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NOTE ON H^1 SPACES RELATED TO DEGENERATE SCHRÖDINGER OPERATORS

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ABSTRACT. Let $\mathcal{L}f(x) = -\frac{1}{w(x)}\sum_{i,j}\partial_i(a_{ij}(\,\cdot\,)\partial_j f)(x) + V(x)f(x)$, where w is a weight from the Muckenhoupt class A_2, V is a nonnegative potential that belongs to a certain reverse Hölder class with respect to the measure $w(x)\,dx$, and $C^{-1}w(x)|\xi|^2 \le \sum_{i,j} a_{ij}(x)\xi_i\bar{\xi}_j \le Cw(x)|\xi|^2$. Let $\{T_t\}_{t>0}$ be the semigroup of linear operators generated by $-\mathcal{L}$. We say that a function f is an element of the space $H^1_{\mathcal{L}}$ if the maximal operator $\mathcal{M}f(x) = \sup_{t>0}|T_t f(x)|$ belongs to $L^1(R^d(w(x)\,dx))$. A special atomic decomposition of $H^1_{\mathcal{L}}$ is proved.

1. Introduction

On $\mathbb{R}^d$ we consider a degenerate Schrödinger operator $\mathcal{L}$ having the form

$$\mathcal{L}f(x) = -\frac{1}{w(x)}\sum_{i,j}\partial_i(a_{ij}(\,\cdot\,)\partial_j f)(x) + V(x)f(x), \tag{1.1}$$

where $a_{ij}(x)$ is a real symmetric matrix satisfying

$$C^{-1}w(x)|\xi|^2 \le \sum_{i,j} a_{ij}(x)\xi_i\bar{\xi}_j \le Cw(x)|\xi|^2, \tag{1.2}$$

with w being a nonnegative weight from the Muckenhoupt class A_2, and $V \ge 0$ belonging to a reverse Hölder class with respect to the measure $d\mu(x) = w(x)\,dx$ (cf. (2.5)). Denote by $\mathcal{E}(f,g)$ the Dirichlet form associated with $\mathcal{L}$, that is,

$$\mathcal{E}(f,g) = \int_{\mathbb{R}^d}\sum_{i,j} a_{ij}(x)\partial_j f(x)\overline{\partial_i g(x)}\,dx + \int_{\mathbb{R}^d} V(x)f(x)\overline{g(x)}\,d\mu(x).$$

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The operator $-\mathcal{L}$ is the infinitesimal generator of the semigroup $\{T_t\}_{t>0}$ of self-adjoint linear operators on $L^2(d\mu)$ having the integral kernels $k_t(x,y)$, that is,

$$T_t f(x) = \int_{\mathbb{R}^d} k_t(x,y) f(y)\, d\mu(y). \tag{1.3}$$

A perturbation formula asserts that

$$0 \le k_t(x,y) \le h_t(x,y), \tag{1.4}$$

where $h_t(x,y)$ are the integral kernels of the semigroup $\{S_t\}_{t>0}$ on $L^2(d\mu)$ generated by $-\mathcal{L}_0$, where

$$\mathcal{L}_0 f(x) = -\frac{1}{w(x)} \sum_{i,j} \partial_i (a_{ij}(\,\cdot\,)\partial_j f)(x). \tag{1.5}$$

It is known that the kernels $h_t(x,y)$ satisfy the Gaussian estimates (3.3). Since the measure μ satisfies the doubling condition (cf. (2.2)), the maximal operator

$$\mathcal{M}f(x) = \sup_{t>0} |T_t f(x)| \tag{1.6}$$

is bounded on the spaces $L^p(d\mu)$ for $1 < p < \infty$, and of weak type $(1,1)$.

In the present paper we study the space of all functions f for which the maximal function $\mathcal{M}f$ is exactly in $L^1(d\mu)$. We shall denote this space by $H^1_{\mathcal{L}}$. The corresponding $H^1_{\mathcal{L}}$ norm is defined by

$$\|f\|_{H^1_{\mathcal{L}}} = \|\mathcal{M}f\|_{L^1(d\mu)}. \tag{1.7}$$

It turns out that, as in the classical theory of real Hardy spaces, every element of $H^1_{\mathcal{L}}$ can be written as a sum of certain basic elements called atoms (cf. Section 2). Our aim is to prove the atomic decomposition of the members of $H^1_{\mathcal{L}}$ (see Theorem 2.1).

The operator $\mathcal{L}_0$ was studied in [7] and [8]. We refer the reader to these articles for a detailed analysis of $\mathcal{L}_0$ and its fundamental solution.

2. Preliminaries and statement of the results

A nonnegative function $w(x)$ is an element of the Muckenhoupt class A_2 if there exists a constant $C > 0$ such that

$$\left(\frac{1}{|B|}\int_B w(x)\, dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-1}\, dx\right) \le C \tag{2.1}$$

for every ball B. Here and subsequently $|B|$ denotes the volume of the ball B with respect to the Lebesgue measure dx. It is well-known that (2.1) implies that the measure $d\mu(x) = w(x)\, dx$ satisfies the doubling condition, that is, there exists a constant $C_0 > 0$ such that

$$\mu\big(B(x,2r)\big) \le C_0 \mu\big(B(x,r)\big) \quad \text{for every} \ \ x \in \mathbb{R}^d,\ r > 0. \tag{2.2}$$

Using the notation from [14], we say that $w \in D_\gamma$, $\gamma > 0$, if there is a constant $C > 0$ such that

$$\mu\big(B(x,tr)\big) \le Ct^\gamma \mu\big(B(x,r)\big), \quad \text{for every} \quad t > 1. \tag{2.3}$$

Let us note that (2.2) guarantees the existence of such a γ.

Similarly $w \in (RD)_\nu$ if

$$t^\nu \mu\big(B(x,r)\big) \le C\mu\big(B(x,tr)\big) \quad \text{for every} \quad t > 1. \tag{2.4}$$

A nonnegative potential V belongs to the reverse Hölder class $(RH)^q_\mu$, $q > 1$, with respect to the measure $d\mu$ if there exists a constant $C > 0$ such that for every Euclidean ball B one has

$$\left(\frac{1}{\mu(B)}\int_B V(y)^q \, d\mu(y)\right)^{1/q} \le C\left(\frac{1}{\mu(B)}\int_B V(y)\, d\mu(y)\right). \tag{2.5}$$

From now on we shall assume that $w \in A_2 \cap D_\gamma \cap (RD)_\nu$, $2 < \nu \le \gamma$, $d\mu(x) = w(x)\,dx$, $V \in (RH)^q_\mu$, $q > \gamma/2$. We set

$$\delta = 2 - \gamma/2.$$

Following [18] (see also [14]) the auxiliary function $m(x,V)$ is defined by

$$r(x) = m(x,V)^{-1} = \sup\left\{ r > 0 : \frac{r^2}{\mu\big(B(x,r)\big)} \int_{B(x,r)} V(y)\, d\mu(y) \le 1 \right\}. \tag{2.6}$$

The function $m(x,V)$ satisfies $0 < m(x,V) < \infty$ (cf. [14]).

We now are in a position to define a notion of $H^1_{\mathcal{L}}$ atom. A function a is an $H^1_{\mathcal{L}}$ atom associated with a ball $B(x,r)$ if

$$r \le r(x), \quad \operatorname{supp} a \subset B(x,r), \text{ and } \|a\|_{L^\infty} \le \mu\big(B(x,r)\big)^{-1}, \tag{2.7}$$

$$\text{if} \quad r \le \frac{r(x)}{4}, \quad \text{then} \quad \int a(y)\, d\mu(y) = 0. \tag{2.8}$$

The atomic norm $\|\ \|_{H^1_{\mathcal{L}}-atom}$ is defined by

$$\|f\|_{H^1_{\mathcal{L}}-atom} = \inf \sum |\lambda_j|,$$

where the infimum is taken over all decompositions $f = \sum_j \lambda_j a_j$, where a_j are $H^1_{\mathcal{L}}$ atoms and λ_j are scalars.

One of the main results of the paper is the following theorem.

THEOREM 2.1. *Assume that* $w \in (RD)_\nu \cap D_\gamma \cap A_2$, $2 < \nu \le \gamma$, *and* $V \in (RH)^q_\mu$ *with* $q > \gamma/2$. *Then there exists a constant* $C > 0$ *such that*

$$\frac{1}{C}\|f\|_{H^1_{\mathcal{L}}-atom} \le \|f\|_{H^1_{\mathcal{L}}} \le C\|f\|_{H^1_{\mathcal{L}}-atom}. \tag{2.9}$$

The atomic decomposition stated in Theorem 2.1 is analogous to those obtained in [3]–[6] for classical Schrödinger operators. The main idea of proving it is based, like in [3]–[6], on an analysis of the local and global behavior of the integral kernels $k_t(x,y)$ of the semigroup $\{T_t\}_{t>0}$. The theorem below (proved in Section 5) provides some control for the behavior of the global part (cf. [6], [15], where estimates for the kernels of classical Schrödinger operators were derived).

