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ARITHMETIC DIFFERENTIAL EQUATIONS AND E-FUNCTIONS

SAID MANJRA

ABSTRACT. Let K be a number field. We give an arithmetic characterization at infinity of the differential operator of $K[x, d/dx]$ with minimal degree in x annihilating a given E-function. Such an operator is called an *E-operator*.

1. Introduction

Let K be a number field and let V_0 be the set of all finite places v of K. For each $v \in V_0$ above a prime number $p = p(v)$, we normalize the corresponding v-adic absolute value so that $|p|_v = p^{-1}$ and we put $\pi_v = p^{-1/(p-1)}$. We denote by K_v the v-adic completion of K. We also fix an embedding $K \hookrightarrow \mathbb{C}$. For a real number $r > 0$, and a differential operator $\phi \in K[x, d/dx]$, we denote by $R_v(\phi, r)$ the generic radius of convergence, bounded above by r, of a basis of solutions of ϕ in a neighborhood of a v-adic generic point of absolute value r. Recall that the *Fourier transform* $\mathcal{F}$ is the K-automorphism of $K[x, d/dx]$ which satisfies $\mathcal{F}(x) = d/dx$ and $\mathcal{F}(d/dx) = -x$. A power series $g = \sum_{n\geq 0} a_n x^n \in K[[x]]$ (resp. $F = \sum_{n\geq 0} a_n x^n/n! \in K[[x]]$) is said to be a *$G$-function* (resp. an *$E$-function*) if there exists a positive constant C such that for any index n, the coefficient a_n and its conjugates over $\mathbb{Q}$ do not exceed C^n in absolute value, and if there exists a common denominator $d_m \geq 1$ for $a_0, \dots, a_n$ which does not exceed C^n. Chudnovsky proved in [CC] that the minimal differential operator of $K[x, d/dx]$ annihilating a given G-function satisfies the *Galochkin condition*. Such an operator is now called a *G-operator* [A1, IV]. E. Bombieri proved in 1982 that the differential operator which has G-function solutions near every regular singularity satisfies the condition $\prod_{v\in V_0} R_v(\phi, 1) \neq 0$ (called *Bombieri's condition*) [Bo, 10]. The equivalence between the condition of Galochkin and that of Bombieri was established in 1989 by Y. André [A1, IV]. In 2000, the latter showed that the differential

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operator of $K[x, d/dx]$ with minimal degree in x annihilating an E-function is the Fourier transform of a certain G-operator and he called such operators *E-operators* [A2, 4]. Recently, in joint work with Remmal [MR], we gave a local p-adic characterization of the E-operators in the neighborhood of 0, which is a regular singularity. This result is given in term of the generic radius of convergence and provides an answer to a conjecture of Y. André [A2, 4.7]. In the present paper, we propose a local arithmetic characterization of the E-operators at infinity (Theorem 3.1), which is in general an irregular singularity of such an operator. This result is the analogue of the *local Bombieri property* for the G-operators [CD, 6]. In the proof of this result, we cannot avoid the case of negative exponents as in [MR, 6]. This requires the standard Laplace transform instead of the formal one used in [MR, 5].

The importance of E-operators comes from the fact that if $y(x)$ is an *arithmetic Gevrey series of non-zero order s* and is a solution of a linear differential equation with coefficients in $K(x)$, then $y(x^{-s})$ is a solution of an E-operator (cf. [A2, 6]).

This article is organized as follows:

In Section 2, we start by giving some preliminaries which will be needed later. In Section 3, we state our main theorem (Theorem 3.1), we give some key lemmas and we prove that the conditions of Theorem 3.1 are necessary. Section 4 is devoted to the Laplace transform $\mathcal{L}$; in §4.1, we summarize main formal properties of $\mathcal{L}$. In §4.2, we give some arithmetic properties of $\mathcal{L}$. For a given differential operator $\psi \in K[x, d/dx]$, we see in Section 5 how we can determine the nature of solutions of $(d/dx)\psi$ at 0 from those of ψ^* at the same point. Using the results of Sections 3, 4 and 5, we prove, in Section 6, that the conditions of Theorem 3.1 are sufficient.

2. Notations and preliminaries

2.1. Differential modules. Let $\mathcal{K}$ be a commutative field equipped with a derivation ∂, let K be the constant field of ∂ in $\mathcal{K}$ and let μ be a positive integer. A differential $\mathcal{K}$-module $\mathcal{M}$ is a free module of rank μ over $\mathcal{K}$ equipped with a K-endomorphism ∇ of $\mathcal{M}$ which satisfies the condition $\nabla(am) = a\nabla(m) + \partial(a)m$ for any $m \in \mathcal{M}$ and $a \in \mathcal{K}$. To each basis $\{e_i\}$ of $\mathcal{M}$ over $\mathcal{K}$ corresponds a matrix $G = (G_{ij}) \in \mathrm{M}_\mu(\mathcal{K})$ satisfying

$$\nabla(e_i) = \sum_{j=1}^{\mu} G_{ij} e_j,$$

called the matrix of ∂ with respect to the basis $\{e_i\}$ (or simply the associated matrix of $\mathcal{M}$), and a differential system $\partial X = GX$, where X denotes a column vector $\mu \times 1$ or $\mu \times \mu$ matrix. A change of bases in $\mathcal{M}$ results in the existence of a matrix $Y \in \mathrm{GL}_\mu(\mathcal{K})$ such that $Y[G] := YGY^{-1} + \partial(Y)Y^{-1}$ is the associated matrix of ∂ in the new basis. If

$$\phi = \sum_{i=0}^{\mu} a_i \partial^i \in \mathcal{K}[\partial]$$

is a differential operator such that $a_\mu \neq 0$, one can associates to it the differential $\mathcal{K}$-module $\mathcal{M}_\phi = \mathcal{K}[\partial]/\mathcal{K}[\partial]\phi$ of rank μ, which corresponds to a system

$$\partial X = A_\phi X,$$

where

$$A_\phi := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ -\frac{a_0}{a_\mu} & \frac{a_1}{a_\mu} & -\frac{a_2}{a_\mu} & \dots & -\frac{a_{\mu-1}}{a_\mu} \end{pmatrix}$$

is called the companion matrix of ϕ. One associates to ϕ the adjoint operator $\phi^* = \sum_{i=0}^{\mu} (-\partial)^i a_i$. One verifies that $-^T A_\phi$ is associated to $\mathcal{M}_{\phi^*} = \mathcal{K}[\partial]/\mathcal{K}[\partial]\phi^*$. More generally, G is associated to $\mathcal{M}_\phi = \mathcal{K}[\partial]/\mathcal{K}[\partial]\phi$ if and only if $-^T G$ is associated to $\mathcal{M}_{\phi^*}$. This comes from the fact that for any $Y \in \mathrm{GL}_\mu(\mathcal{K})$ one has

$$\begin{aligned} (2.1) \qquad -^T(Y[A_\phi]) &=^T Y^{-1}(-^T A_\phi)^T Y -^T Y^{-1}\partial(^T Y) \\ &=^T Y^{-1}(-^T A_\phi)^T Y + \partial(^T Y^{-1})^T Y = (^T Y^{-1})[-^T A_\phi]. \end{aligned}$$

2.2. The Newton-Ramis polygon. Let

$$\phi = \sum_{i=0}^{\mu} a_i(x) \Big(\frac{d}{dx}\Big)^i = \sum_{i=0}^{\mu}\sum_{j=0}^{\nu} a_{i,j} x^j \Big(\frac{d}{dx}\Big)^i \in K[x, \frac{d}{dx}]$$

be a differential operator of rank μ. The Newton polygon in the sense of Ramis of ϕ, which we shall denote by $NR(\phi)$, is the convex hull, in the plane uv, of the horizontal half-lines $\{u \leq i,\ v = j - i \mid a_{i,j} \neq 0\}$ (cf. [Ra]).

With this definition, it is easy to check that $NR(\overline{\phi}) = NR(\phi)$ (where $\overline{\phi}$ denotes the operator obtained from ϕ by the change of variable $x \to -x$). Also, $NR(\phi)$ has a non-vertical side if and only if a_μ is a monomial, in which case ϕ has no non-zero finite singularity.

The part of $NR(\phi)$ located in the half-plane $v \leq \mathrm{ord}_x(a_\mu)$ corresponds to the classic Newton polygon $N(\phi)$ of ϕ. As for the part of $NR(\phi)$ located in the half-plane $v \geq deg(a_\mu)$, it corresponds, by translation, to the transform, by the symmetry $(u, v) \longrightarrow (u, -v)$, of the Newton polygon $N(\phi_\infty)$ of the operator ϕ_∞ obtained from ϕ by the change of variable $x \to 1/x$ (cf. [Ma, V]). The slopes of $N(\phi_\infty)$ are called the slopes of ϕ at infinity.

This implies that the non-vertical slopes of $NR(\phi)$ depend only of $\mathcal{M}_\phi$, since $N(\phi)$ and $N(\phi_\infty)$ depend only of $\mathcal{M}_\phi$ (cf. [VS, 3.3.3]).

The polygon of $\mathcal{F}(\phi)$ may be obtained from $NR(\phi)$ by applying to it the transformation $(u,v) \to (u+v,-v)$ (cf. [Ma, V]). This implies, in particular, the following: *$NR(\phi)$ has no non-zero finite slopes if and only if all slopes of $NR(\mathcal{F}(\phi))$ lie in $\{0,-1\}$.*

If $A_1,\dots,A_\ell$ are square matrices, we denote by $\oplus_{1\le i\le \ell} A_i$ the block diagonal matrix

$$\oplus_{1\le i\le \ell} A_i = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_\ell \end{pmatrix}$$

with blocks $A_1,\dots,A_\ell$ on the diagonal.

2.3. Radius of convergence in neighborhoods of singularities. Consider the differential field $\mathcal{K} = K(x)$ equipped with the derivation $\partial = d/dx$. Let ϕ be a differential operator of rank μ such that the slopes of $N(\phi)$ lie in $\{0,1\}$ and let $G \in \mathrm{M}_\mu(K(x))$ be an associated matrix of $\mathcal{M}_\phi$. The *Turrittin-Levelt* decomposition states that there exist a finite extension K' of K, a matrix $Y_0(x) \in \mathrm{GL}_\mu(K'((x)))$, called a reduction matrix of G (or simply of ϕ if $G = A_\phi$) at 0, an upper triangular matrix $C_0 \in \mathrm{M}_\mu(K')$ and a diagonal matrix $\Delta_0 \in \mathrm{M}_\mu(K')$ commuting with C_0 such that $Y_0(x)[G(x)] = \Delta_0/x^2 + C_0/x$ [Le, 3]. By base change, we may assume that C_0 is in Jordan form.

One observes that the matrix $Y_0(x)^{-1}x^{C_0}\exp(-\Delta_0/x)$ is a solution of the system $\frac{d}{dx}X = G(x)X$. In the particular case where $G = A_\phi$, the first line of $Y_0(x)^{-1}x^{C_0}\exp(-\Delta_0/x)$ form a basis of solutions of ϕ at 0.

$\Delta_0 = 0$ means that $NR(\phi)$ has non-positive slopes. In this case, $\mathcal{M}_\phi$ and ϕ are both called regular at 0, the solutions of ϕ at 0 are called logarithmic, and one verifies that the eigenvalues of C_0 modulo $\mathbb{Z}$ depend only on $\mathcal{M}_\phi$ and are called *exponents* of $\mathcal{M}_\phi$ and ϕ at 0. According to what precedes, if the slopes of $NR(\phi)$ at infinity are in $\{0,1\}$, there exist a finite extension K' of K, a matrix $Y_\infty(x) \in \mathrm{GL}_\mu(K'((x)))$, called a reduction matrix of A_ϕ (or of ϕ) at infinity, an upper triangular matrix $C_\infty \in \mathrm{M}_\mu(K')$ and a diagonal matrix Δ_∞ commuting with C_∞ such that $Y_\infty(\frac{1}{x})[A_\phi(x)] = -\Delta_\infty - \frac{1}{x}C_\infty$. In this case, $Y_\infty(\frac{1}{x})^{-1}(\frac{1}{x})^{C_\infty}\exp(-\Delta_\infty x)$ is a solution of the system $\frac{d}{dx}X = A_\phi(x)X$ at infinity. The exponents of $\mathcal{M}_\phi$ at infinity are those of $\mathcal{M}_{\phi_\infty}$ at 0.

By extension, we attribute to ϕ the properties that $\mathcal{M}_\phi$ has. Then one observes, from §2.1, that ϕ is regular at 0 (resp. infinity) if and only if ϕ^* is regular at the same point, in which case the exponents of ϕ^* at 0 (resp. at infinity) are those of ϕ but with the opposite sign.

In the sequel, we assume K is sufficiently large so that we can take $K' = K$. Also, for any matrix Y of $\mathrm{M}_\mu(K((x)))$ and any finite place v of V_0, we denote by $r_v(Y)$ the upper bound of the reals $r > 0$ for which all entries of Y are analytic in the punctured open disc $D(0,r^-)\setminus\{0\}$ of K_v. If $Y \in$

$\mathrm{GL}_\mu(K((x)))$, we put $R_v(Y) = \min(r_v(Y), r_v(Y^{-1}))$. We end this subsection with the following result due to F. Baldassarri (See Theorem 2 of [Ba, III]):

PROPOSITION 2.1. *If $Y(x)$ is a reduction matrix of a $K(x)$-module $\mathcal{M}$ at 0 or at infinity, then $R_v(Y)$ is non-zero for each finite place v of V_0.*

2.4. $\mathcal{E}$-functions. A formal power series $f = \sum_{n\geq 0} a_n x^n \in K[[x]]$ is said to be an *$\mathcal{E}$-function* if the power series $\sum_{n\geq 0} \frac{a_n}{n!} x^n$ is a G-function.

This definition is motivated by the fact that all power series occurring in the solutions of the E-operators at infinity are $\mathcal{E}$-functions (see Theorem 2.3 below). A simple example of these power series is the Euler series $\sum_{n\geq 0} (-1)^n n! x^n$.

