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ON THE SCHATTEN CLASS MEMBERSHIP OF HANKEL OPERATORS ON THE UNIT BALL

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ABSTRACT. A well-known theorem of K. Zhu [7] asserts that, for $2 \le p < \infty$, the Hankel operators H_f and $H_{\bar f}$ on the Bergman space $L^2_a(B_n, dV)$ of the unit ball belong to the Schatten class $\mathcal{C}_p$ if and only if the mean oscillation $\mathrm{MO}(f)(z) = \{\widetilde{|f|^2}(z) - |\tilde f(z)|^2\}^{1/2}$ belongs to $L^p(B_n, (1-|z|^2)^{-n-1}dV(z))$. It is well known that, for trivial reasons, this theorem cannot be extended to the case $p \le 2n/(n+1)$. This paper fills the gap between $2n/(n+1)$ and 2. More precisely, we prove that, when $2n/(n+1) < p < 2$, the same theorem holds true.

1. Introduction

Let B_n be the open unit ball $\{z \in \mathbf{C}^n : |z| < 1\}$ in $\mathbf{C}^n$ and let dV be the volume measure on B_n normalized in such a way that $V(B_n) = 1$. Recall that the Bergman space $L^2_a(B_n, dV)$ is defined to be the subspace $\{\psi \in L^2(B_n, dV) : \psi \text{ is analytic on } B_n\}$ of $L^2(B_n, dV)$. Let P be the orthogonal projection from $L^2(B_n, dV)$ to $L^2_a(B_n, dV)$. Given a symbol function f, the Hankel operator $H_f \colon L^2_a(B_n, dV) \to L^2_a(B_n, dV)^\perp$ is defined by the formula $H_f = (1-P)M_f P$, where M_f is the operator of multiplication by f.

As usual, we write $\langle z, \zeta\rangle = z_1\bar\zeta_1 + \cdots + z_n\bar\zeta_n$ for $z = (z_1, \ldots, z_n)$ and $\zeta = (\zeta_1, \ldots, \zeta_n)$ in $\mathbf{C}^n$. It is well known that P has $K(z,\zeta) = (1 - \langle z, \zeta\rangle)^{-n-1}$ as its kernel, i.e.,

$$(P\psi)(z) = \int K(z,\zeta)\psi(\zeta)dV(\zeta) = \int \frac{\psi(\zeta)}{(1-\langle z,\zeta\rangle)^{n+1}}\, dV(\zeta).$$

Associated with K are the unit vectors $\{k_z : |z| < 1\}$ in $L^2_a(B_n, dV)$, where

$$k_z(\zeta) = \{K(z,z)\}^{-1/2}\, K(\zeta, z) = \left(1-|z|^2\right)^{(n+1)/2} / \left(1 - \langle \zeta, z\rangle\right)^{n+1}.$$

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For any function $f \in L^2(B_n, dV)$, its *Berezin transform* is defined to be

$$\tilde{f}(z) = \langle f k_z, k_z \rangle = \int f\,|k_z|^2\,dV.$$

Recall that the *mean oscillation* $\mathrm{MO}(f)$ of f is given by the formula

$$\mathrm{MO}(f)(z) = \left\{\widetilde{|f|^2}(z) - \left|\tilde{f}(z)\right|^2\right\}^{1/2} = \left\{\int \left|f - \tilde{f}(z)\right|^2 |k_z|^2\,dV\right\}^{1/2}.$$

Let $d\lambda$ denote the Möbius-invariant measure on B_n, i.e.,

$$d\lambda(z) = \left(1 - |z|^2\right)^{-n-1} dV(z).$$

Recall that, for $1 \le p < \infty$, the Schatten p-class $\mathcal{C}_p$ consists of operators T satisfying the condition $\|T\|_p < \infty$, where the norm $\|.\|_p$ is defined by the formula

$$\|T\|_p = \{\mathrm{tr}\,(|T|^p)\}^{1/p} = \left\{\mathrm{tr}\left((T^*T)^{p/2}\right)\right\}^{1/p}.$$

In the study of Bergman space operators, a natural problem is to determine the membership of H_f in $\mathcal{C}_p$ in terms of function-theoretical data. Indeed there is a very rich literature on this subject (see, e.g., [1]–[3], [6], [8]). Of particular relevance to this paper is [8], in which K. Zhu characterized the simultaneous membership $H_f \in \mathcal{C}_p$ and $H_{\bar{f}} \in \mathcal{C}_p$ in the case $p \ge 2$.

THEOREM 1 ([8]). *Let $2 \le p < \infty$ and $f \in L^2(B_n, dV)$. Then $H_f \in \mathcal{C}_p$ and $H_{\bar{f}} \in \mathcal{C}_p$ if and only if* $\mathrm{MO}(f) \in L^p(B_n, d\lambda)$.