Theorem 2.2. *There exists a constant $c>0$ such that for every $N \geq 0$ there exists a constant C_N such that*

(2.10)

$$k_t(x,y) \leq \frac{C_N}{\mu\big(B(x,\sqrt{t})\big)} \left(1+\frac{\sqrt{t}}{r(x)}\right)^{-N} \left(1+\frac{\sqrt{t}}{r(y)}\right)^{-N} \exp(-c|x-y|^2/t).$$

Remark 2.3. Since $k_t(x,y) = k_t(y,x)$, the factor $C_N/\mu(B(x,\sqrt{t}))$ in (2.10) can be replaced by

$$\frac{C_N}{\mu(B(x,\sqrt{t}))^{1/2}\mu(B(y,\sqrt{t}))^{1/2}}.$$

3. Dirichlet forms

Let $\mathcal{E}_0(f,g)$ be the Dirichlet form on $L^2(d\mu)$ associated with the operator $\mathcal{L}_0 f(x)$, that is,

$$\mathcal{E}_0(f,g) = \int_{\mathbb{R}^d} \sum_{i,j} a_{ij}(x)\partial_j f(x)\partial_i \overline{g(x)}\,dx, \tag{3.1}$$

with the domain $\mathcal{D} = \{f \in L^2(d\mu):\ |\nabla f| \in L^2(d\mu)\}$. The Dirichlet form is regular, that is, there exists a subset $\mathcal{C} \subset \mathcal{D} \cap C_c(\mathbb{R}^d)$ which is dense in $\mathcal{D}$ for the norm $(\|f\|^2_{L^2(d\mu)} + \mathcal{E}_0(f,f))^{1/2}$ and dense in $C_c(\mathbb{R}^d)$ in the uniform norm (cf. [10]). Moreover, it is strictly local, which means that $\mathcal{E}_0(u,v)=0$ for any $u,v \in \mathcal{D}$ such that u,v are supported on compact sets and v is constant in a neighborhood of the support of u. Let us note that the "energy measure" $d(\Gamma(f,g)) = \sum_{i,j} a_{ij}\partial_j f(x)\partial_i \overline{g}(x)\,dx$ for $f,g \in \mathcal{D}$ is absolutely continuous with respect to μ, and the pseudo-distance

$$d(x,y) = \sup\left\{ f(x)-f(y) :\ f \in \mathcal{C}, \frac{d\Gamma(f,f)}{d\mu} \leq 1 \right\}$$

is comparable to $|x-y|$. The Dirichlet space $(\mathcal{E}_0, \mathcal{D}, L^2(d\mu))$ satisfies the Poincaré inequality (cf. [8]), that is, there is a constant $C>0$ such that for every $f \in \mathcal{D}$ and every ball $B = B(y,r)$ we have

$$\int_B |f-f_B|^2\,d\mu \leq Cr^2 \int_B |\nabla f|^2\,d\mu, \tag{3.2}$$

where

$$f_B = \frac{1}{\mu(B)} \int_B f(x)\, d\mu(x).$$

The doubling condition (2.2) and (3.2) imply that the kernels $h_t(x,y)$ of the semigroup $\{S_t\}_{t>0}$ associated with $\mathcal{E}_0$ are continuous and satisfy the following estimates (cf., e.g., Theorems 2.7, 2.3, 2.4, and Corollary 3.4 of [13]):

$$\frac{c_1}{\mu\big(B(x,\sqrt{t})\big)} \exp\left(-\frac{|x-y|^2}{c_2 t}\right) \le h_t(x,y) \le \frac{C_1}{\mu\big(B(x,\sqrt{t})\big)} \exp\left(-\frac{|x-y|^2}{C_2 t}\right), \tag{3.3}$$

$$|h_t(x,y) - h_t(x,z)| \le C\mu\big(B(x,\sqrt{t})\big)^{-1} \left(\frac{|y-z|}{\sqrt{t}}\right)^{\alpha} \exp\big(-(|x-y| - 2|y-z|)_+^2/ct\big) \tag{3.4}$$

with constants $\alpha > 0$, $c > 0$, $C > 0$,

$$|\partial_t^k h_t(x,y)| \le \frac{C_k}{t^k \mu\big(B(x,\sqrt{t})\big)} \exp\left(\frac{-|x-y|^2}{ct}\right). \tag{3.5}$$

For detailed relations between Poincaré, Harnack and Sobolev-type inequalities and Gaussian bounds we refer the reader to [13], [1], [17] and the references therein.

4. Green's functions related to $\mathcal{L}_0$ and $\mathcal{L}$

At the beginning of this section we state some basic properties of the auxiliary function $m(x,V)$. For more details we refer the reader to [14]. The second part of the section is devoted to the fundamental solutions to the operators $i\tau + \mathcal{L}$ and $i\tau + \mathcal{L}_0$. The methods of the proofs of the estimates stated in Proposition 4.9 are borrowed from [18].

LEMMA 4.1 (cf. Lemma 2 of [14]). *Assume that $w \in D_\gamma$, $V \in (RH)^q(\mu)$ with $q > \gamma/2$. Then there exists a constant $C > 0$ such that for every $0 < r < R < \infty$, $y \in \mathbb{R}^d$ we have*

$$\frac{r^2}{\mu\big(B(y,r)\big)} \int_{B(y,r)} V(x)\, d\mu(x) \le C\left(\frac{r}{R}\right)^{\delta} \frac{R^2}{\mu\big(B(y,R)\big)} \int_{B(y,R)} V(x)\, d\mu(x). \tag{4.1}$$

LEMMA 4.2 (cf. Lemma 3 of [14]). *Under the assumptions of Lemma 4.1, for every constant $C_1 > 1$ there exists a constant $C_2 > 1$ such that if*

$$\frac{1}{C_1} \le \frac{r^2}{\mu\big(B(x,r)\big)} \int_{B(x,r)} V(y)\, d\mu(y) \le C_1,$$

then ${C_2}^{-1} \le rm(x,V) \le C_2$.

LEMMA 4.3 (cf. Lemma 4 of [14]). *Under the assumptions of Lemma 4.1, for every constant* $C_1 \ge 1$ *there is a constant* $C_2 \ge 1$ *such that*

$$\frac{1}{C_2} \le \frac{m(x,V)}{m(y,V)} \le C_2 \quad \textit{for} \;\; |x-y| \le C_1 r(x).$$

Moreover, there exists constants $k_0, C, c > 0$ *such that*

$$m(y,V) \le C\big(1+|x-y|m(x,V)\big)^{k_0} m(x,V), \tag{4.2}$$

$$m(y,V) \ge cm(x,V)\big(1+|x-y|m(x,V)\big)^{-k_0/(1+k_0)}. \tag{4.3}$$

LEMMA 4.4. *There exists constants* $l, C > 0$ *such that*

$$\frac{R^2}{\mu(B(x,R))}\int_{B(x,R)} V(y)\,d\mu(y) \le C\big(Rm(x,V)\big)^l \qquad \textit{provided } R \ge m(x,V)^{-1}.$$

Proof. (See [18], Lemma 1.8.) Denote $r_0 = m(x,V)^{-1}$. Let j be such that $2^j r_0 \le R < 2^{j+1} r_0$. Since the measure $V(x)d\mu(x)$ satisfies the doubling condition,

$$\int_{B(x,R)} V(y)\,d\mu(y) \le C_0^{j+1}\int_{B(x,r_0)} V(y)\,d\mu(y) \le C_0^{j+1}\frac{\mu(B(x,r_0))}{r_0^2}.$$

Thus

$$\begin{aligned}\frac{R^2}{\mu(B(x,R))}\int_{B(x,R)} V(y)\,d\mu(y) &\le C_0^{j+1}\frac{R^2}{r_0^2}\frac{\mu(B(x,r_0))}{\mu(B(x,R))}\\ &\le 2^{(j+1)\log_2 C_0}\frac{R^2}{r_0^2}\frac{\mu(B(x,r_0))}{\mu(B(x,R))}\\ &\le C\left(\frac{R}{r_0}\right)^{2+\log_2 C}. \end{aligned}$$

□

From Lemmas 4.1 and 4.4 we deduce:

COROLLARY 4.5. *For any constants* $c, C' > 0$ *there exists a constant* $C > 0$ *such that*

$$\int \frac{e^{-c|x-y|^2/t}V(y)}{\mu\big(B(x,\sqrt{t})\big)}\,d\mu(y) \le Ct^{-1}\big(\sqrt{t}m(x,V)\big)^{\delta} \quad \textit{for} \;\; \sqrt{t} \le C'm(x,V)^{-1}.$$

The following lemma proved in [14] (see also [18]) is a weighted version of the Fefferman-Phong inequality.

LEMMA 4.6. *If* $w \in A_2 \cap D_\gamma$, $V \in (RH)^q$, $q > \gamma/2$, *then there exists a constant* $C > 0$ *such that*

$$\int |u(x)|^2 m(x,V)^2\,d\mu(x) \le C\bigg(\mathcal{E}(u,u) + \int |u(x)|^2 V(x)\,d\mu(x)\bigg). \tag{4.4}$$

LEMMA 4.7. *For every $k \geq 0$ there is a constant C_k such that for every $R_0, \tau \in \mathbb{R}$, if $(\mathcal{L} + i\tau)u = 0$ on $B(x_0, 4R_0)$, then*

$$\sup\{|u(x)| : x \in B(x_0, R_0)\}$$
$$\leq \frac{C_k}{(1 + R_0|\tau|^{1/2})^k (1 + R_0 m(x_0, V))^k} \left[\frac{1}{\mu(B(x, R_0))} \int_{B(x_0, 2R_0)} |u(x)|^2 d\mu(x)\right]^{1/2}.$$

Proof. The proof is modelled after [18]. For completeness we present the details. Let us note that

$$-\mathcal{L}_0 |u|^2(x) = 2V(x)|u(x)|^2 + \frac{2}{w(x)} \sum_{ij} a_{ij}(x) \partial_i u(x) \partial_j \overline{u(x)} \geq 0.$$

Therefore, by Lemma 8 of [14], for every $1 < \kappa < 2$ and every $R_0 \leq R \leq 2R_0$ there is a constant C_κ such that

$$\sup_{x \in B(x_0, R)} |u(x)|^2 \leq \frac{C_\kappa}{\mu\big(B(x_0, \kappa R)\big)} \int_{B(x_0, \kappa R)} |u(x)|^2 \, d\mu(x). \tag{4.5}$$

The Caccioppoli inequality asserts that for every $1 < \kappa \leq 2$ there is a constant C_κ such that for every $R_0 \leq R \leq 2R_0$ we have

(4.6)
$$\int_{B(x_0, R)} \left(|\nabla u(x)|^2 + \big(|\tau| + V(x)\big)|u(x)|^2\right) \, d\mu(x) \leq \frac{C_\kappa}{R^2} \int_{B(x_0, \kappa R)} |u(x)|^2 \, d\mu(x).$$

Hence

$$\int_{B(x_0, R)} |u(x)|^2 \, d\mu(x) \leq \frac{C_\kappa}{|\tau| R^2} \int_{B(x_0, \kappa R)} |u(x)|^2 \, d\mu(x). \tag{4.7}$$

An iteration of (4.7) leads to

$$\int_{B(x_0, R)} |u(x)|^2 \, d\mu(x) \leq \frac{C_k}{(|\tau| R^2)^k} \int_{B(x_0, 2R)} |u(x)|^2 \, d\mu(x). \tag{4.8}$$

