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ON A CONJECTURE OF CONWAY

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ABSTRACT. The purpose of this paper is to prove a conjecture of Conway, which asserts that the class of combinatorial games constitutes a "universally embedding" partially ordered abelian group.

1. Introduction

One of the virtues of Conway's elegant theory of games is the simplicity of its basic notions. Therefore it will not take us too far afield to recall the foundational ideas.

The class of *games* is defined by the following inductive process: given any two sets of previously constructed games $\{G^L\}$ and $\{G^R\}$, there is another game $G = \{G^L|G^R\}$. We call G^L and G^R the left and right options of G. One defines a binary relation $\leq$ on the games by induction: $G \leq H$ unless there is some left option of G such that $H \leq G^L$, or some right option of H such that $H^R \leq G$. An easy induction shows that $\leq$ is reflexive and transitive. It follows that the collection of pairs (G, H) such that $G \leq H$ and $H \leq G$ is an equivalence relation on the class of games; we shall say that G and H are equal if (G, H) belongs to this collection, and write $G = H$.

The sum of two games G and H is defined inductively as follows:

$$G + H = \{G^L + H, G + H^L | G^R + H, G + H^R\}$$

One can show easily by induction that if $G \leq G'$ and $H \leq H'$, then $G + H \leq G' + H'$. It follows that if $G = G'$ and $H = H'$, then $G + H = G' + H'$. The collection of (equivalence classes of) games forms a partially ordered abelian group with respect to this addition. The identity element is the (equivalence class of the) game $\{|\}$ which has no left or right options, and the additive inverse of a game G may be constructed by the following recursion:

$$-G = \{-G^R | -G^L\}.$$

We shall denote the group of equivalence classes of games by $\mathbb{U}$. It is standard to abuse terminology by referring to elements of $\mathbb{U}$ also as games. We shall

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resist this abuse and distinguish between elements of $\mathbb{U}$ and the games which represent them.

We call a map ϕ between partially ordered sets *order-preserving* if $x \leq y$ is equivalent to $\phi(x) \leq \phi(y)$. Now we can state more precisely Conway's conjecture:

THEOREM 1. *Let $S \subseteq S'$ be partially ordered abelian groups. Suppose that $\phi : S \to \mathbb{U}$ is an order-preserving homomorphism. Then there exists an order-preserving homomorphism $\phi' : S' \to \mathbb{U}$ such that $\phi'|S = \phi$.*

Let us make a few comments about this result. First of all, it is essential to note that the games form a proper class (even after identifying games which are equal). It is implicit in the statement of the theorem that S' and therefore S are sets; otherwise we could take $\phi : S \to \mathbb{U}$ to be an isomorphism and S' to be some larger group. In the event that the reader does not like using proper classes, other versions of this embedding theorem can be formulated, provided one has an appropriate dichotomy between "large" and "small". For example, if we change the definition of "game" so as to allow only countably many options, then $\mathbb{U}$ is actually a set and the embedding theorem is true (with the same proof) provided that S' is countable.

We should also mention that Theorem 1 characterizes the partially ordered group $\mathbb{U}$ up to isomorphism, provided that one admits a sufficiently strong version of the axiom of choice (such as the existence of a well-ordering of the universe). This is proved by a standard "back-and-forth" argument familiar to model-theorists. The argument runs something like this: suppose that $\mathbb{U}'$ is another partially ordered abelian group (necessarily with a proper class of elements) for which Theorem 1 holds. We may filter $\mathbb{U}$ and $\mathbb{U}'$ by subgroups $\mathbb{U}_\alpha$ and $\mathbb{U}'_\alpha$, where α runs over all the ordinals, in such a way that each $\mathbb{U}_\alpha$ and $\mathbb{U}'_\alpha$ is a set and $\mathbb{U} = \bigcup_\alpha \mathbb{U}_\alpha$, $\mathbb{U}' = \bigcup_\alpha \mathbb{U}'_\alpha$. (For example, take $\mathbb{U}'_\alpha$ to be the subgroup of $\mathbb{U}'$ generated by all elements of $\mathbb{U}'$ having rank $\leq \alpha$, and similarly for $\mathbb{U}$.) Now we proceed inductively to define subgroups $V_\alpha \subseteq \mathbb{U}$, $V'_\alpha \subseteq \mathbb{U}'$ and order-preserving isomorphisms $\phi_\alpha : V_\alpha \simeq V'_\alpha$ with the following properties:

- Each V_α (and therefore also V'_α) is a set.
- If $\beta < \alpha$, then $\mathbb{U}_\beta \subseteq V_\alpha$ and $\mathbb{U}'_\beta \subseteq V'_\alpha$.
- If $\beta < \alpha$, then $V_\beta \subseteq V_\alpha$ and $\phi_\alpha|V_\beta = \phi_\beta$ (and therefore also $V'_\beta \subseteq V'_\alpha$).

