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O-MINIMAL STRUCTURES: LOW ARITY VERSUS GENERATION

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Abstract. We show that an analogue of Hilbert's Thirteenth Problem fails in the real subanalytic setting. Namely we prove that, for any integer n, the o-minimal structure generated by restricted analytic functions in n variables is strictly smaller than the structure of all global subanalytic sets, whereas these two structures define the same subsets in $\mathbb{R}^{n+1}$.

1. Introduction

The aim of this paper is to prove that, for any fixed $n \in \mathbb{N}$, the o-minimal structure generated by the family of all global subanalytic subsets of $\mathbb{R}^n$ is strictly smaller than the structure of all global subanalytic sets: some subanalytic subsets of $\mathbb{R}^{n+1}$ are "transcendental" over the family of all subanalytic subsets of $\mathbb{R}^n$.

The main motivation for this work was to prove that the statement

> "*Given an o-minimal structure* $\mathcal{S}$ *over* X*, there is an integer* n *such that* $\mathcal{S}$ *and* $\operatorname{str}(\mathcal{S}^{(n)})$*—its reduct generated by* $\mathcal{S}$*-definable subsets of* X^n*—define the same subsets of* X^N*, for all* N*.*"

is false. We now know it fails in the case $\mathcal{S}$ is the structure of global subanalytic sets.

This result can be seen as a negative answer to a generalized real analytic version of the second part of Hilbert's Thirteenth Problem: subanalytic functions do not have the superposition property (see [12] for the positive answer in the continuous setting).

In Section 2, we give the following definitions: o-minimal structure, generated structure, subanalytic sets and sub-n-analytic sets; only the last one is original. We then recall some well known properties.

In Section 3, we show that restricted analytic functions in n variables and subanalytic subsets of $\mathbb{R}^{n+1}$ have the same definability power. This elegant

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proof is due to Daniel J. Miler and is based on Hironaka's Uniformization Theorem for subanalytic sets.

In Section 4-7, we use Gabrielov's "Explicit Fibre Cutting Lemma", a diagonal argument on formal series and metric control on truncation of translated power series, to prove that there is a restricted analytic function $f : [-1,1]^{n+1} \to \mathbb{R}$ whose graph cannot be defined by mean of restricted analytic functions in n variables.

2. Definitions

Definition 2.1. We call $\mathcal{S} = (\mathcal{S}^{(n)})_{n\in\mathbb{N}}$ a structure over $(\mathbb{R}; +, \cdot\,)$ if it has the following properties:

(S1) $\mathcal{S}^{(n)}$ is a boolean subalgebra of $\mathcal{P}(\mathbb{R}^n)$ for each $n \in \mathbb{N}$.
(S2) If n is an integer and A is a semialgebraic subset of $\mathbb{R}^n$ then $A \in \mathcal{S}^{(n)}$.
(S3) If $A \in \mathcal{S}^{(n)}$, then $\mathbb{R} \times A \in \mathcal{S}^{(n+1)}$.
(S4) If $A \in \mathcal{S}^{(n+1)}$ and $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the cartesian projection $\pi(x_1, \ldots, x_{n+1}) = (x_1, \ldots x_n)$ then $\pi(A) \in \mathcal{S}^{(n)}$.

It is said to be an *o*-minimal structure over $(\mathbb{R}; +, \cdot\,)$, if, in addition, it has the following property:

(S5) Every element of $\mathcal{S}^{(1)}$ is a finite union of singletons and open intervals.

In other words, a structure over $(\mathbb{R}; +, \cdot)$ is a collection of real sets, containing the family of all semialgebraic sets and stable under natural set theoretical operations: union, intersection, complementation, cartesian projection and cartesian product. The structure is *o*-minimal if the elements of $\mathcal{S}^{(1)}$ are the simplest possible: finite unions of intervals and points.

Elements of $\bigcup_n \mathcal{S}^{(n)}$ are called $\mathcal{S}$-definable sets; given an $\mathcal{S}$-definable set A, we call the integer n such that $A \in \mathcal{S}^{(n)}$ the *arity* of A.

A function f from some $A \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$ is said to be $\mathcal{S}$-definable if its graph is an $\mathcal{S}$-definable set.

For an introduction to the geometry in *o*-minimal structure, see, for instance, [6] or [7].

Let us now define the notion of *generated structure*.