Zhu [8] further raised the question of what happens when $p < 2$. It is easy to see that this result cannot be extended to the case where $p \le 2n/(n+1)$. That is, if $p \le 2n/(n+1)$, then for trivial reasons the condition $\mathrm{MO}(f) \in L^p(B_n, d\lambda)$ is sufficient, but *not* necessary, for the simultaneous membership $H_f \in \mathcal{C}_p$ and $H_{\bar{f}} \in \mathcal{C}_p$. Indeed, because

$$\begin{aligned}\mathrm{MO}(f)(z) &\ge \left\{\int \left|f - \tilde{f}(z)\right|^2 \frac{\left(1 - |z|^2\right)^{n+1}}{(1+|z|)^{2n+2}}\,dV\right\}^{1/2} \\ &\ge \frac{\left(1 - |z|^2\right)^{(n+1)/2}}{2^{n+1}} \left\{\inf_{\alpha \in \mathbf{C}} \int |f - \alpha|^2 dV\right\}^{1/2},\end{aligned}$$

when $(p(n+1)/2) - (n+1) \le -1$, i.e., when $p \le 2n/(n+1)$, the condition $\mathrm{MO}(f) \in L^p(B_n, d\lambda)$ forces the factor $\inf_{\alpha \in \mathbf{C}} \int |f - \alpha|^2 dV$ in the above expression to be 0, which implies that f is a *constant* a.e. on B_n. But obviously there are non-constant functions f on B_n for which both H_f and $H_{\bar{f}}$ are of trace class. For example, if f is bounded and vanishes outside some $\{z \in \mathbf{C}^n\colon |z| < \eta\}$, $\eta < 1$, then both H_f and $H_{\bar{f}}$ belong to $\mathcal{C}_1$. But this analysis and Theorem 1 still leave us with the gap $2n/(n+1) < p < 2$.

This gap was recently filled in the special case of complex dimension $n = 1$. That is, we showed in [5] that, for $1 < p < 2$, the Hankel operators H_f and $H_{\bar{f}}$ on the Bergman space $L^2_a(D, dA)$ of the unit disc belong to the Schatten class $\mathcal{C}_p$ if and only if $\mathrm{MO}(f) \in L^p(D, (1 - |z|^2)^{-2} dA(z))$. This special case gives us the confidence that Theorem 1 can also be extended to $2n/(n+1) < p < 2$ when $n \geq 2$. The main result of the present paper, where we focus on complex dimensions $n \geq 2$, is that this is indeed true.

Theorem 2. *Let $2n/(n+1) < p < 2$ and $f \in L^2(B_n, dV)$. Then we have $H_f \in \mathcal{C}_p$ and $H_{\bar{f}} \in \mathcal{C}_p$ if and only if $\mathrm{MO}(f) \in L^p(B_n, d\lambda)$.*

When $2n/(n + 1) < p < 2$, the "easy" and "hard" directions in the proof are the exact opposite of the corresponding directions for the case $2 \leq p < \infty$. In other words, the difficulty in the proof of Theorem 2 is to show that the condition $\mathrm{MO}(f) \in L^p(B_n, d\lambda)$ is necessary for $H_f \in \mathcal{C}_p$ and $H_{\bar{f}} \in \mathcal{C}_p$, while the sufficiency of this condition is trivial.

Indeed, when $p/2 \leq 1$, we have $\langle |H_f|^p k_z, k_z\rangle \leq \langle |H_f|^2 k_z, k_z\rangle^{p/2} = \|H_f k_z\|^p$. Now $\|H_f k_z\|^2 = \|f k_z\|^2 - \|P f k_z\|^2 \leq \|f k_z\|^2 - |\langle P f k_z, k_z\rangle|^2 = \{\mathrm{MO}(f)(z)\}^2$. Obviously, $\mathrm{MO}(\bar{f}) = \mathrm{MO}(f)$. Thus

$$\begin{aligned} \mathrm{tr}\left(|H_f|^p + \left|H_{\bar{f}}\right|^p\right) &= \int \left\langle \left(|H_f|^p + \left|H_{\bar{f}}\right|^p\right) k_z, k_z \right\rangle \, d\lambda(z) \\ &\leq 2 \int \{\mathrm{MO}(f)(z)\}^p \, d\lambda(z) \end{aligned} \tag{1.1}$$

(see pages 115–117 in [7]). Therefore, when $p \leq 2$, the condition $\mathrm{MO}(f) \in L^p(B_n, d\lambda)$ implies $H_f \in \mathcal{C}_p$ and $H_{\bar{f}} \in \mathcal{C}_p$. This proves the "if" part of Theorem 2.