To start, we take V_0 and V'_0 to be $\{0\}$. At limit stages, we let

$$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha, \quad V'_\lambda = \bigcup_{\alpha<\lambda} V'_\alpha, \quad \phi_\lambda = \bigcup_{\alpha<\lambda} \phi_\alpha.$$

It remains to handle successor stages. Suppose that V_α, V'_α, and ϕ_α have been defined. Let W be the subgroup generated by V'_α and $\mathbb{U}'_\alpha$. By the universal embedding property for $\mathbb{U}$, we may extend ϕ_α^{-1} to an embedding

$\psi : W \to \mathbb{U}$. Now let $V_{\alpha+1}$ be the subgroup generated by $\mathbb{U}_\alpha$ and $\psi(W)$. By the universal embedding property for $\mathbb{U}'$, we can extend ψ^{-1} to an embedding $\phi_{\alpha+1} : V_{\alpha+1} \to \mathbb{U}'$; now simply take $V'_{\alpha+1} = \phi_{\alpha+1}(V_{\alpha+1})$. This completes the induction. Now the union of the ϕ_α is the desired isomorphism between $\mathbb{U}$ and $\mathbb{U}'$.

Theorem 1 may be considered a complete description of the partially ordered group of games. However, if we impose finiteness conditions on our games, then the issue becomes much more subtle. A game is called *short* if it has only finitely many options, each of which is itself a short game (in other words, the class of short games is defined in exactly the same way as the class of games, except that only finitely many options are permitted). Theorem 1 is not true for the class of short games, even if we require S' to be a finite group. For example, it is shown in [1] that no short game can have odd order. It is not known which partially-ordered abelian groups occur as finitely generated subgroups of the short games. However, this question has been answered for *cyclic* groups: see [2].

2. Embedding for partially ordered sets

If we replace "partially ordered abelian group" by "partially ordered set" in the statement of Conway's conjecture, then the conjecture becomes much easier to prove. In this section, we will give a proof of this easier result, which we will call the *weak embedding theorem*. The purpose for this is twofold. First, the proof of the weak embedding theorem is a good warm-up for the proof of Theorem 1 (which we shall refer to as the *strong embedding theorem*): the structure of the proof is the same, but the details are much simpler. Second, we will actually use the weak embedding theorem in the course of proving the strong one.

Let S be a set of games. We will call S *hereditary* if, for every $x \in S$, all of its options x^L and x^R lie also in S. Note that any set S of games may be enlarged to a hereditary set, simply by adjoining all options of elements of S, together with the options of those options, and so forth.

Lemma 2. *Suppose S is a hereditary set of games, and let $L, R \subseteq S$. Assume that:*

- *The set L is closed downwards. That is, if $x \in L$, $y \in S$, and $y \leq x$, then $y \in L$.*
- *The set R is closed upwards. That is, if $x \in R$, $y \in S$, and $y \geq x$, then $y \in R$.*
- *The inequality $x \leq y$ holds, whenever $x \in L$, $y \in R$.*

Let G be a game having as left options all elements of S which do not lie in R, and as right options those elements of S which do not lie in L. Then for any $x \in S$, $G \leq x$ if and only if $x \in R$, and $x \leq G$ if and only if $x \in L$.

Proof. We prove this by induction on x. By symmetry it suffices to prove the first claim. If $G \leq x$, then x cannot be a left option of G, so $x \in R$. Conversely, if $G \not\leq x$, then we have either $x^R \leq G$ or $x \leq G^L$. In the first case, the inductive hypothesis implies that $x^R \in L$, so that $x^R \not\leq x$ implies $x \notin R$. In the second case, $x \leq y$ for some $y \notin R$, so again $x \notin R$. □

Using this lemma, we now prove the weak embedding theorem.

