Illinois Journal of Mathematics
Volume 46, Number 3, Fall 2002, Pages 929–945
S 0019-2082

INTERPOLATING SEQUENCES FOR HOLOMORPHIC FUNCTIONS OF RESTRICTED GROWTH

ANDREAS HARTMANN AND XAVIER MASSANEDA

ABSTRACT. We show that the interpolating sequences for the algebra of holomorphic functions in the unit disk of order at most $\alpha > 0$ are characterized by a hyperbolic density condition. We also give conditions along the same lines for the analogous problem in the unit ball of $\mathbb{C}^n$.

1. Introduction

In 1956 A.G. Naftalevič defined the interpolating sequences for the Nevanlinna class

$$\mathcal{N} = \{f \in H(\mathbb{D}) : \sup_{r<1} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta < \infty\}$$

in the unit disk $\mathbb{D}$ of $\mathbb{C}$ as those sequences $\{a_k\}_k \subset \mathbb{D}$ such that for every sequence of values $\{v_k\}_k$ with

$$\sup_{k\in\mathbb{N}} (1 - |a_k|) \log^+ |v_k| < \infty \tag{1}$$

there exists $f \in \mathcal{N}$ such that $f(a_k) = v_k$ for all $k \in \mathbb{N}$. With this definition, interpolating sequences for $\mathcal{N}$ are characterized by the following conditions.

THEOREM A (Naftalevič [Na56]). *A sequence $\{a_k\}_k$ is interpolating for the Nevanlinna class $\mathcal{N}$ if and only if:*

(a) *$\{a_k\}_k$ is a Blaschke sequence, i.e., $\sum_k (1 - |a_k|) < \infty$.*
(b) *$\{a_k\}_k$ lies inside a polygon inscribed in the closed unit disk.*
(c) *$\prod_{j\neq k} \left|\frac{a_j - a_k}{1-\bar{a}_k a_j}\right| \geq \delta \exp\left(-\frac{c}{1-|a_k|}\right)$ for some $c, \delta > 0$ and all $k \in \mathbb{N}$.*

Received January 22, 2002; received in final form June 4, 2002.
2000 *Mathematics Subject Classification.* Primary 30E05. Secondary 32A30.
Both authors are supported by a PICS program of Generalitat de Catalunya and the CNRS. The second author is also supported by the DGICYT grant PB98-1242-C02-01 and the CIRIT grant 2000-SGR00059.

The geometric condition (b) is somewhat unnatural. In particular, it implies that there exist interpolating sequences for H^∞ (the space of bounded holomorphic functions) which are not interpolating for $\mathcal{N}$ (just take any $\{a_k\}_k$ with $\inf_k \prod_{j\neq k} \left|\frac{a_j-a_k}{1-\bar{a}_k a_j}\right| > 0$ not inscribed in any polygon). An explanation for the presence of (b) in Theorem A is that the maximal growth of Nevanlinna functions reflected in (1) (if $f\in\mathcal{N}$ then $\sup_{\mathbb{D}}(1-|z|)\log^+|f(z)|<\infty$) can only be attained in a finite number of Stolz angles ([Na56, Lemma 1]).

In our opinion (1) is the natural compatibility condition for interpolation by a different class of functions. Given $\alpha>0$, consider the algebra of holomorphic functions of order at most $\alpha>0$:

$$A_\alpha = \{f\in H(\mathbb{D}) : \sup_{z\in\mathbb{D}}(1-|z|)^\alpha \log^+|f(z)|<\infty\}.$$

Definition 1. A sequence $\{a_k\}_k\subset\mathbb{D}$ is *A_α-interpolating* (denoted by $\{a_k\}_k\in \operatorname{Int} A_\alpha$) if for all sequences of values $\{v_k\}_k$ with

$$\sup_{k\in\mathbb{N}}(1-|a_k|)^\alpha\log^+|v_k|<\infty \tag{2}$$

there exists $f\in A_\alpha$ such that $f(a_k)=v_k$ for all $k\in\mathbb{N}$.

For $z,\zeta\in\mathbb{D}$ consider the hyperbolic pseudodistance $d(z,\zeta)=|\phi_z(\zeta)|$, where $\phi_z(\zeta)=(z-\zeta)/(1-\bar{z}\zeta)$ is the automorphism of $\mathbb{D}$ exchanging z and 0. Also, given $\{a_k\}_k\subset\mathbb{D}$ and $\delta\in(0,1)$, define the pseudodisk $\mathcal{K}(z,\delta)=\{\zeta\in\mathbb{D}: d(z,\zeta)<\delta\}$, and the counting functions

$$\begin{aligned} n(z,\delta) &= \#\,\{a_k\}_k\cap\overline{\mathcal{K}(z,\delta)},\\ N(z,\delta) &= \int_0^\delta \frac{n(z,t)-n(z,0)}{t}\,dt + n(z,0)\log\delta. \end{aligned}$$

Our main result is the following.

Theorem 1. *A sequence $\{a_k\}_k\subset\mathbb{D}$ is A_α-interpolating if and only if*

$$\sup_{k\in\mathbb{N}}\frac{N(a_k,1/2)}{(1-|a_k|)^{-\alpha}}<\infty. \tag{3}$$

The necessity of this condition is an immediate consequence of Jensen's formula, while the sufficiency is proved by an L^2-estimate for the solution to a $\bar\partial$-equation, as in [BeOr]. The constant $1/2$ can be replaced by any other value $\delta\in(0,1)$, as will be clear from the proof.

Henceforth we write $A\preceq B$ when $A\le cB$ for some $c>0$, and $A\simeq B$ when $A\preceq B$ and $B\preceq A$.

Remarks. (a) When $\{a_k\}_k$ is a Blaschke sequence and $\alpha=1$, condition (3) and condition (c) in Naftalevič's theorem are equivalent. Since $\log 1/x\simeq$

$1-x$ for $x \simeq 1$ and $1-|\phi_z(a)|^2 = (1-|z|^2)(1-|a|^2)/|1-\bar{z}a|^2$, we have

$$\sum_{j:|\phi_{a_k}(a_j)|>1/2} \log \frac{1}{|\phi_{a_k}(a_j)|} \preceq \sum_{j:|\phi_{a_k}(a_j)|>1/2} \frac{(1-|a_k|^2)(1-|a_j|^2)}{|1-\bar{a}_k a_j|^2}$$
$$\preceq \frac{\sum_j 1-|a_j|}{1-|a_k|}.$$

Hence (c) in Naftalevič's theorem is equivalent to

$$\sum_{j:0<|\phi_{a_k}(a_j)|\leq 1/2} \log \frac{1}{|\phi_{a_k}(a_j)|} \preceq (1-|a_k|)^{-1},$$

and therefore to (3) with $\alpha = 1$, since

$$N(a_k, 1/2) = \sum_{j:0<|\phi_{a_k}(a_j)|\leq 1/2} \log \frac{1/2}{|\phi_{a_k}(a_j)|} + \log \frac{1}{2}.$$

One could then conjecture that with an appropriate definition of an $\mathcal{N}$-interpolating sequence, conditions (a) and (c) in Naftalevič theorem should be sufficient. This is not the case, as long as we accept that with any such definition one should be able to interpolate bounded sequences of values, or just 1's and 0's (see the Appendix for more details).