The proof of the "only if" part of Theorem 2 amounts to reversing inequality (1.1) up to a constant multiple under the condition $p > 2n/(n+1)$. Since the reversal of (1.1) in the case of complex dimension $n = 1$ was accomplished in [5], one would naturally expect the same method to work in complex dimensions $n \geq 2$. In this sense one might consider this paper as a generalization of [5] to the high-dimensional case.

It has been suggested that generalizations of this kind can range anywhere from a trivial exercise to a breakthrough. While certainly not a breakthrough, it is not clear where on this scale the present paper fits. This is because the method in [5] works only if one has the right *decomposition scheme* for the domain in question. When the domain is the unit disc, this scheme requires the circular sectors

$$\begin{aligned} &\left\{re^{i\theta} : 1 - 2^{-k} < r < 1, 2^{-k}(j-1) < \theta \leq 2^{-k} j\right\}, \\ &\left\{re^{i\theta} : 1 - 2^{-k} < r \leq 1 - 2^{-k-1}, 2^{-k}(j-1) < \theta \leq 2^{-k} j\right\}, \end{aligned} \tag{1.2}$$

where $1 \leq j \leq 2^k$ and $k \geq 1$, and three other types of sectors generated by these [5, pp. 3561–3562]. The non-trivial part of this particular generalization lies in the search for the right analogue of these sets in the case of complex dimensions $n \geq 2$. It is quite clear that the dyadic decomposition $1 - 2^{-k} < r \leq 1 - 2^{-k-1}$ is still the right one for the radial direction of the unit ball. But it is far from obvious what the high-dimensional analogue of (1.2) should look like in the spherical directions.

As it turns out, in complex dimensions $n \geq 2$, the right spherical decomposition does not involve the Euclidian metric or any other isotropic metric as one might extrapolate from (1.2). Rather, the right spherical decomposition involves the *anisotropic* metric

$$d(u, v) = |1 - \langle u, v\rangle|^{1/2}, \qquad u, v \in \partial B_n, \tag{1.3}$$

on the unit sphere. Given $u, v \in \partial B_n$, v has a component $v^{\perp}$ orthogonal to u with $|v^{\perp}| = (1 - |\langle u, v\rangle|^2)^{1/2}$. If $n = 1$, then, of course, $|v^{\perp}| = 0$. But when $n \geq 2$, $|v^{\perp}|$ can be as large as on the order of $|1 - \langle u, v\rangle|^{1/2} = d(u, v)$. This explains why [5] provides no hint for the right spherical decomposition for the high-dimensional case: the anisotropic nature of this decomposition reveals itself only in complex dimensions $n \geq 2$. Indeed our decision to publish the case $n \geq 2$ is mainly based on such considerations. As expected, once the right decomposition is found, the rest of the proof works in much the same way as it did in [5]. But the details are more complicated here.

The rest of the paper is organized as follows. Section 2 contains the aforementioned decomposition and other necessary preliminaries. The main part of the proof consists of Lemmas 6 and 7, which are collected in Section 3.

2. Decomposing the ball

For any u in the unit sphere $\partial B_n = \{z \in \mathbf{C}^n : |z| = 1\}$ and $a > 0$, define

$$\beta(u, a) = \{v \in \partial B_n : |1 - \langle u, v\rangle| < a\}. \tag{2.1}$$

Let $d\sigma$ denote the surface measure on $\partial B_n = S^{2n-1}$ normalized so that $\sigma(\partial B_n) = 1$. Then

$$dV = 2nr^{2n-1} dr\, d\sigma.$$

Fundamental to our subsequent estimates are the following facts about $\beta(u, a)$:

Lemma 3.

(i) *Let $0 < a \leq 1$. If u, $v \in \partial B_n$ and $\beta(u, a) \cap \beta(v, a) \neq \emptyset$, then*

$$\beta(v, a) \subset \beta(u, 9a).$$

(ii) *There exist constants $0 < \alpha_1 < \alpha_2 < \infty$ which depend only on the complex dimension n such that the inequality*

$$\alpha_1 a^n \leq \sigma(\beta(u, a)) \leq \alpha_2 a^n$$

holds for all $u \in \partial B_n$ and $0 < a \leq 1$.

