Illinois Journal of Mathematics
Volume 46, Number 2, Summer 2002, Pages 507–518
S 0019-2082

VANISHING LOGARITHMIC CARLESON MEASURES

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Abstract. Vanishing Carleson-type measures defined with additional logarithmic terms are characterized by using functions in $BMOA$ and the Bloch space. The results are applied to Cesàro type operators on $BMOA$ and the Bloch space.

1. Introduction

Let $D = \{z : |z| < 1\}$ be the unit disk in the complex plane and let $H(D)$ denote the space of all analytic functions on D. Recall that a positive Borel measure μ on D is called a Carleson measure if there is a positive finite constant K such that

$$\mu(S(I)) \leq K|I| \tag{1}$$

for all arcs $I \subset \partial D$, where $|I|$ denotes the normalized arc length of I (so that $|\partial D| = 1$) and $S(I)$ is the Carleson square defined by

$$S(I) = \{z : 1 - |I| < |z| < 1,\ z/|z| \in I\}.$$

Carleson measures are ubiquitous in the study of function-theoretic operator theory. A fundamental property of Carleson measures due to L. Carleson addresses the issue of when the inclusion map is bounded from the Hardy space H^p to $L^p(D, d\mu)$. Recall that for $0 < p < \infty$, H^p consists of the functions $f \in H(D)$ satisfying

$$\|f\|_p^p \equiv \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < \infty.$$

Theorem A (Carleson's Theorem). *For μ a positive Borel measure on D and $0 < p < \infty$, the following are equivalent:*

(i) *μ is a Carleson measure.*

(ii) *There is a constant $C_1 > 0$ such that, for all $f \in H^p$,*

$$\int_D |f(z)|^p \, d\mu(z) \leq C_1 \|f\|_p^p.$$

Received August 1, 2001; received in final form January 22, 2002.
2000 *Mathematics Subject Classification.* Primary 30D50. Secondary 30D45, 47B35.

(iii) *There is a constant $C_2 > 0$ such that, for every $a \in D$,*

$$\int_D |\varphi_a'(z)|\, d\mu(z) \le C_2,$$

where $\varphi_a(z) = (a - z)/(1 - \overline{a}z)$ is an automorphism of D.

For a proof see, for example, [D] or [CM]. If we write $\|\mu\|$ for

$$\sup_{I \subset \partial D} \frac{\mu(S(I))}{|I|},$$

then in Carleson's theorem the quantities C_1, C_2, and $\|\mu\|$ are comparable, meaning that there are absolute constants bounding the ratio of any two of them.

If we have

$$\lim_{|I| \to 0} \frac{\mu(S(I))}{|I|} = 0,$$

then we say that μ is a *vanishing Carleson measure*. For vanishing Carleson measures we have the following well-known analogue of Theorem A:

THEOREM B. *For μ a positive Borel measure on D and $0 < p < \infty$, the following are equivalent:*

(i) *μ is a vanishing Carleson measure.*
(ii) *The identity mapping I from H^p into $L^p(D, \mu)$ is a compact operator.*
(iii) *μ satisfies*

$$\lim_{|a| \to 1^-} \int_D |\varphi_a'(z)|\, d\mu(z) = 0.$$

A proof can be found in [Z, Theorem 8.2.5]; the equivalence of (i) and (iii) is contained in our proof of Theorem 2 below (with $p = 0$ and $s = 1$).

For $0 \le p < \infty$ and $0 < s < \infty$, we define *p-logarithmic s-Carleson measures* by replacing the condition (1) by the following condition:

$$\mu(S(I)) \le K \frac{|I|^s}{(\log \frac{2}{|I|})^p}. \tag{2}$$

If $s = 1$, we call μ a *p-logarithmic Carleson measure*; if moreover $p = 2$, we call μ a *logarithmic Carleson measure*. In [Zh] the second author characterized the p-logarithmic s-Carleson measures by criteria involving $BMOA$ functions when $s = 1$ and Bloch functions when $s > 1$. Recall that $BMOA$ consists of the analytic functions f on D for which

$$\|f\|_* \equiv \sup_{a \in D} \|f \circ \varphi_a - f(a)\|_2 < \infty,$$

where $\varphi_a(z)$ is a disk automorphism as defined in Theorem A. The John-Nirenberg Theorem ensures that $\|f\|_* \approx \sup_{a \in D} \|f \circ \varphi_a - f(a)\|_p$ for $0 < p <$

∞. This means that there is a constant $C > 0$ such that

$$\frac{1}{C}\|f\|_* \le \sup_{a\in D} \|f\circ\varphi_a - f(a)\|_p \le C\|f\|_*.$$

$BMOA$ is a Banach space under the norm $\|f\|_{BMOA} = |f(0)| + \|f\|_*$.