We suppose in the sequel that K contains all the coefficients of G-functions, E-functions and $\mathcal{E}$-functions that we shall meet thereafter.

The *Pochhammer symbol* $(\alpha)_n$ stands for $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$. With this notation, Theorem 2 of [Cl] shows that for any finite place v of K above a prime number p, if α is either an integer ≥ 1 or a non-integer rational number of denominator prime to p, then

$$\lim_{n\to\infty} |(\alpha)_n|_v^{1/n} = \pi_v = p^{-1/(p-1)}. \tag{2.2}$$

For the special case $\alpha = 1$, we get

$$\lim_{n\to\infty} |n!|_n^{1/n} = \pi_v \tag{2.3}$$

Combining this equality with the remark below, we find that any $\mathcal{E}$-function f satisfies

$$\prod_{v\in V_0} \min(r_v(f)\pi_v, 1) \neq 0. \tag{2.4}$$

REMARK 2.2 ([A1, p. 126]). If g is a G-function, then $\prod_{v\in V_0} \min(r_v(g), 1) \neq 0$.

2.5. G-operators and E-operators. We will give here an equivalent definition of the G-operators, called the *local Bombieri property* [A1], which will be useful for the proof of our main theorem.

DEFINITIONS.

(1) An operator ϕ of $K[x, d/dx]$ of rank ν is said to be a G-operator if the differential system $dX/dx = A_\phi X$ (where A_ϕ is the companion matrix of ϕ defined in §2.1) has a solution at 0 of the form $Y(x)x^C$, where $Y(x)$ is a $\nu\times\nu$ invertible matrix with entries in $K((x))$ such that $\prod_{v\in V_0} \min(R_v(Y_\phi), 1) \neq 0$, and where C is a $\nu\times\nu$ matrix with entries in K and with eigenvalues in $\mathbb{Q}$.

(2) An operator $\psi \in K[x, d/dx]$ is said to be E-operator if it is the Fourier transform of a G-operator.

Combining the condition of Bombieri, mentioned in the Introduction, with the properties of the functions of generic radius of convergence, we obtain that if $\phi \in K[x, d/dx]$ is a G-operator, then:

(1) $\overline{\phi}$ and ϕ^* are also G-operators.
(2) ϕ has only regular singularities with rational exponents.

From the fact that $\mathcal{F}(\phi^*) = \left(\overline{\mathcal{F}(\phi)}\right)^*$ (cf. [Ma, V.3.6]), the first statement implies that if ψ is an E-operator then $\overline{\psi}$ and ψ^* are also E-operators. The second statement means that the Newton-Ramis polygon of any G-operator has no slope other than 0 and ∞, and hence, from §2.2, that the slopes of Newton-Ramis polygon of any E-operator are in $\{0, -1\}$.

The following theorem, due to André, describes the nature of solutions of the E-operators at 0 and at infinity:

Theorem 2.3 ([A2]). *Let ψ be an E-operator of rank μ. Then:*

(1) *The slopes of $NR(\psi)$ lie in $\{-1, 0\}$.*
(2) *ψ admits a basis of solutions at 0 of the form*

$$(F_1, \dots, F_\mu) x^{\Gamma_0},$$

where the F_i are E-functions, where Γ_0 is a $\mu \times \mu$ upper triangular matrix with elements in $\mathbb{Q}$.

(3) *ψ admits a basis of solutions at infinity of the form*

$$\left(f_1\left(\frac{1}{x}\right), \dots, f_\mu\left(\frac{1}{x}\right)\right)\left(\frac{1}{x}\right)^{\Gamma} \exp(-\Delta x),$$

where the f_i are $\mathcal{E}$-functions, where Γ is a $\mu \times \mu$ upper triangular matrix with elements in $\mathbb{Q}$, and where Δ is a $\mu \times \mu$ diagonal matrix with elements in K which commutes with Γ.

3. The main theorem

Before stating the main theorem, we recall that for a given differential operator ψ, A_ψ denotes its companion matrix (see §2.1).

Theorem 3.1. *Let ψ be a differential operator of $K[x, d/dx]$. Then ψ is an E-operator if and only if ψ satisfies the following conditions:*

(1) *The coefficients of ψ are not all in K.*
(2) *The slopes of $NR(\psi)$ lie in $\{-1, 0\}$.*
(3) *The differential system $dZ/dx = A_\psi Z$ has a solution of the from*

$$Y\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)^{\Gamma} \exp(-\Delta x),$$

where $Y(x)$ is a $\mu \times \mu$ invertible matrix with entries in $K((x))$ such that $\prod_{v \in V_0} \min(R_v(Y)\pi_v, 1) \neq 0$, where Γ is a $\mu \times \mu$ matrix with entries in

K and with eigenvalues in $\mathbb{Q}$, and where Δ is a $\mu\times\mu$ diagonal matrix with entries in K which commutes with Γ.

The first condition of this theorem is necessary by the definition of the E-operators. Theorem 2.3 above shows that the second one is also necessary. In §3.2 we will prove that the third one is also necessary. The fact that these conditions are sufficient is postponed to Section 6. In the following subsection, we give some preliminary results which will be useful in the rest of this paper.

3.1. Preliminary results. Throughout this subsection, $\phi = a_\mu(d/dx)^\mu + \cdots + a_0$ denotes a differential operator of $K[x, d/dx]$ of rank $\mu \in \mathbb{Z}_{>0}$, $\overline{\phi}$ denotes the differential operator obtained from ϕ by the change of variable $x \to -x$, Γ_1 and Γ_2 denote two $\mu\times\mu$ matrices with entries in K, Δ_1 and Δ_2 denote two $\mu\times\mu$ diagonal matrices with entries in K such that $\Gamma_1\Delta_1 = \Delta_1\Gamma_1$ and $\Gamma_2\Delta_2 = \Delta_2\Gamma_2$, and $y_1, \dots, y_\mu, z_1, \dots, z_\mu$ denote power series of $K((x))$.

Lemma 3.2. *Let G be a $\mu\times\mu$ matrix with entries in K, and let Y_1 and Y_2 be two matrices of $\mathrm{GL}_\mu(K((x)))$ such that $Y_1[G] = \frac{\Delta_1}{x^2} + \frac{\Gamma_1}{x}$ and $Y_2[G] = \frac{\Delta_2}{x^2} + \frac{\Gamma_2}{x}$. Then:*

1. *The matrices Δ_1 and Δ_2 are similar.*
2. *$Y_1Y_2^{-1}[\frac{\Gamma_2}{x}] = \frac{\Gamma_1}{x}$ and $Y_1Y_2^{-1} \in \mathrm{GL}_\mu(K[x, 1/x])$.*
3. *The eigenvalues of Γ_1 coincide, modulo $\mathbb{Z}$, with those of Γ_2.*

Proof. Let $a = (a_1, \dots, a_\mu) \in K^\mu$. Put, for $i = 1, 2$,

$$\mathrm{E}_i(a) = \{\mathrm{v} \in K^\mu \mid \Delta_i \mathrm{v} = a_j \mathrm{v}, \quad 1 \le j \le \mu\},$$

and

$$\Sigma_i = \{a \in K^\mu \mid \mathrm{E}_i(a) \neq 0\}.$$

Then

$$K^\mu = \bigoplus_{a\in\Sigma_i} \mathrm{E}_i(a).$$

Moreover, Γ_i commutes with the projection $K^\mu \longrightarrow \mathrm{E}_i(a)$. Thus, Γ_i can be written as

$$\Gamma_i = \bigoplus_{a\in\Sigma_i} \Gamma_i(a),$$

where

$$\Gamma_i(a) \in \mathrm{M}_{\dim_K(\mathrm{E}_i(a))}(K).$$

In addition, by hypothesis, we have

$$Y_1Y_2^{-1}\left[\frac{\Delta_2}{x^2} + \frac{\Gamma_2}{x}\right] = \frac{\Delta_1}{x^2} + \frac{\Gamma_1}{x}.$$

According to Proposition 6.4 of [BV] we have:

1. The matrices Δ_1 and Δ_2 are similar.
2. $\Sigma := \Sigma_1 = \Sigma_2, \quad \dim_K(\mathrm{E}_1(a)) = \dim_K(\mathrm{E}_2(a)) \quad$ for any $\quad a \in \Sigma$.

(3) $Y := Y_1 Y_2^{-1} = \bigoplus_{a\in\sum} Y(a)$, such that $Y(a) \in \mathrm{M}_{\dim_K(\mathrm{E}_1(a))}(K((x)))$ and $Y(a)[\Gamma_2(a)/x] = \Gamma_1(a)/x$ for any $a \in \Sigma$.

Thus, $Y[\Gamma_2/x] = \Gamma_1/x$, and hence the eigenvalues of Γ_1 coincide, modulo $\mathbb{Z}$, with those of Γ_2 (cf. [DGS, III.8]). Moreover, for any $a \in \Sigma$, we have

$$x\frac{d}{dx}Y(a) = \Gamma_1(a)Y(a) - Y(a)\Gamma_2(a).$$

Therefore, if we write

$$Y(a) = \sum_{m\in\mathbb{Z}} Y(a)_m x^m,$$

we obtain for any $m \in \mathbb{Z} \setminus \{0\}$,

$$mY(a)_m = \Gamma_1(a)Y(a)_m - Y(a)_m\Gamma_2(a).$$

But the eigenvalues of the maps

$$\begin{aligned} T_m(a) : \mathrm{M}_\mu(\overline{\mathbb{Q}}) &\longrightarrow \mathrm{M}_\mu(\overline{\mathbb{Q}}) \\ X &\longmapsto \Gamma_1(a)X - X\Gamma_2(a) - mX \end{aligned}$$

are of the form $\lambda(a) - \gamma(a) - m$, where $\lambda(a)$ and $\gamma(a)$ are, respectively, eigenvalues of $\Gamma_1(a)$ and of $\Gamma_2(a)$. This means that $T_m(a)$ is invertible, except possibly for a finite set of integers m. Hence $Y(a)_m$ is zero except for a finite set of integers m and the conclusion follows. □

Corollary 3.3. *Let $(y_1, \ldots, y_\mu)x^{\Gamma_1}\exp(\Delta_1/x)$ be a basis of solutions of ϕ at 0. Then ϕ has a basis of solutions at 0 of the form $(\widetilde{y}_1, \ldots, \widetilde{y}_\mu)x^{\widetilde{\Gamma}}\exp(\widetilde{\Delta}/x) = (\xi_1, \ldots, \xi_\mu)$, where:*

(1) *$\widetilde{y}_1, \ldots, \widetilde{y}_\mu$ are formal power series of $K((x))$ and are $K[x, 1/x]$-linear combinations of $y_1, \ldots, y_\mu$ and of their derivatives.*
(2) *$\widetilde{\Gamma}$ is a $\mu\times\mu$ matrix, in Jordan form, whose entries lie in K and whose eigenvalues coincide, modulo $\mathbb{Z}$, with those of Γ_1.*
(3) *$\widetilde{\Delta}$ is a $\mu \times \mu$ diagonal matrix similar to Δ_1.*

Moreover, if $\gamma_1, \ldots, \gamma_\mu$ denote the eigenvalues of Γ_1 and $\delta_1, \ldots, \delta_\mu$ denote the diagonal terms of Δ_1, then $\xi_1, \ldots, \xi_\mu$ lie in

$$\Big\langle \widetilde{y}_i\, x^{\gamma_j}\, (\ln x)^{k-1}\exp(\delta_\ell/x), \quad 1 \le i, j, k, \ell \le \mu \Big\rangle_{K[x,1/x]}.$$

Proof. Let W_1 be the Wronskian matrix of $(y_1, \ldots, y_\mu)x^{\Gamma_1}\exp(\Delta_1/x)$. Thus, W_1 is a solution of the system $dX/dx = A_\phi X$. Moreover, W_1 can be written in the form $Y_1x^{\Gamma_1}\exp(\Delta_1/x)$, where Y_1 is a matrix in $\mathrm{GL}_\mu(K((x)))$ whose entries are $K[x, 1/x]$-linear combinations of $y_1, \ldots, y_\mu$ and of their derivatives. Therefore, $Y_1^{-1}[A_\phi] = -\Delta_1/x^2 + \Gamma_1/x$. The Turrittin-Levelt decomposition states, in this case, that there exists a $\mu \times \mu$ invertible matrix $\widetilde{Y} = (\widetilde{y}_{ij}) \in \mathrm{GL}_\mu(K((x)))$, a $\mu \times \mu$ matrix $\widetilde{\Gamma}$ in Jordan form with entries in K, and a $\mu \times \mu$ diagonal matrix $\widetilde{\Delta} = (\widetilde{\delta}_{ij})$ with entries in K commuting with

$\widetilde{\Gamma}$ such that $\widetilde{Y}^{-1}[A_\phi] = -\widetilde{\Delta}/x^2 + \widetilde{\Gamma}/x$. Hence, by the previous lemma, the matrices Δ_1 and $\widetilde{\Delta}$ are similar, the eigenvalues of $\widetilde{\Gamma}$ coincide, modulo $\mathbb{Z}$, with those of Γ_1, and there exists $L \in \mathrm{GL}_\mu(K[x, 1/x])$ such that $\widetilde{Y} = LY_1$. In particular, the entries $(\widetilde{y}_{ij})$ of $\widetilde{Y}$ are $K[x, 1/x]$-linear combinations of $y_1, \dots, y_\mu$ and of their derivatives. In addition, since the matrix $\widetilde{Y}x^{\widetilde{\Gamma}}\exp(\widetilde{\Delta}/x)$ is a solution of the system $dX/dx = A_\phi X$, it is the Wronskian matrix of the μ-tuple $(\widetilde{y}_{11}, \dots, \widetilde{y}_{1\mu})x^{\widetilde{\Gamma}}\exp(\widetilde{\Delta}/x)$. Thus, the coefficients of $(\widetilde{y}_{11}, \dots, \widetilde{y}_{1\mu})x^{\widetilde{\Gamma}}\exp(\widetilde{\Delta}/x)$ form a basis of solutions of ϕ at 0. Hence, by putting $\widetilde{y}_i = \widetilde{y}_{1i}$ for $i = 1, \dots, \mu$, we find that $(\widetilde{y}_1, \dots, \widetilde{y}_\mu)x^{\widetilde{\Gamma}}\exp(\widetilde{\Delta}/x)$ is a basis of solutions of ϕ at 0 which meets the conditions (1), (2) and (3) of Corollary 3.3. On the other hand, by hypothesis, $\widetilde{\Gamma} = (\widetilde{\gamma}_{ij})$ is of the form $D + N$, where D is a diagonal matrix and N is a nilpotent upper triangular matrix such that $DN = ND$ and $N^\mu = 0$. Thus,

$$x^{\widetilde{\Gamma}} = x^{D+N} = x^D \sum_{0 \le k \le \mu-1} \frac{N^k}{k!}(\ln x)^k = x^D + x^D \sum_{1 \le k \le \mu-1} \frac{N^k}{k!}(\ln x)^k.$$