For $R_0 < R \leq 2R_0$ and $1 < \kappa < 2$ let $\eta \in C_c^\infty(B(x_0, \kappa R))$ be such that $\eta = 1$ on $B(x, R)$, $|\nabla \eta| \leq C_\kappa R^{-1}$. Then applying Lemma 4.6 to the function $u\eta$, we get

$$\int_{B(x_0, R)} m(x, V)^2 |u(x)|^2 \, d\mu(x)$$
$$\leq C \int_{B(x_0, \kappa R)} \left(|\nabla u(x)|^2 + \frac{|u(x)|^2}{R^2} + |u(x)|^2 V(x)\right) d\mu(x).$$

By (4.6) we have

$$\int_{B(x_0, R)} m(x, V)^2 |u(x)|^2 \, d\mu(x) \leq \frac{C_\kappa}{R^2} \int_{B(x_0, \kappa R)} |u(x)|^2 \, d\mu(x). \tag{4.9}$$

Now, using (4.3) and (4.9), we obtain

$$\begin{aligned}\int_{B(x_0,R)}|u(x)|^2\,d\mu(x)&=\int_{B(x_0,R)}m(x,V)^2|u(x)|^2m(x,V)^{-2}\,d\mu(x)\\&\le C_\kappa\frac{(1+Rm(x_0,V))^{2k_0/(1+k_0)}}{m(x_0,V)^2R^2}\int_{B(x_0,\kappa R)}|u(x)|^2\,d\mu(x).\end{aligned}$$

Iterating this procedure, we get

$$\int_{B(x_0,R)}|u(x)|^2\,d\mu(x)\le\frac{C_k}{(1+Rm(x_0,V))^k}\int_{B(x_0,2R)}|u(x)|^2\,d\mu(x).\tag{4.10}$$

Finally, Lemma 4.7 follows from (4.5), (4.8), and (4.10). □

For $\tau\in\mathbb{R}$ let us denote by $G_0(x,y;\tau)$ and $G(x,y;\tau)$ the fundamental solutions of the operators $\mathcal{L}_0+i\tau$ and $\mathcal{L}+i\tau=\mathcal{L}_0+V+i\tau$. Obviously,

$$G_0(x,y;\tau)=\int_0^\infty e^{-it\tau}h_t(x,y)\,dt,\tag{4.11}$$

$$G(x,y;\tau)=\int_0^\infty e^{-it\tau}k_t(x,y)\,dt.\tag{4.12}$$

LEMMA 4.8. *Assume that $w\in A_2\cap D_\gamma\cap(RD)_\nu$, $\gamma\ge\nu>2$, $V\in(RH)^q$, $q>\gamma/2$. Then*

$$\begin{aligned}0\le G_0(x,y;0)&\le C|x-y|^2\mu(B(x,|x-y|))^{-1},\\0\le G(x,y;0)&\le C|x-y|^2\mu(B(x,|x-y|))^{-1},\\|G_0(x,y;\tau)|+|G(x,y;\tau)|&\le C|x-y|^2\mu(B(x,|x-y|))^{-1}.\end{aligned}$$

Proof. The lemma is a consequence of (1.4), (4.11), (4.12), (2.3), (2.4), and (3.3). □

PROPOSITION 4.9. *Assume that $w\in A_2\cap D_\gamma\cap(RD)_\nu$, $2<\nu\le\gamma$, $V\in(RH)^q$, $q>\gamma/2$. Then for every $k\ge0$ there is a constant C_k such that*

$$|G(x,y;\tau)|\le\frac{C_k}{\big(1+|x-y||\tau|^{1/2}\big)^k\big(1+|x-y|m(x,V)\big)^k}\frac{|x-y|^2}{\mu\big(B(x,|x-y|)\big)}.$$

Proof. For fixed $y\in\mathbb{R}^d$ let $u(x)=G(x,y;\tau)$. Then $(\mathcal{L}+i\tau)u(x)=0$ for $x\ne y$. Therefore, applying Lemma 4.7 (with $R_0=|x_0-y|/8$), we obtain

$$\begin{aligned}|G(x_0,y;\tau)|&\le C_k\big(1+|x_0-y||\tau|^{1/2}/8\big)^{-k}\big(1+|x_0-y|m(x_0,V)/8\big)^{-k}\\&\quad\times\left(\frac{1}{\mu\big(B(x_0,|x_0-y|)\big)}\int_{B(x_0,|x_0-y|/4)}|u(x)|^2\,d\mu(x)\right)^{1/2}.\end{aligned}$$

Using Lemma 4.8 we get the required estimate. □

REMARK 4.10. The above arguments can be applied to obtain the following well-known estimate:

$$|G_0(x,y;\tau)| \le \frac{C_k|x-y|^2}{\big(1+|x-y||\tau|^{1/2}\big)^k \mu\big(B(x,|x-y|)\big)}. \tag{4.13}$$

5. Estimates for $k_t(x,y)$

Proof of Theorem 2.2. The semigroup $\{T_t\}_{t>0}$ has an extension to the holomorphic semigroup of contractions on $L^2(d\mu)$. Therefore

$$\|\partial_t^k T_t\|_{L^2(d\mu)\to L^2(d\mu)} \le C_k t^{-k}. \tag{5.1}$$

Let

$$k_t'(x,y) = \frac{d}{dt}k_t(x,y).$$

Since

$$k_{2t}'(x,y) = \left(\frac{d}{dt}T_t\right)\big(k_t(\,\cdot\,,y)\big)(x),$$

we have

$$\|k_t'(x,y)\|_{L^2(d\mu(x))} \le \frac{C}{t}\|k_t(x,y)\|_{L^2(d\mu(x))} \le \frac{C}{t}\mu\big(B(y,\sqrt{t})\big)^{-1/2}, \tag{5.2}$$

and, by the Schwarz inequality,

$$|k_t'(x,y)| \le Ct^{-1}\mu\big(B(x,\sqrt{t})\big)^{-1/2}\mu\big(B(y,\sqrt{t})\big)^{-1/2}. \tag{5.3}$$

The functional calculus asserts that $k_t(x,y) = c\int_{-\infty}^{\infty} e^{it\tau}G(x,y;\tau)\,d\tau$. Thus, using Proposition 4.9, we obtain

$$k_t(x,y) \le C_k\big(1+|x-y|m(x,V)\big)^{-k}\mu\big(B(x,|x-y|)\big)^{-1}. \tag{5.4}$$

Hence

$$k_t(x,y) \le C_k\big(1+\sqrt{t}m(x,V)\big)^{-k}\mu\big(B(x,\sqrt{t})\big)^{-1} \quad \text{for} \quad \sqrt{t} \le |x-y|. \tag{5.5}$$

Since $k_t(x,y) \le h_t(x,y)$, applying (5.5) and (3.3), we conclude

$$k_t(x,y) \le \frac{C_k\exp(-c'|x-y|^2/t)}{\big(1+\sqrt{t}m(x,V)\big)^{k/2}\mu\big(B(x,\sqrt{t})\big)} \quad \text{for} \quad \sqrt{t} \le |x-y|. \tag{5.6}$$

In order to complete the proof, it suffices to consider $\sqrt{t}m(x,V) \ge 1$ and $\sqrt{t} \ge |x-y|$.

Assume that there exists $\beta \ge 0$ and constants $C_\beta, c_\beta > 0$ such that for every $x,y \in \mathbb{R}^d$ and $t>0$ we have

$$k_t(x,y) \le C_\beta\big(1+\sqrt{t}m(y,V)\big)^{-\beta}\mu\big(B(y,\sqrt{t})\big)^{-1}\exp(-c_\beta|x-y|^2/t), \tag{5.7}$$

$$|k_t'(x,y)| \le \frac{C_\beta}{t}\big(1+\sqrt{t}m(y,V)\big)^{-\beta}\mu\big(B(x,\sqrt{t})\big)^{-1/2}\mu\big(B(y,\sqrt{t})\big)^{-1/2}. \tag{5.8}$$

By symmetry,

$$k_t(x,y) \le C_\beta\big(1+\sqrt{t}m(x,V)\big)^{-\beta}\mu\big(B(x,\sqrt{t})\big)^{-1}\exp(-c_\beta|x-y|^2/t), \tag{5.9}$$

$$|k_t'(x,y)| \le \frac{C_\beta}{t}\big(1+\sqrt{t}m(x,V)\big)^{-\beta}\mu\big(B(x,\sqrt{t})\big)^{-1/2}\mu\big(B(y,\sqrt{t})\big)^{-1/2}, \tag{5.10}$$

for every $x, y \in \mathbb{R}^d$, $t > 0$. Applying Proposition 4.9 and (5.8), we get

$$\begin{aligned}
k_t(x,y) &= \int_{\mathbb{R}^d} G(x,z;0)k_t'(z,y)\,d\mu(z)\\
&\le C_\beta \int_{\mathbb{R}^d} \frac{C_l|z-x|^2}{\big(1+|x-z|m(x,V)\big)^l\mu\big(B(x,|x-z|)\big)}\\
&\qquad \times t^{-1}\mu\big(B(y,\sqrt{t})\big)^{-1/2}\mu\big(B(z,\sqrt{t})\big)^{-1/2}\big(1+\sqrt{t}m(y,V)\big)^{-\beta}\,d\mu(z)\\
&\le C\sum_{n\ge 0}\int_{2^n r(x)<|z-x|\le 2^{n+1}r(x)} + C\sum_{n<0}\int_{2^n r(x)<|z-x|\le 2^{n+1}r(x)}\\
&= S_1 + S_2.
\end{aligned}$$

Now,

$$\begin{aligned}
S_1 \le C\sum_{n\ge 0}\int_{2^n r(x)<|z-x|\le 2^{n+1}r(x)} &\frac{C_l 2^{2n} r(x)^2}{(1+2^n)^l\mu\big(B(x,2^n r(x))\big)}\\
&\times t^{-1}\mu\big(B(y,\sqrt{t})\big)^{-1/2}\mu\big(B(z,\sqrt{t})\big)^{-1/2}\big(1+\sqrt{t}m(y,V)\big)^{-\beta}\,d\mu(z).
\end{aligned}$$

Since

$$\mu\big(B(z,\sqrt{t})\big)^{-1} \le C\mu\big(B(x,\sqrt{t})\big)^{-1}\left(1+\frac{2^n r(x)}{\sqrt{t}}\right)^\gamma$$

for $|z-x| \sim 2^n r(x)$, S_1 is bounded by

$$\begin{aligned}
&\sum_{n\ge 0}\frac{C_l 2^{2n} r(x)^2}{t(1+2^n)^l}\mu\big(B(y,\sqrt{t})\big)^{-1/2}\mu\big(B(x,\sqrt{t})\big)^{-1/2}\\
&\qquad\qquad\qquad \times\left(1+\frac{2^n r(x)}{\sqrt{t}}\right)^{\gamma/2}\left(1+\frac{\sqrt{t}}{r(y)}\right)^{-\beta}\\
&\le C\left(\frac{r(x)}{\sqrt{t}}\right)^2\mu\big(B(x,\sqrt{t})\big)^{-1/2}\mu\big(B(y,\sqrt{t})\big)^{-1/2}\big(1+\sqrt{t}m(y,V)\big)^{-\beta}
\end{aligned}$$

for $\sqrt{t}m(x,V) \ge 1$.