The purpose of this paper is to develop analogous criteria for vanishing measures, defined by replacing the condition (2) by the corresponding little-oh condition. Thus a positive Borel measure of D is said to be a *vanishing p-logarithmic s-Carleson measure* if

$$\mu(S(I)) = o\left(\frac{|I|^s}{(\log\frac{2}{|I|})^p}\right) \quad \text{as } |I| \to 0$$

More briefly, if $s = 1$, we call μ a *vanishing p-logarithmic Carleson measure*; if moreover $p = 2$, we call μ a *vanishing logarithmic Carleson measure.*

Our main result in the case $s = 1$ is the following theorem.

THEOREM 1. *Let $0 < p < \infty$ and let μ be a positive Borel measure on D. Then the following conditions are equivalent:*

(i) *μ is a vanishing p-logarithmic Carleson measure.*

(ii) *μ satisfies*

$$\lim_{|a|\to 1}\left(\log\frac{2}{1-|a|}\right)^p \int_D |\varphi_a'(z)|\,d\mu(z) = 0.$$

(iii) *For any bounded sequence $\{f_n\} \subset BMOA$ satisfying $f_n \to 0$ uniformly on compact subsets of D,*

$$\lim_{n\to\infty}\sup_{a\in D}\int_D |f_n(z)|^p|\varphi_a'(z)|\,d\mu(z) = 0.$$

(iv) *For $0 < q < \infty$, and for any bounded sequence $\{f_n\} \subset BMOA$ satisfying $f_n \to 0$ uniformly on compact subsets of D,*

$$\lim_{n\to\infty}\sup_{g\in H^q,\,\|g\|_q=1}\int_D |f_n(z)|^p|g(z)|^q\,d\mu(z) = 0.$$

Theorem 1 will be proved in Section 2, after we provide a general characterization of vanishing p-logarithmic s-Carleson measures. Then we will give an application of Theorem 1 to certain Cesàro type operators on $BMOA$.

In Section 3, we will give the corresponding results on p-logarithmic s-Carleson measures for $s > 1$. Here the role of $BMOA$ will be replaced by the Bloch space, defined below. We will also give a similar application to Cesàro type operators on the Bloch space.

In the following, we use the convention that C will be a finite positive constant whose value may vary from line to line.

2. Vanishing p-logarithmic Carleson measures

Before proving Theorem 1, we will give a general result for any vanishing p-logarithmic s-Carleson measure for $0 \le p < \infty$ and $0 < s < \infty$. The $p = 0$ case of this result was first proved in [ASX].

THEOREM 2. *Let $0 \le p < \infty$ and $0 < s < \infty$. Let μ be a positive Borel measure on D. Then μ is a vanishing p-logarithmic s-Carleson measure if and only if*

$$\lim_{|a|\to 1} \left(\log \frac{2}{1-|a|} \right)^p \int_D |\varphi_a'(z)|^s \, d\mu(z) = 0. \tag{3}$$

Proof. Let (3) be satisfied. Take any $I \subset \partial D$. Let $a = (1-|I|)e^{i\theta}$, where $e^{i\theta}$ is the center of I. Then $1-|a| = |I|$, and since for any $z \in S(I)$, $|\varphi_a'(z)| \ge C/|I|$, we get

$$\frac{(\log \frac{2}{|I|})^p}{|I|^s} \mu(S(I)) \le C \left(\log \frac{2}{1-|a|} \right)^p \int_D |\varphi_a'(z)|^s \, d\mu(z).$$

Taking the limit as $|I| \to 0$ (or, equivalently, $|a| \to 1$) we see that

$$\lim_{|I|\to 0} \frac{(\log \frac{2}{|I|})^p}{|I|^s} \mu(S(I)) = 0.$$

Thus μ is a vanishing p-logarithmic s-Carleson measure.

