The remaining sets still form an internal collection. In particular we discard all standard singleton sets $\{x\}$ for $x \in [0,1]$. This exclusion of the standard points will prevent us from standardizing the generalized derivative, whose domain is ${}^*[0,1]$, by simply restricting it to the standard points in ${}^*[0,1]$. We use a notation introduced by Loeb: for a set $C \in \mathcal{C}$ we define $I_C = \{i \in \{1, \ldots, H\} : A_i \subseteq {}^*C\}$.

2. The nonstandard hull valued derivative

Given the fine nonstandard partition $\{A_1, \ldots, A_H\}$ of ${}^*[0,1]$, for which we assume that ${}^*\lambda(A_i) \neq 0$ for $i = 1, \ldots, H$, we define a nonstandard Radon-Nikodým derivative in the same way as we would construct it in the case of a finite σ-algebra. We then show that the nonstandard derivative is an internal simple S-integrable function. Such functions are $\widehat{\lambda}$-almost everywhere $\mathrm{fin}({}^*E)$-valued. This allows us to compose an S-integrable internal function with the quotient map $\pi : \mathrm{fin}({}^*E) \to \widehat{E}$ to make it a nonstandard hull valued function defined on the Loeb space ${}^*[0,1]$.

In [13] we defined a Banach space $M(\widehat{\lambda}, \widehat{E})$ of extended integrable functions as the set of equivalence classes under equality $\widehat{\lambda}$-almost everywhere of functions $f : {}^*[0,1] \to \widehat{E}$ for which there is an internal simple S-integrable $\varphi : {}^*[0,1] \to {}^*E$ such that $\pi \circ \varphi = f$ except on a set of $\widehat{\lambda}$-measure zero. Such a φ is called a lifting of f. On $M(\widehat{\lambda}, \widehat{E})$ we defined an integral by setting $\int f \, d\widehat{\lambda} = \pi\left(\int \varphi \, d{}^*\lambda\right)$, where the integral of the internal simple function φ is defined in the obvious way. This integral generalizes the Bochner integral in the sense that $M(\widehat{\lambda}, \widehat{E})$ contains $L_1(\widehat{\lambda}, \widehat{E})$ and the integrals agree on that subspace. But $M(\widehat{\lambda}, \widehat{E})$ does also contain functions which fail to be essentially separably valued and hence fail to be measurable. Surprisingly, the extended integral does not coincide with the Pettis intgral.

Lemma 2.1. *Let $\nu : \mathcal{C} \to E$ be a λ-absolutely continuous (countably additive) vector measure of bounded variation. Then the function $\varphi_\nu : {}^*[0,1] \to {}^*E$ defined by*

$$\varphi_\nu(x) = \sum_{i=1}^{H} \frac{{}^*\nu(A_i)}{{}^*\lambda(A_i)}\ 1_{A_i}(x)$$

is S-integrable.

Proof. If ν is of bounded variation, then for all finite standard partitions $\{C_1, \dots, C_n\} \subset \mathcal{C}$ of $[0,1]$ into disjoint sets one has $\sum_{k=1}^{n} \|\nu(C_k)\| \le |\nu|([0,1])$. For our nonstandard partition, it follows that $\sum_{i=1}^{H} \|{}^*\nu(A_i)\| \le |\nu|([0,1])$, that is,

$$\sum_{i=1}^{H} \frac{\|{}^*\nu(A_i)\|}{{}^*\lambda(A_i)}\, {}^*\lambda(A_i) = \int_{{}^*[0,1]} \|\varphi_\nu\|\, d\,{}^*\lambda \le |\nu|([0,1]).$$