Therefore, $\xi_1 = \widetilde{y}_1 x^{\widetilde{\gamma}_{11}}\exp(\widetilde{\delta}_{11}/x)$, and for all $2 \le i \le \mu$,

$$\xi_i = \Big(\widetilde{y}_i x^{\widetilde{\gamma}_{ii}} + \sum_{j=1}^{i-1} \widetilde{y}_j x^{\widetilde{\gamma}_{jj}} \sum_{1 \le k \le \mu-1} \frac{(N^k)_{ji}}{k!}(\ln x)^k\Big)\exp(\widetilde{\delta}_{ii}/x),$$

since $(N^k)_{ji} = 0$ for all $k \ge 1$ and all $1 \le i \le j \le \mu$. Hence the last statement of the corollary results from the fact that Δ_1 and $\widetilde{\Delta}$ are similar and that $\widetilde{\Gamma}$ and Γ_1 have the same eigenvalues modulo $\mathbb{Z}$. $\square$

Corollary 3.4. *Let $(y_1, \dots, y_\mu)x^{\Gamma_1}\exp(\Delta_1/x)$ (respectively $(z_1, \dots, z_\mu)x^{\Gamma_2}\exp(\Delta_2/x)$) be bases of solutions of ϕ (resp. ϕ^*) at 0. Then the differential system $dX/dx = A_\phi X$ has a solution of the form $Y(x)x^{\Gamma_1}\exp(\Delta_1/x)$, where Y is a $\mu \times \mu$ invertible matrix such that the entries of Y (resp. of Y^{-1}) are $K[x, 1/x]$-linear combinations of $y_1, \dots, y_\mu$ (resp. of $z_1, \dots, z_\mu$) and of their derivatives. Moreover, the matrices Δ_1 and $-\Delta_2$ are similar, and the eigenvalues of Γ_1 are those of $-\Gamma_2$ modulo $\mathbb{Z}$.*

Proof. Let $\phi = a_\mu(d/dx)^\mu + \cdots + a_0$. Since $(y_1, \dots, y_\mu)x^{\Gamma_1}\exp(\Delta_1/x)$ is a basis of solutions of ϕ at 0, the Wronskian matrix W of the μ-tuple $(y_1, \dots, y_\mu)x^{\Gamma_1}\exp(\Delta_1/x)$ is then a solution of the system $dX/dx = A_\phi X$. Moreover, W can be written in the form $Yx^{\Gamma_1}\exp(\Delta_1/x)$, where Y is a matrix of $\mathrm{GL}_\mu(K((x)))$ whose entries are $K[x, 1/x]$-linear combinations of $y_1, \dots, y_\mu$ and of their derivatives. Then we have, $Y^{-1}[A_\phi] = -\Delta_1/x^2 + \Gamma_1/x$, which means that Y^{-1} is a reduction matrix of ϕ at 0, or also

$$ {}^TY[-{}^TA_\phi] = {}^T\Delta_1 \frac{1}{x^2} - {}^T\Gamma_1 \frac{1}{x}. \tag{3.1}$$

In addition, the μ-tuple $a_\mu(x)(z_1,\ldots,z_\mu)x^{\Gamma_2}\exp(\Delta_2/x)$ is a basis of solutions of $\phi^* a_\mu^{-1} = (a_\mu^{-1}\phi)^*$ at 0. Therefore, the matrix U whose rows $u_1,\ldots,u_\mu$ are defined recursively by

$$u_\mu = a_\mu(x)(z_1,\ldots,z_\mu)x^{\Gamma_2}\exp(\Delta_2/x),$$
$$u_{\mu-i} = \frac{a_{\mu-i}(x)}{a_\mu(x)}u_\mu - \frac{d}{dx}u_{\mu-i+1} \quad (1\le i\le \mu-1),$$

is a solution of the system $dX/dx = -{}^TA_\phi X$. Moreover, U may be written of the form $Zx^{\Gamma_2}\exp(\Delta_2/x)$, where Z is an invertible matrix $\mu\times\mu$ whose entries are $K[x,1/x]$-linear combinations of $z_1,\ldots,z_\mu$ and of their derivatives. Thus, we have

$$Z^{-1}[-{}^TA_\phi] = -\frac{1}{x^2}\Delta_2 + \frac{1}{x}\Gamma_2. \tag{3.2}$$

Thus, by formulae (3.1), (3.2) and Lemma 3.2, the matrices ${}^T\Delta_1(=\Delta_1)$ and $-\Delta_2$ are similar, the eigenvalues of ${}^T\Gamma_1$(which are also those of Γ_1) are those of $-\Gamma_2$ modulo $\mathbb{Z}$, and there exists $L\in \mathrm{GL}_\mu(K[x,1/x])$ such that ${}^TY = LZ^{-1}$. Consequently, the entries of Y^{-1} are $K[x,1/x]$-linear combinations of $z_1,\ldots,z_\mu$ and of their derivatives. The conclusion follows. $\square$

LEMMA 3.5. *Let $(y_1(x),\ldots,y_\mu(x))x^{\Gamma_1}\exp(\Delta_1/x)$ be a basis of solutions of ϕ at* 0. *Then $(y_1(-x),\ldots,y_\mu(-x))x^{\Gamma_1}\exp(-\Delta_1/x)$ is a basis of solutions of $\overline{\phi}$ at* 0.

Proof. Let W be the Wronskian matrix of $(y_1(x),\ldots,y_\mu(x))x^{\Gamma_1}\exp(\Delta_1/x)$. Thus W can be written in the form $Y(x)x^{\Gamma_1}\exp(\Delta_1/x)$, where $Y(x)$ is a $\mu\times\mu$ invertible matrix with entries in $K((x))$. Thus $Y^{-1}(x)[A_\phi] = -\Delta_1/x^2+\Gamma_1/x$. By the change of variable $x\to -x$, we find

$$Y^{-1}(-x)A_\phi(-x)Y(-x) + Y^{-1}(-x)\frac{d}{dx}(Y(-x)) = -\frac{\Delta_1}{x^2} - \frac{\Gamma_1}{x}.$$

Thus $Y(-x)x^{\Gamma_1}\exp(-\Delta_1/x)$ is solution of the system $\frac{d}{dx}X = -A_\phi(-x)X$. Consequently, the μ-tuple $(y_1(-x),\ldots,y_\mu(-x))x^{\Gamma_1}\exp(-\Delta_1/x)$ is a basis of solutions of $\overline{\phi}$ at 0. $\square$

LEMMA 3.6. *Let $\phi = a_\mu(d/dx)^\mu+\cdots+a_0\in K[x,d/dx]$. Let W be a $\mu\times\mu$ invertible matrix with entries in some Picard-Vessiot extension of K. If W is a solution of $dX/dx = A_\phi X$ at* 0, *then the elements of $\mu-th$ row of $a_\mu^{-1}({}^TW^{-1})$ form a basis of solutions of ϕ^* at* 0.

Proof. Let $W_1,\ldots,W_\mu$ denote the rows of ${}^TW^{-1}$. Since W is a solution of $dX/dx = A_\phi X$ at 0, these rows are then related by

$$W_{\mu-i} = \frac{a_{\mu-i}}{a_\mu}W_\mu - \frac{d}{dx}W_{\mu-i+1}, \quad (1\le i\le\mu-1),$$

and the elements of W_μ are solutions of $\phi^* a_\mu^{-1} = (a_\mu^{-1}\phi)^*$ at 0. We get therefore, by induction on the index i,

$$W_i \in \Big\langle W_\mu, \dots, \Big(\frac{d}{dx}\Big)^\mu (W_\mu)\Big\rangle_{K[x,a_\mu^{-1}]} \quad (1 \le i \le \mu - 1).$$

In addition, since W is an invertible matrix, the elements of W_μ are then linearly independent over K, and hence they form a basis of solutions of $\phi^* a_\mu^{-1}$ at 0, and the conclusion follows. $\square$

LEMMA 3.7. *Let $Y(x)x^{\Gamma_1}\exp(\Delta_1/x)$ be a solution of $dX/dx = A_\phi(x)X$ at 0, where $Y(x) = (y_{ij}(x)) \in \mathrm{GL}_\mu(K((x)))$. Assume that the leading coefficient a_μ of ϕ is a monomial. Then there exists $\widetilde{Y}(x) \in \mathrm{GL}_\mu(K((x)))$ such that $\widetilde{Y}(x)x^{-{}^T\Gamma_1}\exp(\Delta_1/x)$ is a solution of $dX/dx = A_{\overline{\phi^*}}(x)X$ at 0, and $R_v(\widetilde{Y}) \ge R_v(Y)$ for all $v \in V_0$.*

Proof. First, it is easy to check that $Y(x)x^{\Gamma_1}\exp(\Delta_1/x)$ is the Wronskian matrix of the elements of $(y_{11}(x), \dots, y_{1\mu}(x))x^{\Gamma_1}\exp(\Delta_1/x)$ and that the elements of $(y_{11}(x), \dots, y_{1\mu}(x))x^{\Gamma_1}\exp(\Delta_1/x)$ is a basis of ϕ at 0.

In addition, if we write $Y^{-1}(x) = (\widetilde{y}_{ij}(x))$, we find, by Lemma 3.6, that $a_\mu^{-1}(x)((\widetilde{y}_{1\mu}(x), \dots, \widetilde{y}_{\mu\mu}(x))x^{-{}^T\Gamma_1}\exp(-\Delta_1/x)$ is a basis of solutions ϕ^* at 0. According to Lemma 3.5, the μ-tuples $(y_{11}(-x), \dots, y_{1\mu}(-x))x^{\Gamma_1}\exp(-\Delta_1/x)$ and $a_\mu^{-1}(-x)((\widetilde{y}_{1\mu}(-x), \dots, \widetilde{y}_{\mu\mu}(-x))x^{-{}^T\Gamma_1}\exp(\Delta_1/x)$ are, respectively, bases of solutions of $\overline{\phi} = (\overline{\phi^*})^*$ and $\overline{\phi^*}$ at 0. Finally, since a_μ^{-1} is a monomial, Lemma 3.4 states that $dX/dx = A_{\overline{\phi^*}}(x)X$ has a solution at 0 in the form $\widetilde{Y}(x)x^{-{}^T\Gamma_1}\exp(\Delta_1/x)$, where $\widetilde{Y}(x)$ is a $\mu \times \mu$ invertible matrix such that the entries of $\widetilde{Y}(x)$ (resp. of $\widetilde{Y}^{-1}(x)$) are $K[x, 1/x]$-linear combinations of $\widetilde{y}_{1\mu}(-x), \dots, \widetilde{y}_{\mu\mu}(-x)$ (resp. of $y_{11}(-x), \dots, y_{1\mu}(-x)$) and of their derivatives. Hence, for all $v \in V_0$, we have $R_v(\widetilde{Y}) = \min_{1\le i,j\le\mu}\{r_v(y_{1i}), r_v(\widetilde{y}_{j\mu})\} \ge R_v(Y)$. The conclusion follows. $\square$

3.2. Necessary conditions. We conclude this section by proving that the second condition of Theorem 3.1 is necessary:

THEOREM 3.8. *Let ψ be an E-operator of $K[x, d/dx]$ of rank μ. Then, the differential system $d/dxZ = A_\psi Z$ has a solution of the from*

$$Y\Big(\frac{1}{x}\Big)\Big(\frac{1}{x}\Big)^\Gamma \exp(-\Delta x),$$

where $Y(x)$ is a $\mu \times \mu$ invertible matrix with entries in $K((x))$ such that $\prod_{v\in V_0}\min(R_v(Y)\pi_v, 1) \ne 0$, where Γ is a $\mu \times \mu$ upper triangular matrix with entries in $\mathbb{Q}$, and where Δ is a $\mu \times \mu$ diagonal matrix with entries in K which commutes with Γ.

Proof. According to §2.5, the operator ψ^* is also an E-operator. Combining this with Theorem 3.1 and Corollary 3.4 (applied at infinity), we observe that the differential system $d/dxZ = A_\psi Z$ has a solution of the from

$$Y\Big(\frac{1}{x}\Big)\Big(\frac{1}{x}\Big)^{\Gamma}\exp(-\Delta x),$$

where $Y(x)$ is a $\mu\times\mu$ invertible matrix such that the entries of $Y(x)=(y_{ij})$ and those of $Y(x)^{-1}=(\widetilde{y}_{kl})$ are $K[x,1/x]$-linear combinations of $\mathcal{E}$-functions and of their derivatives, where Γ is a $\mu\times\mu$ upper triangular matrix with entries in $\mathbb{Q}$, and where Δ is a $\mu\times\mu$ diagonal matrix with entries in K which commutes with Γ. Thus, by (2.4), we have

$$\begin{aligned}\prod_{v\in V_0}\min(R_v(Y_\psi)\pi_v,1) &= \prod_{v\in V_0}\Big(\min(\min_{i,j}(r_v(y_{ij})\pi_v),\min_{k,l}(r_v(\widetilde{y}_{kl})\pi_v),1)\Big)\\ &\geq \prod_{v\in V_0}\Big(\prod_{ij}\min(r_v(y_{ij})\pi_v,1)\prod_{kl}\min(r_v(\widetilde{y}_{kl})\pi_v,1)\Big)\\ &\geq \prod_{ij}\prod_{v\in V_0}\min(r_v(y_{ij})\pi_v,1)\\ &\quad\times\prod_{kl}\prod_{v\in V_0}\min(r_v(\widetilde{y}_{kl})\pi_v,1)\neq 0. \qquad\square\end{aligned}$$

In the sequel, we fix an embedding of K into $\mathbb{C}$.