Conversely, let μ be a vanishing p-logarithmic s-Carleson measure. For any fixed $\varepsilon > 0$, there is a $\delta > 0$ such that for all arcs $I \subset \partial D$ with $|I| \le \delta$,

$$\frac{(\log \frac{2}{|I|})^p}{|I|^s} \mu(S(I)) < \varepsilon.$$

Suppose $a = re^{i\theta}$ and $r > 1-\delta$. Denote by I_δ the arc centered at $e^{i\theta}$ satisfying $|I_\delta| = \delta$, and by $S(I_\delta)$ the corresponding Carleson box. Then

$$\begin{aligned} K(a) &= \left(\log \frac{2}{1-|a|} \right)^p \int_D |\varphi_a'(z)|^s \, d\mu(z) \\ &= \left(\log \frac{2}{1-|a|} \right)^p \int_{D\setminus S(I_\delta)} |\varphi_a'(z)|^s \, d\mu(z) \\ &\quad + \left(\log \frac{2}{1-|a|} \right)^p \int_{S(I_\delta)} |\varphi_a'(z)|^s \, d\mu(z) \\ &= K_1(a) + K_2(a). \end{aligned}$$

To estimate $K_2(a)$, let $\{I_n\}$ be the arcs centered at $e^{i\theta}$ with $|I_n| = \alpha^{(n-1)}(1-|a|)$, where $1 < \alpha < 2/\delta$, $n = 1, 2, \cdots, N$ and N is the smallest integer such that $\alpha^{(N-1)}(1-|a|) \geq \delta$. Thus

$$\log_\alpha \frac{\delta\alpha}{1-|a|} \leq N \leq \log_\alpha \frac{\delta\alpha}{1-|a|} + 1.$$

Setting $I_0 = \emptyset$, a calculation shows that for $z \in S(I_n) \setminus S(I_{n-1})$,

$$|\varphi_a'(z)|^s \leq \frac{C}{\alpha^{2ns}(1-|a|)^s}.$$

Thus we get

$$\begin{aligned}
\int_{S(I_\delta)} |\varphi_a'(z)|^s \, d\mu(z) &\leq \frac{C}{(1-|a|)^s} \sum_{n=1}^{N-1} \frac{1}{\alpha^{2ns}} \mu(S(I_n) \setminus S(I_{n-1})) \\
&\quad + \frac{C}{(1-|a|)^s} \frac{1}{\alpha^{2Ns}} \mu(S(I_\delta) \setminus S(I_{N-1})) \\
&\leq \frac{C}{(1-|a|)^s} \sum_{n=1}^{N-1} \frac{1}{\alpha^{2ns}} \frac{\varepsilon |I_n|^s}{(\log \frac{2}{|I_n|})^p} \\
&\quad + \frac{C}{(1-|a|)^s \alpha^{2Ns}} \frac{\varepsilon |I_\delta|^s}{(\log \frac{2}{|I_\delta|})^p} \\
&\leq \frac{C\varepsilon}{(1-|a|)^s} \sum_{n=1}^{N} \frac{1}{\alpha^{2ns}} \frac{|I_n|^s}{(\log \frac{2}{|I_n|})^p} \\
&\leq C\varepsilon \sum_{n=1}^{N} \frac{1}{\alpha^{ns}} \frac{1}{(\log \frac{2}{\alpha^{n-1}(1-|a|)})^p}.
\end{aligned}$$

We may bound this last expression by

$$C\varepsilon \frac{1}{(\log \frac{2}{1-|a|)})^p}.$$

When $p = 0$ this is obvious; for $p > 0$ the sum is bounded above by

$$C \int_1^{2+\log_\alpha \delta - \log_\alpha(1-|a|)} \frac{1}{\alpha^{ts}} \frac{1}{(\log \frac{2}{\alpha^{t-1}(1-|a|)})^p} dt,$$

since $N \leq \log_\alpha(\delta\alpha/(1-|a|)) + 1 = 2 + \log_\alpha \delta - \log_\alpha(1-|a|)$. Standard estimates show that for $|a| \geq 3/4$ this integral is bounded above by