Proof. Obviously,

$$\beta(u,a) = Q(u,\sqrt{a}),$$

where

$$Q(\zeta,\delta) = \{\eta \in \partial B_n : d(\zeta,\eta) < \delta\}$$

with d being defined by (1.3). By [4, Proposition 5.1.2], d satisfyies the triangle inequality. Therefore, if $Q(u,\sqrt{a}) \cap Q(v,\sqrt{a}) \neq \emptyset$, then $\beta(v,a) = Q(v,\sqrt{a}) \subset Q(u,3\sqrt{a}) = \beta(u,9a)$, which proves (i). Note that the value of $\sigma(\beta(u,a)) = \sigma(Q(u,\sqrt{a}))$ is independent of the choice of u in ∂B_n. Hence (ii) follows immediately from Proposition 5.1.4 of [4]. $\square$

LEMMA 4. *Suppose that $0 < b \le a \le 1/9$. Suppose that $u \in \partial B_n$ and $v_1, \ldots, v_N \in \partial B_n$ satisfy the conditions that $\beta(u,a) \cap \beta(v_j,b) \neq \emptyset$ for every $1 \le j \le N$ and $\beta(v_i,b) \cap \beta(v_j,b) = \emptyset$ if $1 \le i < j \le N$. Then we have the bound $N \le (\alpha_2/\alpha_1) \cdot 9^n \cdot (a/b)^n$, where α_1 and α_2 are the constants that appear in Lemma* 3(ii).

Proof. Since $b \le a$, Lemma 3(i) tells us that $\beta(v_j,b) \subset \beta(u,9a)$ for every $1 \le j \le N$. By the disjointness of the $\beta(v_j,b)$'s and by Lemma 3(ii), we have

$$N\alpha_1 b^n \le \sum_{j=1}^{N} \sigma\left(\beta(v_j,b)\right) = \sigma\left(\bigcup_{j=1}^{N} \beta\left(v_j,b\right)\right) \le \sigma(\beta(u,9a)) \le \alpha_2(9a)^n.$$

The conclusion follows from this inequality. $\square$

We now decompose ∂B_n according to the facts provided by Lemma 3. Let any integer $k \ge 20$ be given. By the lower bound in Lemma 3(ii) and by virtue of the fact that $\sigma(\partial B_n) < \infty$, there is a *maximal* finite subset $\{u_{k,1}, \ldots, u_{k,m(k)}\}$ of ∂B_n such that

$$\beta\left(u_{k,i}, 2^{-k}/9\right) \cap \beta\left(u_{k,j}, 2^{-k}/9\right) = \emptyset \quad \text{if } 1 \le i < j \le m(k). \tag{2.2}$$

The term "maximal" means, of course, that if $u \in \partial B_n$, then $\beta(u, 2^{-k}/9) \cap \beta(u_{k,j}, 2^{-k}/9) \neq \emptyset$ for some $j \in \{1, \ldots, m(k)\}$. By Lemma 3(i), this implies that if $u \in \partial B_n$, then $\beta(u, 2^{-k}/9) \subset \beta(u_{k,j}, 2^{-k})$ for some $j \in \{1, \ldots, m(k)\}$. Hence

$$\bigcup_{j=1}^{m(k)} \beta\left(u_{k,j}, 2^{-k}\right) = \partial B_n. \tag{2.3}$$

For the rest of the paper, $u_{k,1}, \ldots, u_{k,m(k)}$ will denote the points in ∂B_n chosen above. Keep in mind that these points satisfy conditions (2.2) and (2.3).

For any $k \ge 20$ and $1 \le j \le m(k)$, define the sets

$$T_{k,j} = \left\{ru : 1 - 2^{-k} \le r < 1 - 2^{-k-1}, u \in \beta\left(u_{k,j}, 2^{-k}\right)\right\}, \tag{2.4}$$

$$Q_{k,j} = \left\{ru : 1 - 2^{-k} \le r < 1 - 2^{-k-2}, u \in \beta\left(u_{k,j}, 9^2 \cdot 2^{-k}\right)\right\}. \tag{2.5}$$

Our selection of the points $u_{k,j}$ ensures that

$$\bigcup_{k=20}^{\infty} \bigcup_{j=1}^{m(k)} T_{k,j} = \left\{ z \in \mathbf{C}^n : 1 - 2^{-20} \leq |z| < 1 \right\}. \tag{2.6}$$

Furthermore, it follows from the definition of $d\lambda$ and Lemma 3(ii) that

$$\sup_{k,j} \lambda \left(T_{k,j} \right) \leq \sup_{k,j} 2^{(k+1)(n+1)} V \left(T_{k,j} \right) < \infty. \tag{2.7}$$

To simplify notation, we will write $|E|$ for the volume $V(E)$ of any Borel set $E \subset B_n$. As usual, when $|E| > 0$, the mean value of f on E will be denoted by f_E, i.e., $f_E = \int_E f \, dV / |E|$. Furthermore, we will use the notation

$$V(f; E) = \frac{1}{|E|} \int_E |f - f_E|^2 \, dV, \tag{2.8}$$

which we think of as the "variance" of f over the set E.