THEOREM 3. *Let $S \subseteq S'$ be partially ordered sets, and let ϕ be an order-preserving map from S to the games. Then ϕ admits an extension to an order-preserving map from S' to the games.*

Proof. Consider all partial extensions ψ of ϕ, partially ordered so that $\psi \leq \psi'$ if ψ' extends ψ. This is a nonempty partial order which admits inductive limits. Therefore it has a maximal element by Zorn's lemma. (The fact that this partial order is actually a proper class introduces a technicality, but it is easy to sidestep since every chain in this partial order is bounded in size.) Without loss of generality, we may replace ϕ by this maximal element and S by its domain; it now suffices to show that $S = S'$. Suppose otherwise, and choose $s \in S' - S$. We will extend ϕ to $S \cup \{s\}$, thereby contradicting the maximality of ϕ.

Choose a hereditary set M of games containing $\phi(S)$, and let $L = \{x \in M : (\exists t \in S)[x \leq \phi(t) \wedge t \leq s]\}$ and $R = \{x \in M : (\exists t \in S)[x \geq \phi(t) \wedge t \geq s]\}$. Then Lemma 2 guarantees the existence of a game G such that $t \leq s$ if and only if $\phi(t) \leq G$ and $t \geq s$ if and only if $\phi(t) \geq G$, for any $t \in S$. Thus we can extend ϕ to $S \cup \{s\}$ by setting $\phi(s) = G$. □

Our proof of the strong embedding theorem will proceed in much the same way. Using Zorn's lemma, we can reduce to the problem of extending the definition of our map to a single new element. In other words, we are given a partially ordered abelian group S (which, for simplicity, we will identify with its image in the group of games), together with a description of a new element that we would like to adjoin. We must produce a game G which matches this description. At first, it might not be obvious how to find G. In the preceding argument, we first enlarged S to a hereditary set. Then Lemma 2 guarantees that the naive attempt to construct G actually works.

In the context of the strong embedding theorem, G will need to satisfy more complicated conditions. It will therefore be necessary to enlarge S in two different ways: so that it is hereditary and so that it is "justified" (a notion which we shall introduce later). In order to carry out the second sort of enlargement, it will be necessary to add to S games which satisfy complicated conditions of their own. Fortunately, the properties required of these auxiliary games are *less* complicated than the properties required of G, and so it is possible to construct these games directly. That is the subject of the next section.

3. Construction of auxiliary games

The strong embedding theorem implies that every partially ordered abelian group is a subgroup of the group $\mathbb{U}$. On the other hand, general partially ordered abelian groups can look somewhat unusual. It is therefore inevitable that a proof of Theorem 1 must hide some method for constructing games satisfying unusual-looking inequalities. In our proof, the construction is hidden in this section.

We begin by recalling that there is a natural embedding of the ordinals into the class of games. For every ordinal α, we identify α with the game $\{\beta|\}$. Here β ranges over all of the ordinals smaller than α, which we also identify with games via the same formula. The ordering on the ordinals induced by their inclusion into the games agrees with their usual ordering.

Note that the game-theoretic sum of two ordinals does not agree with their order-type sum (though the former *does* turn out to be another ordinal). Thus there is a potential for confusion in expressions such as $\alpha+\beta$; we will always take this to mean the game-theoretic sum of α and β.

We shall need only the following simple property of the ordinals in our discussion:

LEMMA 4. *Let S be any set of games. Then there exists an ordinal α such that $x<\alpha$ for all $x \in S$.*

(Of course, $x<\alpha$ simply means that $x \leq \alpha$ and $\alpha \not\leq x$.)

Proof. Since the class of games is constructed recursively, it admits an ordinal rank defined (recursively) by the following condition: the rank of G is the smallest ordinal which is larger than the rank of any option of G. One can verify easily by induction that if the rank of G is less than α, then $G<\alpha$. It now suffices to choose α larger than the ranks of all elements of S. □

Now we are well-equipped to produce the games we will need.

LEMMA 5. *Let $\{H_i\}$ be a set of games such that each $H_i \not\leq 0$, and let α be an ordinal. Then there exists a game $G \geq 0$ such that $G \not\geq H_i$ for each i, and yet $nG \geq \alpha$ for any $n>1$.*

Proof. Enlarging α if necessary, we may assume that α is larger than each $-H_i$. Let $G=\{\alpha+\alpha|H_i\}$. Then $G \not\geq H_i$ by construction. Since each $H_i \not\leq 0$, we have $G \geq 0$.