To deal with S_2 we use the fact that if $|z-x| \le 2^{n+1}r(x) < 2r(x) < 2\sqrt{t}$, then $\mu(B(z,\sqrt{t})) \sim \mu(B(x,\sqrt{t}))$. Therefore,

$$S_2 \le C\sum_{n<0}\left(\frac{r(x)}{\sqrt{t}}\right)^2 2^{2n}\mu\big(B(y,\sqrt{t})\big)^{-1/2}\mu\big(B(x,\sqrt{t})\big)^{-1/2}\big(1+\sqrt{t}m(y,V)\big)^{-\beta}.$$

Thus

$$k_t(x,y) \le \frac{C\big(r(x)/\sqrt{t}\big)^2}{\mu\big(B(y,\sqrt{t})\big)^{1/2}\mu\big(B(x,\sqrt{t})\big)^{1/2}\big(1+\sqrt{t}m(y,V)\big)^{\beta}}. \tag{5.11}$$

Taking the geometric mean of (5.7) and (5.11) we have

$$\begin{aligned} k_t(x,y) \le C\big(\sqrt{t}m(x,V)\big)^{-1}\mu\big(B(y,\sqrt{t})\big)^{-3/4}\mu\big(B(x,\sqrt{t})\big)^{-1/4} \\ \times \big(1+\sqrt{t}m(y,V)\big)^{-\beta}\exp(-c_\beta|x-y|^2/2t). \end{aligned}$$

It follows from Lemma 4.3 that for $\sqrt{t} \ge |x-y|$ and $\sqrt{t}m(x,V) \ge 1$ one has

$$\big(\sqrt{t}m(x,V)\big)^{-1} \le C\big(1+\sqrt{t}m(y,V)\big)^{-1/(1+k_0)}.$$

This leads to

(5.12)

$$\begin{aligned} k_t(x,y) &\le C\big(1+\sqrt{t}m(y,V)\big)^{-\beta-\frac{1}{1+k_0}}\frac{\exp(-c'_\beta|x-y|^2/t)}{\mu\big(B(y,\sqrt{t})\big)^{3/4}\big(B(x,\sqrt{t})\big)^{1/4}} \\ &\le C\big(1+\sqrt{t}m(y,V)\big)^{-\beta-\frac{1}{1+k_0}}\mu\big(B(y,\sqrt{t})\big)^{-1}\exp(-c''_\beta|x-y|^2/t). \end{aligned}$$

Similarly, using (5.12) and (5.8), we get

(5.13)

$$\begin{aligned} |k'_t(x,y)| &= \Big|\int_{\mathbb{R}^d} k'_t(x,z)k_t(z,y)\,d\mu(z)\Big| \\ &\le \frac{C}{t}\mu\big(B(y,\sqrt{t})\big)^{-1/2}\mu\big(B(x,\sqrt{t})\big)^{-1/2}\big(1+\sqrt{t}m(y,V)\big)^{-\beta-\frac{1}{1+k_0}}. \end{aligned}$$

Now (2.10) follows from (5.12) and (5.13) by the iteration process presented above. $\square$

We set $q_t(x,y) = h_t(x,y) - k_t(x,y)$. The perturbation formula asserts

(5.14)

$$q_t(x,y) = h_t(x,y) - k_t(x,y) = \int_0^t\int_{\mathbb{R}^d} h_s(x,z)V(z)k_{t-s}(z,y)\,d\mu(z)\,ds.$$

Proposition 5.1. *There are constants $C, c > 0$ such that*

$$0 \le q_t(x,y) \le C\big(\sqrt{t}m(x,V)\big)^{\delta}\mu\big(B(x,\sqrt{t})\big)^{-1}\exp(-c|x-y|^2/t). \tag{5.15}$$

Proof. Cf. [6]. We divide the integral in (5.14) into the sum of the following four integrals:

$$q_t(x,y) = \int_0^{t/2}\int_{|z-x|>|y-x|/2} + \int_0^{t/2}\int_{|z-x|<|y-x|/2} + \int_{t/2}^{t}\int_{|z-x|>|y-x|/2} + \int_{t/2}^{t}\int_{|z-x|<|y-x|/2} = I_1 + I_2 + I_3 + I_4. \tag{5.16}$$

Applying (3.4), (1.4), (2.2), and Corollary 4.5, we have

$$\begin{aligned} I_1 &\le C\int_0^{t/2}\int_{|z-x|>|y-x|/2} \frac{e^{-c'|x-y|^2/t}e^{-c'|x-z|^2/s}V(z)}{\mu\big(B(x,\sqrt{s})\big)\mu\big(B(y,\sqrt{t})\big)}\,d\mu(z)\,ds \\ &\le C\mu(B(x,\sqrt{t}))^{-1}e^{-c'|x-y|^2/t}\int_0^{t/2}\frac{\big(\sqrt{s}m(x,V)\big)^{\delta}}{s}\,ds \\ &\le C\big(\sqrt{t}m(x,V)\big)^{\delta}\mu\big(B(y,\sqrt{t})\big)^{-1}e^{-c'|x-y|^2/t}. \end{aligned} \tag{5.17}$$

Now we turn to estimate I_2. Since $|z-y|\ge|x-y|/2$, using (3.4), (1.4), and Corollary 4.5, we get

$$\begin{aligned} I_2 &\le C\mu\big(B(y,\sqrt{t})\big)^{-1}e^{-c'|x-y|^2/t}\int_0^{t/2}\int p_s(x,z)V(z)\,d\mu(z)\,ds \\ &\le C\big(\sqrt{t}m(x,V)\big)^{\delta}\mu(B(y,\sqrt{t}))^{-1}e^{-c'|x-y|^2/t}. \end{aligned} \tag{5.18}$$

We continue in this fashion to obtain

$$\begin{aligned} I_3 &\le C\int_{t/2}^{t}\int_{|z-x|>|y-x|2}\frac{e^{-c'|y-x|^2/t}}{\mu(B(x,\sqrt{t}))}V(z)k_t(z,y)\,d\mu(z)\,ds \\ &\le C\mu(B(x,\sqrt{t}))^{-1}e^{-c'|y-x|^2/t}\big(\sqrt{t}m(y,V)\big)^{\delta}. \end{aligned} \tag{5.19}$$

The estimate of I_4 is similar to that of I_3. Indeed,

$$\begin{aligned} I_4 &\le C\int_{t/2}^{t}\int_{|z-x|<|y-x|/2}\frac{e^{-c'|z-y|^2/(t-s)}e^{-c'|z-y|^2/(t-s)}V(z)}{\mu\big(B(x,\sqrt{t})\big)\mu\big(B\big(y,\sqrt{t-s}\big)\big)}\,d\mu(z)\,ds \\ &\le C\int_{t/2}^{t}\int_{|z-x|<|y-x|/2}\frac{e^{-c''|x-y|^2/t}e^{-c'|z-y|^2/(t-s)}V(z)}{\mu\big(B(x,\sqrt{t})\big)\mu\big(B(y,\sqrt{t-s})\big)}\,d\mu(z)\,ds \\ &\le C\mu(B(x,\sqrt{t}))^{-1}e^{-c''|x-y|^2/t}\big(\sqrt{t}m(y,V)\big)^{\delta}. \end{aligned} \tag{5.20}$$

Now (5.15) is a consequence of (5.17)–(5.20) and the following inequalities:

$$\mu(B(y,\sqrt{t}))^{-1}e^{-c|x-y|^2/t} \leq C\mu(B(x,\sqrt{t}))^{-1}e^{-c'|x-y|^2/t},$$
$$\frac{(\sqrt{t}m(y,V))^\delta}{\mu\big(B(x,\sqrt{t})\big)}e^{-c|x-y|^2/t} \leq C\frac{(\sqrt{t}m(x,V))^\delta}{\mu\big(B(x,\sqrt{t})\big)}e^{-c'|x-y|^2/t}. \qquad \square$$

6. Hardy spaces and local Hardy spaces on the space of homogeneous type

At the beginning of this section we present basic results from the theory of Hardy spaces on spaces of homogeneous type (cf. [2], [16], [19]). Then these will be used to introduce the notion and properties of local Hardy spaces. The local Hardy spaces on the spaces of homogeneous type that we consider here are analogues of the classical local Hardy spaces studied originally on $\mathbb{R}^d$ by D. Goldberg in [12].