Throughout the paper, universal constants will be denoted by $C_1, C_2, \ldots,$ which may represent different values in the proofs of different lemmas. Let us emphasize that these are constants which do not depend on anything other than n and p, and some of them may even be independent of n or p or both.

Suppose that $E_1, \ldots, E_m$ are subsets of a set X which have the property that, for any $1 \leq j \leq m$, the cardinality of the set

$$\{i \in \{1, \ldots, m\} : E_i \cap E_j \neq \emptyset\}$$

is at most N. Then there exists a partition

$$\{1, \ldots, m\} = P_1 \cup \cdots \cup P_{N+1}$$

of the index set $\{1, \ldots, m\}$ such that, for any P_ν, if $i, j \in P_\nu$ and $i \neq j$, then $E_i \cap E_j = \emptyset$. This follows from an induction argument on the cardinality m of the index set. In fact, this is trivial if $m \leq N + 1$. Now suppose that $j \geq N + 1$ and that the set $\{1, \ldots, j\}$ has a partition $P_1^j, \ldots, P_{N+1}^j$ with the desired property. Since E_{j+1} intersects at most N of the E_i's, there is a $\mu \in \{1, \ldots, N+1\}$ such that $E_{j+1} \cap E_i = \emptyset$ if $i \in P_\mu^j$. Thus if we set $P_\nu^{j+1} = P_\nu^j$ for $\nu \neq \mu$ and $P_\mu^{j+1} = P_\mu^j \cup \{j+1\}$, then $P_1^{j+1}, \ldots, P_{N+1}^{j+1}$ is a desired partition for $\{1, \ldots, j, j+1\}$.

Lemma 5. *Let $1 < p \leq 2$ and let $f \in L^2(B_n, dV)$ be such that $H_f \in \mathcal{C}_p$ and $H_{\bar{f}} \in \mathcal{C}_p$. Then*

$$\sum_{k=20}^{\infty} \sum_{j=1}^{m(k)} \left\{ V \left(f; Q_{k,j} \right) \right\}^{p/2} < \infty.$$

Proof. We begin the proof with the inequality

$$C_1 \left| Q_{k,j} \right|^{-1} \leq |1 - \langle z, \zeta \rangle|^{-n-1} \leq C_2 \left| Q_{k,j} \right|^{-1} \quad \text{if } z, \zeta \in Q_{k,j}. \tag{2.9}$$

To prove this, observe that $C_3 2^{-(n+1)k} \le |Q_{k,j}| \le C_4 2^{-(n+1)k}$ by (2.5) and Lemma 3(ii). On the other hand, we have $2^{-k-2} \le |1-\langle z,\zeta\rangle| \le (2+9^3)\cdot 2^{-k}$ when $z,\zeta \in Q_{k,j}$. Indeed the lower bound holds because $1-|z| \le |1-\langle z,\zeta\rangle|$. For the upper bound, we write $z = ru$ and $\zeta = \rho v$ with u, $v \in \beta(u_{k,j}, 9^2\cdot 2^{-k})$ and $1-2^{-k} \le r, \rho < 1-2^{-k-2}$. Then

$$|1-\langle z,\zeta\rangle| \le |1-r\rho| + r\rho\,|1-\langle u,v\rangle| \le 2\cdot 2^{-k} + |1-\langle u,v\rangle|\,.$$

Since $u \in \beta(u_{k,j}, 9^2\cdot 2^{-k})$, it follows from Lemma 3(i) that $\beta(u, 9^3\cdot 2^{-k}) \supset \beta(u_{k,j}, 9^2\cdot 2^{-k})$, which contains v. Therefore $|1-\langle u,v\rangle| \le 9^3\cdot 2^{-k}$. This proves (2.9).

For any $k \ge 20$ and $1 \le j \le m(k)$, define the integral operator

$$(K_{k,j}\psi)(z) = \chi_{Q_{k,j}}(z)\int_{Q_{k,j}} \frac{\overline{f(\zeta)} - \overline{f(z)}}{(1-\langle z,\zeta\rangle)^{n+1}}\psi(\zeta)\,dV(\zeta), \qquad \psi \in L^2(B_n, dV)\,.$$