To prove $nG \geq \alpha$ for $n>1$, it suffices by the positivity of G to prove that $\alpha \leq 2G$. Suppose otherwise. Then either $2G \leq \alpha^L$ or $(2G)^R \leq \alpha$. In the first case, we obtain $2G \leq \beta$, which is impossible since it implies $\beta \leq G+(\alpha+\alpha)$.

In the second case we have $G+H_i \leq \alpha$, so that $\alpha \not\leq (\alpha+\alpha)+H_i$, or $\alpha \not\geq -H_i$, contrary to our assumption. $\square$

We now prove a stronger version of the preceding lemma.

Lemma 6. *Let $A, \{B_n\}_{n>0}$, and $\{C_n\}_{n>0}$ be sets of games. Suppose that $a \not\leq b_1$ for all $a \in A$, $b_1 \in B_1$. Then there exists a game x with the following properties:*

- *$a \not\leq x$ for all $a \in A$.*
- *$b_n \leq nx$ for all $b_n \in B_n$.*
- *$nx \not\leq c_n$ for all $c_n \in C_n$.*

Proof. Consider first the easier task of finding a game satisfying the conditions listed above, but only for $n = 1$. In this case, the arithmetic of games does not enter into the question: we only need x to satisfy certain inequalities. Moreover, the hypotheses guarantee that these inequalities are consistent with one another. It follows from the weak embedding theorem that there is a game x_0 satisfying these inequalities. Without loss of generality, we may replace x by $x - x_0$, A by $\{a - x_0 : a \in A\}$, B_n by $\{b - nx_0 : b \in B_n\}$, and C_n by $\{c - nx_0 : c \in C_n\}$. After carrying out this replacement, we know the following:

- Each $a \in A$ is $\not\leq 0$.
- Each $b_1 \in B_1$ is ≤ 0.
- Each $c_1 \in C_1$ is $\not\geq 0$.

Furthermore, since every set of games is bounded by some ordinal, there is an ordinal α such that $\alpha > b_n, c_n$ for all $b_n \in B_n$, $c_n \in C_n$. Thus it suffices to find a game x such that $x \geq 0$, $nx \geq \alpha$ for $n > 1$, and $x \not\geq a$ for $a \in A$. The existence of this game follows from Lemma 5. $\square$

4. Framings

We have said earlier that the proof of Theorem 1 can be reduced to the problem of finding a single game G with certain properties which relate it to some set S of previously constructed games. In the case of the weak embedding theorem, these properties were purely order-theoretic: we had specified for us which elements of S should be smaller than G, and which should be larger than G. In the case of the strong embedding theorem, the arithmetic of the games is also involved: we must specify how the elements of S compare to all multiples of G. This motivates the following definition:

Definition 7. Let S be a subgroup of $\mathbb{U}$. A *framing* of S is a collection of subsets $S_i \subseteq S$, indexed by the integers, with the following properties:

- $S_i + S_j \subseteq S_{i+j}$.
- $g \in S_0$ if and only if $g \geq 0$.

If S' is a subgroup of $\mathbb{U}$ containing S, then a framing of S' *extends* a framing of S if $S_i = S'_i \cap S$.

The idea is that S_i should be the set of elements s such that $s \geq iG$, where G is the game which has yet to be constructed.

LEMMA 8. *Let $S \subseteq S'$ be subgroups of $\mathbb{U}$. Then any framing of S extends to a framing of S'.*

Proof. Set $S'_n = \{g' \in S' : (\exists g \in S_n)[g' \geq g]\}$. □

DEFINITION 9. Let S be a framed subgroup of $\mathbb{U}$, and suppose $g \notin S_n$ for some $g \in S$, $n > 1$. We say that (g, n) is *justified* if there exists $x \in S_{-1}$ such that $g + x \notin S_{n-1}$. Similarly, if $g \notin S_{-n}$, we say $(g, -n)$ is justified if there exists $x \in S_1$ such that $g + x \notin S_{-n+1}$.

Here is the idea behind the definition. We would like to find a game G with the property that $S_n = \{g \in S : nG \leq g\}$. Now the general philosophy by which the games are constructed asserts that an inequality $nG \leq g$ will hold *unless there is a reason to the contrary.* If the pair (g, n) is justified, then we can provide such a reason by looking at the S_k for $|k| < n$.

Suppose that $S \subseteq S'$ are (compatibly) framed subgroups of $\mathbb{U}$. If $g \in S$, $g \notin S_n$, and (g, n) is justified with respect to S, then (g, n) is justified with respect to S'. On the other hand, if (g, n) is not justified with respect to S, we shall now show that S' can be chosen so that (g, n) becomes justified in S'.