(b) Set $\Phi_\alpha(z) = (1-|z|^2)^{-\alpha}$. A direct calculation (see also (8)) shows that (3) can also be viewed as the following hyperbolic density with respect to the metric $\Delta\Phi_\alpha$:

$$\sup_{k\in\mathbb{N}} \frac{N(a_k, 1/2)}{\int\limits_{\mathcal{K}(a_k,1/2)} \Delta\Phi_\alpha} < \infty.$$

The problem of interpolation by functions in A_α can also be considered in higher dimensions. Let $\mathbb{B}_n$ denote the unit ball in $\mathbb{C}^n$.

The analogous L^2-estimate for the $\bar{\partial}$-equation provides a sufficient condition which is formally identical to that in the disk.

THEOREM 2. *Let $\{a_k\}_k \subset \mathbb{B}_n$. If* (3) *holds then $\{a_k\}_k$ is A_α-interpolating.*

This cannot be improved, in the following sense: no condition of type

$$\sup_{k\in\mathbb{N}} \frac{N(a_k, 1/2)}{\Lambda(|a_k|)} < \infty,$$

where $\Lambda : [0,1) \longrightarrow \mathbb{R}_+$ is an increasing function with $\lim_{r\to 1} \Lambda(r)(1-r)^\alpha = +\infty$, can be sufficient. This is an easy consequence of Theorem 1 and the obvious fact that for sequences $\{a_k\}_k \subset \mathbb{B}_n$ with $a_k = (\alpha_k, 0) \in \mathbb{D} \times \{0\}^{n-1}$ one has $\{a_k\}_k \in \operatorname{Int} A_\alpha(\mathbb{B}_n)$ if and only if $\{\alpha_k\}_k \in \operatorname{Int} A_\alpha(\mathbb{D})$.

To conclude, and for the sake of completeness, we state a necessary density condition which is an adaptation to the ball of a result by Li and Taylor [LiTa] for some spaces of entire functions. Let $\Phi_\alpha(z) = (1-|z|^2)^{-\alpha}$ be as above.

THEOREM 3. *Let $\{a_k\}_k \subset \mathbb{B}_n$ be A_α-interpolating. Then:*

(a) *$\{a_k\}_k$ is weakly separated, that is, there exist $\varepsilon, p > 0$ such that*

$$d(a_k, a_j) \geq 2\varepsilon \max[e^{-p\Phi_\alpha(a_k)}, e^{-p\Phi_\alpha(a_j)}].$$

(b) *There exists $C > 0$ such that*

$$n(z,r) \leq \frac{C}{(1-r)^n}[\Phi_\alpha(r)\ \Phi_\alpha(z)]^n \qquad \text{for all } r \in (0,1) \text{ and } z \in \mathbb{B}_n.$$

In particular,

$$\sup_{z\in\mathbb{B}_n} \frac{n(z,1/2)}{\Phi_\alpha^n(z)} < \infty.$$

This can be rewritten as a density with respect to the Monge-Ampère mass associated to Φ_α. Consider

$$i\partial\bar\partial\Phi_\alpha(z) = \frac{\alpha\ \Phi_\alpha(z)}{(1-|z|^2)^2}\left[(\alpha+1)i\partial|z|^2 \wedge \bar\partial|z|^2 + (1-|z|^2)i\partial\bar\partial|z|^2\right] \tag{4}$$

and the fundamental form of the Bergman metric in $\mathbb{B}_n$:

$$\Psi(z) := i\partial\bar\partial \log\left(\frac{1}{1-|z|^2}\right) = \frac{(1-|z|^2)i\partial\bar\partial|z|^2 + i\partial|z|^2\wedge\bar\partial|z|^2}{(1-|z|^2)^2}.$$

Then $\alpha\Phi_\alpha \cdot \Psi \leq i\partial\bar\partial\Phi_\alpha \leq (\alpha+1)\Phi_\alpha\cdot\Psi$, and $\int_{\mathcal{K}(z,1/2)}(i\partial\bar\partial\Phi_\alpha)^n$ is comparable to $(\Phi_\alpha(z))^n$. This shows that Theorem 3(b) is equivalent to

$$\frac{n(z,r)}{\int\limits_{\mathcal{K}(z,r)}(i\partial\bar\partial\Phi_\alpha)^n} \leq C\ (\Phi_\alpha(r))^n \qquad \text{for all } r\in(0,1) \text{ and } z \in \mathbb{B}_n.$$

See [Ln] for more precise Monge-Ampère density conditions for interpolation and sampling in some spaces of entire functions.

The paper is organized as follows. In Section 2 we prove Theorem 1. Sections 3 and 4 contain the proofs of Theorems 2 and 3, respectively, while in Section 5 we state analogous theorems for slightly more general versions of the weight Φ_α. Finally, in an appendix we collect some remarks on the interpolation problem for the Nevanlinna class.

A final word about notation: C will always denote a positive constant and its actual value may change from one occurrence to the next.

2. Proof of Theorem 1

The topological structure of the algebra A_α plays an important role in the proof of Theorem 1. It is clear from the definition that $A_\alpha = \bigcup_{p>0} A_\alpha^{-p}$, where

$$A_\alpha^{-p} = \left\{ f \in H(\mathbb{D}) : \|f\|_{A_\alpha^{-p}} := \sup_{\mathbb{D}} |f|\, e^{-p\Phi_\alpha} < \infty \right\}.$$

A_α is the inductive limit of the Banach spaces A_α^{-p}, and it enjoys the structure of a Fréchet (LF)-space.

Similarly, given $S = \{a_k\}_k$, we consider the Fréchet (LF)-space $A_\alpha(S) = \bigcup_{p>0} A_\alpha^{-p}(S)$, where

$$A_\alpha^{-p}(S) = \left\{ \{v_k\}_k : \sup_{k\in\mathbb{N}} |v_k|\, e^{-p\Phi_\alpha(a_k)} < \infty \right\}.$$

In these terms, S is A_α-interpolating if and only if the restriction operator

$$\begin{array}{rll} \mathcal{R}_S: & A_\alpha & \longrightarrow A_\alpha(S) \\ & f & \to \{f(a_k)\}_k \end{array}$$

is onto. An application of the open mapping theorem to R_S yields the following result.