By the absolute continuity of the total variation $|\nu|$ with respect to λ, if $A \in {}^*\mathcal{C}$ with ${}^*\lambda(A) \approx 0$, then $|{}^*\nu|(A) \approx 0$. We use this fact to show that the integral of $\|\varphi_\nu\|$ over a set of infinitesimal measure is infinitesimal. Assume that $A \in {}^*\mathcal{C}$ with ${}^*\lambda(A) \approx 0$. As the variation of ν is also countably additive (see [4], Proposition I.1.9), the internal variation of ${}^*\nu$ is internally hyperfinitely additive; hence

$$\int_A \|\varphi_\nu\|\, d\,{}^*\lambda = \int_A \left\| \sum_{i=1}^{H} \frac{{}^*\nu(A_i)}{{}^*\lambda(A_i)} 1_{A_i}(x) \right\| d\,{}^*\lambda(x)$$
$$= \sum_{i=1}^{H} \|{}^*\nu(A_i \cap A)\| \le |{}^*\nu|(A) \approx 0. \qquad \square$$

Assume from now on that ν is λ-absolutely continuous and of bounded variation. By the previous lemma, $\|\varphi_\nu\|$ is S-integrable and therefore finite $\widehat{\lambda}$ almost everywhere on ${}^*[0,1]$. Hence we can compose φ_ν with π, the projection from $\mathrm{fin}({}^*E)$ onto the nonstandard hull $\widehat{E}$. We thus define a function $f_\nu : {}^*[0,1] \to \widehat{E}$ which has φ_ν as a lifting and which is extended integrable in the sense that it is in $M(\widehat{\lambda}, \widehat{E})$ and the integral defined in [13] can be used for it. We call the function f_ν the *generalized Radon-Nikodým derivative of* ν. From the generalized Radon-Nikodým derivative we can recover the vector measure using the extended integral as follows.

THEOREM 2.2. *Let ν be an λ-absolutely continuous vector measure of bounded variation and construct $f_\nu : {}^*[0,1] \to \widehat{E}$ as above. Then for any set $C \in \mathcal{C}$*

(1) $\int_{^*C} f_\nu \, d\widehat{\lambda} = \nu(C)$ *and*
(2) $|\nu|(C) = \int_{^*C} \|f_\nu\| \, d\widehat{\lambda}$.

Proof. (1) The nonstandard extension of any set $C \in \mathcal{C}$ equals (up to a null set that was discarded from the fine partition) $\bigcup_{i \in I_C} A_i$. Therefore

$$\begin{aligned}\int_{^*C} f_\nu \, d\widehat{\lambda} &= \pi\left(\int_{^*C} \varphi_\nu \, d^*\lambda\right) = \pi\left(\int_{^*C} \frac{^*\nu(A_i)}{^*\lambda(A_i)} \cdot 1_{A_i} \, d^*\lambda\right) \\ &= \pi\left(\sum_{i \in I_C} {}^*\nu(A_i)\right) = \pi\left({}^*\nu(\bigcup_{i \in I_C} A_i)\right) \\ &= \pi({}^*\nu({}^*C)) = \nu(C).\end{aligned}$$

(2) We already know from the proof of Lemma 2.1 that ${}^\circ\int_{^*[0,1]} \|\varphi_\nu\| \, d^*\lambda \leq |\nu|([0,1])$. The same argument works with the set C in place of $[0,1]$. Hence $\int_{^*C} \|f_\nu\| \, d\widehat{\lambda} \leq |\nu|(C)$. By the definition of the total variation, the other inequality is obvious. □

REMARK 2.3. By a result of Anderson (see Theorem 3.6 in [2]), $\widehat{\lambda}({}^*C \triangle \mathrm{st}^{-1}(C)) = 0$ for every set $C \in \mathcal{C}$. Hence it is also true that

$$\int_{\mathrm{st}^{-1}(C)} f_\nu \, d\widehat{\lambda} = \nu(C).$$

EXAMPLE 2.4. Look at the measure $\nu : \mathcal{C} \to L_1(\lambda)$ defined by $\nu(C) = 1_C$. The generalized Radon-Nikodým derivative f_ν of this measure is given by

$$f_\nu(x) = \pi\left(\sum_{i=1}^{H} \frac{1_{A_i}}{^*\lambda(A_i)} \; 1_{A_i}(x)\right).$$

Integrating f_ν over a set ${}^*C$ we get

$$\int_{^*C} f_\nu \, d\widehat{\lambda} = \pi\left(\sum_{i \in I_C} 1_{A_i}\right) = \pi(1_{^*C}) = 1_C,$$

where 1_C is identified with its image in the nonstandard hull of $L_1(\lambda)$.