If $\mathcal{U} = (\mathcal{U}^{(n)})_{n\in\mathbb{N}}$ and $\mathcal{V} = (\mathcal{V}^{(n)})_{n\in\mathbb{N}}$ are such that $\mathcal{U}^{(n)} \subseteq \mathcal{P}(\mathbb{R}^n)$ and $\mathcal{V}^{(n)} \subseteq \mathcal{P}(\mathbb{R}^n)$, we will denote by $\mathcal{U} \sqsubseteq \mathcal{V}$ the property "$\mathcal{U}^{(n)} \subseteq \mathcal{V}^{(n)}$ for all $n \in \mathbb{N}$".

If $\mathcal{A} = (\mathcal{A}^{(n)})_{n\in\mathbb{N}}$ is such that $\mathcal{A}^{(n)} \subseteq \mathcal{P}(\mathbb{R}^n)$, there exists a smallest element (for the partial order $\sqsubseteq$ on $\prod_{n\in\mathbb{N}} \mathcal{P}(\mathcal{P}(\mathbb{R}^n))$) among the $\mathcal{S} = (\mathcal{S}^{(n)})_{n\in\mathbb{N}}$ forming a structure over $(\mathbb{R}; +, \cdot\,)$ and satisfying $\mathcal{A} \sqsubseteq \mathcal{S}$. We will denote this structure by $\operatorname{str}(\mathcal{A})$, and call it the *structure generated by* $\mathcal{A}$.

REMARK 2.2. Let n_0 be an integer and $\mathcal{F}^{(n_0)}$ a subset of $\mathcal{P}(\mathbb{R}^{n_0})$; when no confusion is possible, we will identify $\mathcal{F}^{(n_0)}$ and the family

$$\mathcal{G} = (\mathcal{G}^{(n)})_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathcal{P}(\mathcal{P}(\mathbb{R}^n)),$$

where $\mathcal{G}^{(n)} = \emptyset$ if $n \neq n_0$ and $\mathcal{G}^{(n_0)} = \mathcal{F}^{(n_0)}$.

In such a case $\mathrm{str}\,(\mathcal{F}^{(n_0)})$ stands for $\mathrm{str}\,(\mathcal{G})$.

Given an $n \in \mathbb{N}$, we let $\mathcal{B}(n)$ be the algebra of all functions $f : [-1,1]^n \to \mathbb{R}$ such that f admits an analytical continuation in a neighbourhood of $[-1,1]^n$. We call such a function f a *restricted analytic function* (in n variables).

Let $\mathcal{E} = (\mathcal{E}^{(n)})_{n\in\mathbb{N}^*}$ be the element of $\prod_{n\in\mathbb{N}^*} \mathcal{P}(\mathcal{P}(\mathbb{R}^n))$ defined by

$$\mathcal{E}^{(n+1)} := \{\mathrm{graph}(f)\,,\, f \in \mathcal{B}(n)\}.$$

With the previous notation, we denote by $\mathbb{R}_{\mathrm{an}}$ the structure $\mathrm{str}\,(\mathcal{E})$.

THEOREM 2.3 (Gabrielov). *$\mathbb{R}_{\mathrm{an}}$ is an o-minimal structure.*

An element A in $\mathbb{R}_{\mathrm{an}}$ is called a *global subanalytic set*.

DEFINITION 2.4. Given an integer n we let

$$\mathbb{R}_{\mathrm{an}(n)} := \mathrm{str}\,(\mathcal{E}^{(n+1)});$$

$\mathbb{R}_{\mathrm{an}(n)}$-definable sets are called *global sub-n-analytic sets*.

In other words, $\mathbb{R}_{\mathrm{an}(n)}$ is the structure generated by the graphs of all restricted analytic functions *in at most n variables* (whereas there is no bound on the number of variables for the restricted analytic functions used to generate $\mathbb{R}_{\mathrm{an}}$).

For instance,

$$\{(x_1,x_2,x_3) \in [-1,1]^3; \cos\frac{x_1+x_2}{2} + \sin\frac{x_3-\cos x_2}{2} > 0\}$$

is a $\mathbb{R}_{\mathrm{an}(1)}$-definable subset of $\mathbb{R}^3$.

PROPOSITION 2.5. *$\mathbb{R}_{\mathrm{an}(n)}$ is model complete (as a $\mathcal{B}(n)$-structure).*

Let p be an integer; we will denote by $A_p(\mathcal{B}(n))$ the subalgebra of $\mathcal{B}(p)$ generated by all the functions

$$(x_1,\ldots,x_p) \mapsto f(x_{\sigma(1)},\ldots,x_{\sigma(n)}),$$

as σ ranges over $\{1,\ldots,p\}^{\{1,\ldots,n\}}$ (the set of functions from $\{1,\ldots,n\}$ to $\{1,\ldots,p\}$) and f ranges over $\mathcal{B}(n)$ (the set of restricted analytic functions in n variables).