4. The Laplace transform

4.1. The Laplace transform $\mathcal{L}$. In this subsection, we summarize the main properties of the formal Laplace transform due in part to Y. André.

Let α be an element of K with real part >-1, let k and n be two nonnegative integers, and let $h_{\alpha,k}$ denote the function defined by $h_{\alpha,k}(x)=x^\alpha(\ln x)^k$; $x>0$. The standard Laplace transform of $h_{\alpha,0}$, denoted by $\mathcal{L}(h_{\alpha,0})$, is given by (cf. [DB, 2.30])

$$\mathcal{L}(h_{\alpha,0})(z)=\int_0^\infty e^{-zx}x^\alpha dx=\Gamma(\alpha+1)z^{-\alpha-1}. \tag{4.1}$$

This implies, in particular,

$$\Big(\frac{d}{d\alpha}\Big)^k\Big(\Gamma(\alpha+1)z^{-\alpha-1}\Big)=\int_0^\infty e^{-zx}x^\alpha(\ln x)^k dx=\mathcal{L}(h_{\alpha,k})(z).$$

The Leibniz formula gives

$$\Big(\frac{d}{d\alpha}\Big)^k\Big(\Gamma(\alpha+1)z^{-\alpha-1}\Big)=\sum_{j=0}^{k}\binom{k}{j}\Gamma^{(j)}(\alpha+1)z^{-\alpha-1}(-1)^{k-j}(\ln z)^{k-j}.$$

Thus,

$$\mathcal{L}(h_{\alpha,k})(z) = \sum_{j=0}^{k} \binom{k}{j} \Gamma^{(j)}(\alpha+1) z^{-\alpha-1} (-1)^{k-j} (\ln z)^{k-j}. \tag{4.2}$$

From the fact that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, we obtain, by induction on $j \geq 1$, the relations

$$\Gamma'(\alpha+1) = \Gamma(\alpha) + \alpha\Gamma'(\alpha) \text{ and } \quad \Gamma^{(j+1)}(\alpha+1) = j\Gamma^{(j)}(\alpha) + \alpha\Gamma^{(j+1)}(\alpha),$$

which implies

$$\mathcal{L}(h_{\alpha,k})(z) \in z^{-\alpha-1} \langle \Gamma(\alpha), \ldots, \Gamma^{(k)}(\alpha) \rangle_{\mathbb{Q}[\alpha, \ln z]}, \tag{4.3}$$

and, in the case where α is a non-zero positive integer, gives

$$\Gamma'(\alpha) = \alpha!. \tag{4.4}$$

On the other hand, the function $h_{\alpha,k}$ satisfies the following equalities (cf. [DB, 2.21, 2.40]):

$$\frac{d}{dz}\mathcal{L}(h_{\alpha,k})(z) = \mathcal{L}(-xh_{\alpha,k})(z), \tag{4.5}$$

$$\begin{aligned} \mathcal{L}\Big(\frac{d}{dx}h_{\alpha,k}\Big)(z) &= z\mathcal{L}(h_{\alpha,k})(z) + \lim_{x \longrightarrow 0^+} h_{\alpha,k}(x) \\ &= z\mathcal{L}(h_{\alpha,k})(z) \quad \text{if} \quad \Re e(\alpha) > 0. \end{aligned} \tag{4.6}$$

To extend the Laplace transform $\mathcal{L}$ of $h_{\alpha,k}$ to any α, we have to introduce the finite parts of $h_{\alpha,k}$ in the following manner: Putting

$$\Phi(x, \alpha, k) = \int h_{\alpha,k}(x)dx,$$

we find therefore

$$\Phi(x, \alpha, k) = \frac{x^{\alpha+1}}{\alpha+1} \sum_{\ell=0}^{k} \frac{(-1)^{k-\ell} k!}{(\alpha+1)^{k-\ell} \ell!} (\ln x)^{\ell} \quad \text{if} \quad \alpha \neq -1,$$

$$\Phi(x, -1, k) = \frac{(\ln x)^{k+1}}{k+1}.$$

The finite part of the integral

$$\int_0^x \sum_{\alpha,k} \lambda_{\alpha,k} h_{\alpha,k}(t)dt,$$

where the $\lambda_{\alpha,k}$ are complex numbers, is defined, for $x > 0$, by

$$\begin{aligned} \text{p.f.} \int_0^x \sum_{\alpha,k} \lambda_{\alpha,k} h_{\alpha,k}(t)dt &= \sum_{\alpha,k} \lambda_{\alpha,k} \lim_{\epsilon \longrightarrow 0^+} \Big(\Phi(\epsilon, \alpha, k) + \int_\epsilon^x h_{\alpha,k}(t)dt \Big) \\ &= \sum_{\alpha,k} \lambda_{\alpha,k} \Phi(x, \alpha, k). \end{aligned}$$

With this definition, we get

$$h^0_{\alpha,k} := \begin{cases} h_{\alpha,k} - \text{p.f.} \displaystyle\int_0^x \Big(\frac{d}{dt} h_{\alpha,k}(t)\Big) dt = 0 & \text{if} \quad (\alpha,k) \neq (0,0), \\ h_{0,0} & \text{otherwise.} \end{cases} \tag{4.7}$$

Now, fix $\alpha \in \mathbb{C}$, $k \in \mathbb{Z}_{\geq 0}$ and put for $n \in \mathbb{Z}_{\geq 0}$

$$F_n(x) = \text{p.f.} \int_0^x \frac{(x-t)^n}{n!} h_{\alpha,k}(t) dt.$$

Then,

$$F_n(x) = \begin{cases} \displaystyle\sum_{m=0}^{n} \frac{(-1)^m}{m!(n-m)!} \frac{x^{\alpha+n+1}}{m+\alpha+1} \sum_{\ell=0}^{k} \frac{k!(-1)^{k-\ell}}{\ell!(m+\alpha+1)^{k-\ell}} (\ln x)^\ell, \\ \qquad \text{if} \quad (-\alpha-1) \neq 0,1,\dots,n, \\ \displaystyle\sum_{\substack{m=0 \\ m \neq -\alpha-1}}^{n} \frac{(-1)^m}{m!(n-m)!} \frac{x^{\alpha+n+1}}{m+\alpha+1} \sum_{\ell=0}^{k} \frac{k!(-1)^{k-\ell}}{\ell!(m+\alpha+1)^{k-\ell}} (\ln x)^\ell \\ \qquad + \displaystyle\frac{(-1)^{\alpha+1}}{(-\alpha-1)!(n+\alpha+1)!} \, x^{\alpha+n+1} \, \frac{(\ln x)^{k+1}}{k+1}, \\ \qquad \text{otherwise.} \end{cases} \tag{4.8}$$

Moreover, these functions satisfy, for any integer $n \geq 1$, the equality (cf. [DB, 5.35])

$$\frac{d}{dx} F_n = F_{n-1}.$$

This means, by (4.6), that the function $z^{n+1}\mathcal{L}(F_n)$ is independent of the choice of n for $n \geq -\Re e(\alpha) - 1$. By this remark we can extend the Laplace transform to any α by putting

$$\mathcal{L}(h_{\alpha,k})(z) = z^{n+1}\mathcal{L}(F_n)(z), \quad \text{for} \quad n \geq -\Re e(\alpha) - 1. \tag{4.9}$$

For simplicity, we write $\mathcal{L}(.)$ instead of $\mathcal{L}(.)(z)$. Then, using formula (4.5) and the linearity of $\mathcal{L}$, it follows that for any α, k and $n \geq -\Re e(\alpha) - 1$,

(4.10)
$$\begin{aligned}\frac{d}{dz}(\mathcal{L}(h_{\alpha,k})) &= (n+1)z^n\mathcal{L}(F_n) + z^{n+1}\frac{d}{dz}(\mathcal{L}(F_n)\\ &= (n+1)z^n\mathcal{L}(F_n) + z^{n+1}\mathcal{L}(-xF_n)\\ &= (n+1)z^n\mathcal{L}(F_n) - z^{n+1}\mathcal{L}\Big((n+1)\text{p.f.}\int_0^x \frac{(x-t)^{n+1}}{(n+1)!}h_{\alpha,k}(t)dt\\ &\qquad\qquad - \text{p.f.}\int_0^x \frac{(x-t)^n}{n!}(-t)h_{\alpha,k}(t)dt\Big)\\ &= (n+1)z^n\mathcal{L}(F_n) - (n+1)z^{n+1}\mathcal{L}(F_{n+1}) + \mathcal{L}(-xh_{\alpha,k})\\ &= \mathcal{L}(-xh_{\alpha,k}).\end{aligned}$$

On the other hand, for $n \geq -\Re e(\alpha)$,

$$\begin{aligned}\mathcal{L}\Big(\frac{d}{dx}h_{\alpha,k}\Big) &= z^{n+1}\mathcal{L}\Big(\text{p.f.}\int_0^x \frac{(x-t)^n}{n!}\Big(\frac{d}{dt}h_{\alpha,k}(t)\Big)dt\Big)\\ &= z^{n+1}\mathcal{L}\Big(\text{p.f.}\int_0^x \frac{d}{dt}\Big(\frac{(x-t)^n}{n!}h_{\alpha,k}(t)\Big)dt\\ &\qquad + \text{ p.f.}\int_0^x \frac{(x-t)^{n-1}}{(n-1)!}h_{\alpha,k}(t)dt\Big)\\ &= z^{n+1}\mathcal{L}\Big(\text{p.f.}\int_0^x \frac{d}{dt}\Big(\frac{(x-t)^n}{n!}h_{\alpha,k}(t)\Big)dt\Big) + z\mathcal{L}(h_{\alpha,k}),\end{aligned}$$

where the last line results from (4.9). But

$$\text{p.f.}\int_0^x \frac{d}{dt}\Big(\frac{(x-t)^n}{n!}h_{\alpha,k}(t)\Big)dt = \sum_{m=0}^n \frac{(-1)^m x^{n-m}}{(n-m)!m!}\ \text{p.f.}\int_0^x \frac{d}{dt}\Big(t^m h_{\alpha,k}(t)\Big)dt.$$

We get therefore, by (4.7),

$$\begin{aligned}\text{p.f.}\int_0^x \frac{d}{dt}\Big(\frac{(x-t)^n}{n!}h_{\alpha,k}(t)\Big)dt &= \sum_{m=0}^n \frac{(-1)^m x^{n-m}}{(n-m)!m!}(h^0_{\alpha+m,k} - x^m h_{\alpha,k})\\ &= \sum_{m=0}^n \frac{(-1)^m x^{n-m}}{(n-m)!m!}(x^{n-m}h^0_{\alpha+m,k} - x^n h_{\alpha,k})\\ &= \sum_{m=0}^n \frac{(-1)^m x^{n-m}}{(n-m)!m!}h^0_{\alpha+m,k}.\end{aligned}$$

The last line results from the fact that

$$\sum_{0\leq m\leq n} \frac{(-1)^m}{(n-m)!m!} = 0.$$

Hence, by (4.7), we obtain

$$\text{p.f.} \int_0^x \frac{d}{dt}\Big(\frac{(x-t)^n}{n!} h_{\alpha,k}(t)\Big) dt = \begin{cases} 0 & \text{if} \quad \alpha \neq 0, -1, \dots, -n \quad \text{or} \quad k \neq 0, \\ \dfrac{(-1)^{-\alpha} x^{n+\alpha}}{(n+\alpha)!(-\alpha)!} & \text{otherwise.} \end{cases}$$

We conclude that for any α and k,

$$\mathcal{L}\Big(\frac{d}{dx} h_{\alpha,k}\Big) - z\mathcal{L}(h_{\alpha,k}) = \begin{cases} 0 \quad \text{if} \quad \alpha \notin \mathbb{Z}_{<0} & \text{or} \quad k \neq 0, \\ \dfrac{(-1)^{-\alpha} z^{-\alpha}}{(-\alpha)!} & \text{otherwise.} \end{cases} \tag{4.11}$$

To simplify the notations, we will denote in the sequel z by x. Finally, using the formula

$$\sum_{m=0}^{n} \frac{(-1)^m}{m!(n-m)!} \frac{1}{X+m} = \frac{1}{X(X+1)\dots(X+n)},$$

we deduce from (4.2), (4.3), (4.4) and (4.9) (with $n \geq -\Re e(\alpha)$) that, if α is not a negative integer, then

$$\mathcal{L}(h_{\alpha,k}) = \Gamma(\alpha+1) x^{-\alpha-1} \sum_{j=0}^{k} \rho_{\alpha,j}^{(k)} (\ln x)^j, \tag{4.12}$$

with

$$\rho_{\alpha,k}^{(k)} = (-1)^k$$

and

$$\rho_{\alpha,j}^{(k)} \in \langle \Gamma(\alpha), \dots, \Gamma^{(k)}(\alpha) \rangle_{\mathbb{Q}[\alpha]} \quad \text{for} \quad j = 0, \dots, k-1,$$

and, if α is a negative integer, then

$$\mathcal{L}(h_{\alpha,k}) = x^{-\alpha-1} \sum_{j=0}^{k+1} \rho_{\alpha,j}^{(k)} (\ln x)^j, \tag{4.13}$$

with

$$\rho_{\alpha,k+1}^{(k)} = \frac{(-1)^{\alpha+1}}{(k+1)(-\alpha-1)!},$$

$$\rho_{\alpha,k}^{(k)} = \frac{(-1)^{\alpha+1+k}(\alpha+n+2)}{(-\alpha-1)!} + \sum_{\substack{m=0 \\ m \neq -\alpha-1}}^{n} \frac{(-1)^{m+k}(\alpha+n+1)}{m!(n-m)!(\alpha+m+1)},$$

and

$$\rho_{\alpha,j}^{(k)} \in \langle \Gamma(\alpha), \dots, \Gamma^{(k)}(\alpha) \rangle_{\mathbb{Q}[\alpha]} \quad \text{for} \quad j = 0, \dots, k-1.$$

Combining (4.10) and (4.11), we get by x-adic completion the following lemma:

LEMMA 4.1. *Let f be a finite sum*

$$\sum_i f_i x^{\alpha_i}(\ln x)^{k_i} x,$$

where $\alpha_i \in K$, $k_i \in \mathbb{Z}_{\geq 0}$ and $f_i \in K((x))$. If f is a solution of some operator $\phi \in K[x, \frac{d}{dx}]$, then there exists a positive integer m such that $(\frac{d}{dx})^m \mathcal{F}(\phi)(\mathcal{L}(f)) = 0$. In other words, $\mathcal{L}(f)$ is a logarithmic solution of $(\frac{d}{dx})^m \mathcal{F}(\phi)$ at infinity.