3. Radon-Nikodým derivatives

As a next step we show that the generalized Radon-Nikodým derivatives we construct are infinitesimally close to the nonstandard extension of the traditional Radon-Nikodým derivative on each set A_i in the fine partition, provided that the traditional Radon-Nikodým derivative exists.

LEMMA 3.1. *Let $f : [0,1] \to E$ be Bochner integrable and let $\{A_1, \ldots, A_H\}$ be the fine partition of ${}^*[0,1]$ described above. Then $\pi \circ {}^*f$ is constant on each partition set A_i in the fine partition.*

Proof. A function f is λ-measurable if there exists a sequence of simple functions (f_n) such that $\lim_{n\to\infty} \|f(x) - f_n(x)\| = 0$ outside a set $C \in \mathcal{C}$ with $\lambda(C) = 0$. Since the nonstandard partition $\{A_1, \ldots, A_h\}$ refines the standard partition into C and its complement and contains no sets of ${}^*\lambda$-measure zero, we have $A_i \cap {}^*C = \emptyset$ for all $i \in \{1, \ldots, H\}$. Assume that $\|{}^*f(x) - {}^*f(y)\| > \epsilon$ for some positive real number ϵ and some $x, y \in A_i$. Then for some f_n we must have ${}^*f_n(x) \neq {}^*f_n(y)$. Therefore there are disjoint measurable sets C_1 and C_2 in $\mathcal{C}$ such that $x \in {}^*C_1$ and $y \in {}^*C_2$, contradicting the assumption that A_i is a set in a partition which is finer than the image under $*$ of any measurable partition of $[0,1]$. □

THEOREM 3.2. *Let $\nu : \mathcal{C} \to E$ be a countably additive vector measure with Radon-Nikodým derivative $f : [0,1] \to E$. Define the generalized Radon-Nikodým derivative $f_\nu : {}^*[0,1] \to \widehat{E}$ as above. Then*

$$\pi \circ {}^*f = f_\nu \quad \text{on each set } A_i \text{ in the fine partition of } {}^*[0,1].$$

Proof. Lemma 3.1 shows that $\pi \circ {}^*f$ is constant on all sets A_i. By construction f_ν is constant on each set A_i. For each $i \in \{1, \ldots, H\}$ choose a $c_i \in A_i$. By assumption $\nu(B) = \int_B f\,d\lambda$ for all $B \in \mathcal{C}$, and hence ${}^*\nu(A_i) = \int_{A_i} {}^*f\,d{}^*\lambda$ for all i. On the other hand ${}^*\nu(A_i) = \int_{A_i} \varphi_\nu\,d{}^*\lambda$ for all i. Hence $\pi\left(\int_{A_i} ({}^*f - \varphi_\nu)\,d{}^*\lambda\right) = 0$. As both functions are constant on the set A_i, this implies that $\pi({}^*f - \varphi_\nu) = 0$ on each set A_i . □

REMARK 3.3. This result seems strange at first, as a Radon-Nikodým derivative is only unique up to equality λ-almost everywhere. In the statement of Theorem 3.2 there is no mention of "almost everywhere". The reason is that in the fine partition $\{A_1, \ldots, A_H\}$ the images under $*$ of all λ-nullsets were discarded.

Given a generalized Radon-Nikodým derivative f_ν, the converse question arises: can we recover the traditional Radon-Nikodým derivative f from the generalized derivative f_ν, provided that f exists? The difficulty here is that the set $\bigcup_{i=1}^H A_i$ does not contain any standard points, hence $f_\nu(x) = 0$ for each standard point $x \in {}^*[0,1]$. For this reason, we cannot recover f from f_ν by just restricting f_ν to the standard points.