LEMMA 10. *Let S be a framed subgroup of $\mathbb{U}$, and suppose $g \notin S_n$. Then there is a framed subgroup $\mathbb{U}$ extending S in which (g, n) is justified.*

Proof. Without loss of generality we assume $n > 1$. We will obtain the desired extension by adjoining a single game x to S. Let x be an arbitrary game, and let S' be the group generated by S and x. In order to guarantee that (g, n) is justified, we will frame S' so as to ensure that $x \in S'_{-1}$ and $g + x \notin S'_{n-1}$.

Set
$$S'_k = \{y \in S' : (\exists i \geq 0)(\exists z \in S_{k+i})[z + ix \leq y]\}$$
Note that $x \in S'_{-1}$. To complete the proof, it suffices to show that it is possible to choose x such that the $\{S'_k\}$ give a framing of S' extending the framing of S, and such that $g + x \notin S'_{n-1}$. We will now examine more carefully what is required of x to make this so.

First, let us determine when the formula above describes a framing on S'. It is obvious that $S'_k + S'_l \subseteq S'_{k+l}$ and that $\{s \in S' : s \geq 0\} \subseteq S'_0$. Thus, to have a framing, we need only verify that $y \geq 0$ for every $y \in S'_0$. Equivalently, we must have $ix \geq -z$ for each $z \in S_i$, $i \geq 0$. For $i = 0$, this is immediate.

We need to know not only that S' is framed, but that its framing is compatible with the framing on S. It is clear that $S_i \subseteq S'_i$; we must show the reverse inequality $S'_i \cap S \subseteq S_i$. In other words, we must show that $z \in S_{k+i}$, $y \in S$, $y \geq z + ix$ implies $y \in S_k$. This will hold if $z \in S_{k+i}$, $y \in S - S_k$ implies $ix \not\leq y - z$. If $i = 0$, then this follows automatically since S_k is closed upwards.

Finally, to ensure that $g + x \notin S'_{n-1}$, we must guarantee three things:

- $g + x \not\geq y$ for $y \in S_{n-1}$.
- $g + x \not\geq y + x$ for $y \in S_n$. This is automatic since $g \notin S_n$.
- $g + x \not\geq y + ix$ for $y \in S_{n+i-1}$, $i > 1$.

To rephrase, we must have $y - g \not\leq x$ for $y \in S_{n-1}$, and $(i-1)x \not\leq g - y$ for $y \in S_{n+i-1}$, $i > 1$.

We want to apply Lemma 6 with $A = \{y - g : y \in S_{n-1}\}$, $B_i = \{-z : z \in S_i\}$, $C_i = \{g - y : y \in S_{n+i}\} \cup \{y - z : z \in S_{k+i}, y \in S - S_k\}$. This will guarantee the existence of x having the desired properties, provided that there is no outright contradiction due to an inequality $a \leq b_1$ for $a \in A$, $b_1 \in B_1$. In this case, we would have $y - g \leq -z$ for some $y \in S_{n-1}$, $z \in S_1$. But then $y + z \leq g$ so that $g \in S_n$, contrary to our assumption. □

5. The end of the proof

Now that we have Lemma 10, the hard work is essentially done.

Lemma 11. *Let S be a framed subgroup of $\mathbb{U}$. Then there exists a framed subgroup S' extending S such that for any $g \notin S_n$, $g \in S$, $(n \neq -1, 0, 1)$, the pair (g, n) is justified in S'.*

Proof. Choose a well-ordering of the set of all pairs (g, n) as above, and index them by an ordinal α. We define a transfinite sequence of framed subgroups $S(\beta)$, so that $S(\beta)$ extends $S(\gamma)$ for $\beta > \gamma$, by induction. For $\beta = 0$, let $S(\beta) = S$. If β is a limit ordinal, we let $S(\beta)$ be the union of all $S(\gamma)$ for $\gamma < \beta$ (with the induced framing). Finally, let $S(\beta + 1)$ be a framed subgroup of the games extending $S(\beta)$ in which the βth pair (g, n) is justified. The framed subgroup $S(\alpha)$ has the desired property. □

Let us call a framed subgroup S of $\mathbb{U}$ *justified* if, whenever $g \notin S_n$, $n \neq -1, 0, 1$, we have (g, n) justified in S. Call a subset S of $\mathbb{U}$ *hereditary* if there exists a hereditary set of games $\widetilde{S}$ having image S in $\mathbb{U}$. Note that a union of hereditary subsets of $\mathbb{U}$ is itself hereditary.