4.2. Arithmetic properties of $\mathcal{L}$. In this subsection, we shall investigate the relations between the radius of convergence of a power series $f \in K((x))$ and those of the formal factors of $\mathcal{L}(fx^\alpha(\ln x)^k)$, where $\alpha \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ and $k \in \mathbb{Z}_{\geq 0}$.

LEMMA 4.2. *Let $\alpha \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ and $k \in \mathbb{Z}_{\geq 0}$. Then, for each $j = 0, \ldots, k$, there exist sequences $\left(r^{(k,\ell)}_{\alpha+n,j}\right)_{n\geq 0}$ of elements of $\mathbb{Q}(\alpha)$, with $\ell = j, \ldots, k$, such that*

$$\rho^{(k)}_{\alpha+n,j} = \sum_{\ell=j}^{k} \rho^{(k)}_{\alpha,\ell}\, r^{(k,\ell)}_{\alpha+n,j} \quad (n \geq 0),$$

where the $\rho^{(k)}_{\alpha,\ell}$ were defined in (4.12) and (4.13). Moreover, for any place v of V_0, these sequences satisfy

$$\limsup_{n \longrightarrow \infty} \left| r^{(k,\ell)}_{\alpha+n,j} \right|_v^{1/n} \leq 1.$$

Proof. We will prove this lemma by downward induction on the index j. In the case $j = k \geq 0$, by (4.12) and (4.13), it suffices to take $r^{(k,k)}_{\alpha+n,k} = 1$ for any $n \in \mathbb{N}$. Suppose now that the lemma is true for some index j with $1 \leq j \leq k$. From the formulas (4.12), (4.13) and (4.10), we obtain the recurrence relation

$$\rho^{(k)}_{\alpha+1,j-1} = -\rho^{(k)}_{\alpha,j-1} + \frac{j}{\alpha+1}\, \rho^{(k)}_{\alpha,j},$$

and by iteration on $n \geq 1$ we find

$$\begin{aligned}
\rho^{(k)}_{\alpha+n,j-1} &= (-1)^n \rho^{(k)}_{\alpha,j-1} + j \sum_{i=0}^{n-1} \frac{(-1)^{n+i+1}}{\alpha+i+1}\, \rho^{(k)}_{\alpha+i,j}. \\
&= (-1)^n \rho^{(k)}_{\alpha,j-1} + j \sum_{i=0}^{n-1} \frac{(-1)^{n+i+1}}{\alpha+i+1} \sum_{\ell=j}^{k} \rho^{(k)}_{\alpha,\ell}\, r^{(k,\ell)}_{\alpha+i,j} \\
&= (-1)^n \rho^{(k)}_{\alpha,j-1} + j \sum_{\ell=j}^{k} \rho^{(k)}_{\alpha,\ell} \sum_{i=0}^{n-1} \frac{(-1)^{n+i+1}}{\alpha+i+1}\, r^{(k,\ell)}_{\alpha+i,j}.
\end{aligned}$$

Thus,

$$(4.14)\quad r^{(k,\ell)}_{\alpha+n,j-1} = j\sum_{i=0}^{n-1} \frac{(-1)^{n+i+1}}{\alpha+i+1}\, r^{(k,\ell)}_{\alpha+i,j}, \quad \ell = j,\dots,k, \quad r^{(k,j-1)}_{\alpha+n,j-1} = (-1)^n,$$

and we get

$$\rho^{(k)}_{\alpha+n,j-1} = \sum_{\ell=j-1}^{k} \rho^{(k)}_{\alpha,\ell}\, r^{(k,\ell)}_{\alpha+n,j-1},$$

where

$$r^{(k,\ell)}_{\alpha+n,j-1} \in \mathbb{Q}(\alpha).$$

Let $v \in V_0$. By the induction hypothesis, we have

$$\limsup_{n\longrightarrow\infty} \left| r^{(k,\ell)}_{\alpha+n,j} \right|_v^{1/n} \le 1$$

for $\ell = j,\dots,k$. Since α is an element of K, hence algebraic over $\mathbb{Q}$, it is non-Liouville for $p(v)$ and consequently we have

$$\limsup_{n\longrightarrow\infty} \left| \frac{1}{\alpha+n} \right|_v^{1/n} = 1$$

(cf. [DGS, VI.1.1]). We deduce that

$$\limsup_{n\longrightarrow\infty} \Big(\max_{0\le m\le n-1} \left| r^{(k,\ell)}_{\alpha+m,j} \right|_v^{1/n} \Big) \le 1, \quad \ell = j,\dots,k,$$

and

$$\limsup_{n\longrightarrow\infty} \Big(\max_{0\le m\le n-1} \left| \frac{1}{\alpha+m+1} \right|_v^{1/n} \Big) \le 1.$$

Combining these estimations with (4.14) we get for $\ell = j,\dots,k$,

$$\limsup_{n\longrightarrow\infty} \left| r^{(k,\ell)}_{\alpha+n,j-1} \right|_v^{1/n} \le 1.$$

The case $\ell = j-1$ is trivial. $\square$

NOTATIONS. If $Y \in \mathrm{GL}_\mu(K((x)))$, we will denote, for $s \in \mathbb{Z}$,

$$\mathcal{R}_s(Y) = \{ y \in K((x)) \mid r_v(y) \ge R_v(Y)\, \pi_v^s, \text{ for almost all } v \in V_0 \},$$
$$\mathcal{R}_s^\infty(Y) = \{ y(x) \in K((1/x)) \mid y(1/x) \in \mathcal{R}_s(Y) \}.$$

Here, "almost all" means with at most finitely many exceptions. It is clear that $\mathcal{R}_s(Y)$ (resp. $\mathcal{R}_s^\infty(Y)$) is a K-subalgebra of $K((x))$ (resp. of $K((1/x))$). For instance, if $f \in K((x))$, then $\mathcal{R}_0(f)$ denotes the K-algebra of the power series $y \in K((x))$ such that $r_v(y) \ge r_v(f)$ for almost all $v \in V_0$.

PROPOSITION 4.3. *Let $f \in K[[x]]$ with $f \neq 0$, $\alpha \in \mathbb{Q}$ and $k \in \mathbb{Z}_{\geq 0}$. Then there exist power series $h_{\alpha,k,j} \in \mathbb{C} \otimes_K \mathcal{R}_{-1}(f)$, $j = 0, \ldots, k$, which satisfy the conditions*

$$\mathcal{L}\Big(fx^{\alpha}(\ln x)^k\Big) = \begin{cases} x^{-\alpha-1}\Gamma(\alpha)\sum_{j=0}^{k} h_{\alpha,k,j}\Big(\frac{1}{x}\Big)(\ln x)^j & \text{if} \quad \alpha \in \mathbb{Q} \setminus \mathbb{Z}_{<0}, \\ \sum_{j=0}^{k+1} h_{\alpha,k,j}\Big(\frac{1}{x}\Big)(\ln x)^j & \text{if} \quad \alpha \in \mathbb{Z}_{<0}. \end{cases}$$

with $h_{\alpha,k,k+1} \in K[x] \setminus \{0\}$ and $h_{\alpha,k,k} \in K[[x]] \setminus \{0\}$ such that $r_v(h_{\alpha,k,k}) = r_v(f)\pi_v^{-1}$ for almost all $v \in V_0$. In particular, $\mathcal{L}\Big(fx^{\alpha}(\ln x)^k\Big) \neq 0$.

Proof. Suppose $\alpha \in \mathbb{Q} \setminus \mathbb{Z}_{<0}$. By (4.12), we may write

$$\mathcal{L}\Big(fx^{\alpha}(\ln x)^k\Big) = x^{-\alpha-1}\Gamma(\alpha)\sum_{j=0}^{k} h_{\alpha,k,j}\Big(\frac{1}{x}\Big)(\ln x)^j;$$

where

$$h_{\alpha,k,j} = \sum_{n\geq 0} a_n \, \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha)} \, \rho^{(k)}_{\alpha+n,j} x^n = \sum_{n\geq 0} a_n (\alpha)_{n+1} \rho^{(k)}_{\alpha+n,j} x^n.$$

For $j = k$, we have $\rho^{(k)}_{\alpha+n,j} = (-1)^k$, Thus $h_{\alpha,k,k} \in K[[x]]$, and since $\alpha \in \mathbb{Q}$, we also have $\alpha \in \mathbb{Z}_{p(v)}$ for almost all $v \in V_0$. Hence, using (2.2), we get

$$r_v(h_{\alpha,k,k})^{-1} = \limsup_{n\longrightarrow\infty} |a_n(\alpha)_{n+1}|_v^{1/n} = r_v(f)^{-1}\pi_v \quad \text{for almost all} \quad v \in V_0.$$

For $j = 0, \ldots, k-1$, Lemma 4.2 gives

$$h_{\alpha,k,j} = \sum_{\ell=j}^{k} \rho^{(k)}_{\alpha,\ell} h^{(k,\ell)}_{k,j}$$

with

$$h^{(k,\ell)}_{\alpha,k,j} = \sum_{n\geq 0} a_n(\alpha)_{n+1} r^{(k,\ell)}_{\alpha+n,j} x^n \in K[[x]],$$

and

$$r_v(h^{(k,\ell)}_{\alpha,k,j})^{-1} \leq \limsup_{n\longrightarrow\infty} |a_n(\alpha)_{n+1}|_v^{1/n} = r_v(f)^{-1}\pi_v$$

for almost all $v \in V_0$. This ends the proof in the case $\alpha \in \mathbb{Q} \setminus \mathbb{Z}_{<0}$. Now, suppose $\alpha \in \mathbb{Z}_{<0}$, write

$$f = \sum_{n\geq 0} a_n x^n$$

and put

$$f_{\alpha} = \sum_{n\geq -\alpha} a_n x^n.$$

Then, by (4.13),

$$\mathcal{L}\Big((f-f_\alpha)x^\alpha(\ln x)^k\Big)=\sum_{j=0}^{k+1}P_j\Big(\frac{1}{x}\Big)(\ln x)^j,$$

with $P_j\in\mathbb{C}[x]$, and $P_{k+1},P_k\in K[x]\setminus\{0\}$. On the other hand, the first assertion, applied to $f_\alpha x^\alpha(\ln x)^k$, shows that there exist $h_{0,k,j}\in\mathbb{C}\otimes_K\mathcal{R}_{-1}(f),\ j=0,\dots,k$, such that

$$\mathcal{L}(f_\alpha x^\alpha(\ln x)^k)=\sum_{j=0}^{k}x^{-1}h_{0,k,j}\Big(\frac{1}{x}\Big)(\ln x)^j,$$

where $h_{0,k,k}\in K[[x]]$ and $r_v(h_{0,k,k})=r_v(f)\pi_v$ for almost all $v\in V_0$. Finally, by the linearity of the Laplace transform, it suffices to take $h_{\alpha,k,j}=xh_{0,k,j}+P_j$ for $j=0,\dots,k$ and $h_{\alpha,k,k+1}=P_{k+1}$ to obtain the second part of the proposition. □

5. Solutions of ψ^* and solutions of $\frac{d}{dx}\psi$

Let ψ be a differential operator of $K[x,d/dx]$ such that all slopes of $NR(\psi)$ lie in $\{1,0\}$, and v be a fixed finite place of V_0. Let K_v be the v-adic completion of K and $\Omega_{p(v)}$ be a v-adic complete field and algebraically closed containing $\mathbb{C}_{p(v)}$ such that its value group is $\mathbb{R}_{\geq 0}$. We fix an embedding $K\hookrightarrow K_v\hookrightarrow\mathbb{C}_{p(v)}\hookrightarrow\Omega_{p(v)}$. In this section, we see how we can determine the nature of the solutions of $(d/dx)\psi$ at 0 from those ψ^* at the same point. For this, we shall begin with the following key lemma.