Lemma 1 ([Gro, Theorem 2, p. 148]). *If S is A_α-interpolating, then for every $q \geq 0$ there exist $p, C > 0$ such that for all $k \in \mathbb{N}$ there is $g_k \in A_\alpha^{-p}$ with $g_k(a_j) = \delta_{jk} e^{q\Phi_\alpha(a_k)}$ and $\|g_k\|_{A_\alpha^{-p}} \leq C$.*

Proof of the necessity. Take the functions g_k given by Lemma 1 and define $f_k := g_k \circ \phi_{a_k}$. Since

$$\Phi_\alpha(\phi_{a_k}(z)) \leq C_\alpha \Phi_\alpha(z)\Phi_\alpha(a_k), \tag{5}$$

we have $\log|f_k(z)| \preceq \Phi_\alpha(z)\Phi_\alpha(a_k)$, and an application of Jensen's formula yields

$$\int_0^r \frac{n(a_k,t)-1}{t}\,dt \leq \frac{1}{2\pi}\int_0^{2\pi} \log|f_k(re^{i\theta})|\,d\theta \leq C_\alpha\ \Phi_\alpha(r)\ \Phi_\alpha(a_k)$$

for all $r < 1$. In particular, $N(a_k, r) \leq C_\alpha\ \Phi_\alpha(r)\ \Phi_\alpha(a_k)$.

Proof of the sufficiency. Given $\{v_k\}_k$ satisfying (2), we will show first that it is possible to construct a smooth interpolating function having the characteristic growth of A_α. Then, by solving a $\bar\partial$ equation with estimates, we will see that the interpolating function can be taken to be holomorphic. For this purpose it will be convenient to express A_α as a union of weighted Bergman spaces:

$$A_\alpha = \bigcup_{p>0} B_{\alpha,p},$$

where

$$B_{\alpha,p} = \left\{ f \in H(\mathbb{D}) : \|f\|^2_{B_{\alpha,p}} := \int_{\mathbb{D}} |f|^2 e^{-p\Phi_\alpha} dm < \infty \right\}$$

and dm denotes the Lebesgue measure.

Condition (3) implies that $\{a_k\}_k$ is weakly separated, i.e., for some $\varepsilon, q > 0$, and $\delta_k = \varepsilon e^{-q\Phi_\alpha(a_k)}$, the hyperbolic pseudoballs $\mathcal{K}_k := \mathcal{K}(a_k, \delta_k)$ are pairwise disjoint.

Let $\mathcal{X}$ be a smooth cut-off function of one real variable, with derivative $\mathcal{X}'$ uniformly bounded, $\mathcal{X}(t) \equiv 1$ for $t < 1/2$ and $\mathcal{X}(t) \equiv 0$ for $t > 1$. Define the smooth interpolating function

$$F(z) = \sum_{k=1}^{\infty} v_k \, \mathcal{X}\left(\frac{|\phi_z(a_k)|^2}{\delta_k^2} \right).$$

The support of F is contained in $\cup_k \mathcal{K}_k$, and for $z \in \mathcal{K}_k$

$$\begin{aligned} |F(z)| &\leq |v_k|, \\ \left|\bar{\partial} F(z)\right| &\preceq |v_k| \frac{1}{1-|a_k|} \frac{1}{\delta_k}. \end{aligned}$$

Using the estimate $(1-|a_k|)^{-1} \preceq e^{\Phi_\alpha(a_k)}$ and (2) we see that there exist a constant C independent of k and s big enough so that

$$|F(z)|^2 \, e^{-s\Phi_\alpha} \leq C \quad \text{and} \quad |\bar{\partial} F(z)|^2 \, e^{-s\Phi_\alpha} \leq C.$$

Because of the weak separation, there exists $M > 0$ such that $\sum_k e^{-M\Phi_\alpha(a_k)} < \infty$. Hence, taking again s big enough, we have

$$\int_{\mathbb{D}} |F|^2 \, e^{-s\Phi_\alpha} < \infty \quad \text{and} \quad \int_{\mathbb{D}} |\bar{\partial} F|^2 \, e^{-s\Phi_\alpha} < \infty. \tag{6}$$

Now, when looking for a holomorphic interpolating function of the form $f := F - u$ we are led to the $\bar{\partial}$-problem

$$\bar{\partial} u = \bar{\partial} F,$$

which we solve by Hörmander's theorem [Hör, Theorem 4.2.1]: Given a subharmonic function ψ in $\mathbb{D}$, there exists a solution u to the above equation such that

$$2 \int_{\mathbb{D}} |u|^2 \frac{e^{-\psi}}{(1+|z|^2)^2} \leq \int_{\mathbb{D}} |\bar{\partial} F|^2 \, e^{-\psi}.$$

We apply this estimate to the weight

$$\psi_\beta(z) = \beta \, \Phi_\alpha(z) + v(z),$$

where $\beta > 0$ will be chosen later and

$$v(z) = \sum_{k=1}^{\infty} \left[\log|\phi_z(a_k)|^2 - \frac{1}{\pi/8^2} \int_{D(0,1/8)} \log|\phi_{\phi_z(a_k)}(\zeta)|^2 dm(\zeta) \right].$$

Changing to polar coordinates and using the identity

$$\frac{1}{2\pi}\int_0^{2\pi} \log|\zeta - re^{i\theta}|^2\, d\theta = \max(\log|\zeta|^2, \log r^2)$$

we see that

$$v(z) = \sum_{k:|\phi_z(a_k)|\leq 1/8} \log\left|\frac{\phi_z(a_k)}{1/8}\right|^2 + 1 - \left|\frac{\phi_z(a_k)}{1/8}\right|^2. \tag{7}$$

Lemma 2. *If* (3) *holds, there exists* $C > 0$ *such that*

$$\Delta v(z) \geq -C(1-|z|^2)^{-2}\Phi_\alpha(z).$$

Proof. From (7) we have

$$\begin{aligned}\Delta v(z) \quad &\geq \quad -\sum_{k:|\phi_z(a_k)|\leq 1/8} \Delta\left|\frac{\phi_z(a_k)}{1/8}\right|^2 = -8^2 \sum_{k:|\phi_z(a_k)|\leq 1/8} \frac{(1-|a_k|^2)^2}{|1-\bar{a}_k z|^4} \\ &\succeq \quad -8^2 \frac{n(z,1/8)}{(1-|z|^2)^2}.\end{aligned}$$

In order to see that $n(z,1/8)$ is controlled by $\Phi_\alpha(z)$ take any $a_k \in \mathcal{K}(z,1/8)$ (if $n(z,1/8) = 0$ there is nothing to prove) and notice that (3) implies

$$n(a_k,1/4) - 1 \preceq \int_{1/4}^{1/2} \frac{n(a_k,t)-1}{t}\, dt \preceq N(a_k,1/2) \preceq \Phi_\alpha(a_k).$$