Once we have noted that, for every $p \in \mathbb{N}$, the algebra $A_p(\mathcal{B}(n))$ is stable under the action of partial derivation operators, Proposition 2.5 easily follows from Gabrielov's "Explicit Model Completeness" ([11, Theorem 1 and Corollary]).

We will use a more precise version of this result in Sections 4 and 5 to show how $\mathbb{R}_{\mathrm{an}(n)}$-definable functions are controlled by restricted analytic functions in at most n variables.

3. Sub-n-analytic sets

PROPOSITION 3.1. *$\mathbb{R}_{\mathrm{an}(n)}$ is the structure generated by global subanalytic sets of arity $n+1$.*

The following proof is due to Daniel J. Miller.

The inclusion $\mathbb{R}_{\mathrm{an}(n)} \sqsubseteq \mathrm{str}\,(\mathbb{R}_{\mathrm{an}}^{(n+1)})$ is easy.

Let us prove the other inclusion by induction on n. The case $n = 0$ is clear, so let $n > 0$ and assume the results holds for $n - 1$.

Denote by K the set $[-1, 1]^n$. By the cell decomposition theorem ([7, Theorem 2.11]), it is enough to prove that, given an $\mathbb{R}_{\mathrm{an}}$-definable function $f : C \to \mathbb{R}$, where C is an $\mathbb{R}_{\mathrm{an}}$-cell either included in or disjoint from K, then f is $\mathbb{R}_{\mathrm{an}(n)}$-definable.

Note that the mapping $i : (x_1, \dots, x_n) \mapsto (1/x_1, \dots, 1/x_n)$ is $\mathbb{R}_{\mathrm{an}(n)}$-definable and sends $\mathbb{R} \setminus K$ to K; we thus can suppose that $A \subseteq K$.

Up to a finer cell decomposition, we can furthermore suppose that $|f(\overline{x})| - 1$ has constant sign on C and, since $y \mapsto 1/y$ is $\mathbb{R}_{\mathrm{an}(n)}$-definable, we can assume that $|f(\overline{x})| \leq 1$ for all $\overline{x} \in C$.

Let G be the closure of the graph of f; G is a compact subanalytic set of dimension $d \leq n$.

Hironaka's uniformization theorem ([1, Theorem 0.1]) gives a d-dimensional analytic manifold Y and a surjective analytic proper mapping $\psi : Y \to G$.

Since G is compact and ψ surjective and proper, Y is compact; we then easily get a finite family $\{\phi_i : [-1, 1]^d \to Y\}_{i=1,\dots,s}$ of restricted analytic functions such that the union of their images is covering Y.

Hence $G = \bigcup_{i=1}^{s} \psi \circ \phi_i([-1, 1]^d)$ is an $\mathbb{R}_{\mathrm{an}(d)}$-definable set and thus an $\mathbb{R}_{\mathrm{an}(n)}$-definable set.

By the induction hypothesis, C is an $\mathbb{R}_{\mathrm{an}(n-1)}$-definable set and thus an $\mathbb{R}_{\mathrm{an}(n)}$-definable set. The function f is $\mathbb{R}_{\mathrm{an}(n)}$-definable, for its graph, $G \cap (C \times \mathbb{R})$, is.

4. n-regularity

In the following sections, we prove that there are some $\mathbb{R}_{\mathrm{an}}$-definable analytic functions in $n + 1$ variables which are not $\mathbb{R}_{\mathrm{an}(n)}$-definable. We first show how each $\mathbb{R}_{\mathrm{an}(n)}$-definable function is "controlled", through the notion of *n-regularity*, by the restricted analytic functions in n variables used to define it.

Let n and p be two integers; in the proof of Proposition 2.5 we have defined the algebra $A_p(\mathcal{B}(n))$.

By definition, each $g \in A_p(\mathcal{B}(n))$ can be written in the form

$$g(x_1, \dots, x_p) = Q\big(h_1(x_{\sigma_1(1)}, \dots, x_{\sigma_1(n)}), \dots, h_q(x_{\sigma_q(1)}, \dots, x_{\sigma_q(n)})\big),$$

where q is an integer, Q is a polynomial in q variables with integer coefficients, the h_i's are restricted analytic functions in n variables and the σ_i's are mappings from $\{1, \dots, n\}$ to $\{1, \dots, p\}$.