Set $q = p/(p-1)$. Because $q \ge 2$, we have $\|K_{k,j}\|_q \le \|K_{k,j}\|_2$ and

$$\begin{aligned} \|K_{k,j}\|_q^2 &\le \|K_{k,j}\|_2^2 \\ &= \int_{Q_{k,j}}\int_{Q_{k,j}} \frac{|f(\zeta)-f(z)|^2}{|1-\langle z,\zeta\rangle|^{2n+2}}\,dV(\zeta)\,dV(z) \\ &\le C_2^2 \int_{Q_{k,j}}\int_{Q_{k,j}} \frac{|f(\zeta)-f(z)|^2}{|Q_{k,j}|^2}\,dV(\zeta)\,dV(z) \\ &= 2C_2^2 V(f; Q_{k,j})\,. \end{aligned} \tag{2.10}$$

For any $L \ge 20$, let

$$K_L = \sum_{k=20}^{L}\sum_{j=1}^{m(k)} c_{k,j}K_{k,j},$$

where

$$c_{k,j} = \{V(f;Q_{k,j})\}^{(p-2)/2}$$

in the case $V(f;Q_{k,j}) > 0$ and $c_{k,j} = 0$ in the case $V(f;Q_{k,j}) = 0$. We claim that

$$\|K_L\|_q \le C_5\left(\sum_{k=20}^{L}\sum_{j=1}^{m(k)}\{V(f;Q_{k,j})\}^{p/2}\right)^{1/q}. \tag{2.11}$$

To prove this, we note that the intersection $Q_{k,j}\cap Q_{k',i}$ can be non-empty only when $k' \in \{k-1, k, k+1\}$. Also, if $Q_{k,j}\cap Q_{k',i} \ne \emptyset$, then $\beta(u_{k,j}, 9^2\cdot 2^{-k}) \cap \beta(u_{k',i}, 9^2\cdot 2^{-k'}) \ne \emptyset$, which guarantees $\beta(u_{k,j}, 9^3\cdot 2\cdot 2^{-k}) \supset \beta(u_{k',i}, 2^{-k'}/9)$ by virtue of Lemma 3(i). And for each fixed k', the $\beta(u_{k',i}, 2^{-k'}/9)$'s are pairwise disjoint. Thus, according to Lemma 4, for any given (k,j), the total number of pairs (k',i) such that $Q_{k,j}\cap Q_{k',i} \ne \emptyset$ is at most

$$N = 3\cdot\left[(\alpha_2/\alpha_1)\cdot 9^n\cdot\left(9^3\cdot 2/(1/9)\right)^n\right] = 3\cdot\left[(\alpha_2/\alpha_1)\cdot 9^{5n}\cdot 2^n\right].$$

According to the paragraph preceding the lemma, there exists a partition

$$\{(k,j) : 20 \le k \le L, 1 \le j \le m(k)\} = P_{L,1} \cup \cdots \cup P_{L,N+1}$$

of the index set such that, for any $\nu \in \{1, \ldots, N+1\}$, if (k,j), $(k',j') \in P_{L,\nu}$ and $(k,j) \neq (k',j')$, then $Q_{k,j} \cap Q_{k',j'} = \emptyset$. Thus, for each $1 \le \nu \le N+1$, the subspaces $\{L^2(Q_{k,j}, dV) : (k,j) \in P_{L,\nu}\}$ are mutually orthogonal. Let

$$A_{L,\nu} = \sum_{(k,j)\in P_{L,\nu}} c_{k,j} K_{k,j}, \qquad 1 \le \nu \le N+1.$$

Then, by (2.10),

$$\begin{aligned} \|A_{L,\nu}\|_q^q &= \sum_{(k,j)\in P_{L,\nu}} c_{k,j}^q \, \|K_{k,j}\|_q^q \\ &\le C_6 \sum_{(k,j)\in P_{L,\nu}} c_{k,j}^q \, \{V(f; Q_{k,j})\}^{q/2} \\ &= C_6 \sum_{(k,j)\in P_{L,\nu}} \{V(f; Q_{k,j})\}^{(q(p-2)/2)+(q/2)} \\ &= C_6 \sum_{(k,j)\in P_{L,\nu}} \{V(f; Q_{k,j})\}^{p/2}. \end{aligned}$$

Now (2.11) follows from the identity $K_L = A_{L,1} + \cdots + A_{L,N+1}$ and the fact that N depends only on the complex dimension n.