Since $\mathcal{K}(z,1/8) \subset \mathcal{K}(a_k,1/4)$, by (5) we have then

$$n(z,1/8) \leq n(a_k,1/4) \preceq \Phi_\alpha(a_k) \preceq \Phi_\alpha(z). \qquad \square$$

A straightforward calculation gives

$$\Delta\Phi_\alpha(z) = \frac{\alpha(1+\alpha|z|^2)}{(1-|z|^2)^2}\,\Phi_\alpha(z). \tag{8}$$

Thus taking β big enough and applying Lemma 2 we see that ψ_β is subharmonic and $\Delta\psi_\beta$ is bounded below:

$$\Delta\psi_\beta = \beta\Delta\Phi_\alpha + \Delta v \geq \frac{\Phi_\alpha}{(1-|z|^2)^2}.$$

Since v is negative (by definition) and $(1+|z|^2)^2$ is comparable to a constant, the L^2-estimate for the $\bar\partial$ solution yields

$$\int_{\mathbb{D}} |u|^2 e^{-\beta\Phi_\alpha} \preceq 2\int_{\mathbb{D}} |u|^2 \frac{e^{-\psi_\beta}}{(1+|z|^2)^2} \leq \int_{\mathbb{D}} |\bar\partial F|^2 e^{-\beta\Phi_\alpha} e^{-v}.$$

If z is in the support of $\bar\partial F$, it belongs to one of the annuli $A_k = \{z : \delta_k/2 < |\phi_z(a_k)| < \delta_k\}$. Then, by (7),

$$\begin{aligned} -v(z) &\leq \sum_{j:|\phi_z(a_j)|\leq 1/8} \log \frac{1/8^2}{|\phi_z(a_j)|^2} \\ &\preceq \log \frac{1}{|\phi_z(a_k)|} + \sum_{\substack{|\phi_z(a_j)|\leq 1/8 \\ j\neq k}} \log \frac{1/8}{|\phi_z(a_j)|} \\ &\preceq \log \frac{1}{\delta_k} + \int_{2\delta_k}^{1/8} \frac{n(z,t)-1}{t}\, dt. \end{aligned}$$

The first term is dominated by $\Phi_\alpha(a_k)$, by definition of δ_k. In order to estimate the integral we use the inclusion $\mathcal{K}(z,t) \subset \mathcal{K}(a_k, \frac{t+\delta_k}{1+t\delta_k})$ and perform the change of variable $s = \frac{t+\delta_k}{1+t\delta_k}$. Since $\{\delta_k\} \searrow 0$, we obtain

$$\begin{aligned} \int_{2\delta_k}^{1/8} \frac{n(z,t)-1}{t}\, dt &\leq \int_{\frac{3\delta_k}{1+2\delta_k^2}}^{\frac{1/8+\delta_k}{1+\delta_k/8}} \frac{n(a_k,s)-1}{s-\delta_k} \frac{1-\delta_k^2}{(1-\delta_k s)^2}\, ds \\ &\preceq \int_{\frac{3\delta_k}{1+2\delta_k^2}}^{\frac{1/8+\delta_k}{1+\delta_k/8}} \frac{n(a_k,s)-1}{s-\delta_k}\, ds. \end{aligned}$$

We can assume δ_k sufficiently small so that $\frac{1/8+\delta_k}{1+\delta_k/8} \leq \frac{1}{2}$. Since $s - \delta_k > s/2$ when $s > \frac{3\delta_k}{1+2\delta_k^2}$, we have

$$-v(z) \preceq \Phi_\alpha(a_k) + \int_0^{1/2} \frac{n(a_k,s)-1}{s/2}\, ds \preceq \Phi_\alpha(a_k) \preceq \Phi_\alpha(z).$$

Thus $e^{-v} \preceq e^{c\,\Phi_\alpha}$ for some $c > 0$, and

$$\int_{\mathbb{D}} |u|^2 e^{-\beta\Phi_\alpha} \preceq \int_{\mathbb{D}} |\bar\partial F|^2 e^{-(\beta-c)\Phi_\alpha}.$$

This and (6) show that if $\beta \geq c + s$ then $f = F - u \in A_\alpha$. Moreover $e^{-\psi_\beta} \simeq |\phi_z(a_k)|^{-2}$ around each a_k. Hence $\int |u|^2 e^{-\psi_\beta} < \infty$ implies $u(a_k) = 0$ for all $k \in \mathbb{N}$ and $f(a_k) = v_k$, as required. □

3. Proof of Theorem 2

The proof in higher dimension goes along the same lines. We only indicate the minor changes required to adapt the proof given above to the ball.

As before, given $\{v_k\}_k$ verifying (2) we can find a smooth interpolating function F satisfying (6). Then we solve the corresponding $\bar\partial$ equation by

Hörmander's theorem [Hör, Theorem 4.2.6]: Given a plurisubharmonic function ψ in $\mathbb{B}_n$, there exists a solution u to $\bar\partial u = \bar\partial F$ such that

$$2\int_{\mathbb{B}_n} |u|^2 \frac{e^{-\psi}}{(1+|z|^2)^2} \leq \int_{\mathbb{B}_n} |\bar\partial F|^2 \, e^{-\psi}.$$

Consider the $\mathcal{C}^1$ function

$$\omega(z) = \begin{cases} \log|z|^2 + 1 - |z|^2 & \text{if} \quad |z| \leq 1, \\ \quad 0 & \text{if} \quad |z| > 1, \end{cases}$$

and the weight

$$v(z) = n \sum_{k=1}^{\infty} \omega\left(\frac{\phi_z(a_k)}{1/8}\right),$$

which is formally the same as in (7). Define $\psi_\beta = \beta\Phi_\alpha + v$.

Let Ψ denote the fundamental form of the Bergman metric in $\mathbb{B}_n$, as in the introduction. We use the notations

$$\begin{aligned} N(\phi_z(a_k)) &= \frac{i\partial\bar\partial|\phi_z(a_k)|^2}{1-|\phi_z(a_k)|^2}, \\ T(\phi_z(a_k)) &= \frac{i\partial|\phi_z(a_k)|^2 \wedge \bar\partial|\phi_z(a_k)|^2}{(1-|\phi_z(a_k)|^2)^2}. \end{aligned}$$

As a consequence of the invariance by automorphisms of Ψ (or by a direct calculation) we have

$$N(\phi_z(a_k)) + T(\phi_z(a_k)) = \Psi(z),$$

and, since $T(\phi_z(a_k))$ is a positive form,

$$\begin{aligned} i\partial\bar\partial v(z) &\geq -\sum_{k:|\phi_z(a_k)|\leq 1/8} i\partial\bar\partial\left|\frac{\phi_z(a_k)}{1/8}\right|^2 \\ &\succeq -\sum_{k:|\phi_z(a_k)|\leq 1/8} N(\phi_z(a_k))(1-|\phi_z(a_k)|^2) \\ &\succeq -n(z,1/8)\Psi(z). \end{aligned}$$

On the other hand, (4) shows that $i\partial\bar\partial\Phi_\alpha(z) \geq \alpha\, \Phi_\alpha(z)\Psi(z)$, and therefore

$$i\partial\bar\partial\psi_\beta \geq [\beta\alpha\, \Phi_\alpha(z) - n(z,1/8)]\, \Psi(z).$$

As in the proof of Lemma 2, the hypothesis implies $n(z,1/8) \preceq \Phi_\alpha(z)$. Hence for β big enough ψ_β is plurisubharmonic. Therefore

$$\int_{\mathbb{B}_n} |u|^2 e^{-\beta\Phi_\alpha} \preceq 2\int_{\mathbb{B}_n} |u|^2 \frac{e^{-\psi_\beta}}{(1+|z|^2)^2} \leq \int_{\mathbb{B}_n} |\bar\partial F|^2 e^{-\beta\Phi_\alpha} e^{-v}.$$

From here we finish as in the proof of Theorem 1.