Let X be a topological space endowed with a Borel measure μ and a quasi-distance ρ satisfying

$$\rho(x,y)=\rho(y,x)\geq 0, \quad \text{for} \ \ x,y\in X, \tag{6.1}$$
$$\rho(x,y)=0 \ \text{ if and only if } \ x=y, \tag{6.2}$$
$$\rho(x,z)\leq A\big(\rho(x,y)+\rho(y,z)\big) \quad \text{for} \ \ x,y,z\in X, \tag{6.3}$$
$$A^{-1}r\leq \mu(B_\rho(x,r))\leq r \quad \text{for} \ \ x\in X, \ r>0, \tag{6.4}$$

with a constant $A>0$, $B_\rho(x,r)=\{y\in X:\rho(x,y)<r\}$. We also assume that there exists a nonnegative continuous function $\tilde{K}(r,x,y)$ on $\mathbb{R}^+\times X\times X$ that for some $\xi>0$ satisfies

$$\tilde{K}(r,x,y)=0 \quad \text{if} \quad \rho(x,y)>r, \tag{6.5}$$
$$\tilde{K}(r,x,x)>A^{-1}>0, \tag{6.6}$$
$$\tilde{K}(r,x,y)\leq A, \tag{6.7}$$
$$|\tilde{K}(r,x,y)-\tilde{K}(r,x,z)|\leq (\rho(y,z)/r)^\xi. \tag{6.8}$$

For later purposes it is convenient to consider a nonnegative continuous kernel $K(r,x,y)$ for which the condition (6.5) is relaxed to the condition (6.9) below, that is, $K(r,x,y)$ satisfies

$$K(r,x,y)\leq \left(1+\frac{\rho(x,y)}{r}\right)^{-1-\xi}, \tag{6.9}$$
$$K(r,x,x)\geq A^{-1}>0, \tag{6.10}$$
$$|K(r,x,y)-K(r,x,z)|\leq \left(\frac{\rho(y,z)}{r}\right)^\xi\left(1+\frac{\rho(x,y)}{r}\right)^{-1-2\xi} \tag{6.11}$$

for

$$\rho(y,z) < \frac{r+\rho(x,y)}{4A}.$$

Set

$$K_r f(x) = \int_X K(r,x,y) f(y)\,\frac{d\mu(y)}{r}.$$

Following [19] we define the maximal function

$$f^{(+)}(x) = \sup_{r>0} |K_r f(x)| = \sup_{r>0} \Big| \int_X K(r,x,y) f(y)\,\frac{d\mu(y)}{r}\Big|, \tag{6.12}$$

and the grand maximal function

$$f^*(x) = \sup\Bigg\{ \Big| \int_X f(y)\varphi(y)\frac{d\mu(y)}{r}\Big| : r>0,\ \operatorname{supp}\varphi \subset B_\rho(x,r), \tag{6.13}$$
$$|\varphi(x)-\varphi(y)| \le \left(\frac{\rho(x,y)}{r}\right)^{\xi},\ \|\varphi\|_{L^\infty} \le 1\Bigg\}.$$

The following theorem was proved by A. Uchiyama (see [19], Theorem 1').

THEOREM 6.1. *There exists* $0 < p_0 < 1$ *such that for* $p > p_0$

$$\|f^*\|_{L^p(X)} \le C_p \|f^{(+)}\|_{L^p(X)},$$

where C_p *is a positive constant depending only on* p *and* X.

We shall denote by $H^1(X)$ the set of all functions f such that

$$\|f\|_{H^1(X)} = \|f^{(+)}\|_{L^1(X)} < \infty.$$

We say that a function a is an atom for $H^1(X)$ associated with a ball $B_\rho(x,r)$ if

$$\operatorname{supp} a \subset B_\rho(x,r), \quad \|a\|_{L^\infty} \le r^{-1}, \tag{6.14}$$

$$\int_X a(x)\,d\mu(x) = 0. \tag{6.15}$$

The atomic norm in the space $H^1(X)$ is defined by

$$\|f\|_{H^1_{atom}} = \inf\left\{\sum_j |\lambda_j|\right\}, \tag{6.16}$$

where the infimum is taken over all decompositions $f = \sum_j \lambda_j a_j$, where λ_j are scalars and a_j are the $H^1(X)$ atoms.

The theorem below is a consequence of Theorem 6.1 and results of [16] and [2] (see [19], Corollary 1').

THEOREM 6.2. *There exist constants $C_1, C_2, C_3 > 0$ depending only on X such that*

$$\|f^{(+)}\|_{L^1(X)} \leq C_1 \|f\|_{H^1_{atom}} \leq C_2 \|f^*\|_{L^1(X)} \leq C_3 \|f^{(+)}\|_{L^1(X)}. \tag{6.17}$$

For $l > 0$ we define the local maximal functions $f^{(*,l)}$ and $f^{(+,l)}$ by

$$f^{(*,l)}(x) = \sup\left\{ \Big| \int_X f(y)\varphi(y) \frac{d\mu(y)}{r} \Big| : l > r > 0,\ \operatorname{supp}\varphi \subset B_\rho(x,r), \right. \tag{6.18}$$

$$\left. |\varphi(x) - \varphi(y)| \leq \left(\frac{\rho(x,y)}{r} \right)^{\xi},\ \|\varphi\|_{L^\infty} \leq 1 \right\},$$

$$f^{(+,l)}(x) = \sup_{l > r > 0} \Big| \int_X K(r,x,y) f(y)\, \frac{d\mu(y)}{r} \Big|. \tag{6.19}$$

It turns out that the proof of Theorem 1' of [19] can be easily adapted to obtain the following theorem.

THEOREM 6.3. *There exists a constant $0 < p_0 < 1$ such that for $p > p_0$*

$$\|f^{(*,l)}\|_{L^p(X)} \leq C_p \|f^{(+,l)}\|_{L^p(X)}. \tag{6.20}$$

The constant C_p depends on p and X, but it is independent of l.

A function f is an element of the local Hardy space $\mathbf{h}^1_l$ if

$$\|f\|_{\mathbf{h}^1_l} = \|f^{(+,l)}\|_{L^1(X)} < \infty.$$

We say that a function a is an atom for the local Hardy space $\mathbf{h}^1_l(X)$ associated with a ball $B_\rho(x,r)$ if

$$\operatorname{supp} a \subset B_\rho(x,r), \quad r < l, \tag{6.21}$$

$$\|a\|_{L^\infty} \leq r^{-1}, \tag{6.22}$$

$$\text{if } \ r < \frac{l}{4}, \ \text{ then } \ \int_X a(x)\, d\mu(x) = 0. \tag{6.23}$$

The atomic norm $\|f\|_{\mathbf{h}^1_l, atom}$ is given by

$$\|f\|_{\mathbf{h}^1_l, atom} = \inf \sum |\lambda_j|, \tag{6.24}$$

where the infimum is taken over all decompositions $f = \sum_j \lambda_j a_j$, where λ_j are scalars and a_j are $\mathbf{h}^1_l(X)$ atoms.

LEMMA 6.4. *Assume that the kernel $\tilde{K}(r,x,y)$ additionally satisfies*

$$\int \tilde{K}(r,x,y)\, d\mu(x) = r. \tag{6.25}$$

Then there exist constants $C > 0$, $\theta > 0$ such that for every $l > 0$

$$\|f - \tilde{K}_l f\|_{H^1} \leq C \|f^{(+,\theta l)}\|_{L^1(X)}. \tag{6.26}$$

Proof. The lemma will be proved if we show that, for constants C and θ that are large enough and independent of l,

$$\Big\| \sup_{0<r<l} |\tilde{K}_r f(x)| \Big\|_{L^1(d\mu(x))} + \Big\| \sup_{0<r<l} |\tilde{K}_r \tilde{K}_l f(x)| \Big\|_{L^1(d\mu(x))} + \Big\| \sup_{r\geq l} |\tilde{K}_r f(x) - \tilde{K}_r \tilde{K}_l f(x)| \Big\|_{L^1(d\mu(x))} \leq C\|f\|_{\mathbf{h}^1_{\theta l}}.$$

It follows from Theorem 6.3 that the first two summands are bounded by $C\|f\|_{\mathbf{h}^1_{\theta l}}$. The task is now to estimate the third one. Since

$$\tilde{K}_r f(x) - \tilde{K}_r \tilde{K}_l f(x) = \iint \Big(\tilde{K}(r,x,z) - \tilde{K}(r,x,u)\Big) \tilde{K}(l,u,z) \frac{d\mu(u)}{r} f(z) \frac{d\mu(z)}{l},$$

it suffices to prove that

$$J = \int \Big| \Big(\tilde{K}(r,x,z) - \tilde{K}(r,x,u)\Big) \tilde{K}(l,u,z) \Big| \frac{d\mu(u)}{r} \leq C\left(1 + \frac{\rho(x,z)}{l}\right)^{-1-\xi},$$

with a constant C independent of r, l provided $r \geq l$. We only need to consider the case where $\rho(x,z) > Cl$. It follows from (6.5) and (6.8) that

$$J \leq \begin{cases} \int_{\rho(z,u)<l} \left(\frac{\rho(z,u)}{r}\right)^{\xi} \tilde{K}(l,u,z) \frac{d\mu(u)}{r} & \text{if } \rho(x,z) < Cr, \\ 0 & \text{if } \rho(x,z) \geq Cr, \end{cases} \leq C \frac{l^{\xi+1}}{\rho(x,z)^{\xi+1}}, \tag{6.27}$$

which completes the proof. □

REMARK 6.5. Let us note that if for $\tilde{K}(r,x,y)$ (6.5)–(6.8) hold, then the kernel $rc_r(y)\tilde{K}(r,x,y)$, where $c_r(y)^{-1} = \int_X \tilde{K}(r,x,y)\, d\mu(x)$, satisfies (6.5)–(6.8) and (6.25).

PROPOSITION 6.6. *There exists a constant $C > 0$ such that for every $l > 0$ one has*

$$C^{-1}\|f\|_{\mathbf{h}^1_{l,atom}} \leq \|f\|_{\mathbf{h}^1_l} \leq C\|f\|_{\mathbf{h}^1_{l,atom}}. \tag{6.28}$$

Proof. The second inequality in (6.32) follows by standard arguments. We shall prove the first one. By Theorem 6.3 and Remark 6.5 there is no loss of generality in assuming that $\tilde{K}$ satisfies (6.25). Assume that $f \in \mathbf{h}^1_{\theta l}$, $f \not\equiv 0$. Then, by Lemma 6.4,

$$\|f - \tilde{K}_l f\|_{H^1(X)} \leq C\|f\|_{\mathbf{h}^1_{\theta l}}.$$

Consequently, $f - \tilde{K}_l f = \sum \lambda_j b_j$, $\sum |\lambda_j| \leq C\|f\|_{\mathbf{h}^1_{\theta l}}$, where b_j are $H^1(X)$ atoms (cf. Theorem 6.2). Let U_n^l be a family of subsets of X such that $U_j^l \cap U_n^l = \emptyset$ if $n \neq j$, $\bigcup_j U_j^l = X$, and each of U_j^l is contained in a ball $B_\rho(x_j^{\{l\}}, l)$. Since

$$\|\tilde{K}_l(\chi_{U_j^l} f)\|_{L^\infty} \leq C l^{-1} \|\chi_{U_j^l} f\|_{L^1}$$

and

$$\operatorname{supp} \tilde{K}_l(\chi_{U_j^l} f) \subset B_\rho(x_j^{\{l\}}, \theta l),$$

we get that

$$\frac{1}{C\|\chi_{U_j^l} f\|_{L^1}} \tilde{K}_l(\chi_{U_j^l} f)(x)$$

is an $\mathbf{h}^1_{\theta l}$ atom. Hence

$$\|f\|_{\mathbf{h}^1_{\theta l, atom}} \leq C\|f\|_{\mathbf{h}^1_{\theta l}}.$$

Obviously,

$$\|f\|_{\mathbf{h}^1_{l,\, atom}} \leq C\|f\|_{\mathbf{h}^1_{\theta l, atom}}.$$