Anderson observed that the nonstandard version of Lusin's Theorem can be described as follows (Theorem 3.7 in [2]):

THEOREM. *Let Y have a countable base of open sets, and let $f : [0,1] \to Y$ be λ-measurable. Then ${}^\circ({}^*f(x)) = f({}^\circ x)$ for $\widehat{\lambda}$-almost all $x \in {}^*[0,1]$.*

It only takes a very minor modification of Anderson's proof to apply in our setting, as a λ-measurable function is essentially separably valued. In our setting the theorem reads as follows:

THEOREM 3.4. *Let $f : [0,1] \to E$ be λ-measurable. Then there is a set $A \in L_\lambda({}^*\mathcal{C})$ such that $\widehat{\lambda}(A) = 0$ and $^\circ({}^*f(x)) = f({}^\circ x)$ for $x \in {}^*[0,1] \setminus A$.*

Combining Theorems 3.2 and 3.4 we have:

COROLLARY 3.5. *If the measure ν has a traditional Radon-Nikodým derivative $f : [0,1] \to E$, then the generalized derivative $f_\nu : {}^*[0,1] \to \widehat{E}$ equals $f \circ \mathrm{st} : {}^*[0,1] \to E \subset \widehat{E}$ $\widehat{\lambda}$-almost everywhere on ${}^*[0,1]$.*

Proof. The equality holds on $\bigcup_{i=1}^H A_i$, a set of $\widehat{\lambda}$-measure one. □

As noted above, we cannot obtain f from f_ν by restricting f_ν to the standard points. What we can do is find a function $f \in M(\widehat{\lambda}, \widehat{E})$ in the equivalence class of f_ν such that f is constant on monads and then restrict f to $[0,1] \subseteq {}^*[0,1]$. Theorem 3.4 implies that the elements of the image of $L_1(\lambda, E)$ in $M(\widehat{\lambda}, \widehat{E})$ have a representative that is constant on monads. The converse also holds; all E-valued functions in $M(\widehat{\lambda}, \widehat{E})$ which are constant on monads are already (in an equivalence class) in the image of $L_1(\lambda, E)$:

THEOREM 3.6. *Let ν be a countably additive, λ-continuous, E-valued measure of bounded variation. Assume that the equivalence class of the generalized Radon-Nikodým derivative f_ν in $M(\widehat{\lambda}, \widehat{E})$ contains an E-valued function $\tilde{f}_\nu$ which is constant on monads. Then the measure ν has a Radon-Nikodým derivative f, namely the restriction of $\tilde{f}_\nu$ to the standard points in ${}^*[0,1]$.*

Proof. Define f to be the restriction of $\tilde{f}_\nu$ to the standard points. The functions $f \circ \mathrm{st}$ and f_ν agree $\widehat{\lambda}$-almost everywhere on ${}^*[0,1]$. By construction, φ_ν is an S-integrable lifting of $f \circ \mathrm{st}$, hence $f \circ \mathrm{st} \in M(\widehat{\lambda}, \widehat{E})$. By Theorem 6.5 in [10], any E-valued function in $M(\widehat{\lambda}, \widehat{E})$ is already an element of $L_1(\widehat{\lambda}, E)$, i.e., $f \circ \mathrm{st}$ is Bochner integrable and in particular $\widehat{\lambda}$-measurable. We need to show that f is λ-measurable. The Pettis Measurability Theorem implies that $f \circ \mathrm{st}$ and therefore also f is essentially separably valued. All that is left to show is that the function f is weakly λ-measurable. For real-valued functions on the unit interval Theorem 3.11 in [2] asserts the following: If $g : {}^*[0,1] \to \mathbb{R}$ is $L_\lambda({}^*C)$-measurable and $g(x) = g({}^\circ x)$ for $\widehat{\lambda}$-almost all x, then the restriction $f = g|_{[0,1]}$ is Lebesgue-measurable and $^\circ({}^*f(x)) = g(x)$ for $\widehat{\lambda}$-almost all x. Composing $f \circ \mathrm{st}$ with any functional $e' \in E'$, we get that $e' \circ f \circ \mathrm{st}$ is real-valued, $\widehat{\lambda}$-measurable and constant on monads. By Anderson's theorem the restriction of $e' \circ f \circ \mathrm{st}$ to the standard points is a