4. Proof of Theorem 3

Part (a) is an easy consequence of Lemma 1 and a gradient estimate; it can be proved in the same way as Theorem 2 in [Ma97].

The proof of (b) is just an adaptation to the ball of the proof of Theorem 3.6 in [LiTa]. We include a sketch for the sake of completeness.

First we see that $\{a_k\}_k$ can be included in the zero set of a suitable map. Let $D_u F$ denote the derivative of F in the unitary direction u.

Theorem 4. *A sequence $\{a_k\}_k$ is A_α-interpolating if and only if there exists a map $F=(F_1,\dots,F_n):\mathbb{B}_n\longrightarrow\mathbb{C}^n$, $F_j\in A_\alpha$, and constants $\delta,p>0$ such that $F(a_k)=0$ for all $k\in\mathbb{N}$ and*

$$\sum_{j=1}^{n}|D_uF_j(a_k)|\geq\delta e^{-p\Phi_\alpha(a_k)}\quad\forall k\in\mathbb{N}\quad\forall u\quad \textit{unitary.}$$

This can be proved in the same way as the Main Theorem in [Ma98]. We give the proof of the necessity part, which is the one we will use.

Proof of the necesity. Because of (a), there is $M>0$ with $\sum_k e^{-M\Phi_\alpha(a_k)}<\infty$. By Lemma 1, there exist $p,K>0$ and functions $g_k\in A_\alpha^{-p}$ such that $\|g_k\|_{A_\alpha^{-p}}\leq K$, $g_k(a_j)=\delta_{jk}e^{M\Phi_\alpha(a_k)}$. Define

$$F_j(z)=\sum_{k=1}^{\infty}e^{-2M\Phi_\alpha(a_k)}g_k^2(z)(z^j-a_k^j),\quad j=1,\dots,n,$$

where z^j denotes the jth coordinate of z. Then F vanishes on $\{a_k\}_k$, $F\in A_\alpha^{-2p}$, and

$$\frac{\partial F_j}{\partial z_l}(a_k)=\delta_{jl}\;g_k(a_k)e^{-M\Phi_\alpha(a_k)}=\delta_{jl}. \qquad \square$$

Theorem 3 will then be a consequence of the following proposition and the invariance by automorphisms of interpolating sequences. Given a map $F=(F_1,\dots,F_n)$, let $\|F\|_{A_\alpha^{-p}}=\sum_j\|F_j\|_{A_\alpha^{-p}}$.

Proposition 1. *Let $\{a_k\}_k\subset\mathbb{B}_n$. Assume $\gamma_0>0$ and $F=(F_1,\dots,F_n)\colon\mathbb{B}_n\longrightarrow\mathbb{C}^n$, $F_j\in A_\alpha^{-p}$, are such that $F(a_k)=0$ for all $k\in\mathbb{N}$ and*

$$\sum_{j=1}^{n}|D_uF_j(a_k)|\geq\gamma_0\quad\forall k\in\mathbb{N}\quad\forall u\quad \textit{unitary.}$$

Then there exists $C=C(n,p)>0$ such that

$$n(0,r)\leq\frac{C}{(1-r)^n}\left[\log\|F\|_{A_\alpha^{-p}}+\log\frac{1}{\gamma_0}+\Phi_\alpha(r)\right]^n.$$

The following lemma is crucial to the proof of Proposition 1. Given a map $F = (F_1, \dots, F_n)$ let $n_F(z, r) = \# F^{-1}(0) \cap \mathcal{K}(z, r)$. For $\tau = (\tau_1, \dots, \tau_n)$, $\tau_j > 0$, and $\varphi = (\varphi_1, \dots, \varphi_n) \in \mathbb{T}^n$ denote $\tau e^{i\varphi} = (\tau_1 e^{i\varphi_1}, \dots, \tau_n e^{i\varphi_n})$.

Lemma 3 ([Gr, Theorem 2.9]). *Let $F = (F_1, \dots, F_n) : \mathbb{B}_n \longrightarrow \mathbb{C}^n$ be nondegenerate and let $M(F_j, t) = \sup_{|z|=t} |F_j(z)|$. For $\tau = (\tau_1, \dots, \tau_n)$, $\tau_j > 0$, and $\varphi \in \mathbb{T}^n$ let $F_{\tau,\varphi} = F - \tau e^{i\varphi}$. Then, for any $r \in (0, 1)$ and $\beta > 1$ such that $\beta^n r < 1$*

$$\int_{\mathbb{T}^n} n_{F_{\tau,\varphi}}(0, r)\, d\varphi \leq \frac{\sigma(\beta^n r)}{(\beta^2 - 1)^n r^{2n}} \prod_{j=1}^{n} \left(\log^+ M(F_j, \beta^{n-j+1} r) + \log^- \tau_j \right),$$

where $d\varphi$ is the product measure in $\mathbb{T}^n$ and $\sigma(\beta^n r)$ the volume of the ball $B(0, \beta^n r)$.

Proof of Proposition 1. Let F the map given by Theorem 4 and fix a_k with $|a_k| \leq r$. Because of the estimate on $D_u F(a_k)$, there exist constants $c_i, A_i > 0$, $i = 1, 2$, depending only on n, p and $\|F\|_{A_\alpha^{-p}}$ such that

$$d(a_k, F^{-1}(0) \setminus \{a_k\}) > d_r := \gamma_0\ c_1 e^{-A_1 \Phi_\alpha(r)}$$

and

$$|F(w)| > \delta_r := \gamma_0\ c_2 e^{-A_2 \Phi_\alpha(r)} \quad \text{for } w \in \partial\mathcal{K}(a_k, d_r). \tag{9}$$

This is proved using the analog for the ball of Lemma 3.9 in [LiTa].