Thus the proposition is proved. □

Let $X = \mathbb{R}^d$ and $d\mu(x) = w(x)\, dx$. We set

$$\rho(x,y) = C_0 \inf\{\mu(B) : B \text{ is a Euclidean ball, } x, y \in B\},$$

where C_0 is the constant in (2.2). It is not difficult to check that

$$C_0^{-1} \mu\big(B(x, |x-y|)\big) \leq \rho(x,y) \leq C_0 \mu\big(B(x,|x-y|)\big). \tag{6.29}$$

Moreover, $X = \mathbb{R}^d$, μ, ρ satisfy (6.1)–(6.4). We put

$$K(r,x,y) = \frac{1}{C_4} \mu\big(B(x,\sqrt{t})\big) h_t(x,y), \quad \text{where} \quad r = \mu\big(B(x,\sqrt{t})\big), \tag{6.30}$$

$$\tilde{K}(r,x,y) = \frac{1}{C_4} \mu\big(B(x,\sqrt{t})\big) h_t(x,y) \varphi_x(y), \tag{6.31}$$

where $r = \mu\big(B(x,\sqrt{t})\big)$, $\varphi_x(y) = \varphi((x-y)/\sqrt{t})$, $\varphi \in C_c^\infty(B(0, 1/C_5))$, $\varphi \geq 0$, $\varphi(x) = 1$ for $|x| \leq 1/(2C_5)$.

We are going to show that for $\tilde{K}(r,x,y)$, $K(r,x,y)$ the estimates (6.5)–(6.8) and (6.9)–(6.11), respectively, hold, provided the constants C_4, C_5 are large enough. Let us note that

$$\frac{\rho(x,y)}{r} \sim \frac{\mu\big(B(x,|x-y|)\big)}{\mu\big(B(x,\sqrt{t})\big)}. \tag{6.32}$$

Obviously (3.3) implies (6.10). In order to prove (6.9) it suffices, by (3.3) and (6.32), to show that

$$\exp(-|x-y|^2/C_2 t) \leq C \left(1 + \frac{\mu\big(B(x,|x-y|)\big)}{\mu\big(B(x,\sqrt{t})\big)}\right)^{-1-\xi}.$$

But this is a consequence of (2.3).

We shall show that (6.11) holds with A being sufficiently large and $\xi = \alpha/\gamma$ (see (2.3) and (3.4)). We assume (cf. (6.11)) that

$$\rho(y,z) \le \frac{r+\rho(x,y)}{4A}, \quad \text{where} \quad r = \mu\big(B(x,\sqrt{t})\big). \tag{6.33}$$

Case 1: $|x-y| < C\sqrt{t}$, where C is a large constant. Let us note that in this case $\mu\big(B(y,\sqrt{t})\big) \sim \mu\big(B(x,\sqrt{t})\big)$, and, by (6.29), (6.33), we have $|y-z| \le C\sqrt{t}$. Applying (3.4) we get

$$|K(r,x,y) - K(r,x,z)| \le \frac{C}{C_4}\left(\frac{|y-z|}{\sqrt{t}}\right)^{\alpha}. \tag{6.34}$$

On the other hand, by virtue of (6.32), we obtain

$$\begin{aligned} &\left(\frac{\rho(y,z)}{r}\right)^{\xi}\left(1+\frac{\rho(x,y)}{r}\right)^{-1-2\xi} \\ &\qquad \ge c\left(\frac{\mu(B(y,|y-z|))}{\mu(B(y,\sqrt{t}))}\right)^{\xi}\left(1+\frac{\mu(B(x,|x-y|))}{\mu(B(x,\sqrt{t}))}\right)^{-1-2\xi} \\ &\qquad \ge c\left(\frac{\mu\big(B(y,|y-z|)\big)}{\mu\big(B(y,\sqrt{t})\big)}\right)^{\xi} \ge c\left(\frac{|y-z|}{\sqrt{t}}\right)^{\xi\gamma}. \end{aligned} \tag{6.35}$$

Now (6.11) follows from (6.34) and (6.35).

Case 2: $|x-y| > C\sqrt{t}$. Then using (6.29) and (6.33), we have $|y-z| < |x-y|/4$.

If $|y-z| < \sqrt{t}$, then, by (3.4),

$$|K(r,x,y) - K(r,x,z)| \le \frac{C}{C_4}\left(\frac{|y-z|}{\sqrt{t}}\right)^{\alpha}\exp(-|x-y|^2/ct). \tag{6.36}$$

Applying (6.32) and (2.3), we get

$$\begin{aligned} &\left(\frac{\rho(y,z)}{r}\right)^{\xi}\left(1+\frac{\rho(x,y)}{r}\right)^{-1-2\xi} \\ &\quad \ge c\left(\frac{\mu\big(B(y,|y-z|)\big)}{\mu\big(B(y,\sqrt{t})\big)}\right)^{\xi}\left(\frac{\mu\big(B(y,\sqrt{t})\big)}{\mu\big(B(x,|x-y|)\big)}\right)^{\xi} \\ &\qquad\qquad \times\left(1+\frac{\mu\big(B(x,|x-y|)\big)}{\mu\big(B(x,\sqrt{t})\big)}\right)^{-1-2\xi} \\ &\quad \ge c\left(\frac{|y-z|}{\sqrt{t}}\right)^{\xi\gamma}\left(\frac{|x-y|}{\sqrt{t}}\right)^{-\xi\gamma}\left(1+\frac{\mu\big(B(x,|x-y|)\big)}{\mu\big(B(x,\sqrt{t})\big)}\right)^{-M}. \end{aligned}$$

Thus (6.11) follows from (6.36) and (2.3).

If $|y-z| \geq \sqrt{t}$, then from (3.3) we conclude

$$\begin{aligned} |K(r,x,y)-K(r,x,z)| &\leq |K(r,x,y)|+|K(r,x,z)| \\ &\leq \frac{C}{C_4}\exp(-|x-y|^2/ct). \end{aligned} \tag{6.37}$$

Using arguments similar to those above, we obtain

$$\begin{aligned} &\left(\frac{\rho(y,z)}{r}\right)^{\xi}\left(1+\frac{\rho(x,y)}{r}\right)^{-1-2\xi} \\ &\qquad \geq c\left(\frac{\mu\big(B(x,|y-z|)\big)}{\mu\big(B(y,\sqrt{t})\big)}\right)^{\xi}\left(1+\frac{\mu\big(B(x,|x-y|)\big)}{\mu\big(B(x,\sqrt{t})\big)}\right)^{-1-2\xi} \\ &\qquad \geq c\left(\frac{\mu\big(B(y,\sqrt{t})\big)}{\mu\big(B(x,|x-y|)\big)}\right)^{\xi}\left(1+\frac{\mu\big(B(x,|x-y|)\big)}{\mu\big(B(x,\sqrt{t})\big)}\right)^{-1-2\xi}. \end{aligned}$$

Since $\mu(B(x,|x-y|)) \sim \mu(B(y,|x-y|))$ and $C\sqrt{t} < |x-y|$, we conclude (6.11) from (2.3) and (6.37).

In the same manner we can see that $\tilde{K}$ satisfies (6.5)–(6.8).

It turns out that compactly supported elements of $\mathbf{h}_l^1$ admit localized atomic decomposition, that is, the following proposition holds.

PROPOSITION 6.7. *There exists constants $C, c > 0$ such that for every $l > 0$, every $x_0 \in \mathbb{R}^d$, and every $f \in \mathbf{h}_l^1$ such that* $\operatorname{supp} f \subset B_\rho(x_0, l)$*, one has*

$$f = \sum_j \lambda_j a_j, \quad \sum_j |\lambda_j| \leq C\|f\|_{\mathbf{h}_l^1},$$

where a_j are $\mathbf{h}_l^1$ atoms that are supported by the ball $B_\rho(x_0, cl)$.

Proof. Let R be such that $\mu(B(x_0,R)) = l$. Then

$$B(x_0, \frac{R}{A_6}) \subset B_\rho(x_0, l) \subset B(x_0, A_6 R).$$

Let $\psi(x) = \psi_0((x-x_0)/A_6R)$, where $\psi_0 \in C_c^\infty(B(0,2))$, $\psi_0 = 1$ on $B(0,1)$. It suffices to prove that there is a constant $C > 0$ such that if a is an $\mathbf{h}_l^1$ atom associated with a ball $B_\rho(y_0, r)$, $r < l/4$, then $\psi a = \sum_{j=0}^N \lambda_j a_j$, where a_j are $\mathbf{h}_l^1$ atoms, $\operatorname{supp} a_j \subset B_\rho(x_0, cl)$, $\sum_j |\lambda_j| \leq C$. Let τ be such that $\mu(B(y_0,\tau)) = r$. Then

$$B(y_0, \frac{\tau}{A_6}) \subset B_\rho(y_0, r) \subset B(y_0, A_6\tau).$$

We define

$$\begin{aligned} a(x)\psi(x) = \big(a(x)\psi(x) - \alpha_1\chi_{B_\rho(y_0,2r)}(x)\big) \\ + \sum_{j=2}^{N}\big(\alpha_j\chi_{B_\rho(y_0,2^jr)}(x) - \alpha_{j+1}\chi_{B_\rho(y_0,2^{j+1}r)}(x)\big) \\ + \alpha_{N+1}\chi_{B_\rho(y_0,2^{N+1}r)}(x), \end{aligned}$$

where $N = [\log_2 \frac{l}{r}]$,

$$\begin{aligned} \alpha_1 &= \mu(B_\rho(y_0,2r))^{-1}\int a(x)(\psi(x)-\psi(y_0))\,d\mu(x), \\ \alpha_j &= \alpha_1\mu(B_\rho(y_0,2r))\mu(B_\rho(y_0,2^jr))^{-1}. \end{aligned}$$

One can check that this is the required decomposition, since $|\alpha_1| \le C\tau r^{-1}R^{-1}$, $\log_2 l/r \le C\log_2 R/\tau$. □

7. Proof of Theorem 2.1

We set $Q_tf(x) = \int q_t(x,y)f(y)\,d\mu(y)$; see (5.14).