$$C \frac{1}{(\log \frac{2}{1-|a|})^p}$$

for some constant C. Thus

$$K_2(a) < \varepsilon \tag{4}$$

for $|a|$ sufficiently close to 1. To estimate $K_1(a)$, notice that for $z \in D \setminus S(I_\delta)$, $|1-\bar{a}z| \geq \delta$. Thus

$$K_1(a) \leq \frac{1}{\delta^{2s}}(1-|a|^2)^s \left(\log \frac{2}{1-|a|}\right)^p \mu(D) < \varepsilon \tag{5}$$

if $|a|$ is sufficiently close to 1. Combining (4) and (5) we see that

$$\lim_{|a|\to 1} K(a) = 0,$$

which finishes the proof. □

In order to prove Theorem 1, we need the following lemma.

Lemma 1. *If μ is a vanishing p-logarithmic Carleson measure, and $\{f_n\}$ is a bounded sequence in $BMOA$ such that $f_n \to 0$ uniformly on compact subsets of D as $n \to \infty$, then*

$$\lim_{n\to\infty} \sup_{a\in D} \int_D |f_n(a)|^p |\varphi_a'(z)|\, d\mu = 0. \tag{6}$$

Proof. By Theorem 2, given any $\varepsilon > 0$, we may find $\delta > 0$ such that

$$\sup_{a\in D\setminus \overline{D}_\delta} \left(\log \frac{2}{1-|a|}\right)^p \int_D |\varphi_a'(z)|\, d\mu(z) < \varepsilon,$$

where $D_\delta = \{\, z \in D : |z| < \delta\}$. Since point evaluation at a is a bounded linear functional on $BMOA$, with uniformly bounded norm as a ranges over $\overline{D_\delta}$ (specifically $|f_n(a)| \leq C\|f_n\|_* \log(2/(1-|a|))$), and $\{f_n\}$ is bounded in $BMOA$, we have

$$\sup_{a\in D\setminus\overline{D_\delta}} \int_D |f_n(a)|^p |\varphi_a'(z)|\, d\mu \leq C \sup_{a\in D\setminus\overline{D_\delta}} \left(\log \frac{2}{1-|a|}\right)^p \int_D |\varphi_a'(z)|\, d\mu < C\varepsilon. \tag{7}$$

Also, since $f_n \to 0$ uniformly on compact subsets of D, we see that for n sufficiently large,

$$\sup_{a\in\overline{D}_\delta} \int_D |f_n(a)|^p |\varphi_a'(z)|\, d\mu \leq \varepsilon \sup_{a\in\overline{D}_\delta} \int_D |\varphi_a'(z)|\, d\mu \leq C\|\mu\|\varepsilon. \tag{8}$$

Combining (7) and (8) we see we can make

$$\sup_{a\in D} \int_D |f_n(a)|^p |\varphi_a'(z)|\, d\mu$$

as small as desired by choosing n sufficiently large, and so (6) is proved. □

Proof of Theorem 1. The equivalence of (i) and (ii) is the special case $s = 1$ of Theorem 2. Now we prove that (ii)⇒(iii). Let $\{f_n\}$ be a bounded sequence in $BMOA$ satisfying $f_n \to 0$ uniformly on compact subsets of D. Consider

$$\int_D |f_n(z) - f(a)|^p |\varphi_a'(z)|\, d\mu.$$

Since (ii) clearly guarantees that μ is a vanishing Carleson measure, for any $\varepsilon > 0$, we may find $r \in (0,1)$ such that $\mu|_{D\setminus\overline{D}_r} = \mu_r$ is a Carleson measure with Carleson constant $\|\mu_r\| < \varepsilon$ (see, [CM, p. 130]). For fixed $a \in D$ let

$$g_{n,a}(z) = (f_n(z) - f_n(a))(\varphi_a'(z))^{1/p}.$$

Since $\{f_n\}$ is a bounded sequence in $BMOA$, $\{g_{n,a}\}$ is a bounded sequence in H^p, with H^p norms bounded independently of $a \in D$. Thus

$$\begin{aligned}(9)\qquad &\sup_{a\in D}\int_{D\setminus\overline{D}_r} |f_n(z) - f_n(a)|^p |\varphi_a'(z)|\, d\mu(z)\\ &= \sup_{a\in D}\int_{D\setminus\overline{D}_r} |g_{n,a}(z)|^p\, d\mu_r(z) \le C\sup_{a\in D}\|\mu_r\|\,\|g_{n,a}\|_p^p < C\varepsilon.\end{aligned}$$