By Sard's lemma there exists $\tau \in \mathbb{R}^n$ with

$$\frac{\delta_r}{4\sqrt{n}} < \tau_j < \frac{\delta_r}{2\sqrt{n}}, \quad j = 1, \dots, n, \tag{10}$$

and a zero measure set $E \subset \mathbb{T}^n$ such that for all $\varphi \in \mathbb{T}^n \setminus E$ the value $\tau e^{i\varphi}$ is regular for F. Hence $F^{-1}(\tau e^{i\varphi})$ is a discrete variety in $\mathbb{C}^n$.

By (9) we have on $\partial\mathcal{K}(a_k, d_r)$

$$|F - F_{\tau,\varphi}|^2 = \sum_{j=1}^{n} |\tau_j|^2 < n \frac{\delta_r^2}{4n} < |F|^2.$$

From the above and Rouché's lemma [BGVY, Theorem 2.12] one deduces that $n_F(\mathcal{K}(a_k, d_r)) = n_{F_{\tau,\varphi}}(\mathcal{K}(a_k, d_r))$ (where $n_F(U)$ is the number of zeros of a map F in U, counted with multiplicities). Hence

$$n(0, r) \leq n_{F_{\tau,\varphi}}(B(0, r)) \quad \text{for all } \varphi \in \mathbb{T}^n \setminus E.$$

Now take $\beta > 1$ such that $\beta^n r = (1+r)/2$ and apply Lemma 3. If $R > 1/2$, then

$$\begin{aligned} n(0,r) &\leq \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} n_{F_\varphi}(B(0,r))\,d\varphi \\ &\leq \frac{c_n}{(\beta^2-1)^n}\prod_{j=1}^n\left[\log^+ M(F_j,\beta^n r) + \log\frac{4\sqrt{n}}{\delta_r}\right] \\ &\leq \frac{c_n}{(\beta-1)^n}\Bigg[\log\|F\|_{A_\alpha^{-p}} + \Phi_\alpha(\beta^n r) + \log(4\sqrt{n}c_2^{-1}) \\ &\qquad\qquad + A_2\Phi_\alpha(r) + 3\log\frac{1}{\gamma_0}\Bigg]^n. \end{aligned}$$

Since $\beta-1 > (1-r)/(2n)$ and $1-\beta^n r = (1-r)/2$, there exists $C > 0$ depending on n, p and α such that $\Phi_\alpha(\beta^n r) \leq C\Phi_\alpha(r)$, and the proof is finished. $\square$

Proof of Theorem 3. Let F be the map provided by Lemma 4 and take $p > 0$ such that $F \in A_\alpha^{-p}$.

Fix $z \in \mathbb{B}_n$. Define $S_z = \{\phi_z(a_k)\}_k$ and $F^z = F\circ\phi_z$. Then $F^z \in A_\alpha^{-p}$ and, because of (5),

$$\log|F^z(\zeta)| \leq \log\|F\|_{A_\alpha^{-p}} + p\,C_\alpha\Phi_\alpha(z)\Phi_\alpha(\zeta).$$

Also $F^z_{|S_z} \equiv 0$, and letting $b_k = \phi_z(a_k)$, we have

$$\sum_{j=1}^n |D_u F_j^z(b_k)| \geq \sum_{j=1}^n |D_u F_j(a_k)|\|\mathcal{J}\phi_z(\phi_z(a_k))\cdot\mathcal{J}\phi_z(a_k)\| \geq \frac{(1-|z|^2)^{2(n+1)}}{\sqrt{n}}.$$

Then we get the desired result as an application of Proposition 1. $\square$

5. Generalizations

The results above can be immediately generalized in two different directions: the spaces of functions and the multiplicity of the interpolation.

Definition 2. A radial increasing subharmonic function $\Phi : \mathbb{D} \longrightarrow \mathbb{R}^+$ is called a *weight* if for some $C, D > 0$:

(i) $\Phi(\phi_a(z)) \leq C\,\Phi(a)\Phi(z)$ for all $z, a \in \mathbb{D}$.
(ii) $(1-|z|^2)^{-2}\Phi(z) \preceq \Delta\Phi(z)$ for all $z \in \mathbb{D}$.

Associated with these weights we consider the spaces

$$A_\Phi := \left\{ f \in H(\mathbb{D}) : \sup_{z\in\mathbb{D}} \Phi^{-1}(z)\log^+|f(z)| < \infty \right\}.$$

Notice that assuming $\Phi \in \mathcal{C}^2$ is no restriction. Otherwise we can replace Φ by the average $\tilde{\Phi}(z) := \frac{1}{m(\mathcal{K}(z,1/2))}\int_{\mathcal{K}(z,1/2)} \Phi\,dm$ and observe that $A_\Phi = A_{\tilde{\Phi}}$.

Examples of such weights are

$$\Phi(z) = (1 - |z|^2)^{-\alpha} \log^{\beta}\left(\frac{1}{1 - |z|^2}\right), \quad \alpha, \beta > 0,$$

$$\Phi(z) = (1 - |z|^2)^{-\alpha} \overbrace{\log \cdots \log}^{n}\left(\frac{1}{1 - |z|^2}\right), \quad \alpha > 0,\ n \in \mathbb{N}.$$

Also, given $m \in \mathbb{N}$, we consider the variety $X = \{(a_k, m)\}_k$, in which with each a_k there is associated a fixed multiplicity m.

Definition 3. A variety $X = \{(a_k, m)\}_k$ is A_Φ-*interpolating* if for every sequence of values $\{v_k^l\}_{k,l}$, $k \in \mathbb{N}$, $l = 0, \ldots, m-1$, with

$$\sup_{k \in \mathbb{N}} \Phi^{-1}(a_k) \log^+ \left(\sum_{l=0}^{m-1} |v_k^l| \right) < \infty$$

there exists $f \in A_\Phi$ with

$$\frac{f^{(l)}(a_k)}{l!} = v_k^l, \quad k \in \mathbb{N}, \quad l = 0, \ldots, m-1.$$

With these definitions the following result follows by a straightforward modification of the proof of Theorem 1.

Theorem. *Let Φ be a weight and let $X = \{(a_k, m)\}$ be a variety in $\mathbb{D}$. Then X is A_Φ-interpolating if and only if*

$$\sup_{k \in \mathbb{N}} \frac{N(a_k, 1/2)}{\Phi(a_k)} < \infty.$$

A similar generalization of Theorem 2 can be obtained (see Section 5 of [HaMa] for the case of entire functions). As for Theorem 3, there is no difficulty in generalizing this result to the weights defined above, but the higher multiplicity case is more delicate (see [LiVi] for a treatment of this problem in the case $\Phi(z) = -\log(1 - |z|^2)$).

6. Appendix. Some remarks on free interpolation in $\mathcal{N}$

It is not clear what growth condition one should impose on the values $\{v_k\}_k$ in order to obtain an appropriate definition of interpolating sequences for $\mathcal{N}$. We can, however, give the following definition.