LEMMA 7.1. *For every $\theta > 0$ there exists a constant C_θ such that for every $x_0 \in \mathbb{R}^d$ we have*

$$\begin{aligned} \Big\| \sup_{0<t\le\theta r(x_0)^2} |Q_t\big(\chi_{B(x_0,r(x_0))}f\big)(x)| \Big\|_{L^1(d\mu(x))} \\ \le C_\theta \big\| |\big(\chi_{B(x_0,r(x_0))}f\big)(x)| \big\|_{L^1(d\mu(x))}. \end{aligned} \tag{7.1}$$

Proof. There is no loss of generality in assuming that $\theta \ge 2$. Since $q_t(x,y) = q_t(y,x)$ it follows from (2.3), (2.4), Lemma 4.3 and Proposition 5.1 that for $x \in B(x_0,\theta r(x_0))$, $y \in B(x_0,r(x_0))$ one has

$$\begin{aligned} |q_t(x,y)| &\le \frac{Ce^{-c|x-y|^2/t}}{\mu\big(B(y,|x-y|)\big)}\left(\frac{\sqrt{t}}{|x-y|}\right)^{\delta} \\ &\qquad\times\left[\left(\frac{|x-y|}{\sqrt{t}}\right)^{\nu} + \left(\frac{|x-y|}{\sqrt{t}}\right)^{\gamma}\right]\left(\frac{|x-y|}{r(y)}\right)^{\delta} \\ &\le \frac{C}{\mu\big(B(x,|x-y|)\big)}\left(\frac{|x-y|}{r(x_0)}\right)^{\delta}. \end{aligned}$$

Therefore

$$\begin{aligned}
&\Big\| \sup_{0<t\le\theta r(x_0)^2} |Q_t(\chi_{B(x_0,r(x_0))}f)(x)| \Big\|_{L^1(B(x_0,\theta r(x_0)),d\mu(x))} \\
&\le C\frac{\|\chi_{B(x_0,r(x_0))}f\|_{L^1(d\mu)}}{r(x_0)^\delta} \sup_{y\in B(x_0,r(x_0))} \int_{B(x_0,\theta r(x_0))} \frac{|x-y|^\delta}{\mu\big(B(x,|x-y|)\big)}\,d\mu(x) \\
&\le C\frac{\|\chi_{B(x_0,r(x_0))}f\|_{L^1(d\mu)}}{r(x_0)^\delta} \sup_{y\in B(x_0,r(x_0))} \int_{B(y,2\theta r(x_0))} \frac{|x-y|^\delta}{\mu\big(B(x,|x-y|)\big)}\,d\mu(x) \\
&\le C\frac{\|\chi_{B(x_0,r(x_0))}f\|_{L^1(d\mu)}}{r(x_0)^\delta} \sum_{j\ge0} \int_{|y-x|\sim 2^{-j}\theta r(x_0)} \frac{\big(2^{-j}\theta r(x_0)\big)^\delta}{\mu\big(B(x,2^{-j}\theta r(x_0)\big)}\,d\mu(x) \\
&\le C_\theta \|\chi_{B(x_0,r(x_0))}f\|_{L^1(d\mu)}.
\end{aligned}$$

If $x\notin B(x_0,\theta r(x_0))$, then, by Proposition 5.1, we get

$$\begin{aligned}
&\big|Q_t\big(\chi_{B(x_0,r(x_0))}f\big)(x)\big| \\
&\quad\le C\int_{B(x_0,r(x_0))} \frac{e^{-c|x-y|^2/t}}{\mu\big(B(y,\sqrt{t})\big)} f(y)\,d\mu(y) \\
&\quad\le C\int_{B(x_0,r(x_0))} \frac{\mu\big(B(y,\sqrt{\theta}r(x_0))\big)e^{-c|x-y|^2/t}f(y)}{\mu\big(B(y,\sqrt{t})\big)\mu\big(B(y,\sqrt{\theta}r(x_0))\big)}\,d\mu(y).
\end{aligned}$$

Using (2.3) and the fact that $0<t\le\theta r(x_0)^2$, we have

$$\begin{aligned}
&\big|Q_t\big(\chi_{B(x_0,r(x_0))}f\big)(x)\big| \\
&\quad\le C\left(\frac{\sqrt{\theta}r(x_0)}{\sqrt{t}}\right)^\gamma \frac{e^{-c'|x-x_0|^2/t}}{\mu\big(B(x_0,\sqrt{\theta}r(x_0))\big)} \big\|\chi_{B(x_0,r(x_0))}f\big\|_{L^1(d\mu)} \\
&\quad\le C\frac{e^{-c''|x-x_0|^2/(\theta r(x_0)^2)}}{\mu\big(B(x_0,\sqrt{\theta}r(x_0))\big)} \big\|\chi_{B(x_0,r(x_0))}f\big\|_{L^1(d\mu)}.
\end{aligned}$$

Thus

$$\begin{aligned}
\text{(7.2)}\qquad &\Big\| \sup_{0<t\le\theta r(x_0)^2} |Q_t(\chi_{B(x_0,r(x_0))}f)(x)| \Big\|_{L^1(B(x_0,2r(x_0))^c)} \\
&\qquad\le C_\theta \|\chi_{B(x_0,r(x_0))}f\|_{L^1(d\mu)}.
\end{aligned}$$

This completes the proof of Lemma 7.1. ☐

LEMMA 7.2. *Under assumptions of Theorem 2.1 we have*

$$\|f\|_{H^1_{\mathcal{L}}} \le C\|f\|_{H^1_{\mathcal{L}-atom}}. \tag{7.3}$$

Proof. It suffices to show (7.3) for the case when f is an $H^1_{\mathcal{L}}$ atom. Let a be an $H^1_{\mathcal{L}}$ atom associated with a ball $B(y_0,r_0)$. Then $r_0 \le r(y_0)$. Obviously, by (2.10),

$$\sup_{t>0} |T_t a(x)| \le C\mu\big(B(y_0,r_0)\big)^{-1}.$$

Thus

$$\int_{B(y_0,4r_0)} |\mathcal{M}a(x)|\,d\mu(x) \le C.$$

It remains to estimate $\mathcal{M}a(x)$ on $B(y_0,4r_0)^c$. Assume first that $\int a(x)\,d\mu(x) = 0$. Fix $\theta > 2$ (large enough). Using (5.14) we have

$$\begin{aligned}\mathcal{M}a(x) &\le \sup_{0<t<\theta r(y_0)^2} |T_t a(x)| + \sup_{t\ge\theta r(y_0)^2} |T_t a(x)|\\ &\le \sup_{0<t<\theta r(y_0)^2} |Q_t a(x)| + \sup_{0<t<\theta r(y_0)^2} |S_t a(x)| + \sup_{t\ge\theta r(y_0)^2} |T_t a(x)|\\ &= M_1(x) + M_2(x) + M_3(x).\end{aligned}$$

Lemma 7.1 yields

$$\int_{\mathbb{R}^d} M_1(x)\,d\mu(x) \le C. \tag{7.4}$$

Applying (3.4), we obtain

$$\begin{aligned}|S_t a(x)| &= \left|\int_{\mathbb{R}^d} \big(h_t(x,y) - h_t(x,y_0)\big) a(y)\,d\mu(y)\right|\\ &\le C\int_{\mathbb{R}^d} \left(\frac{|y-y_0|}{\sqrt{t}}\right)^{\alpha} \frac{e^{-c|x-y|^2/t}}{\mu\big(B(x,\sqrt{t})\big)} |a(y)|\,d\mu(y)\\ &\le C\frac{r_0^\alpha}{t^{\alpha/2}} \frac{e^{-c|x-y_0|^2/t}}{\mu\big(B(x,\sqrt{t})\big)} \le C\frac{r_0^\alpha}{|x-y_0|^\alpha \mu\big(B(y_0,|x-y_0|)\big)}.\end{aligned}$$

Thus

$$\int_{B(y_0,4r_0)^c} M_2(x)\,d\mu(x) \le C. \tag{7.5}$$

We turn to estimate M_3. If $t \ge \theta r(y_0)^2$, then, by (2.10) and Lemma 4.3, we get

$$|T_t a(x)| \le \frac{C_N \mu\big(B(y_0,r(y_0))\big) e^{-c|x-y_0|^2/t}}{\mu\big(B(y_0,r(y_0))\big)\mu\big(B(y_0,\sqrt{t})\big)\big(\sqrt{t}m(y_0,V)\big)^N}.$$

Hence,

$$\begin{aligned}&|T_t a(x)| \le C\mu\big(B(y_0,r(y_0))\big)^{-1} \quad \text{for } \ x \in B(y_0,4r(y_0)) \setminus B(y_0,4r_0),\\ &|T_t a(x)| \le C_N\big(m(y_0,V)|x-y_0|\big)^{-N} \mu\big(B(y_0,r(y_0))\big)^{-1} \quad \text{for } x \notin B(y_0,4r(y_0)).\end{aligned}$$

Consequently,

$$\int_{|x-y_0|>4r_0} M_3(x)\,d\mu(x) \le \int_{4r_0<|x-y_0|<4r(y_0)} + \int_{|x-y_0|\ge 4r(y_0)} \le C. \tag{7.6}$$

It remains to consider the case where $\int a(x)\,d\mu(x) \neq 0$. Then, by the definition of $H^1_{\mathcal{L}}$ atoms, $r_0 \sim r(y_0)$. If $x \notin B(y_0, 4r_0)$, then (2.10), (2.3), (2.4) imply

$$\begin{aligned} |T_t a(x)| &\le C \int k_t(x,y)|a(y)|\,d\mu(y) \\ &\le C_N \big(m(y_0,V)|x-y_0|\big)^{-N} \mu\big(B(y_0, r(y_0))\big)^{-1}. \end{aligned} \tag{7.7}$$

This leads to

$$\int_{|x-y_0|>4r_0} \mathcal{M}a(x)\,d\mu(x) \le C. \qquad \square$$

Now we turn to the proof of the atomic decomposition. It is not difficult to prove using Lemma 4.3 that there is a sequence $x_j \in \mathbb{R}^d$ and a family of smooth functions φ_j such that

$$\begin{aligned} &\operatorname{supp}\varphi_j \subset B_j = B(x_j, r_j), \quad r_j = r(x_j), \\ &0 \le \varphi_j \le 1, \quad |\nabla\varphi_j| \le C r_j^{-1}, \\ &\sum_j \varphi_j = 1, \quad \sum_j \chi_{B(x_j,4r_j)} \le C. \end{aligned} \tag{7.8}$$