Lemma 5.1. *Assume $y\in K((x))$, $\alpha\in K\cap\mathbb{Z}_{p(v)}$, $\delta\in K$ and $k\in\mathbb{Z}_{\geq 0}$. Then the differential equation $(d/dx)(z)=yx^\alpha(\ln x)^k\exp(\delta/x)$ has a solution of the form $\sum_{0\leq i\leq k+1}y_ix^\alpha(\ln x)^i\exp(\delta/x)$ at 0, where for $i=0,\dots,k+1$, $y_i\in K((x))$ is such that*

$$r_v(y_i)\geq\begin{cases}r_v(y) & \text{if}\quad \delta=0,\\ \min(|\delta|_v\pi_v^{-1},r_v(y)) & \text{otherwise.}\end{cases}$$

Proof. Let $m=\min(0,\mathrm{ord}_x(y))$ and write $y=\sum_{n\geq m}a_nx^n\in K((x))$. Let us consider

$$z=\sum_{0\leq i\leq k+1}\sum_{n\geq m}a_{i,n}x^{n+\alpha}(\ln x)^i\exp(\delta/x),$$

where $a_{i,n} \in K$ for all $i = 0, \ldots, k+1$ and all $n \geq m$. Then

$$\begin{aligned}\frac{d}{dx} z = \sum_{0 \leq i \leq k} \sum_{n \geq m+1} \Big(&(n-1+\alpha) a_{i,n-1} \\ &+ (i+1) a_{i+1,n-1} - \delta a_{i,n} \Big) x^{n+\alpha-2} (\ln x)^i \exp(\delta/x) \\ + \sum_{n \geq m+1} &\Big((n-1+\alpha) a_{k+1,n-1} - \delta a_{k+1,n} \Big) x^{n+\alpha-2} (\ln x)^{k+1} \exp(\delta/x) \\ + \sum_{0 \leq i \leq k+1} &-\delta a_{i,m} x^{m+\alpha-2} (\ln x)^i \exp(\delta/x).\end{aligned}$$

z is then a solution of the differential equation $(d/dx)(z) = yx^\alpha (\ln x)^k \exp(\delta/x)$ if and only if, the coefficients $a_{i,n}$ satisfy the following relations for all $n \geq m$:

$$\delta a_{0,m} = \delta a_{1,m} = \cdots = \delta a_{k+1,m} = 0, \tag{5.1}$$

$$(n+\alpha) a_{k+1,n} - \delta a_{k+1,n+1} = 0, \tag{5.2}$$

$$(n+\alpha+1) a_{k,n+1} + (k+1) a_{k+1,n+1} - \delta a_{k,n+2} = a_n, \tag{5.3}$$

$$(m+\alpha) a_{k,m} + (k+1) a_{k+1,m} - \delta a_{k,m+1} = 0, \tag{5.4}$$

$$(n+\alpha) a_{i,n} + (i+1) a_{i+1,n} - \delta a_{i,n+1} = 0, \quad \text{for} \quad 0 \leq i < k. \tag{5.5}$$

This means:

Case 1: If $\delta = 0$, we have from (5.2) and (5.3),

$$\sum_{n \geq m} a_{k+1,n} x^n = \begin{cases} 0 & \text{if } \alpha \text{ is a non-integer } < -m, \\ a_{k+1,-\alpha} x^{-\alpha} = \dfrac{a_{-\alpha-1}}{k+1} x^{-\alpha} & \text{otherwise,} \end{cases} \tag{5.6}$$

and therefore, for all $n \geq m+1$, and all $0 \leq i \leq k$, we get from (5.3) and (5.5),

$$\begin{cases} a_{1,-\alpha} = \ldots = a_{k,-\alpha} = 0 & \text{if } \alpha \text{ is an integer } \leq -m, \\ a_{k,n} = \dfrac{a_{n-1}}{n+\alpha} \quad \text{for all} \quad n \neq -\alpha, \\ a_{i,n} = -\dfrac{(i+1) a_{i+1,n}}{n+\alpha} = \dfrac{(-1)^{k-i} k! a_{n-1}}{i! (n+\alpha)^{k-i+1}} \quad \text{for all} \quad n \neq -\alpha. \end{cases} \tag{5.7}$$

Hence the finite sum

$$\sum_{0 \leq i \leq k+1} y_i x^\alpha (\ln x)^i,$$

where the coefficients of the power series $y_i = \sum_{n \geq m} a_{i,n} x^n$ are defined by (5.6), (5.7), where $a_{0,-\alpha} = 0$ if α is an integer $\leq -m$, and where $a_{0,m} = a_{1,m} = \cdots = a_{k+1,m} = 0$, is a solution of the equation $dz/dx = yx^\alpha (\ln x)^k$ at 0. In addition, since $\alpha \in K \cap \mathbb{Z}_{p(v)}$, it is non-Liouville for $p(v)$ and consequently we have

$$\limsup_{n \longrightarrow \infty} \left| \frac{1}{\alpha + n} \right|_v^{1/n} = 1$$

(cf. [DGS, VI.1.1]). Therefore, by (5.7), we find for $i = 0, \dots, k+1$,

$$\limsup_{n \longrightarrow \infty} |a_{i,n}|_v^{1/n} \leq \limsup_{n \longrightarrow \infty} |a_{n-1}|_v^{1/n} = \limsup_{n \longrightarrow \infty} |a_n|_v^{1/n}.$$

This implies that $r_v(y_i) \geq r_v(y)$ for $i = 0, \dots, k+1$, and hence the lemma is proved in the case $\delta = 0$.

Case 2: If $\delta \neq 0$, we find, from (5.1) and (5.4), that $a_{0,m} = a_{1,m} = \cdots = a_{k+1,m} = a_{k,m+1} = 0$, and therefore, by induction on $n \geq m$ and by (5.2), that $\sum_{n \geq m} a_{k+1,n} x^n = 0$. In addition, from (5.3) and (5.5), we get for any $n \geq m$,

$$\begin{cases} a_{k,n+2} = \dfrac{(n+\alpha+1)}{\delta} a_{k,n+1} - \dfrac{1}{\delta} a_n, \\ a_{i,n+1} = \dfrac{(n+\alpha)}{\delta} a_{i,n} + \dfrac{i+1}{\delta} a_{i+1,n} \quad \text{for any} \quad 0 \leq i < k. \end{cases} \tag{5.8}$$

Hence the finite sum

$$\sum_{0 \leq i \leq k+1} y_i x^\alpha (\ln x)^i,$$

where the coefficients of the power series $y_i = \sum_{n \geq m} a_{i,n} x^n$ are defined recursively by (5.8), and where $a_{0,m} = a_{1,m} = \cdots = a_{k,m} = \sum_{n \geq m} a_{k+1,n} x^n = 0$, is a solution of the equation $dz/dx = yx^\alpha (\ln x)^k$ at 0. It remains to prove that the power series y_i satisfy the condition of Lemma 5.1.

From (5.8) we find, for any $n \geq 3$ and any $0 \leq i < k$, that

$$\begin{aligned} a_{k,n+2} = {} & \frac{(n+\alpha+1)(n+\alpha)\dots(2+\alpha)}{\delta^n} a_{k,2} - \frac{1}{\delta} a_n - \frac{(n+\alpha+1)}{\delta^2} a_{n-1} \\ & - \frac{(n+\alpha+1)(n+\alpha)}{\delta^3} a_{n-2} - \cdots - \frac{(n+\alpha+1)(n+\alpha)\dots(3+\alpha)}{\delta^n} a_1, \end{aligned}$$

$$\begin{aligned} a_{i,n+2} = {} & \frac{(n+\alpha+1)(n+\alpha)\dots(2+\alpha)}{\delta^n} a_{i,2} + \frac{i+1}{\delta} a_{i+1,n+1} \\ & + \frac{(i+1)(n+\alpha+1)}{\delta^2} a_{i+1,n} + \frac{(i+1)(n+\alpha+1)(n+\alpha)}{\delta^3} a_{i+1,n-1} \\ & + \cdots + \frac{(i+1)(n+\alpha+1)(n+\alpha)\dots(3+\alpha)}{\delta^n} a_{i+1,2}. \end{aligned}$$

Consequently, if α is a non-integer ≤ -2, we have for any $n \geq 1$ and any $0 \leq i < k$,

$$\begin{cases} a_{k,n+2} = \dfrac{(\alpha+2)_n}{\delta^n} a_{k,2} - \dfrac{(\alpha+2)_n}{\delta^{n+1}} \displaystyle\sum_{1\leq j\leq n} \dfrac{\delta^j a_j}{(\alpha+2)_j}, \\ a_{i,n+2} = \dfrac{(\alpha+2)_n}{\delta^n} a_{i,2} + \dfrac{(i+1)(\alpha+2)_n}{\delta^{n+2}} \displaystyle\sum_{2\leq j\leq n+1} \dfrac{\delta^j a_{i+1,j}}{(\alpha+2)_{j-1}}, \end{cases} \tag{5.9}$$

and, if α is an integer ≤ -2, we have for $n \geq -\alpha$ and any $0 \leq i < k$,

$$\begin{cases} a_{k,n+2} = \dfrac{(n+\alpha+1)!}{\delta^n} a_{k,1-\alpha} - \dfrac{(\alpha+1+n)!}{\delta^{n+1}} \displaystyle\sum_{-\alpha\leq j\leq n} \dfrac{\delta^j a_j}{(\alpha+1+j)!}, \\ a_{i,n+2} = \dfrac{(n+\alpha+1)!}{\delta^n} a_{i,1-\alpha} + \dfrac{(i+1)(\alpha+1+n)!}{\delta^{n+2}} \displaystyle\sum_{-\alpha\leq j\leq n+1} \dfrac{\delta^j a_{i+1,j}}{(\alpha+j)!}. \end{cases} \tag{5.10}$$

Now, in the case where α is a non-integer ≤ -2, we have to study two subcases:

Case 2.1.a: If α is a non-integer ≤ -2, and if $r_v(y) \geq \pi_v^{-1}|\delta|_v$, or in other words, $\limsup_{n\to\infty} |a_n|_v^{1/n} \leq \pi_v |\delta|_v^{-1}$, we have

$$\limsup_{n\to\infty} \Big(\Big| \frac{\delta^n a_n}{(\alpha+2)_n} \Big|_v^{1/n} \Big) \leq 1,$$

since $\lim_{n\to\infty} |(\alpha+2)_n|_v^{1/n} = \pi_v$. Then

$$\limsup_{n\to\infty} \Big(\max_{1\leq i\leq n} \Big| \frac{\delta^i a_i}{(\alpha+2)_i} \Big|_v^{1/n} \Big) \leq 1.$$

This implies that the power series

$$\sum_{n\geq 2} \left(\frac{(\alpha+2)_n}{\delta^{n+1}} \sum_{1\leq j\leq n} \frac{\delta^j a_j}{(\alpha+2)_j} \right) x^n$$

has a radius of convergence at least $\pi_v^{-1}|\delta|_v$. Thus, by (5.9), we get $r_v(y_k) \geq \pi_v^{-1}|\delta|_v$. Using the same argument, we prove, by downward induction on the index i and by (5.9), that $r(y_i) \geq \pi_v^{-1}|\delta|_v$ for any $0 \leq i \leq k$. This concludes the proof of Lemma 5.1 in Case 2.1.a.

Case 2.1.b: If α is a non-integer ≤ -2, and if $r_v(y) < \pi_v^{-1}|\delta|_v$. We will prove the lemma in this case by downward induction on the index i. First, let l be an element of $\Omega_{p(v)}$ such that $|l|_v = \pi_v^{-1}|\delta|_v \limsup_{n\to\infty} |a_n|_v^{1/n} > 1$. Since $\lim_{n\to\infty} |(\alpha+2)_n|_v^{1/n} = \pi_v$, we have

$$\limsup_{n\to\infty} \Big| \frac{\delta^n a_n}{l^n (\alpha+2)_n} \Big|_v^{1/n} = 1,$$

and hence

$$\limsup_{n\to\infty}\Big(\max_{1\le i\le n}\Big|\frac{\delta^i a_i}{l^i(\alpha+2)_i}\Big|_v^{1/n}\Big)\le 1.$$

Since $|l|_v>1$, we obtain

$$\limsup_{n\to\infty}\Big(\max_{1\le i\le n}\Big|\frac{\delta^i a_i}{l^n(\alpha+2)_i}\Big|_v^{1/n}\Big)\le 1,$$

and

$$\limsup_{n\to\infty}\Big(\max_{1\le i\le n}\Big|\frac{\delta^i a_i}{(\alpha+2)_i}\Big|_v^{1/n}\Big)\le |l|_v=\pi_v^{-1}|\delta|_v\limsup_{n\to\infty}|a_n|_v^{1/n}.$$

This shows that the power series

$$\sum_{n\ge 2}\Big(\frac{(\alpha+2)_n}{\delta^{n+1}}\sum_{1\le j\le n}\frac{\delta^j a_j}{(\alpha+2)_j}\Big)x^n$$

has a radius of convergence at least $r_v(y)$. Thus, by (5.9), we get

$$r_v(y_k)\ge\min(\pi_v^{-1}|\delta|_v,r_v(y))=r_v(y).$$

Suppose now that $r_v(y_{i+1})\ge\min(\pi_v^{-1}|\delta|_v,r_v(y))$ for some index $1\le i\le k-1$. If $r_v(y_{i+1})<\pi_v^{-1}|\delta|_v$, we find, with the same argument as above and by (5.9), that

$$r_v(y_i)\ge\min(\pi_v^{-1}|\delta|_v,r_v(y_{i+1}))\ge\min(\pi_v^{-1}|\delta|_v,r_v(y)).$$

If $r_v(y_{i+1})\ge\pi_v^{-1}|\delta|_v$, we get, with the same argument as in Case 2.1.a and by (5.9),

$$r_v(y_i)\ge\pi_v^{-1}|\delta|_v=\min(\pi_v^{-1}|\delta|_v,r_v(y)).$$

This shows that, for all $0\le i\le k$, $r_v(y_i)\ge\pi_v^{-1}|\delta|_v$. This ends the proof of the lemma in Case 2.1.b.