We say that a sequence space l is *ideal* if for every sequence $\{v_k\}_k \in l$ and every sequence $\{u_k\}_k$ with $|u_k| \le |v_k|$, $k \in \mathbb{N}$, we also have $\{u_k\}_k \in l$. This can also be expressed in terms of multipliers. Indeed, a sequence space l is ideal if it is stable under ℓ^∞ multiplication: $\ell^\infty l \subset l$, i.e., if $\{\mu_k v_k\}_k \in l$ for all $\{\mu_k\}_k \in \ell^\infty$ and $\{v_k\}_k \in l$.

We can now introduce free interpolation sequences in the following way (we refer the reader to [Nik] for more information on free interpolation).

Definition 4. Given a space X of holomorphic functions on $\mathbb{D}$, a sequence $S \subset \mathbb{D}$ is of *free interpolation* for X if $X|S$ is ideal. We denote this by $S \in \mathrm{Int}_{\ell^\infty} X$ (to distinguish it from the notion of interpolating sequences defined earlier).

Proposition. *Let* $S = \{a_k\}_k \subset \mathbb{D}$.

(i) *If* $S \in \mathrm{Int}_{\ell^\infty} \mathcal{N}$ *then conditions* (a) *and* (c) *in Theorem A hold.*

(ii) *There exist sequences* $S \in \mathrm{Int}_{\ell^\infty} \mathcal{N}$ *that are not* $\mathcal{N}$*-interpolating in the sense of Naftalevič.*

(iii) *Conditions* (a) *and* (c) *in Theorem A are not sufficient for free interpolation in* $\mathcal{N}$ *(and hence they are not sufficient for being* $\mathcal{N}$*-interpolating in the sense of Naftalevič).*

An example of (ii) has already been given in the introduction. Any sequence $\{a_k\}_k$ not inscribed in any polygon and with $\inf_k \prod_{j\neq k} \left|\frac{a_j - a_k}{1-\bar{a}_k a_j}\right| > 0$ is obviously not interpolating in the sense of Naftalevič, but it is interpolating for the space $H^\infty(\mathbb{D})$ of bounded holomorphic functions, and therefore it is of free interpolation for $\mathcal{N}$. In particular, this suggests that the trace chosen in Naftalelevič's result is somewhat unnatural, in the sense that it is not compatible with free interpolation.

Part (i) of the proposition could be proved similarly to the necessity part of Theorem 1. We prove it instead by showing that free interpolation sequences for $\mathcal{N}$ are also interpolating for A_1. This is an immediate consequence of the following lemma, which also indicates that the above definition is reasonable.

Lemma 4. *Let* $S \subset \mathbb{D}$ *and* $\alpha > 0$. *Then* $S \in \mathrm{Int}\, A_\alpha$ *if and only if* $S \in \mathrm{Int}_{\ell^\infty} A_\alpha$.

Observe that if X is an algebra containing the constants (or just a function with trace 1 on S), then the definition of free interpolation is in fact equivalent to $\ell^\infty \subset X|S$. The spaces under consideration here (A_α or $\mathcal{N}$) are algebras containing the constants, so this lemma actually shows that a sequence is A_α-interpolating (in the sense of Definition 1) if and only if $A_\alpha|S$ contains ℓ^∞.

Proof. It is clear from Definition 1 that if S is A_α-interpolating, then the trace space $A_\alpha|S$ is ideal.

For the converse, assume that $A_\alpha|S$ is ideal. It will be enough to show that in this situation the conclusion of Lemma 1 holds, since then we can proceed as in the proof of Theorem 1 and obtain (3).

We will use a standard argument based on Baire's theorem (see, for example, [BeLi, Lemma 3.3]). Define for $n \in \mathbb{N}$

$$\ell_n^\infty = \left\{\mu \in \ell^\infty : \exists\, f \in A_\alpha^{-n} \text{ with } |f(z)| \le n e^{n\Phi_\alpha(z)}, z \in \mathbb{D}, \text{ and } f|S = \mu\right\}.$$

Clearly $\ell_n^\infty \neq \emptyset$. Also $\ell_n^\infty \subset \ell^\infty$, and hence $\bigcup_{n\in\mathbb{N}} \ell_n^\infty \subset \ell^\infty$. To obtain the inverse inclusion, assume that $\mu \in \ell^\infty$. By the interpolation condition there exists $f \in A_\alpha$, and so $f \in A_\alpha^{-n}$ for some $n \in \mathbb{N}$, such that $f|S = \mu$. Moreover, for some $m \geq n$, we will get $|f(z)| \leq ce^{n\Phi_\alpha(z)} \leq me^{m\Phi_\alpha(z)}$ so that $\mu \in \ell_m^\infty$. Hence we also have $\ell^\infty \subset \bigcup_{n\in\mathbb{N}} \ell_n^\infty$ and $\ell^\infty = \bigcup_{n\in\mathbb{N}} \ell_n^\infty$.

Let us show that ℓ_n^∞ is closed. Let $\{\mu^{(k)}\}_k$ be a sequence in ℓ_n^∞ converging in ℓ^∞ to μ. By definition there exist $f_k \in A_\alpha^{-n}$ such that $f_k|S = \mu^{(k)}$ and $|f_k(z)| \leq ne^{n\Phi_\alpha(z)}$, $z \in \mathbb{D}$. In particular, the sequence $\{f_k\}_k$ is uniformly bounded on every compact set, and by a normal family argument we can extract a subsequence $\{f_{k_j}\}_j$ converging uniformly on every compact set to some function $f \in H(\mathbb{D})$. Clearly $|f(z)| \leq ne^{-n\Phi_\alpha(z)}$ for every $z \in \mathbb{D}$, and since $\mu_l = \lim_{k\to\infty} \mu_l^{(k)} = \lim_{j\to\infty} \mu_l^{(k_j)} = \lim_{j\to\infty} f_{k_j}(a_l) = f(a_l)$, we also have $f|S = \mu$.

By Baire's theorem we can now conclude that at least one ℓ_n^∞ contains a ball of ℓ^∞: $B_{\ell^\infty}(0,\varepsilon) \subset \ell_n^\infty$ for some $\varepsilon > 0$ and some $n \in \mathbb{N}$. In particular, $((\varepsilon/2)\delta_{kj})_j \in \ell_n^\infty$. Hence there exist functions g_k with $g_k(a_j) = (\varepsilon/2)\delta_{kj}$ and $|g_k(z)| \leq ne^{-n\Phi_\alpha(z)}$ for all $z \in \mathbb{D}$, i.e., $\|g_k\|_{A_\alpha^{-n}} \leq n$, $k \in \mathbb{N}$. So, the conclusion of Lemma 1 is still valid under the a priori weaker condition of free interpolation. As mentioned above, we can proceed as in the proof of Theorem 1 to get the result. $\square$

Proof of the Proposition. (i) That $\{a_k\}$ must be a Blaschke sequence is immediate: Define $w_1 = 1$ and $w_k = 0$ for all $k > 1$. Since $\{w_k\}_k \in \ell^\infty\mathcal{N}|S \subset \mathcal{N}|S$ (observe that $1 \in \mathcal{N}$), there exists $f \in \mathcal{N}$ with $f(a_k) = 0$ for all $k > 1$, and so $(z - a_k)f \in \mathcal{N}$ vanishes on S. It is well known that zero sequences of Nevanlinna functions satisfy the Blaschke condition.

