Illinois Journal of Mathematics
Volume 45, Number 4, Winter 2001, Pages 1347–1350
S 0019-2082

DEFINABLE BOOLEAN COMBINATIONS OF OPEN SETS ARE BOOLEAN COMBINATIONS OF OPEN DEFINABLE SETS

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Abstract. We show that, in any topological space, boolean combinations of open sets have a canonical representation as a finite union of locally closed sets. As an application, if $\mathfrak{M}$ is a first-order topological structure, then sets definable in $\mathfrak{M}$ that are boolean combinations of open sets are boolean combinations of open definable sets.

Let X be a topological space. We show that boolean combinations of open sets have a canonical representation. We then give an application of this result to first-order definability theory.

Given $A \subseteq X$, denote the closure of A by $\mathrm{cl}(A)$. The *frontier* of A, denoted by $\mathrm{fr}(A)$, is the set $\mathrm{cl}(A) \setminus A$. Given $x \in A$, A is *locally closed* at x if there exists an open neighborhood U of x such that $A \cap U = \mathrm{cl}(A) \cap U$; A is *locally closed* if A is locally closed at each $x \in A$. It is easy to check that the following are equivalent:

- A is locally closed.
- $A = \mathrm{cl}(A) \cap U$ for some open U.
- $A = F \cap U$ for some open U and closed F.
- $\mathrm{fr}(A)$ is closed.
- $A \cap \mathrm{cl}(\mathrm{fr}(A)) = \emptyset$.

Note also that A is a boolean combination of open sets if and only if A is a finite union of locally closed sets. (The forward implication follows easily from passing to a disjunctive normal form.)

We define the *locally closed points of A*, denoted by $\mathrm{lc}(A)$, to be the set $A \setminus \mathrm{cl}(\mathrm{fr}(A))$, that is, $\mathrm{lc}(A)$ is the relative interior of A in $\mathrm{cl}(A)$. Note that $\mathrm{lc}(A)$ is locally closed and $A \setminus \mathrm{lc}(A) = A \cap \mathrm{cl}(\mathrm{fr}(A)) = \mathrm{fr}(\mathrm{fr}(A))$. Inductively define sets $A^{(k)}$ as follows:

$$A^{(0)} = A; \quad A^{(k+1)} = A^{(k)} \setminus \mathrm{lc}\big(A^{(k)}\big) \; .$$

Received November 1, 2000.
2000 *Mathematics Subject Classification.* Primary 03C99, 54A99. Secondary 03C64.
The second author was supported by NSF Grant No. DMS-9896225.

(The construction is extended transfinitely in the obvious way.) The set $A^{(1)}$ ($= A \setminus \mathrm{lc}(A)$) is often called the *residue* of A.

THEOREM. *The set A is a boolean combination of open sets if and only if there exists $k \in \mathbb{N}$ such that $A^{(k)} = \emptyset$.*

Surprisingly, we were unable to locate any published statement of this theorem. (Compare this with the classical result that if X is a Polish space, then $A \in \mathrm{F}_\sigma \cap \mathrm{G}_\delta$ if and only if there exists a countable ordinal α such that $A^{(\alpha)} = \emptyset$; see, e.g., Hausdorff [3, §30].)

The Theorem is immediate from the following result, which we will prove below.

PROPOSITION. *For $k \in \mathbb{N}$, A is a union of k locally closed sets if and only if $A^{(k)} = \emptyset$.*

Hence, if k is minimal such that $A^{(k)} = \emptyset$, then A is a disjoint union of k distinct locally closed sets, and A cannot be represented as a union of fewer than k locally closed sets.

First, we have two easy lemmas, the proofs of which we leave to the reader.

LEMMA 1. *Let $A_1, \ldots, A_n \subseteq X$. Then*

$$\mathrm{fr}\left(\bigcup_{i=1}^n A_i\right) = \bigcup_{i=1}^n \left(\mathrm{fr}(A_i) \setminus \bigcup_{j \neq i} A_j\right)$$

and

$$\mathrm{fr}\left(\bigcap_{i=1}^n (X \setminus A_i)\right) \subseteq \bigcup_{i=1}^n A_i \ .$$

LEMMA 2. *If A is closed, then $\mathrm{fr}(A \cap B) \subseteq A \cap \mathrm{fr}(B)$ for every $B \subseteq X$.*

Notation. Define $\mathrm{fr}^k(A)$ inductively for $k \in \mathbb{N}$ as follows:

$$\mathrm{fr}^0(A) = A; \quad \mathrm{fr}^{k+1}(A) = \mathrm{fr}(\mathrm{fr}^k(A)) \ .$$

Note that $A^{(k)} = \mathrm{fr}^{2k}(A)$.

LEMMA 3. *Let $k, m \in \mathbb{N}$. Let $\mathcal{F}$ be a finite collection of closed sets and A be in the boolean algebra generated by $\mathcal{F}$. If each point of A belongs to at least m elements of $\mathcal{F}$, then each point of $\mathrm{fr}^k(A)$ belongs to at least $k+m$ elements of $\mathcal{F}$.*

Proof. For $k = 0$, the result is trivial. We now proceed by induction on $k \geq 1$.