Lemma 12. *Any framed subgroup of $\mathbb{U}$ can be extended to a justified, hereditary framed subgroup.*

Proof. Let S be a framed subgroup of $\mathbb{U}$. We define a sequence of framed subgroups

$$S(0) \subseteq S(1) \subseteq S(2) \subseteq \cdots$$

as follows. Let $S(0) = S$.

Now assume that the framed group $S(i)$ has been defined, let $\widetilde{S(i)}$ be a hereditary set of games whose image in $\mathbb{U}$ contains $S(i)$, and let $S'(i)$ denote the subgroup generated by the image of $\widetilde{S(i)}$. By Lemma 8, we can extend the framing of $S(i)$ to a framing of $S'(i)$. Now let $S(i+1)$ be a framed subgroup of the games containing $S'(i)$ in which for every $g \in S'(i) - S'(i)_n$ ($n \neq -1, 0, 1$), (g, n) is justified in $S(i+1)$. Then the union of the $S(i)$ is the desired framed subgroup of $\mathbb{U}$. □

We can now show that it is possible to find the extensions we need to adjoin a single element.

LEMMA 13. *Let S be a framed subgroup of $\mathbb{U}$. Then there exists a game x such that $S_n = \{y \in S : nx \leq y\}$ for every integer n.*

Proof. Without loss of generality, we may enlarge S so that S is hereditary and justified. Let $\widetilde{S}$ denote a hereditary set of games with image S in $\mathbb{U}$, and let $\widetilde{S}_i$ denote the preimage of S_i in $\widetilde{S}$. We define the game x to be such that its left options are the elements of $\widetilde{S}$ not contained in $\widetilde{S}_1$, and its right options are the elements of $\widetilde{S}$ not contained in $-\widetilde{S}_{-1}$.

We now prove that $S_n = \{y : nx \leq y\}$ by induction on $|n|$. For $n = 0$, the assertion follows from the definition of a framing. For $|n| = 1$, we simply apply Lemma 2. Thus we may assume $|n| > 1$. We will assume that $n > 0$; the proof in the case where $n < 0$ is the same.

First suppose that $y \notin S_n$. Then, since (y, n) is justified, there exists $z \in S_{-1}$ such that $y + z \notin S_{n-1}$. By the inductive hypothesis, we have $-x \leq z$ and $(n-1)x \nleq y + z$. Therefore $nx \nleq y$.

Now suppose that $y \in S_n$, and identify y with one of its representatives in $\widetilde{S}$. We must prove that $nx \leq y$. Otherwise, we have either $y \leq x^L + (n-1)x$ or $y^R \leq nx$. In the first case, the inductive hypothesis implies that $x^L - y \in S_{-n+1}$, so that $x^L \in S_1$, a contradiction. In the second case, what we have shown implies $-y^R \in S_{-n}$, so that $y - y^R \in S_0$, so that $y^R \leq y$, also a contradiction. □

We now have the tools we need to establish the main theorem of this paper.

Proof of Theorem 1. Consider the collection of all partial extensions of ϕ, partially ordered by extension. By Zorn's lemma, this collection possesses a maximal element (as in the proof of the weak embedding theorem, the fact that this collection forms a proper class does not pose any real difficulty). Replacing ϕ by this maximal element, we may assume that ϕ is itself maximal.

It now suffices to show that $S = S'$. Suppose otherwise, and choose $s \in S' - S$. We will show that it is possible to extend ϕ to the subgroup of S' generated by S and s. This will contradict the maximality of ϕ and complete the proof.

Let us identify S with its image under ϕ. Let $S_n = \{t \in S : ns \leq t\}$. This is a framing of S. By Lemma 13, there exists a game x such that $S_n = \{t \in S : nx \leq t\}$. Then we can extend ϕ by setting $\phi(s) = x$. $\square$

Remark 14. We used the axiom of choice several times in the course of proving the embedding theorem. The constructively minded reader may note that we can eliminate all appeals to the axiom of choice if we are provided with a well-ordering of S' and a set of games containing representatives for each element of $\phi(S)$. With these additional hypotheses, the proof of Theorem 1 that we have given is valid in any admissible set satisfying the axiom of infinity.

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