Since $f_n \to 0$ uniformly on compact subsets of D, we have

$$\sup_{a\in D}\int_{\overline{D}_r} |f_n(z)|^p|\varphi_a'(z)|\, d\mu \le \varepsilon \sup_{a\in D}\int_{\overline{D}_r} |\varphi_a'(z)|\, d\mu \le C\|\mu\|\varepsilon$$

for n sufficiently large. By Lemma 1 we know that, for n large enough,

$$\sup_{a\in D}\int_{\overline{D}_r} |f_n(a)|^p|\varphi_a'(z)|\, d\mu < C\varepsilon.$$

Thus, for n sufficiently large,

$$(10)\qquad \sup_{a\in D}\int_{\overline{D}_r} |f_n(z) - f_n(a)|^p|\varphi_a'(z)|\, d\mu(z) < C\varepsilon.$$

Combining (9) and (10) with Lemma 1, we get

$$\lim_{n\to\infty}\sup_{a\in D}\int_D |f_n(z)|^p|\varphi_a'(z)|\, d\mu = 0.$$

Thus (iii) is true.

Next we prove that (iii)⇒(i). Suppose (i) does not hold. Then there is a sequence of arcs $\{I_n\} \subset \partial D$ with $|I_n| \to 0$ and $\varepsilon > 0$ such that

$$(11)\qquad \mu(S(I_n)) \ge \varepsilon\frac{|I_n|}{(\log\frac{2}{|I_n|})^p}.$$

Let $a_n = (1 - |I_n|)e^{i\theta_n}$, where $e^{i\theta_n}$ is the center of I_n. Consider the sequence of functions $\{g_n\}$ defined by

$$g_n(z) = \left(\log\frac{2}{1-|a_n|}\right)^{-1}\left(\log\frac{2}{1-\bar{a}_n z}\right)^2.$$

Then it is easy to check that $\{g_n\}$ is a bounded sequence in $BMOA$, and $g_n \to 0$ uniformly on compact subsets of D as $n \to \infty$.

By (iii),

$$\sup_{a\in D} \int_D |g_n(z)|^p |\varphi_a'(z)|\, d\mu(z) \to 0, \qquad \text{as } n \to \infty.$$

Thus

$$\frac{C}{|I_n|} \int_{S(I_n)} |g_n(z)|^p \, d\mu(z) \to 0, \qquad \text{as } n \to \infty, \tag{12}$$

since $|\varphi_{a_n}'(z)| \geq C/|I_n|$ for any $z \in S(I_n)$, for some absolute constant C. But on $S(I_n)$,

$$|g_n(z)|^p \geq \left[\left(\log \frac{2}{1-|a_n|} \right)^{-1} \left(k \log \frac{2}{|I_n|} \right)^2 \right]^p$$

for some constant k. Using the estimate in (12) we see that

$$\frac{1}{|I_n|} \left(\log \frac{2}{|I_n|} \right)^p \mu(S(I_n)) \to 0, \qquad \text{as } n \to \infty,$$

which contradicts (11). Thus (i) must hold.

Finally, we prove that (iii)⇔(iv). Let X_μ^p be the space of analytic functions f on the unit disk D such that

$$\|f\|_{X_\mu^p}^p \equiv \sup_{a\in D} \int_D |f(z)|^p |\varphi_a'(z)|\, d\mu(z) < \infty,$$

and for $0 < q < \infty$, let $Y_\mu^{p,q} = Y_\mu^p$ the space of analytic functions f on D such that

$$\|f\|_{Y_\mu^p}^p \equiv \sup_{g\in H^q, \|g\|_q = 1} \int_D |f(z)|^p |g(z)|^q \, d\mu(z) < \infty.$$

Then $f \in X_\mu^p$ if and only if $d\mu_f(z) = |f(z)|^p \, d\mu(z)$ is a Carleson measure, which by Theorem A is equivalent to the condition

$$\sup_{g\in H^q, \|g\|_q = 1} \int_D |f(z)|^p |g(z)|^q \, d\mu(z) < \infty.$$

Thus $X_\mu^p = Y_\mu^p$, and in fact, by the equivalence of the various constants in the statement of Carleson's theorem we know $\|f\|_{X_\mu^p}$ and $\|f\|_{Y_\mu^p}$ are comparable. Consequently, (iii) and (iv) are equivalent. The proof is completed. □