We will call an element of $A_p(\mathcal{B}(n))$ a *restricted analytic function in p variables which essentially depends on at most n variables.*

In some sense, the graph of an $\mathbb{R}_{\mathrm{an}(n)}$-definable function looks almost everywhere like an analytic variety defined as a zero-set of restricted analytic functions depending on at most n variables.

Let us make this statement more precise: we first recall a special case of Gabrielov's "Explicit Fibre Cutting Lemma" (see [11, Lemma 3 and Theorem 1]):

THEOREM 4.1 (Gabrielov). *Given a d-dimensional sub-n-analytic set $Y \subseteq \mathbb{R}^m$, there is a $p \in \mathbb{N}$, a finite family $\{X_\nu\}$ of sub-n-analytic subsets of $\mathbb{R}^{m+p}$ and a sub-n-analytic subset V of $\mathbb{R}^{m+p}$ such that, if $\pi : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^m$ is given by $\pi(x_1, \dots, x_{m+p}) = (x_1, \dots, x_m)$, one has:*

1. *$Y = \pi(V) \cup \bigcup \pi(X_\nu)$;*
2. *$\dim \pi(V) < d$;*
3. *for each ν, $\dim X_\nu = d$ and $\pi_{|X_\nu} : X_\nu \to Y$ has rank d at every point of X_ν;*
4. *for each $\overline{s} \in X_\nu$, $\{\overline{x} - \overline{s}\,; \overline{x} \in X_\nu\}$ is near $\overline{0}$ the zero-set of $m+p-d$ elements $f_i : \mathbb{R}^{m+p} \to \mathbb{R}$ of $A_{m+p}(\mathcal{B}(n))$, $(df_i)_i$ having rank $m+p-d$ at $\overline{0}$;*
5. *$X_\lambda \cap X_\mu = \emptyset$ for $\lambda \neq \mu$.*

This theorem leads us to the following definition:

DEFINITION 4.2. Let f be a function from a neighbourhood U of $\overline{0}$ in $\mathbb{R}^{n+1}$, to $\mathbb{R}$. f is said to be n-regular at $\overline{0}$ if there exist

- an integer p,
- a $(p+1)$-tuple $(g_1, \cdots, g_{p+1})$ of elements of $A_{n+p+2}(\mathcal{B}_n)$,
- a neighbourhood $V \subseteq U$ of $\overline{0} \in \mathbb{R}^{n+1}$, and
- for each $\overline{x} \in V$, a point $(y_1(\overline{x}), \dots, y_p(\overline{x}))$ in $\mathbb{R}^p$,

such that

- $g_i(\overline{x}, y_1(\overline{x}), \dots, y_p(\overline{x}), f(\overline{x})) = 0$, for all i, and

- the rank of

$$\left(\frac{\partial g_i}{\partial z_j}\right)_{\substack{1\le i\le p+1\\ n+2\le j\le n+p+2}}$$

is full at the point $(\overline{x}, y_1(\overline{x}), \ldots, y_p(\overline{x}), f(\overline{x}))$.

A function f from a neighbourhood U of $\overline{a} \in \mathbb{R}^{n+1}$ is said to be n-subregular at $\overline{a}$ if $\overline{x} \mapsto f(\overline{a} + \overline{x})$ is n-regular at $\overline{0}$.

In other words, f is n-regular at $\overline{0}$ if, as in Theorem 4.1, the germ of its graph is the germ of the projection $\pi(X)$ of an analytic manifold X given as the zero-set of some functions depending essentially on at most n variables, and $\pi_{|X}$ is locally a diffeomorphism.

Proposition 4.3. *Given an $\mathbb{R}_{\mathrm{an}(n)}$-definable function $f : [-1,1]^{n+1} \to \mathbb{R}$, there is a point $\overline{a} \in]-1,1[^n$ such that f is n-regular at $\overline{a}$.*

This proposition follows from an easy dimensional argument and Theorem 4.1.

5. Diagonalization

In the sequel we will build a function $h : [-1,1]^{n+1} \to \mathbb{R}$ such that

- there is no $\overline{a} \in]-1,1[^{n+1}$, at which h is n-regular (and thus h cannot be $\mathbb{R}_{\mathrm{an}(n)}$-definable),
- but h is a restriction to $[-1,1]^{n+1}$ of some analytic function from $\mathbb{R}^{n+1}$ to $\mathbb{R}$ (and consequently is $\mathbb{R}_{\mathrm{an}}$-definable).

We will now "enumerate" the germs (above $\overline{0} \in \mathbb{R}^{n+1}$) of n-regular (at $\overline{0}$) functions $f : \mathbb{R}^{n+1} \to \mathbb{R}$.