Moreover, there exists a constant $N > 0$ such that

$$\sup_{y\in\mathbb{R}^d} \left\{ \sum_j \left(1 + \frac{|x_j - y|}{r_j}\right)^{-N} \right\} < \infty. \tag{7.9}$$

Set

$$\mathcal{M}_j f(x) = \sup_{0<t<\theta r_j^2} |\varphi_j(x) T_t f(x) - T_t(\varphi_j f)(x)|. \tag{7.10}$$

Lemma 7.3. *There exists a constant $C_\theta > 0$ such that*

$$\sum_j \|\mathcal{M}_j f\|_{L^1(d\mu)} \le C_\theta \|f\|_{L^1(d\mu)}.$$

Proof. Write $B_j^* = B(x_j, 2r_j)$, $B_j^{**} = B(x_j, 4r_j)$. Then

$$\begin{aligned} |\varphi_j(x)T_t f(x) - T_t(\varphi_j f)(x)| &\le \int_{\mathbb{R}^d} \big|\big(\varphi_j(x) - \varphi_j(y)\big)k_t(x,y) f(y)\big|\,d\mu(y) \\ &= \int_{(B_j^*)^c} + \int_{B_j^*} = I_{t,j}(x) + J_{t,j}(x). \end{aligned}$$

Since $t < \theta r_j^2$, applying (7.8), (2.3) and (2.10), we have

$$|I_{t,j}(x)| \le \frac{C_\theta \varphi_j(x)}{\mu\big(B(x_j,r_j)\big)} \int_{(B_j^*)^c} e^{-c_\theta |x_j-y|^2/r_j^2} |f(y)|\, d\mu(y). \tag{7.11}$$

Observe that

$$\int \frac{\varphi_j(x)}{\mu(B(x_j,r_j))} e^{-c_\theta|x_j-y|^2/r_j^2}\, d\mu(x) \sim e^{-c_\theta|x_j-y|^2/r_j^2}.$$

Therefore, by (7.9), we get

$$\sum_j \Big\| \sup_{0<t<\theta r_j^2} |I_{t,j}| \Big\|_{L^1} \le C_\theta \|f\|_{L^1(d\mu)}. \tag{7.12}$$

To estimate $J_{t,j}(x)$ for $x \in B_j^{**}$, we apply (7.8), (2.3), (2.4) and get

$$\begin{aligned}
|J_{t,j}(x)| &\le C \int_{B_j^*} \frac{|x-y|}{r_j} \frac{e^{-c|x-y|^2/t}}{\mu(B(x,\sqrt{t}))} |f(y)|\, d\mu(y) \\
&\le C \int_{B_j^*} \frac{|x-y|}{r_j} \left(\left(\frac{|x-y|}{\sqrt{t}}\right)^\gamma + \left(\frac{|x-y|}{\sqrt{t}}\right)^\nu \right) \\
&\qquad\qquad \times \frac{e^{-c|x-y|^2/t}|f(y)|}{\mu\big(B(y,|x-y|)\big)}\, d\mu(y).
\end{aligned}$$

Thus

$$\begin{aligned}
\Big\| \sup_{0<t<\theta r_j^2} |J_{t,j}| \Big\|_{L^1(B_j^{**},(d\mu))} \tag{7.13}
&\le C_\theta \|f\|_{L^1(B_j^*,(d\mu))} \sup_{y\in B_j^*} \int_{B_j^{**}} \frac{|x-y|}{r_j \mu(B(y,|x-y|))} d\mu(x) \\
&\le C_\theta \|f\|_{L^1(B_j^*,(d\mu))}.
\end{aligned}$$

Recall that $t < \theta r_j^2$. Thus, if $x \notin B_j^{**}$, then, by (2.3), we get

$$\begin{aligned}
|J_{t,j}(x)| &\le C \int \frac{1}{\mu\big(B(x,\sqrt{t})\big)} e^{-c|x-x_j|^2/t} \varphi_j(y)|f(y)|\, d\mu(y) \tag{7.14} \\
&\le C \left(\frac{r_j}{\sqrt{t}}\right)^\gamma \int \frac{e^{-c|x-x_j|^2/t}}{\mu(B(x,r_j))} \varphi_j(y)|f(y)|\, d\mu(y) \\
&\le C_\theta \frac{1}{\mu\big(B(x,r_j)\big)} \int e^{-c_\theta|x-x_j|^2/r_j^2} \varphi_j(y)|f(y)|\, d\mu(y) \\
&\le C_\theta \|\varphi_j f\|_{L^1(d\mu)} \frac{e^{-c_\theta|x-x_j|^2/r_j^2}}{\mu\big(B(x,r_j)\big)}.
\end{aligned}$$

Finally, (7.8), (7.13), and (7.14) imply

$$\begin{aligned}(7.15)\quad \sum_j \left(\sup_{0<t<\theta r_j^2} |J_{t,j}(x)|\right) d\mu(x) &\le C_\theta \sum_j \Big(\|\varphi_j f\|_{L^1(d\mu)} + \|f\|_{L^1(B_j^*,d\mu)}\Big) \\ &\le C_\theta \|f\|_{L^1(d\mu)}.\end{aligned}$$

This combined with (7.12) ends the proof of Lemma 7.3. □

The proof of the atomic decomposition will be complete if we show

LEMMA 7.4. *There exists a constant $C>0$ such that*

$$\|f\|_{H^1_{\mathcal{L}-atom}} \le C\|\mathcal{M}f\|_{L^1(d\mu)}. \tag{7.16}$$

Proof. Assume that $\mathcal{M}f \in L^1(d\mu)$. From Lemma 7.1 we conclude

$$\begin{aligned}(7.17)\quad &\Big\|\sup_{0<t<\theta r_j^2} |S_t(\varphi_j f)(x)|\Big\|_{L^1(d\mu(x))} \\ &\le \Big\|\sup_{0<t<\theta r_j^2} |Q_t(\varphi_j f)(x)|\Big\|_{L^1(d\mu(x))} \\ &\quad + \Big\|\sup_{0<t<\theta r_j^2} |T_t(\varphi_j f)(x)|\Big\|_{L^1(d\mu(x))} \\ &\le C\left(\|\varphi_j f\|_{L^1(d\mu)} + \|\varphi_j(\mathcal{M}f)\|_{L^1(d\mu)} + \|\mathcal{M}_j f\|_{L^1(d\mu)}\right).\end{aligned}$$

Hence, by (7.8) and Lemma 7.3,

$$\sum_j \Big\|\sup_{0<t<\theta r_j^2} |S_t(\varphi_j f)(x)|\Big\|_{L^1(d\mu(x))} \le C_\theta \|\mathcal{M}f\|_{L^1(d\mu)}. \tag{7.18}$$

According to the definitions of atoms for $H^1_{\mathcal{L}}$ and $\mathbf{h}^1_l$ spaces Lemma 7.4 will be proved, by using (7.18), (6.29), and Propositions 6.6, 6.7, 6.3, if we show that there exist constants $C>0$, $\theta>2$ such

(7.19)

$$\|(\varphi_j f)^{(+,l)}\|_{L^1(d\mu)} \le C\left(\Big\|\sup_{0<t<\theta r_j^2} |S_t(\varphi_j f)(x)|\Big\|_{L^1(d\mu(x))} + \|\varphi_j f\|_{L^1(d\mu)}\right),$$

where $l=\mu\big(B(x_j,r_j)\big)$. For $x \in B_j^* = B(x_j, 2r_j)$ and $r \le l$, we have $K(r,x,y) = \mu(B(x,\sqrt{t}))h_t(x,y)$, where $\mu\big(B(x,\sqrt{t})\big) = r \le \mu\big(B(x_j,r_j)\big)$. Thus $\sqrt{t} \le Cr_j$, and, consequently,

$$\Big\|(\varphi_j f)^{(+,l)}\Big\|_{L^1(B_j^*,d\mu)} \le C\Big\|\sup_{0<t<\theta r_j^2} |S_t(\varphi_j f)(x)|\Big\|_{L^1(d\mu(x))}, \tag{7.20}$$

with a constant $\theta > 2$ independent of j. If $x \notin B_j^*$ and $r \le l$, then, as above, $K(r,x,y) = \mu\big(B(x,\sqrt{t})\big)h_t(x,y)$, where $\mu\big(B(x,\sqrt{t})\big) = r \le l$. Set

$$\tau_j(x) = r_j\big(|x - x_j|/r_j\big)^{(\gamma-\nu)/\gamma}.$$

One can prove using (2.3) and (2.4) that $t^{1/2} \le C\tau_j(x) \le C|x-x_j|$. Therefore, by (2.3),

$$\begin{aligned}
|(\varphi_j f)^{(+,l)}(x)| &\le C \sup_{0<t\le C\tau(x)^2} \int_{\mathbb{R}^d} |(\varphi_j f)(y)| \frac{e^{-c|x-x_j|^2/t}}{\mu\big(B(x,\sqrt{t})\big)}\, d\mu(y) \\
&\le C\|\varphi_j f\|_{L^1(d\mu)} \sup_{0<t\le C\tau(x)^2} \frac{\mu\big(B(x,|x-x_j|)\big)e^{-c|x-x_j|^2/t}}{\mu\big(B(x,\sqrt{t})\big)\mu\big(B(x,|x-x_j|)\big)} \\
&\le C\|\varphi_j f\|_{L^1(d\mu)} \sup_{0<t\le C\tau(x)^2} \left(\frac{|x-x_j|}{\sqrt{t}}\right)^{\gamma} \frac{e^{-c|x-x_j|^2/t}}{\mu\big(B(x,|x-x_j|)\big)} \\
&\le C\|\varphi_j f\|_{L^1(d\mu)} \frac{e^{-(c'|x-x_j|^2/r_j^2)^{\nu/\gamma}}}{\mu\big(B(x,|x-x_j|)\big)}.
\end{aligned}$$

This leads to

$$\|(\varphi_j f)^{(+,l)}\|_{L^1((B_j^*)^c,d\mu)} \le C\|\varphi_j f\|_{L^1(d\mu)}. \tag{7.21}$$

This finishes the proof of Lemma 7.4. $\square$

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