λ-measurable function. Therefore $e' \circ f$ is λ-measurable for all $e' \in E'$. We showed that f is essentially separably valued and weakly measurable. By the Pettis Measurability Theorem, f is measurable. The integrability of f follows from the integrability of $f \circ \mathrm{st}$. □

4. Bidual-valued derivatives

Theorem 1 in [5] translates to the setting of this paper as follows:

THEOREM. Let $\nu : \mathcal{C} \to E$ be a countably additive vector measure of bounded variation such that $\nu \ll \lambda$. Then there is a function $f : [0,1] \to E''$ having the following properties:

(1) For every $e' \in E'$, the function $\langle e', f \rangle$ is Lebesgue integrable and

$$\langle \nu(A), e' \rangle = \int_A \langle e', f \rangle \, d\lambda, \text{ for any } A \in \mathcal{C}.$$

(2) The function $\|f\|$ is Lebesgue integrable and for the variation of ν we have

$$|\nu|(A) = \int_A \|f\| \, d\lambda \text{ for any } A \in \mathcal{C}.$$

Our generalized derivative $f_\nu : {}^*[0,1] \to \widehat{E}$ falls short of this derivative of Dinculeanu and Uhl on two counts: its domain is too big and it has $\widehat{E}$ instead of E'' as its range. In this section we will show how to restrict the domain and the range of our generalized derivative f_ν to obtain the standard weak*-derivative of Dinculeanu and Uhl. We start with the range space.

It is a consequence of the principle of local reflexivity (see [6], Prop. 3.13) that the nonstandard hull $\widehat{E}$ contains an isometric copy of the second dual E'' of the Banach space E, provided that the nonstandard extension is at least κ-saturated, where $\mathrm{card}(E'') < \kappa$. Since we assumed this amount of saturation, Proposition 3.13 in [6] produces is an isometry T of E'' into $\widehat{E}$ that satisfies

(1) $Tx = x$ for all $x \in E$ and
(2) $\langle y, x \rangle = \langle Tx, y \rangle$ for all $x \in E''$ and $y \in E'$.

Hence we can regard E'' as a subspace of $\widehat{E}$. Furthermore, with the usual identification of a space as a subspace of its second dual, we see that every element of $\widehat{E}$ defines a continuous linear functional on $\widehat{E}'$. By [6], E' is a subspace of $\widehat{E}'$ and this allows us to define a contractive projection $P : \widehat{E} \to E''$ that assigns to every element in $\widehat{E}$ its restriction to E'. This makes E'' a complemented subspace of $\widehat{E}$.

Composing our generalized derivative with this projection $P : \widehat{E} \to E''$ does restrict the range of f_ν. The price to pay is that the integral gets weaker in the process and the generalized integral becomes a weak*-integral.

Proposition 4.1. *If f_ν is a generalized Radon-Nikodým derivative of $\nu : \mathcal{C} \to E$, then $P \circ f_\nu : {}^*[0,1] \to E''$ is a weak*-measurable function such that for all $C \in \mathcal{C}$ and $e' \in E'$*

$$\langle \nu(C), e' \rangle = \int_{^*C} \langle e', P \circ f_\nu \rangle \, d\widehat{\lambda}.$$