Under the assumption $H_f \in \mathcal{C}_p$ and $H_{\bar{f}} \in \mathcal{C}_p$ of the lemma, we have

$$[M_f, P] = [M_f, P]\, P + [M_f, P]\,(1-P) = H_f - \left(H_{\bar{f}}\right)^* \in \mathcal{C}_p.$$

Furthermore,

$$\begin{aligned} (2.12)\quad \|[M_f, P]\|_p \, \|K_L\|_q & \\ &\ge \operatorname{tr}\left([M_f, P]\, K_L\right) \\ &= \sum_{k=20}^{L} \sum_{j=1}^{m(k)} c_{k,j} \int_{Q_{k,j}} \int_{Q_{k,j}} \frac{|f(\zeta) - f(z)|^2}{|1 - \langle z, \zeta\rangle|^{2n+2}} \, dV(\zeta)\, dV(z) \\ &\ge C_1^2 \sum_{k=20}^{L} \sum_{j=1}^{m(k)} c_{k,j} \int_{Q_{k,j}} \int_{Q_{k,j}} \frac{|f(\zeta) - f(z)|^2}{|Q_{k,j}|^2} \, dV(\zeta)\, dV(z) \\ &= 2C_1^2 \sum_{k=20}^{L} \sum_{j=1}^{m(k)} c_{k,j} V(f; Q_{k,j}) \\ &= 2C_1^2 \sum_{k=20}^{L} \sum_{j=1}^{m(k)} \{V(f; Q_{k,j})\}^{p/2}, \end{aligned}$$

where we used (2.9) and the definition of $c_{k,j}$. Since $[M_f, P] \in \mathcal{C}_p$, the conclusion of the lemma follows from (2.11) and (2.12). $\square$

We conclude this section with the recollection of the following elementary fact, which will be used in the paper without further reference: If ω is a probability measure and if $\varphi \in L^2(\omega)$, then

$$\int \left|\varphi - \int \varphi \, d\omega\right|^2 d\omega \le \int |\varphi - c|^2 d\omega \quad \text{for any } c \in \mathbf{C}.$$

3. Reconstructing MO(f)

To control the kernel function $|k_z|^2$, we need another family of subsets of B_n. Given any $k \ge 20$ and $1 \le j \le m(k)$, we define

$$F_{k,j} = \left\{(\ell, i) : \ell > k, 1 \le i \le m(\ell), \beta\left(u_{\ell,i}, 2^{-\ell}\right) \cap \beta\left(u_{k,j}, 9 \cdot 2^{-k}\right) \ne \emptyset\right\}$$

and

$$S_{k,j} = Q_{k,j} \bigcup \left\{ \bigcup_{(\ell,i) \in F_{k,j}} Q_{\ell,i} \right\}. \tag{3.1}$$

Obviously,

$$S_{k,j} \supset \left\{ru : 1 - 2^{-k} \le r < 1, u \in \beta\left(u_{k,j}, 9 \cdot 2^{-k}\right)\right\}.$$

This implies that

$$|1 - \langle z, \zeta\rangle| \ge 2^{-k-1} \quad \text{if } z/|z| \in \beta\left(u_{k,j}, 2^{-k}\right) \text{ and } \zeta \in B_n \backslash S_{k,j}. \tag{3.2}$$

Indeed, given a $\zeta = \rho v \in B_n \backslash S_{k,j}$, there are two possibilities. Either $\rho < 1 - 2^{-k}$, which implies $|1 - \langle z, \zeta\rangle| \ge 1 - \rho \ge 2^{-k}$ and, therefore, (3.2). Or $\rho \ge 1 - 2^{-k}$, which necessitates $v \notin \beta(u_{k,j}, 9 \cdot 2^{-k})$. In the latter case we write $z = ru$ with $0 < r < 1$ and $u \in \beta(u_{k,j}, 2^{-k})$. According to Lemma 3(i), $\beta(u, 2^{-k}) \subset \beta(u_{k,j}, 9 \cdot 2^{-k})$. Hence $v \notin \beta(u, 2^{-k})$, i.e., $|1 - \langle u, v\rangle| \ge 2^{-k}$. Now for any $0 \le t < 1$, $0 \le a < 1$ and $\theta \in \mathbf{R}$, we have

$$\left|1 - tae^{i\theta}\right|^2 = t\left|1 - ae^{i\theta}\right|^2 + (1-t)\left(1 - ta^2\right) \ge t\left|1 - ae^{i\theta}\right|^2.$$

Hence

$$|1 - \langle z, \zeta\rangle| \ge \sqrt{r\rho}\,|1 - \langle u, v\rangle| \ge \sqrt{r\rho} \cdot 2^{-k}.$$

If $r \ge 1 - 2^{-20}$, then, of course, $|1 - \langle z, \zeta\rangle| \ge 2^{-1} \cdot 2^{-k}$. If $r < 1 - 2^{-20}$, then $|1 - \langle z, \zeta\rangle| \ge 1 - r \ge 2^{-20}$. In any case, (3.2) holds as promised.