Case 2.2: The case where α is an integer ≤ -2 can be proved with the same arguments employed in Cases 2.1.a and 2.1.b, using (5.10), since $\lim_{n\to\infty}|n!|_v^{1/n}=\pi_v$. This concludes the proof of Lemma 5.1. □

NOTATIONS. Let $y_1,\dots,y_s$ be elements of $K((x))$, and $\Delta=(\delta_{ij})$ be a $t\times t$ diagonal matrix with entries in K. We denote by $\mathfrak{R}_v(y_1,\dots,y_s,\Delta)$ the K-subalgebra of $K((x))$ consisting of power series $y\in K((x))$ satisfying

$$r_v(y)\ge\begin{cases}\min\limits_{1\le h\le s}\{r_v(y_h)\} & \text{if}\quad \Delta=0,\\ \min(\min\limits_{1\le i\le t}\{|\delta_{ii}|_v\mid\delta_{ii}\ne 0\}\pi_v^{-1},\ \min\limits_{1\le h\le s}\{r_v(y_h)\}) & \text{otherwise.}\end{cases}$$

Also, we denote by $\mathfrak{R}(y_1,\dots,y_s,\Delta)$ the K-subalgebra of $K((x))$ consisting of power series $y\in K((x))$ belonging to $\mathfrak{R}_v(y_1,\dots,y_s,\Delta)$ for almost all v in V_0. Again here and in the sequel, "almost all" means with at most finitely many exceptions.

PROPOSITION 5.2. *Let $\psi \in K[x, d/dx]$ be a differential operator of rank μ such that all slopes of $NR(\psi)$ lie in $\{0,1\}$. Assume that ψ^* has a basis of solutions at 0 with elements in*

$$\mathfrak{R}_v(y_1,\dots,y_s,\Delta)[\ln x,\ x^{\gamma_1},\dots,x^{\gamma_s},\exp(\delta_{11}/x),\dots,\exp(\delta_{ss}/x)],$$

where $y_1,\dots,y_s$ are elements of $K((x))$, where $\gamma_1,\dots,\gamma_s$ are elements of $\mathbb{Z}_{p(v)}\cap K$, and where $\Delta=(\delta_{ij})$ is a $s\times s$ diagonal matrix with entries in K. Then $\frac{d}{dx}\psi$ and $(\frac{d}{dx}\psi)^*$ have bases of solutions at 0 with elements, respectively, in

$$\mathfrak{R}_v(y_1,\dots,y_s,\Delta)[\ln x,\ x^{\pm\gamma_1},\dots,x^{\pm\gamma_s},\exp(\pm\delta_{11}/x),\dots,\exp(\pm\delta_{ss}/x)]$$

and in

$$\mathfrak{R}_v(y_1,\dots,y_s,\Delta)[\ln x,\ x^{\gamma_1},\dots,x^{\gamma_s},\exp(\delta_{11}/x),\dots,\exp(\delta_{ss}/x)].$$

Proof. Write

$$\psi = a_\mu(x)(d/dx)^\mu + a_{\mu-1}(x)(d/dx)^{\mu-1}+\cdots+a_0(x)\in K[x,d/dx].$$

Since all slopes of $NR(\psi)$ lie in $\{0,1\}$, a_μ is a monomial, say $a_\mu = x^\nu$ with $\nu\in\mathbb{Z}_{\geq 0}$, and ψ is regular at infinity. If we denote by ψ_∞ the operator obtained from ψ by the change of variable $x\to 1/x$, we find

$$\begin{aligned}\psi_\infty = x^{-\nu}(-x^2)^\mu(d/dx)^\mu &+ \frac{\mu(\mu-1)}{2}(-2x)(-x^2)^{\mu-1}(d/dx)^{\mu-1}\\ &+(-x^2)^{\mu-1}a_{\mu-1}(1/x)(d/dx)^{\mu-1}\\ &+\quad \text{terms with lower degree in } (d/dx),\end{aligned}$$

because for all $a\in K(x)$ and all integer $h\geq 1$, we have

$$\begin{aligned}(a\cdot(d/dx))^h = a^h(d/dx)^h &+ \frac{h(h-1)}{2}a^{h-1}\cdot(d/dx)(a)\cdot(d/dx)^{h-1}\\ &+\quad \text{terms with lower degree in } (d/dx).\end{aligned}$$

The regularity of ψ_∞ at 0 implies, in particular, that $\deg(a_{\mu-1}(x))\leq \nu-1$. In addition, it is easy to check that

(5.11)

$$\begin{aligned}((d/dx)\psi)^* &= \psi^*(d/dx)\\ &= (-1)^\mu x^\nu(d/dx)^{\mu+1} + ((-1)^\mu\nu x^{\nu-1}+(-1)^{\mu-1}a_{\mu-1}(x))(d/dx)^\mu\\ &\qquad + \text{ terms with lower degree in } (d/dx)\\ &= (-1)^\mu x^\nu\Big[(d/dx)^{\mu+1}+(\nu x^{-1}-x^{-\nu}a_{\mu-1}(x))(d/dx)^\mu\\ &\qquad + \text{ terms with lower degree in } (d/dx)\Big],\end{aligned}$$

because

$$(d/dx)^{\mu} \cdot x^{\nu} = x^{\nu}(d/dx)^{\mu} + \nu x^{\nu-1}(d/dx)^{\mu-1} \\ + \text{ terms with lower degree in } (d/dx).$$

This shows that

$$\operatorname{tr}(A_{((d/dx)\psi)^*}) = -(\nu x^{-1} - x^{-\nu}a_{\mu-1}(x)) \in \frac{1}{x}K\left[\frac{1}{x}\right]. \tag{5.12}$$

On the other hand, by hypothesis, ψ^* has a basis of solutions $(u_1, \dots, u_\mu)$ at 0 such that the u_i are of the form

$$u_i = \sum_{\text{finite sum on } j} \widehat{y}_{i_j} x^{\gamma_{i_j}} (\ln x)^{k_{i_j}} \exp(\delta_{i_j}/x) \\ \in \mathfrak{R}_v(y_1, \dots, y_s, \Delta)[\ln x,\ x^{\gamma_1}, \dots, x^{\gamma_s}, \exp(\delta_{11}/x), \dots, \exp(\delta_{ss}/x)].$$

By Lemma 5.1, for $i = 1, \dots, \mu$, the differential equation $(d/dx)(z) = u_i$ has a solution of the form

$$z_i = \sum_{\text{finite sum on } j} \Bigg(\sum_{\text{finite sum on } \ell} \widetilde{y}_{i_{j_\ell}} (\ln x)^{k_{i_{j_\ell}}} \Bigg) x^{\gamma_{i_j}} \exp(\delta_{i_j}/x) \\ \in K((x))[\ln x,\ x^{\gamma_1}, \dots, x^{\gamma_s}, \exp(\delta_{11}/x), \dots, \exp(\delta_{ss}/x)]$$

such that

$$r_v(\widetilde{y}_{i_{j_\ell}}) \geq \begin{cases} \{r_v(\widehat{y}_{i_j})\} & \text{if} \quad \delta_{i_j} = 0, \\ (|\delta_{i_j}|_v \pi_v^{-1}, r_v(\widehat{y}_{i_j})) & \text{otherwise.} \end{cases}$$

Thus, the elements $1, z_1, \dots, z_\mu$ form a basis of solutions of $\left(\frac{d}{dx}\psi\right)^* = \psi^* \frac{d}{dx}$ at 0. Moreover, $1, z_1, \dots, z_\mu$ lie in

$$\mathfrak{R}_v(y_1, \dots, y_\mu, \Delta)[\ln x,\ x^{\gamma_{11}}, \dots, x^{\gamma_{\mu\mu}}, \exp(\delta_{11}/x), \dots, \exp(\delta_{\mu\mu}/x)].$$

Now, denote by W the Wronskian matrix of $1, z_1, \dots, z_\mu$. Thus, the matrix W is solution of $\frac{d}{dx}X = A_{(\frac{d}{dx}\psi)^*}X$, and all entries of W lie in

$$\mathfrak{R}_v(y_1, \dots, y_\mu, \Delta)[\ln x,\ x^{\gamma_1}, \dots, x^{\gamma_\mu}, \exp(\delta_{11}/x), \dots, \exp(\delta_{\mu\mu}/x)].$$

On the other hand, $\det(W)$ satisfies the differential equation $(d/dx)(\det(W)) = \operatorname{tr}(A_{((d/dx)\psi)^*})\det(W)$. By (5.12), $\det(W)$ is of the form $x^\alpha \exp(P(1/x))$, where $P \in K[x]$ and where $\alpha \in K$. By definition of W, we find $\alpha \in \langle 1, \gamma_1, \dots, \gamma_\mu \rangle_{\mathbb{Z}}$ and $P(x) = \delta x$ for some $\delta \in \langle \delta_{11}, \dots, \delta_{\mu\mu} \rangle_{\mathbb{Z}}$. This implies that all entries of W^{-1} lie in

$$\mathfrak{R}_v(y_1, \dots, y_\mu, \Delta)[\ln x,\ x^{\pm\gamma_1}, \dots, x^{\pm\gamma_\mu}, \exp(\pm\delta_{11}/x), \dots, \exp(\pm\delta_{\mu\mu}/x)].$$

Hence, by Lemma 3.6 and the fact that leading coefficient of $(d/dx)\psi)^*$ is monomial (see (5.11)), the differential operator $\frac{d}{dx}\psi$ has a basis of solutions at 0, with elements in

$$\mathfrak{R}_v(y_1, \dots, y_\mu, \Delta)[\ln x,\ x^{\pm\gamma_1}, \dots, x^{\pm\gamma_\mu}, \exp(\pm\delta_{11}/x), \dots, \exp(\pm\delta_{\mu\mu}/x)].$$

This conclues the proof of Proposition 5.2. $\square$

Corollary 5.3. *Under hypotheses of Proposition 5.2, for all positive integers $m \geq 1$, the differential operators $(\frac{d}{dx})^m\psi$ and $((\frac{d}{dx})^m\psi)^*$ have bases of solutions at 0 with elements, respectively, in*

$$\mathfrak{R}_v(y_1,\dots,y_\mu,\Delta)[\ln x,\ x^{\pm\gamma_1},\dots,x^{\pm\gamma_\mu},\exp(\pm\delta_{11}/x),\dots,\exp(\pm\delta_{\mu\mu}/x)]$$

and

$$\mathfrak{R}_v(y_1,\dots,y_\mu,\Delta)[\ln x,\ x^{\gamma_1},\dots,x^{\gamma_\mu},\exp(\delta_{11}/x),\dots,\exp(\delta_{\mu\mu}/x)].$$

Proof. First, the differential operator $(\frac{d}{dx})^m\psi$ has the same leading coefficient as ψ which is a monomial. In addition, the properties of Newton polygon ([Ma, III.1]) lead to

$$\begin{aligned}\{\text{slopes of } N\Big(\Big(\frac{d}{dx}\Big)^m\psi\Big)\} &= \{\text{slopes of } N\Big(\frac{d}{dx}\Big)\}\\ &\qquad\cup\{\text{slopes of } N(\psi)\}\in\{0,1\},\\ \{\text{slopes of } N\Big(\Big(\Big(\frac{d}{dx}\Big)^m\psi\Big)_\infty\Big)\} &= \{\text{slopes of } N\Big(\Big(\frac{d}{dx}\Big)_\infty\Big)\}\\ &\qquad\cup\{\text{slopes of } N(\psi_\infty)\}\\ &= \{0\}.\end{aligned}$$

Thus, the slopes of $NR((\frac{d}{dx})^m\psi)$ lie in $\{0,1\}$ for all integer $m\geq 1$. Hence the corollary can be proved by induction on m, using Proposition 5.2. $\square$

Let f_E denote the Euler series $\sum_{n\geq 0}(-1)^n n! x^n$. With the notations of §4.2, we obtain:

Corollary 5.4. *Let $\psi\in K[x,d/dx]$ be a differential operator of rank μ such that all slopes of $NR(\psi)$ lie in $\{0,1\}$. Assume that the differential system $dX/dx = A_\psi X$ has solution at 0 of the form $Y(x)x^\Gamma\exp(\Delta/x)$, where $Y(x)$ is a $\mu\times\mu$ invertible matrix with entries in $K((x))$, where Γ is a $\mu\times\mu$ matrix with entries in K and eigenvalues $\gamma_1,\dots,\gamma_\mu$ in $\mathbb{Q}$, and where $\Delta=(\delta_{ij})$ is a $\mu\times\mu$ diagonal matrix with entries in K. Then, for all positive integers $m\geq 1$, the differential operators $(\frac{d}{dx})^m\psi$ and $((\frac{d}{dx})^m\psi)^*$ have bases of solutions at 0 with elements, respectively, in*

$$\Big(\mathcal{R}_0(Y)\cap\mathcal{R}_0(f_E)\Big)[\ln x,\ x^{\pm\gamma_1},\dots,x^{\pm\gamma_\mu},\exp(\pm\delta_{11}/x),\dots,\exp(\pm\delta_{\mu\mu}/x)]$$

and

$$\Big(\mathcal{R}_0(Y)\cap\mathcal{R}_0(f_E)\Big)[\ln x,\ x^{-\gamma_1},\dots,x^{-\gamma_\mu},\exp(-\delta_{11}/x),\dots,\exp(-\delta_{\mu\mu}/x)].$$

Proof. Since all slopes of $NR(\psi)$ lie in $\{0,1\}$, the leading coefficient a_μ of ψ is a monomial. Let $\widetilde{Y}_\mu(x)=(\widetilde{y}_1(x),\dots,\widetilde{y}_\mu(x))\in \mathrm{M}_{\mu\times 1}(K((x)))$ denote the μ-th row of the matrix $a_\mu^{-1}({}^TY(x)^{-1})$. By Lemma 3.7, the elements of

$\widetilde{Y}_\mu(x)(x)^{(-^T\Gamma)}\exp(-^T\Delta/x)$ form a basis of solutions of ψ^* at 0. According to Corollary 3.3, the elements of this basis lie, for all $v\in V_0$, in

$$\mathfrak{R}_v(\widetilde{y}_1,\dots,\widetilde{y}_\mu,\Delta)[\ln x,\ x^{-\gamma_1},\dots,x^{-\gamma_\mu},\exp(-\delta_{11}/x),\dots,\exp(-\delta_{\mu\mu}/x)].$$