Proof. The weak*-measurability follows as $e' \circ f_\nu$ is a real-valued extended integrable function, which by Theorem 6.5 in [10] already is an integrable function. The rest is proved as follows:

$$\begin{aligned}\langle \nu(C), e' \rangle &= e' \left(\int_{^*C} f_\nu \, d\widehat{\lambda} \right) \\ &= e' \left(\pi \left(\int_{^*C} \varphi_\nu \, d^*\lambda \right) \right) \\ &= {}^\circ \left({}^*e' \left(\int_{^*C} \varphi_\nu \, d^*\lambda \right) \right) \\ &= {}^\circ \left(\int_{^*C} \langle \varphi_\nu, {}^*e' \rangle \, d^*\lambda \right) \\ &= \int_{^*C} {}^\circ \langle \varphi_\nu, {}^*e' \rangle \, d\widehat{\lambda} \\ &= \int_{^*C} \langle e', P \circ f_\nu \rangle \, d\widehat{\lambda}.\end{aligned}$$

We used the S-integrability of φ_ν to interchange the standard part map with the integral in the second but last step. □

To restrict the domain of the generalized derivative appropriately, we resort to liftings. In [3], Bliedtner and Loeb have obtained a strong multiplicative lifting of $\mathcal{L}^\infty([0,1])$ by using a base choice function $S : [0,1] \to {}^*[0,1]$ with the following three properties:

(1) $S(x) \approx x$.
(2) For each set E for which x is a point of density $S(x) \in {}^*E$.
(3) For each standard null set B, $S(x) \notin {}^*B$.

The lifting of a bounded measurable function f is then given by $\rho(f)(x) = {}^\circ\, {}^*f(S(x))$.

In a remark it is pointed out that this even gives a lifting for functions in $\mathcal{L}^1(\lambda)$ if one allows infinities and does not require multiplicativity at those points. This follows by approximating an integrable function by bounded functions.

In the nonstandard partition $\{A_1, \dots, A_H\}$ of ${}^*[0,1]$ we have discarded all ${}^*\lambda$ nullsets. The union of these nullsets is still an ${}^*\lambda$ nullset. To avoid the possibility that $S(x)$ lies in any of the discarded sets of ${}^*\lambda$-measure zero, we need to make an additional assumption on the lower density e used in the construction of S. We assume that its base generating function only takes its values in the power set of $\bigcup_{i=1}^{H} A_i$, instead of the power set of ${}^*[0,1]$.

THEOREM 4.2. *Assume that $f_\nu : {}^*[0,1] \to \widehat{E}$ is the generalized Radon-Nikodým derivative of the measure $\nu : \mathcal{C} \to E$. Then the function $f : [0,1] \to E''$ defined as $f(x) = P\left(f_\nu(S(x))\right)$ has the following properties:*

(1) *For every $e' \in E'$, the function $\langle e', f\rangle$ is Lebesgue integrable and*

$$\langle \nu(A), e'\rangle = \int_A \langle e', f\rangle \, d\lambda, \text{ for any set } A \in \mathcal{C}.$$

(2) *The function $\|f\|$ is Lebesgue integrable and for the variation of ν we have*

$$|\nu|(A) = \int_A \|f\| \, d\lambda \text{ for any } A \in \mathcal{C}.$$

Proof. (1) Let $e' \in E'$. Assume that $g : [0,1] \to \mathbb{R}$ is a Radon-Nikodým derivative of the real-valued measure $e' \circ \nu$. For the existence of g we use the Radon-Nikodým theorem for real-valued measures without proving it again. Our methods do allow a proof, but it does not differ significantly from the well-known proof of the Radon-Nikodým theorem for real-valued measures that uses the Hahn decomposition. Ross gives a nonstandard proof of the Radon-Nikodým theorem in [12] in which he uses a conditional expectation in the final step of reducing the domain to a standard domain. By Theorem 3.2, the function $e' \circ f_\nu$ agrees with the standard part of the nonstandard extension of g on each partition set A_i, and in particular at each point $S(x)$. Since $\rho(g)$ is also a Radon-Nikodým derivative for $\langle \nu, e'\rangle$ and $\rho(g)(x) = {}^\circ({}^*g(S(x)))$, we get for every $x \in [0,1]$

$$\langle e', f(x)\rangle = \langle e', P(f_\nu(S(x)))\rangle = {}^\circ({}^*g(S(x))) = \rho(g)(x).$$

Hence $\int_A \langle e', f\rangle \, d\lambda = \int_A g \, d\lambda = \langle \nu(A), e'\rangle$.