We first have to choose a value y for $f(0, \ldots, 0)$.

By the definition of n-regularity, it is enough to consider, as p ranges over $\mathbb{N}$, all $(p+1)$-tuples $(g_1, \ldots, g_{p+1})$ of elements in $\mathcal{A}_{n+p+2}(\mathcal{B}_n)$ such that $g_i(0, \ldots, 0, y) = 0$ and the rank of

$$\left(\frac{\partial g_i}{\partial z_j}\right)_{\substack{1\le i\le p+1\\ n+2\le j\le n+p+2}}$$

is full at points $(0, \ldots, 0, y)$.

Let us fix such a $p \in \mathbb{N}$.

By definition, each $g \in \mathcal{A}_{n+p+2}(\mathcal{B}_n)$ is of the following form:

- there is a $q \in \mathbb{N}$ and a $Q \in \mathbb{Z}[T_1, \ldots, T_q]$,
- there are some $h_1, \ldots, h_q \in \mathcal{B}(n)$,
- for each $i \in \{1, \ldots, q\}$, there is a mapping σ_i from $\{1, \ldots, n\}$ to $\{1, \ldots, n+p+2\}$,

such that

$$g(x_1,\dots,x_{n+p+2}) = Q(h_1(x_{\sigma_1(1)},\dots,x_{\sigma_1(n)}),\dots,h_q(x_{\sigma_q(1)},\dots,x_{\sigma_q(n)})).$$

So let us fix a $q \in \mathbb{N}$, a $(p+1)$-tuple of elements in $\mathbb{Z}[T_1,\dots,T_q]$ and, for each $1 \le j \le q$ and $1 \le i \le p+1$, a mapping σ_j^i from $\{1,\dots,n\}$ to $\{1,\dots,n+p+2\}$.

The only parameters left free are now

- the value of y of $f(0,\dots,0)$,
- the $(p+1)q$-tuple of restricted analytic functions h in n variables.

All those germs are thus built by choosing a set of "assembly instructions" (the integers p and q, polynomials Q and mappings σ) and then by assembling "pieces" (the restricted analytic functions h in n variables) that fit this set of instructions.

Let

$$s \mapsto \left((p(s),q(s)),(Q_k(s))_{1\le k\le p(s)+1},(\sigma_j^i(s))_{\substack{1\le j\le q(s)\\ 1\le k\le p(s)+1}}\right)$$

be a surjective mapping from $\mathbb{N}$ to

$$\coprod_{(p,q)\in\mathbb{N}^2}\{(p,q)\}\times(\mathbb{Z}[T_1,\dots,T_q])^{p+1}\times\left(\left(\{1,\dots,n+p+2\}^{\{1,\dots,n\}}\right)^q\right)^{p+1}.$$

Fix an $s \in \mathbb{N}$, and thus some integers $p(s)$, $q(s)$, some polynomials

$$(Q_k(s))_{1\le k\le p(s)+1}$$

and some mappings

$$(\sigma_j^k(s))_{\substack{1\le j\le q(s)\\ 1\le k\le p(s)+1}}.$$

Then let M_s be the subset of

$$\mathbb{R}\times\left(\mathbb{R}\{X_1,\dots,X_n\}^{q(s)}\right)^{p(s)+1}$$

consisting of the elements

$$\left(y,\left((g_j^k)_{1\le j\le q(s)}\right)_{1\le k\le p(s)+1}\right)$$

that satisfy the conditions in Definition 4.2:

(1) $h_i(0,\dots,0,y)=0, \forall i\in\{1,\dots,p(s)+1\}$,

(2) the rank of

$$\left(\frac{\partial h_i}{\partial x_j}\right)_{\substack{1\le i\le p+1\\ n+1\le j\le n+p+2}}$$

at $(0,\dots,0,y)$ is full, with

$$h_k(x_1,\dots,x_{n+p(s)+2}) = Q_k(s)\big(g_1^k(\overline{x}^{\sigma(s)_1^k}),\dots,g_{q(s)}^k(\overline{x}^{\sigma(s)_{q(s)}^k})\big),$$

and

$$\overline{x}^{\sigma(s)_j^k} = (x_{\sigma(s)_j^k(1)},\dots,x_{\sigma(s)_j^k(n)}).$$

Then, by the Implicit Function Theorem, we have a mapping

$$\Phi^s : M_s \longrightarrow \mathbb{R}\{Y_1, \ldots, Y_{n+1}\}$$

which sends

$$\left(y, (g_j^k)_{\substack{1\leq j\leq q(s)\\ 1\leq k\leq p(s)+1}} \right)$$

to an analytic function f defined in a neighbourhood of $\overline{0} \in \mathbb{R}^{n+1}$ and satisfying

- $f(0, \ldots, 0) = y$;
- there are analytic functions $(f_1, \ldots, f_{p(s)})$ in a neighbourhood of $(0, \ldots, 0)$ such that the graph of $(f_1, \ldots, f_{p(s)}, f)$ is, in a neighbourhood of $(0, \ldots, 0, y)$, the zero-set of the h_i's.