Let us now prove condition (c) of Naftalevič. Since $\mathcal{N} \subset A_1$, it is clear that if $\mathcal{N}|S$ is ideal then $\ell^\infty \subset \mathcal{N}|S$. Hence $\ell^\infty \subset A_1|S$, and by the remark made after Lemma 4 we know that S is A_1-interpolating. The result follows now from Lemma 4 and the fact that (3) together with the Blaschke condition are equivalent to condition (c) of Naftalevič (see Remark (a) in the Introduction).

It only remains to prove (iii). To do this, we give an example of a sequence satisfying (a) and (c) of Naftalevič' theorem, which is not of free interpolation for $\mathcal{N}$.

For a given Blaschke sequence $S = \{b_k\}_k$, denote its associated Blaschke product by

$$B^S(z) = \prod_{k=1}^{\infty} \frac{z - b_k}{1 - \bar{b}_k z}$$

and let $B_k^S = B^S / \frac{z-b_k}{1-\bar{b}_k z}$.

Take $\sigma = \{\sigma_k\}$ to be a Carleson sequence (i.e., $\inf_k |B_k^\sigma(\sigma_k)| > 0$) approaching the unit circle tangentially at some point. Take also $\sigma' = \{\sigma'_k\}$ such that $d(\sigma_k, \sigma'_k) = \exp(-\frac{1}{1-|\sigma_k|})$ and define $S = \sigma \cup \sigma'$. Since σ and σ' are Carleson, we have for any $a_k \in S$

$$|B_k^S(a_k)| \simeq \exp\left(-\frac{1}{1-|a_k|}\right).$$

Hence conditions (a) and (c) in Naftalevič' result are satisfied.

Assuming that $\{a_k\}_k$ is of free interpolation for $\mathcal{N}$ there exists $f \in \mathcal{N}$ such that

$$f(a_k) = \begin{cases} 1 & \text{if} \quad a_k \in \sigma, \\ 0 & \text{if} \quad a_k \in \sigma', \end{cases}$$

and so f can be factorized into $f = B^{\sigma'} g$, with $g \in \mathcal{N}$. Since

$$\exp\left(-\frac{1}{1-|\sigma_k|}\right) \simeq |B_k^S(\sigma_k)| \leq |B^{\sigma'}(\sigma_k)| \leq \frac{1}{\delta}|B_k^S(\sigma_k)| \simeq \exp\left(-\frac{1}{1-|\sigma_k|}\right),$$

we get

$$|g(\sigma_k)| \simeq \exp\left(\frac{1}{1-|\sigma_k|}\right) \quad \forall k \in \mathbb{N}, \tag{11}$$

which contradicts the fact that Nevanlinna functions can only attain such a growth within a finite union of Stolz angles (see [Na56, Lemma 1]).

□

Remark. Notice that part (iii) of the Proposition will be true with any other possible definition of $\mathcal{N}$-interpolating sequence, as long as such a definition permits interpolation of bounded sequences of values (or just 0's and 1's).

References

[BGVY] C.A. Berenstein, R. Gay, A. Vidras, and A. Yger, *Residue currents and Bézout identities*, Progr. Math., vol. 114, Birkhäuser, Basel, 1993. MR **94m:**32006.

[BeLi] C.A. Berenstein and B.Q. Li, *Interpolating varieties for spaces of meromorphic functions*, J. Geom. Anal. **5** (1995), 1–48. MR **96b:**30064.

[BeOr] B. Berndtsson and J. Ortega Cerdà, *On interpolation and sampling in Hilbert spaces of analytic functions*, J. Reine Angew. Math. **464** (1995), 109–128. MR **96g:**30070.

[Gro] A. Grothendieck. *Topological vector spaces*, Gordon & Breach, London, 1973. MR 51#8772.

[Gr] L. Gruman, *The area of analytic varieties in* $\mathbb{C}^n$, Math. Scand. **41** (1977), 365–397. MR 57#16670.

[HaMa] A. Hartmann and X. Massaneda, *On interpolating varieties for weighted spaces of entire functions*, J. Geom. Anal. **10(4)** (2000), 683–696. MR **2001m:**30032.

[Hör] L. Hörmander, *Notions of convexity*, Progr. Math., vol. 127, Birkhäuser, Boston, MA, 1994. MR **95k:**00002.

[LiTa] B.Q. Li and B.A. Taylor, *On the Bézout problem and area of interpolating varieties in* $\mathbb{C}^n$, Amer. J. Math. **118** (1996), 989–1010. MR **97h:**32002.

[LiVi] B.Q. Li and E. Villamor, *Interpolation in the unit ball of* $\mathbb{C}^n$, Israel J. Math. **123** (2001), 341–358. MR **2003a:**32018.

[Ln] N. Lindholm, *Sampling in weighted L^p spaces of entire functions in $\mathbb{C}^n$ and estimates of the Bergman kernel*, J. Funct. Anal. **182** (2001), 390–426. MR **2002g:**32007.

[Ma97] X. Massaneda, *Interpolation by holomorphic functions in the unit ball with polynomial growth*, Ann. Fac. Sci. Toulouse **VI** (1997), 277–296. MR **99b:**32014.

[Ma98] ———, *$A^{-\infty}$ interpolation in the ball*, Proc. Edinburgh. Math. Soc. **41** (1998), 359–367. MR **99e:**32017.

[Na56] A.G. Naftalevič, *On interpolation by functions of bounded characteristic (Russian)*, Vilniaus Valst. Univ. Mokslų Darbai. Mat. Fiz. Chem. Mokslų **Ser. 5** (1956), 5–27. MR 22#11141.

[Nik] N.K. Nikol'skiĭ, *Treatise on the shift operator*, Springer-Verlag, Berlin, 1986. MR **87i:**47042.

[ShSh] J. Shapiro and A. Shields, *Unusual topological properties of the Nevanlinna class*, Amer. J. Math. **97** (1975), 915–936. MR 52#11053.

Andreas Hartmann, Laboratoire de Mathématiques Pures, Université Bordeaux I, 351 cours de la Libération, 33405 Talance, France

E-mail address: `hartmann@math.u-bordeaux.fr`

Xavier Massaneda, Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Gran Via 585, 08071 Barcelona, Spain

E-mail address: `xavier@mat.ub.es`