As an application of Theorem 1 we characterize the compactness of a Cesàro type integral operator on $BMOA$. For $f, g \in H(D)$, the integral operator J_f with symbol f is defined by

$$J_f g(z) = \int_0^z g(\zeta) f'(\zeta)\, d\zeta.$$

If $f(z) = -\log(1-z)$ then J_f is the well-known Cesàro operator. It was first shown by Ch. Pommerenke [Po] that J_f is bounded on H^2 if and only if $f \in BMOA$. This operator was systematically studied by A. Aleman and A. Siskakis [AS1][AS2]. In [AS1], it was proved that J_f is bounded on the Hardy space H^p for any $p \geq 1$, if and only if $f \in BMOA$. The boundedness and compactness of J_g on $BMOA$ was characterized by Siskakis and the second author in [SZ]. Here we derive the criterion of compactness of J_g on $BMOA$ from Theorem 1. This result was first proved in [SZ].

Corollary 1. *For $f \in H(D)$, J_f is compact on $BMOA$ if and only if*

$$\lim_{|a|\to 1} \left(\log \frac{2}{1-|a|}\right)^2 \int_D |f'(z)|^2 (1-|\varphi_a(z)|^2)\, dA(z) = 0, \tag{13}$$

where $dA(z) = dxdy/\pi$ is the normalized Lebesgue measure on D.

Proof. We will use the following criterion for a function in $BMOA$: if $f \in H(D)$, then $f \in BMOA$ if and only if

$$B(f) = \sup_{a\in D} \int_D |f'(z)|^2 (1-|\varphi_a(z)|^2)\, dA(z) < \infty,$$

and $B(f)$ is comparable to $\|f\|_*^2$ (see, for example, [AXZ]).

Since $(J_f g)' = gf'$, and $1-|\varphi_a(z)|^2 = (1-|z|^2)|\varphi_a'(z)|$, we know that J_f is compact on $BMOA$ if and only if, for any bounded sequence $\{g_n\} \subset BMOA$ with $g_n \to 0$ uniformly on compact subsets of D,

$$\lim_{n\to\infty} \sup_{a\in D} \int_D |g_n(z)|^2 |f'(z)|^2 (1-|z|^2)|\varphi_a'(z)|\, dA(z) = 0.$$

By Theorem 1, this means that $d\mu_f(z) = |f'(z)|^2(1-|z|^2)\, dA(z)$ is a vanishing logarithmic Carleson measure, or (13) is satisfied. The proof is complete. □

3. The case $s > 1$

When $s > 1$, s-Carleson measures are closely related, by analogues of Theorems A and B, to the weighted Bergman spaces $L_a^{p,\alpha}$ defined for $0 < p < \infty$ and $\alpha > -1$ as those $f \in H(D)$ for which

$$\|f\|_{L_a^{p,\alpha}}^p \equiv \int_D |f(z)|^p (1-|z|^2)^\alpha\, dA(z) < \infty.$$

The analogue of Theorem A asserts the equivalence of the conditions

(i) μ is an s-Carleson measure.
(ii) The identity map from $L_a^{p,s-2}$ to $L^p(D, d\mu)$ is bounded.
(iii) There is a finite constant $C > 0$ such that, for every $a \in D$,

$$\int_D |\varphi_a'(z)|^s\, d\mu(z) \leq C.$$

(See, for example, Theorem 2.36 in [CM].) Similarly there is a characterization of vanishing s-Carleson measures, analogous to Theorem B, with H^p replaced by $L_a^{p,s-2}$.