Remark 5.1. By the definition of n-regularity, if $f : U \to \mathbb{R}$ is n-regular at $\overline{0} \in \mathbb{R}^{n+1}$, then the germ of f at $\overline{0}$ is in $\bigcup_{s\in\mathbb{N}} \Phi^s(M_s)$.

Let us denote by $\mathbb{R}_{D,E}[X_1, \ldots X_m]$ the set of polynomials in k variables, of degree $< D$ and of order $\geq d$ at the origin, with real coefficients.

Definition 5.2. We denote the truncation mapping by

$$\begin{array}{rccl} T_{DE}^m : & \mathbb{R}\{X_1, \ldots, X_m\} & \to & \mathbb{R}_{D,E}[X_1, \ldots X_m] \\ & h & \mapsto & \displaystyle\sum_{D\leq|\nu|<E} \frac{\partial^{|\nu|}h}{\partial \overline{X}^{\nu}}(\overline{0}) \cdot \overline{X}^{\nu} \end{array}.$$

The chain rule for derivatives and an easy induction on E gives us the next proposition, which will be useful in deducing the non-surjectivity of the map Φ^s from the non-surjectivity of some rational mapping Φ_{DE}^s between finite dimensional spaces.

Proposition 5.3. *Given three integers s, D and E with $D < E$, let $\widetilde{M}_s$ be the image of M_s by the truncation*

$$\Pi := \mathrm{Id} \otimes ({T_{0E}^n}^{\otimes q(s)})^{\otimes(p(s)+1)}$$

of power series

$$\Pi : \mathbb{R} \times (\mathbb{R}\{X_1, \ldots, X_n\}^{q(s)})^{p(s)+1} \to \mathbb{R} \times (\mathbb{R}_{0,E}[X_1, \ldots, X_n]^{q(s)})^{p(s)+1}.$$

Then there is a rational mapping Φ_{DE}^s such that the diagram

$$\begin{array}{ccc} M_s & \xrightarrow{\Phi^s} & \mathbb{R}\{Y_1, \ldots, Y_{n+1}\} \\ \Pi \downarrow & & \downarrow T_{DE}^{n+1} \\ \widetilde{M}_s & \xrightarrow[\Phi_{DE}^s]{} & \mathbb{R}_{D,E}[Y_1, \ldots, Y_{n+1}] \end{array}$$

is commutative.

This proposition simply says that the derivatives at the origin of order $< E$ of an element ξ in the image of Φ^s depend only on y and on the derivatives at the origin of order $< E$ of the g_j^k used to define ξ in the source space of Φ^s, and that this dependence is in a rational manner.

6. Translation in the source space

The previous section would help us to produce, by a diagonal argument, an analytic function which is outside of the image of each Φ^s and thus is not n-regular at $\overline{0} \in \mathbb{R}^{n+1}$.

However, we want to construct a function which is *nowhere* n-regular in a neighbourhood of $\overline{0}$. Hence we have to consider $\overline{x} \mapsto h(\overline{\alpha} + \overline{x})$ as $\overline{\alpha}$ ranges over a neighbourhood (let us say $]-1,1[^{n+1}$) of $\overline{0}$; unfortunately, we then lose the finite dimensional dependency we found in the previous section.

More precisely, for $\overline{\alpha} \in]-1,1[^{n+1}$, if we let $\tau_{\overline{\alpha}}$ be the function that assigns to an analytic function h near $[-1,1]^{n+1}$ the function $\overline{x} \mapsto h(\overline{x}+\overline{\alpha})$ (which is analytic near $\overline{0}$), we do not have the equality

$$T^{n+1}_{DE}(\tau_{\overline{\alpha}}(h)) = T^{n+1}_{DE}\big(\tau_{\overline{\alpha}}\big(T^{n+1}_{DE}(h)\big)\big);$$

each partial derivative of $h_{\overline{\alpha}}$ at the origin depends on *all* partial derivatives of h at zero.

The aim of this section is to show that this dependency can, however, be handled by metric arguments.