In addition, for almost all $v\in V_0$, the eigenvalues of Γ lie in $\mathbb{Z}_{p(v)}$. Hence, by Corollary 5.3, the differential operators $(\frac{d}{dx})^m\psi$ and $((\frac{d}{dx})^m\psi)^*$ have bases of solutions at 0 with elements, respectively, in

$$\mathfrak{R}(\widetilde{y}_1,\dots,\widetilde{y}_\mu,\Delta)[\ln x,\ x^{\pm\gamma_1},\dots,x^{\pm\gamma_\mu},\exp(\pm\delta_{11}/x),\dots,\exp(\pm\delta_{\mu\mu}/x)]$$

and

$$\mathfrak{R}(\widetilde{y}_1,\dots,\widetilde{y}_\mu,\Delta)[\ln x,\ x^{-\gamma_1},\dots,x^{-\gamma_\mu},\exp(-\delta_{11}/x),\dots,\exp(-\delta_{\mu\mu}/x)].$$

The corollary results therefore from the following observation:

$$\mathfrak{R}(\widetilde{y}_1,\dots,\widetilde{y}_\mu,\Delta)\subseteq\mathcal{R}_0(\widetilde{y}_1)\cap\dots\mathcal{R}_0(\widetilde{y}_\mu)\cap\mathcal{R}_0(f_E)\subseteq\mathcal{R}_0(Y)\cap\mathcal{R}_0(f_E).\quad\square$$

6. Sufficient conditions

Let $\overline{\mathcal{F}}$ denote the inverse of $\mathcal{F}$, that is the K-automorphism of $K[x,d/dx]$ satisfying $\overline{\mathcal{F}}(x)=-d/dx$ and $\overline{\mathcal{F}}(d/dx)=x$. In this section, we will prove that the conditions in Theorem 3.1 are sufficient:

THEOREM 6.1. *Let $\psi\in K[x,d/dx]$ be an operator of rank μ satisfying the following conditions:*

(1) *The coefficients of ψ are not all in K.*
(2) *The slopes of $NR(\psi)$ lie in $\{-1,0\}$.*
(3) *The differential system $d/dxZ=A_\psi Z$ has a solution of the from*

$$Y_\psi\Big(\frac{1}{x}\Big)\Big(\frac{1}{x}\Big)^\Gamma\exp(-\Delta x),$$

where $Y_\psi(x)$ is a $\mu\times\mu$ invertible matrix with entries in $K((x))$ such that $\prod_{v\in V_0}\min(R_v(Y_\psi)\pi_v,1)\neq 0$, where Γ is a $\mu\times\mu$ matrix with entries in K and with eigenvalues $\gamma_1,\dots,\gamma_\mu$ in $\mathbb{Q}$, and where $\Delta=(\delta_{ij})$ is a diagonal $\mu\times\mu$ matrix with entries in K which commutes with Γ.

Then ψ is an E-operator.

Note that condition (1) means that the differential operator $\phi:=\overline{\mathcal{F}}(\psi)$ is not a polynomial.

LEMMA 6.2. *Under the hypotheses of Theorem 5.2, the differential operator $\phi:=\overline{\mathcal{F}}(\psi)$ has a basis of solutions at 0 of the form $(f_1,\dots,f_\nu)x^C$, where $f_1,\dots,f_\nu$ are power series of $K[[x]]$ such that $\prod_{v\in V_0}\min(r_v(f_i),1)\neq 0$ for $i=1,\dots,\nu$, and where C is a $\nu\times\nu$ upper triangular matrix with entries in $\mathbb{Q}$.*

Proof. By §2.2, ϕ is regular at 0 and admits a basis of solutions at 0 of the form

$$(\zeta_1, \zeta_2, \dots, \zeta_\nu) := (f_1, f_2, \dots, f_\nu)x^C$$

such that

(1) $f_1, \dots, f_\nu \in K[[x]]$,
(2) $C = D + N$ is an $\nu \times \nu$ matrix, where D is a diagonal matrix whose diagonal entries $D_{ii} := \alpha_i \in K$ are the exponents of ϕ at 0, and $N = (N_{ij})$ is an upper triangular nilpotent matrix with entries in $\mathbb{Q}$ such that $DN = ND$.

Since

$$x^C = x^{D+N} = x^D \sum_{k\geq 0} \frac{N^k}{k!}(\ln x)^k = x^D + x^D \sum_{k=1}^{\nu} \frac{N^k}{k!}(\ln x)^k,$$

we obtain

$$\zeta_1 = f_1 x^{\alpha_1}, \tag{6.1}$$

and for $1 < i \leq \nu$

$$\zeta_i = f_i x^{\alpha_i} + \sum_{j=1}^{i-1} f_j x^{\alpha_j} \sum_{k=1}^{\nu} \frac{(N^k)_{ji}}{k!}(\ln x)^k, \tag{6.2}$$

since $(N^k)_{ji} = 0$ for $j \geq i$.

In addition, by Lemma 4.1, there exists a positive integer m such that $\left(\frac{d}{dx}\right)^m \psi$ annihilates $\mathcal{L}(\zeta_i)$ for $i = 1, \dots, \nu$. We then define, $\Psi = \left(\frac{d}{dx}\right)^m \psi$. Applying Corollary 5.4 to ψ at infinity, we find that Ψ has a basis of solutions $\xi_1, \dots, \xi_{\mu+m}$ at infinity with elements in

$$\Big(\mathcal{R}_0^\infty(Y_\psi) \cap \mathcal{R}_0^\infty(f_E)\Big)[\ln x,\ x^{\pm\gamma_1}, \dots, x^{\pm\gamma_\mu}, \exp(\pm\delta_{11}x), \dots, \exp(\pm\delta_{\mu\mu}x)].$$

Now, let $\mathcal{A}_0$ denote the set

$$\mathbb{C}\otimes_K \Big(\mathcal{R}_0^\infty(Y_\psi)\cap\mathcal{R}_0^\infty(f_E)\Big)[\ln x, x^{\pm\gamma_1}, \dots, x^{\pm\gamma_\mu}, \exp(\pm\delta_{11}x), \dots, \exp(\pm\delta_{\mu\mu}x)].$$

Therefore, we have for all $1 \leq i \leq \nu$, $\mathcal{L}(\zeta_i) \in \mathcal{A}_0$. By induction on i, we deduce from (6.1) and (6.2) that

$$\mathcal{L}\Big(f_i x^{\alpha_i}\Big) \in \mathcal{A}_{i-1} \quad (i = 1, \dots, \nu), \tag{6.3}$$

where $\mathcal{A}_1, \dots, \mathcal{A}_{\nu-1}$ are the $\mathbb{C}[\ln x]$-modules of finite type defined recursively by

$$\mathcal{A}_i = \mathcal{A}_{i-1} + \Big\langle \mathcal{L}(f_i x^{\alpha_i}(\ln x)^j);\ 0 \leq j \leq \nu \Big\rangle_{\mathbb{C}[\ln x]}.$$

This shows, by iteration on i and by (4.12) and (4.13), that the exponents α_i are rational numbers. Thus, by Proposition 4.3, the Laplace transform of $f_i x^{\alpha_i}(\ln x)^k$ (for $i = 1, \ldots, \nu$ and $k \in \mathbb{Z}_{\geq 0}$) can be written as

$$\mathcal{L}\Big(f_i x^{\alpha_i}(\ln x)^k\Big) = \begin{cases} x^{-\alpha_i - 1}\Gamma(\alpha)\displaystyle\sum_{j=0}^{k} h_{i,k,j}\Big(\frac{1}{x}\Big)(\ln x)^j \\ \qquad\qquad \text{if} \quad \alpha_i \in \mathbb{Q}\setminus\mathbb{Z}_{<0}, \\ \displaystyle\sum_{j=0}^{k+1} h_{i,k,j}\Big(\frac{1}{x}\Big)(\ln x)^j \\ \qquad\qquad \text{if} \quad \alpha_i \in \mathbb{Z}_{<0}, \end{cases} \tag{6.4}$$

where $h_{i,k,j} \in \mathbb{C}\otimes_K \mathcal{R}_{-1}(f_i)$, $j = 0, \ldots, k$, $h_{i,k,k+1} \in K[x]\setminus\{0\}$ and $h_{i,k,k} \in K[[x]]\setminus\{0\}$ are such that $r_v(h_{i,k,k}) = r_v(f_i)\pi_v^{-1}$ for almost all $v \in V_0$. To conclude, it suffices to prove, by induction on i, that

$$f_i \in \mathcal{R}_1(Y_\psi)\cap\mathcal{R}_1(f_E), \quad (i = 1, \ldots, \nu). \tag{6.5}$$

Combining (6.3) with (6.4) for $i = 1$ and $k = 0$, we find that $\alpha_1 \in \{\pm\gamma_j + m \mid m \in \mathbb{Z},\ j = 1, \ldots, \mu\}$ and that $h_{1,0,0} \in \mathcal{R}_0(Y_\psi)\cap\mathcal{R}_0(f_E)$. Thus $f_1 \in \mathcal{R}_1(Y_\psi)\cap\mathcal{R}_1(f_E)$ and for any $0 \leq j \leq k$, we have

$$h_{1,k,j} \in \mathbb{C}\otimes_K (\mathcal{R}_0(Y_\psi)\cap\mathcal{R}_0(f_E)),$$

and hence, for any $k \geq 0$,

$$\mathcal{L}\Big(f_1 x^{\alpha_1}(\ln x)^k\Big) \in x^{-\alpha_1}\mathbb{C}\otimes_K (\mathcal{R}_0(Y_\psi)\cap\mathcal{R}_0(f_E))[\ln x].$$

This implies

$$\mathcal{A}_1 \ \subseteq\ \mathcal{A}_0.$$

Suppose now that, for some integer τ with $1 \leq \tau - 1 < \nu$, we have $f_i \in \mathcal{R}_1(Y_\psi)\cap\mathcal{R}_1(f_E)$, and $\alpha_i \in \{\pm\gamma_j + m \mid m \in \mathbb{Z},\ j = 1, \ldots, \mu\}$, for $i = 1, \ldots, \tau - 1$. Then, by (6.4),

$$h_{i,k,j} \in \mathbb{C}\otimes_K \mathcal{R}_0(Y_\psi)\cap\mathcal{R}_0(f_E)) \quad \text{for} \quad 1 \leq i \leq \tau - 1, \text{ and } \ 0 \leq j \leq k.$$

This implies $\mathcal{A}_{\tau-1} \ \subseteq\ \mathcal{A}_0$. In particular, by (6.3), we get $\mathcal{L}\Big(f_\tau x^{\alpha_\tau}\Big) \ \in \mathcal{A}_0$. Therefore, by (6.4), we find $\alpha_\tau \in \{\pm\gamma_j + m \mid m \in \mathbb{Z},\ j = 1, \ldots, \mu\}$ and $h_{\tau,0,0} \in \mathcal{R}_0(Y_\psi)\cap\mathcal{R}_0(f_E)$, and consequently $f_\tau \in \mathcal{R}_1(Y_\psi)\cap\mathcal{R}_1(f_E)$. This proves that $f_i \in \mathcal{R}_1(Y_\psi)\cap\mathcal{R}_1(f_E)$ and $\alpha_i \in \{\pm\gamma_j + m \mid m \in \mathbb{Z},\ j = 1, \ldots, \mu\}$ for $i = 1, \ldots, \nu$. On the other hand, by Corollary 3.4, the power series f_i are entries of the inverse of a reduction matrix of A_ϕ. Therefore, by Proposition 2.1, they satisfy $r_v(f_i) \neq 0$ for any $v \in V_0$. Combining this with the fact that $f_i \in \mathcal{R}_1(Y_\psi)\cap\mathcal{R}_1(f_E)$ for $i = 1, \ldots, \nu$, we get

$$\prod_{v\in V_0} \min(r_v(f_i), 1) \neq 0 \quad \text{ for } \quad i = 1, \ldots, n.$$

The lemma follows therefore since $\alpha_1, \dots, \alpha_\nu \in \mathbb{Q}$. $\square$

Proof of Theorem 5.2. First, by §2.3, the differential operator ϕ^* is regular at 0. In addition, by Corollary 3.7, the differential system $dX/dx = A_{\overline{\psi^*}}X$ has a solution of the form $\widetilde{Y}(\frac{1}{x})(\frac{1}{x})^{-{}^T\Gamma}\exp(\Delta x)$, where $\widetilde{Y}(x) \in \mathrm{GL}_\mu(K((x)))$ such that $\prod_{v \in V_0} \min(r_v(\widetilde{Y}), 1) \neq 0$. Moreover, we have $\mathcal{F}(\phi^*) = (\mathcal{F}\overline{\phi})^* = \overline{\psi^*}$ (cf. [Ma, V.3.6])). Then, by the same proof as in Lemma 6.2, we find that ϕ^* has also a basis of solutions at infinity of the form $(z_1, \dots, z_\nu)x^\Lambda$, where $z_1, \dots, z_\nu$ are power series of $K[[x]]$ such that $\prod_{v \in V_0} \min(r_v(z_i), 1) \neq 0$ for $i = 1, \dots, \nu$, and where Λ is a $\nu \times \nu$ upper triangular matrix with entries in $\mathbb{Q}$. Combining this with Lemma 6.2 and Lemma 3.4, we find that the differential system $dX/dx = A_\phi X$ has a solution at 0 of the form $Y_\phi(x)x^C$, where $Y(x) \in GL_\nu(K((x)))$ such that $\prod_{v \in V_0} \min(r_v(Y_\phi), 1) \neq 0$, and where $C \in M_\nu(\mathbb{Q})$ is an upper triangular matrix (see proof of Theorem 3.8). Hence ϕ is a G-operator and consequently ψ is an E-operator. $\square$

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Department of Mathematics, University of Ottawa, 585 King Edward, Ottawa, Ontario K1N 6N5, Canada

Current address: Dipartimento di Matematica Pura ed Applicata, Universita degli Studi di Padova, Via G.B. Belzoni, 7 - 35131 Padova, Italy

E-mail address: `manjra@math.unipd.it`