First, let $k = 1$. There exist a nonnegative integer l and collections

$$\mathcal{B}_1, \dots, \mathcal{B}_l, \mathcal{C}_1 \dots, \mathcal{C}_l \subseteq \mathcal{F}$$

such that

$$A = \bigcup_{i=1}^{l} \left(\bigcap \mathcal{B}_i \cap \bigcap \{ X \setminus C : C \in \mathcal{C}_i \} \right),$$

where, for $i = 1, \dots, l$, we have $\mathcal{B}_i \cap \mathcal{C}_i = \emptyset$ and $\operatorname{card}(\mathcal{B}_i) \geq m$. Each $\bigcap \mathcal{B}_i$ is closed, so by Lemmas 1 and 2 we have $\operatorname{fr}(A) \subseteq \bigcup_{i=1}^{l} (\bigcap \mathcal{B}_i \cap \bigcup \mathcal{C}_i)$. Hence, if $x \in \operatorname{fr}(A)$ then there exists $i \in \{1, \dots, l\}$ and $F \in \mathcal{F} \setminus \mathcal{B}_i$ such that $x \in \bigcap \mathcal{B}_i \cap F$. Then x belongs to at least $m + 1$ elements of $\mathcal{F}$.

Assume now that the result holds for some $k \geq 1$; we will prove it for $k + 1$. By the previous paragraph, each point of $\operatorname{fr}(A)$ belongs to at least $m + 1$ elements of $\mathcal{F}$. If $\operatorname{cl}(A) \in \mathcal{F}$, then $\operatorname{fr}(A)$ belongs to the boolean algebra generated by $\mathcal{F}$. By applying the inductive assumption to $\operatorname{fr}(A)$, we see that each point of $\operatorname{fr}^{k+1}(A)$ belongs to at least $m + k + 1$ elements of $\mathcal{F}$. On the other hand, if $\operatorname{cl}(A) \notin \mathcal{F}$, then each point of $\operatorname{fr}(A)$ belongs to at least $m + 2$ elements of the collection $\mathcal{F} \cup \{\operatorname{cl}(A)\}$. Applying the inductive assumption, each point of $\operatorname{fr}^{k+1}(A)$ belongs to at least $m + k + 2$ elements of $\mathcal{F} \cup \{\operatorname{cl}(A)\}$. Hence, each point of $\operatorname{fr}^{k+1}(A)$ belongs to at least $m + k + 1$ elements of $\mathcal{F}$. $\square$

Proof of the Proposition. Let $k \in \mathbb{N}$ and suppose that A is a union of k locally closed sets. Then A is a boolean combination of $2k$ closed sets, and each point of A belongs to at least one of these closed sets. By Lemma 3, $\emptyset = \operatorname{fr}^{2k}(A) = A^{(k)}$. Conversely, if $A^{(k)} = \emptyset$, then $A = \bigcup_{i=0}^{k-1} \operatorname{lc}\left(A^{(i)}\right)$. $\square$

We now give some applications to first-order definability theory.

Let $\mathfrak{M}$ be a first-order structure, with underlying set M, in a language L. Suppose there exist a positive integer l and an $(l+1)$-ary L-formula ϕ such that the collection $\left\{ \{ t \in M : \mathfrak{M} \models \phi(x, t) \} : x \in M^l \right\}$ is a basis for a topology on M. For each $n \in \mathbb{N}$, equip the cartesian product M^n with the product topology induced by the topology on M. (Regard M^0 as the one-point space $\{\emptyset\}$.) Following Pillay [6], we call $\mathfrak{M}$—more precisely, $(\mathfrak{M}, \phi)$—a *first-order topological structure.*

A familiar example is the case that $\mathfrak{M}$ expands a dense linearly ordered set $(M, <)$ with no first or last element: The formula $x_1 < t < x_2$ yields a basis for the usual order topology on M as (x_1, x_2) ranges over M^2. (Indeed, the results of this paper were motivated mainly by the prospects of investigating, via topological methods, expansions of the real line $(\mathbb{R}, <)$; for examples, see Friedman and Miller [2] and Miller and Speissegger [5].) For a discussion of other examples, see Pillay [6] or Mathews [4].

Let $A \subseteq M^n$ be definable; then so are $\operatorname{cl}(A)$, $\operatorname{fr}(A)$, $\operatorname{lc}(A)$, and each $A^{(k)}$ for $k \in \mathbb{N}$. (We take "definable" to mean "definable, in $\mathfrak{M}$, with parameters from some fixed $C \subseteq M$".) If A is locally closed, then there is an open definable set U (namely, $M \setminus \mathrm{cl}(\mathrm{fr}(A))$) such that $A = \mathrm{cl}(A) \cap U$. Hence:

Corollary 1. *Let $A \subseteq M^n$ be definable and a boolean combination of open sets. Then A is a boolean combination of open definable sets.*

(The above answers a question raised by van den Dries [private communication] in connection with dense pairs of o-minimal structures [1].)

Corollary 2. *Let $k, n \in \mathbb{N}$ and $A \subseteq M^n$ be definable and a union of k locally closed subsets of M^n. Then A is a disjoint union of k locally closed definable subsets of M^n.*

Corollary 3. *Let $k, m, n \in \mathbb{N}$ and $A \subseteq M^{m+n}$ be definable. Then the set of all $x \in M^m$ such that the fiber $\{y \in M^n : (x, y) \in A\}$ is a union of k locally closed sets is definable.*

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