We first equip each $\mathbb{R}_{D,E}[Y_1, \ldots, Y_{n+1}]$ with the norm

$$\left\| \sum_{\nu} a_\nu \overline{Y}^{\nu} \right\|_\infty = \max_{\nu}\{|a_\nu|\}.$$

Proposition 6.1. *Let (D_k) be a increasing sequence of integers, η a positive real number, $\overline{\alpha}$ a point in $]-1,1[^{n+1}$, h an analytic function in a neighbourhood of $[-1,1]^{n+1}$ and K an integer.*

If for all $k > K$ we have

$$\left\| T^{n+1}_{D_k D_{k+1}}(h) \right\|_\infty \le \frac{\eta}{2^k (D_{k+1}!)^{\,n+1}},$$

then

$$\left\| T^{n+1}_{D_K D_{K+1}}(\tau_{\overline{\alpha}}(h)) - T^{n+1}_{D_K D_{K+1}}\left(\tau_{\overline{\alpha}}\left(T^{n+1}_{D_K D_{K+1}}(h)\right)\right) \right\|_\infty \le \eta.$$

This is an easy consequence of the fact that, if $D_k \le |\mu| < D_{k+1}$, then

$$\frac{\partial^{|\mu|}\big(\tau_\alpha(h)\big)}{\partial Y_1^{\mu_1} \ldots Y_{n+1}^{\mu_{n+1}}}(\overline{0}) = \sum_{j \ge k} \sum_{\substack{\nu_i \ge \mu_i \\ D_j \le |\nu| < D_{j+1}}} \frac{\partial^{|\nu|} h}{\partial Y_1^{\nu_1} \ldots Y_{n+1}^{\nu_{n+1}}}(\overline{0}) \cdot \prod_i \binom{\nu_i}{\mu_i} {\alpha_i}^{\nu_i - \mu_i}$$

and

$$\left|\prod_i \binom{\nu_i}{\mu_i} {\alpha_i}^{\nu_i-\mu_i}\right| \leq (D_{k+1}!)^{n+1}$$

if $|\nu| < D_{k+1}$ and $|\overline{\alpha}| \leq 1$.

REMARK 6.2. The linear mapping $L^k_{\overline{\alpha}}$ on $\mathbb{R}_{D_k,D_{k+1}}[Y_1,\dots,Y_{n+1}]$ defined by $L^k_{\overline{\alpha}}(P) = T_{D_kD_{k+1}}(\tau_{\overline{\alpha}}(P))$ is an isomorphism, since the image of a monomial $\overline{X}^\nu$ is the sum of $\overline{X}^\nu$ and some lower degree monomials.

Furthermore we have the identity

$$\left\|(L^k_{\overline{\alpha}})^{-1}\right\|_\infty = \max\left\{1/\|L^k_{\overline{\alpha}}(P)\|_\infty\,;\|P\|_\infty = 1\right\}$$

and the mapping $(P,\overline{\alpha}) \mapsto 1/\|L^k_{\overline{\alpha}}(P)\|_\infty$ is continuous on the compact set $\{\|P\|_\infty = 1\} \times [-1,1]^{n+1}$.

Thus we have a bound S_k for the norm of $(L^k_{\overline{\alpha}})^{-1}$ that is independent of $\overline{\alpha} \in]-1,1[^{n+1}$.

7. Construction

We will use the good behaviour through truncation of the Φ^s to build a sequence of integers (D_s) and, for each $s \in \mathbb{N}$, a polynomial P_s in $\mathbb{R}_{D_s,D_{s+1}}[Y_1,\dots Y_n]$, such that the formal power series $h(Y_1,\dots,Y_{n+1}) = \sum_s P_s(Y_1,\dots,Y_{n+1})$ is the power expansion of an analytic function on $\mathbb{R}^{n+1}$, while $\tau_{\overline{\alpha}}(h)$ is outside of the image of Φ^s for each $s \in \mathbb{N}$ and $\overline{\alpha} \in]-1,1[^{n+1}$. The restriction to $[-1,1]^{n+1}$ of this function (which is clearly $\mathbb{R}_{\mathrm{an}(n+1)}$-definable) will thus not be $\mathbb{R}_{\mathrm{an}(n)}$-definable as announced in Section 5.

As we noted before Proposition 5.3, the lack of surjectivity of each Φ^s will follow from the lack of surjectivity of some mapping $\Phi^s_{D_sD_{s+1}}$ between finite dimensional spaces.