Correspondingly we may obtain a version of Theorem 1 characterizing vanishing p-logarithmic s-Carleson measures for $s > 1$ and $p > 0$, where the role of $BMOA$ is now played by the Bloch space B defined as follows. A function $f \in H(D)$ is said to be in the Bloch space B if

$$\|f\|_B \equiv |f(0)| + \sup_{z \in D} |f'(z)|(1 - |z|^2) < \infty.$$

For any $s > 1$ we have the following result:

Theorem 3. *Let $1 < s < \infty$, $0 < p < \infty$, and let μ be a positive Borel measure on D. Then the following conditions are equivalent:*

(i) *μ is a vanishing p-logarithmic s-Carleson measure.*

(ii)
$$\lim_{|a| \to 1} \left(\log \frac{2}{1 - |a|} \right)^p \int_D |\varphi_a'(z)|^s \, d\mu(z) = 0.$$

(iii) *For any bounded sequence $\{f_n\} \subset B$ satisfying $f_n \to 0$ uniformly on compact subsets of D,*
$$\lim_{n \to \infty} \sup_{a \in D} \int_D |f_n(z)|^p |\varphi_a'(z)|^s \, d\mu(z) = 0$$

(iv) *For $0 < q < \infty$, and for any bounded sequence $\{f_n\} \subset B$ satisfying $f_n \to 0$ uniformly on compact subsets of D,*
$$\lim_{n \to \infty} \sup_{g \in L_a^{q,s-2},\, \|g\| = 1} \int_D |f_n(z)|^p |g(z)|^q \, d\mu(z) = 0.$$

The proof of Theorem 3 follows by arguments quite similar to those used in Theorem 1, beginning with a version of Lemma 1 for the Bloch space, in which condition (6) of that lemma is replaced by

$$\lim_{n \to \infty} \sup_{a \in D} \int_D |f_n(a)|^p |\varphi_a'(z)|^s \, d\mu = 0$$

for every $s > 1$. While the details of the proof of Theorem 3 are left to the interested reader, we make a few comments about the relevant changes in the proof of Theorem 1.

For the implication (ii)$\Rightarrow$(iii), we begin with the fact that when (ii) holds, the norm of the restriction measure $\mu_r = \mu|_{D \setminus \overline{D}_r}$ (defined by setting $\|\mu_r\| = \sup\{\mu_r(S(I))/|I|^s : I \subset \partial D\}$) can be made as small as desired by choosing r sufficiently close to 1. This is obtained for $s > 1$ by an easy adaptation of the argument in [CM, p. 130] in the case $s = 1$. Then assuming $\{f_n\}$ is a bounded sequence in the Bloch space, the functions

$$g_{n,a}(z) = (f_n(z) - f_n(a))(\varphi_a'(z))^{s/p}$$

will be bounded in $L_a^{p,s-2}$ for all n and $a \in D$. This follows from the fact that for $f \in H(D)$, the quantities

$$\sup_{a\in D} \|f \circ \varphi_a - f(a)\|_{L_a^{p,s-2}}$$

and

$$\sup_{z\in D} |f'(z)|(1-|z|^2)$$

are comparable (see [A]). The remainder of the argument for (ii)⇒(iii) then proceeds as in the proof of Theorem 1.

For (iii)⇒(i) we make use of exactly the same test functions as in equation (12); these are also a bounded sequence in B.

The equivalence of (iii) and (iv) follows from the chain of equivalences

$$\begin{aligned}\sup_{a\in D} \int_D |f|^p |\varphi_a'|^s d\mu < \infty \\ &\Leftrightarrow |f|^p d\mu \text{ is an } s\text{-Carleson measure} \\ &\Leftrightarrow \sup\left\{ \int_D |f|^p |g|^q d\mu : g \in L_a^{q,s-a}, \|g\| = 1 \right\} < \infty,\end{aligned}$$

with the two supremums being comparable for fixed $f \in H(D)$.

We may apply Theorem 3 to study compactness of the integral operator J_f on B.

Corollary 2. *For $f \in H(D)$, J_f is compact on B if and only if for some (all) $s > 1$*

$$\lim_{|a|\to 1} \left(\log \frac{2}{1-|a|} \right)^2 \int_D |f'(z)|^2 (1-|\varphi_a(z)|^2)^s \, dA(z) = 0.$$

Since for f analytic in the disk and any $\alpha > -1$, $\sup_{a\in D} \|f \circ \varphi_a - f(a)\|_{L_a^{2,\alpha}}$ and $\sup_{z\in D} |f'|^2(1-|z|^2)$ are comparable, $f \in B$ if and only if

$$\sup_{a\in D} \int_D |f'|^2 (1-|\varphi_a|^2)^s dA < \infty$$

for some (all) $s > 1$. Thus Corollary 2 is proved in the same manner as Corollary 1.

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