(2) By Theorem 2.2, part (2), $\|f_\nu\|$ is the generalized derivative of $|\nu|$ with respect to λ and $|\nu|(A) = \int_{{}^*A} \|f_\nu\| \, d\widehat{\lambda}$ for all sets $A \in \mathcal{C}$. Since P is a contractive projection, $\int_{{}^*A} \|f_\nu\| \, d\widehat{\lambda} \geq \int_{{}^*A} \|P \circ f_\nu\| \, d\widehat{\lambda}$, so all that remains to prove is that $\int_{{}^*A} \|f_\nu\| \, d\widehat{\lambda} = \int_A \|f_\nu(S(x))\| \, d\lambda$. Using the Radon-Nikodým theorem for real-valued measures, there is a real-valued derivative $\frac{d|\nu|}{d\lambda} : [0,1] \to \mathbb{R}$. Theorem 3.2 asserts that $\pi \circ {}^*\frac{d|\nu|}{d\lambda} = \|f_\nu\|$ on each set A_i in the fine partition. Since ${}^\circ\,{}^*\frac{d|\nu|}{d\lambda}(S(x))$ is a lifting of $\frac{d|\nu|}{d\lambda}$, we get

$$|\nu|(A) = \int_A \frac{d|\nu|}{d\lambda} \, d\lambda = \int_A {}^\circ\,{}^*\left(\frac{d|\nu|}{d\lambda}\right)(S(x)) \, d\lambda = \int_A \|f_\nu(S(x))\| \, d\lambda.$$

Hence $\int_{{}^*A} \|f_\nu\| \, d\widehat{\lambda} = \int_A \|f_\nu(S(x))\| \, d\lambda$, finishing the proof, as by the definition of the total variation $|\nu|(A) \leq \int_A \|P \circ f_\nu \circ S\| \, d\lambda$. □

One quick application is a well-known result:

Corollary 4.3. *Every separable dual space has the Radon-Nikodým property.*

Proof. If E is a separable dual space, then we can obtain E as a complemented subspace of $\widehat{E}$. The contractive projection is obtained by restricting elements of $\widehat{E}$, which we may regard as functionals on $\widehat{E}'$, to functionals on the predual of E. We denote the projection again by $P : \widehat{E} \to E$. Hence we can construct a derivative $P \circ f_\nu \circ S : [0,1] \to E$ as above. This derivative is weak*-measurable and separably valued. Since the predual of E is norming for E and by the separability of E, it follows (see II.1.4 in [4], a corollary of the Pettis Measurability Theorem) that $P \circ f_\nu \circ S$ is λ-measurable. The integrability follows from Theorem 4.2, which establishes the integrability of $\|P \circ f_\nu \circ S\|$. □

5. Conclusion

Unifying the Banach space-valued and bidual-valued Radon-Nikodým derivatives is useful, since the drawback in the existing general bidual-valued derivatives was the weakness of the measurability conditions of the derivative, whereas the nonstandard hull valued derivatives are still extended integrable.

We opted to use the base choice function of Bliedtner and Loeb to restrict the domain of the generalized derivative. Ross uses a conditional expectation with respect to the smallest σ-algebra containing the nonstandard extension of all standard measurable sets for his proof of the Radon-Nikodým theorem in [12]. The existence of conditional expectations for Bochner integrable functions can be proved independently of the Radon-Nikodým theorem (see Theorem V.4 in [4]) and the proof which starts with simple functions can be extended to internal $*$-simple functions or functions in $M(\widehat{\lambda}, \widehat{E})$. We could also have used Ross' idea for changing the domain of our generalized derivatives from $\bigcup_{i=1}^{H} A_i$ to $[0,1]$.

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