More precisely, if we fix s and D, the function

$$E \mapsto \dim(\mathbb{R} \times (\mathbb{R}_{0,E}[X_1,\dots,X_n]^{q(s)})^{p(s)+1})$$

is a polynomial of degree n in E, whereas

$$E \mapsto \dim(\mathbb{R}_{D,E}[Y_1,\dots,Y_{n+1}])$$

is a polynomial of degree $n+1$.

We thus can build an increasing sequence of integers (D_s) such that

$$\dim(\mathbb{R} \times (\mathbb{R}_{0,D_{s+1}}[X_1,\dots,X_n]^{q(s)})^{p(s)+1}) + n + 1$$

is smaller than

$$\dim(\mathbb{R}_{D_s,D_{s+1}}[Y_1,\dots,Y_{n+1}]),$$

for each s.

Suppose we have built for $r < s$ some $P_r \in \mathbb{R}_{D_r,D_{r+1}}[Y_1,\dots Y_{n+1}]$ and $\eta_r > 0$ such that

(A_r) $\forall t < r\,,\ \|P_r\|_\infty \leq \frac{\eta_t}{2^r(D_{r+1}!)^{n+1}}$;

(B_r) the ball of center P_r and radius $\eta_r S_r$, where S_r is such that

$$\forall \overline{\alpha} \in]-1,1[^{n+1},\ S_r \geq \|(T^{n+1}_{D_r D_{r+1}} \circ \tau_{\overline{\alpha}})^{-1}\|_\infty$$

(see Remark 6.2), does not meet the image of

$$\rho_r : (\alpha,\xi) \mapsto (T^{n+1}_{D_r D_{r+1}} \circ \tau_{\overline{\alpha}})^{-1} \circ \Phi^r_{D_r D_{r+1}}(\xi),$$

where α ranges over $]-1,1[^{n+1}$ and ξ over $\widetilde{M}_s$.

We can then choose $P_s \in \mathbb{R}_{D_s,D_{s+1}}[Y_1,\dots Y_{n+1}]$ and $\eta_s > 0$ satisfying (A_s) and (B_s) as follows:

Let

$$\delta = \min\left\{\frac{\eta_t}{2^r(D_{r+1}!)^{n+1}}\,;t<s\right\};$$

by the dimensional inequality of source and image space (due to the choice of D_{s+1}) and the rationality of $\rho_s : (\alpha,\xi) \mapsto (T^{n+1}_{D_s D_{s+1}} \circ \tau_{\overline{\alpha}})^{-1} \circ \Phi^s_{D_s D_{s+1}}(\xi)$, we know that the image of ρ_s is nowhere dense in $\mathbb{R}_{D_s,D_{s+1}}[Y_1,\dots Y_{n+1}]$. We thus can choose P_s and η_s such that $\|P_s\| < \delta$ and

$$B(P_s\,;\eta_s S_s) \cap \rho_s(]-1,1[^{n+1} \times \widetilde{M}_s) = \emptyset.$$

Let $h(Y_1,\dots,Y_{n+1})$ be the formal series $\sum_s P_s(Y_1,\dots,Y_{n+1})$. We easily get from the inequalities (A_r) that h is the power expansion of an analytic function on $\mathbb{R}^{n+1}$.

Let $\overline{\alpha}$ be a point in $]-1,1[^{n+1}$. From condition (B_r) we get that

$$(T^{n+1}_{D_r D_{r+1}} \circ \tau_{\overline{\alpha}})\big(B(T^{n+1}_{D_r D_{r+1}} h\,;\eta_r S_r)\big) \cap T^{n+1}_{D_r D_{r+1}} \Phi_r(M_r) = \emptyset$$

and then, by the definition of S_r,

$$B((T^{n+1}_{D_r D_{r+1}} \circ \tau_{\overline{\alpha}} \circ T^{n+1}_{D_r D_{r+1}})\,h\,;\eta_r) \cap T^{n+1}_{D_r D_{r+1}} \Phi_r(M_r) = \emptyset.$$

By (A_s) for $s > r$, we get from Proposition 6 that

$$\left\|(T^{n+1}_{D_r D_{r+1}} \circ \tau_{\overline{\alpha}})\,h - (T^{n+1}_{D_r D_{r+1}} \circ \tau_{\overline{\alpha}} \circ T^{n+1}_{D_r D_{r+1}})\,h\right\|_\infty \leq \eta_r;$$

thus

$$(T^{n+1}_{D_r D_{r+1}} \circ \tau_{\overline{\alpha}})\,h \notin T^{n+1}_{D_r D_{r+1}} \Phi_r(M_r).$$

Hence

$$\tau_{\overline{\alpha}}\,h \notin \Phi_r(M_r).$$

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