Suppose that E and F are measurable subsets of B_n such that $|E \cap F| > 0$. Then

$$\begin{aligned} |f_E - f_{E\cap F}| &= \left|\int_{E\cap F} (f_E - f)\, dV\right| / |E \cap F| \\ &\le (|E|/|E \cap F|) \int_E |f_E - f|\, dV/|E|. \end{aligned}$$

By the Cauchy-Schwartz inequality, we have

$$|f_E - f_{E\cap F}| \le (|E|/|E\cap F|)\sqrt{V(f;E)}.$$

Therefore

$$\begin{aligned}(3.3) \qquad |f_E - f_F| &\le |f_E - f_{E\cap F}| + |f_{E\cap F} - f_F| \\ &\le \frac{|E|}{|E\cap F|}\sqrt{V(f;E)} + \frac{|F|}{|E\cap F|}\sqrt{V(f;F)}.\end{aligned}$$

Lemma 6. *Let $1 \le p \le 2$ and $f \in L^2(B_n, dV)$ be such that*

$$\sum_{k=20}^{\infty}\sum_{j=1}^{m(k)}\{V(f;Q_{k,j})\}^{p/2} < \infty.$$

Then

$$\sum_{k=20}^{\infty}\sum_{j=1}^{m(k)}\{V(f;S_{k,j})\}^{p/2} < \infty.$$

Proof. Given $k \ge 20$ and $1 \le j \le m(k)$, it follows from (2.8) and (3.1) that

$$\begin{aligned}V(f;S_{k,j}) &\le \frac{1}{|S_{k,j}|}\int_{S_{k,j}}\left|f - f_{Q_{k,j}}\right|^2 dV \\ &\le \frac{|Q_{k,j}|}{|S_{k,j}|}V(f;Q_{k,j}) + \sum_{(\ell,i)\in F_{k,j}}\frac{|Q_{\ell,i}|}{|S_{k,j}|}\cdot\frac{1}{|Q_{\ell,i}|}\int_{Q_{\ell,i}}\left|f - f_{Q_{k,j}}\right|^2 dV.\end{aligned}$$

By (2.5) and Lemma 3(ii),

$$|Q_{\ell,i}| \le C_1 2^{-(n+1)\ell} \text{ and } |S_{k,j}| \ge |Q_{k,j}| \ge C_2 2^{-(n+1)k}.$$

Setting $C_3 = C_1/C_2$, we have

$$\begin{aligned}(3.4) \qquad V(f;S_{k,j}) &\le V(f;Q_{k,j}) \\ &\quad + C_3\sum_{(\ell,i)\in F_{k,j}}2^{-(n+1)(\ell-k)}\frac{1}{|Q_{\ell,i}|}\int_{Q_{\ell,i}}\left|f - f_{Q_{k,j}}\right|^2 dV.\end{aligned}$$

Let us consider a pair $(\ell,i) \in F_{k,j}$ for a moment. Pick an $x \in \beta(u_{k,j}, 9\cdot 2^{-k}) \cap \beta(u_{\ell,i}, 2^{-\ell})$, which is possible since the intersection is non-empty by the definition of $F_{k,j}$. Then there is a chain of indices $\{(t,i(t)) : k \le t \le \ell\}$ such that $(\ell,i(\ell)) = (\ell,i)$, $(k,i(k)) = (k,j)$, and $x \in \beta(u_{t,i(t)}, 2^{-t})$ if $k < t \le \ell$. This implies that

$$Q_{t,i(t)} \cap Q_{t+1,i(t+1)} \supset T_{t+1,i(t+1)} \quad \text{if } k \le t < \ell.$$

Indeed, since $\beta(u_{t,i(t)}, 9\cdot 2^{-t}) \cap \beta(u_{t+1,i(t+1)}, 2^{-t-1})$ contains x, it follows from Lemma 3(i) that $\beta(u_{t+1,i(t+1)}, 2^{-t-1}) \subset \beta(u_{t,i(t)}, 9^2\cdot 2^{-t})$. The above assertion now follows from (2.4) and (2.5). Since

$$\left|T_{t+1,i(t+1)}\right| \ge C_4 2^{-(t+1)(n+1)} = 2^{-n-1}C_4 2^{-(n+1)t},$$

we have $|Q_{t,i(t)}|/|Q_{t,i(t)}\cap Q_{t+1,i(t+1)}|\le 2^{n+1}C_1/C_4$. By (3.3), we now have

$$\left|f_{Q_{t,i(t)}}-f_{Q_{t+1,i(t+1)}}\right|\le C_5\left(\left\{V\left(f;Q_{t,i(t)}\right)\right\}^{1/2}+\left\{V\left(f;Q_{t+1,i(t+1)}\right)\right\}^{1/2}\right)$$

if $k\le t<\ell$. Therefore

$$\begin{aligned}\left|f_{Q_{k,j}}-f_{Q_{\ell,i}}\right|^2&\le\left(\sum_{t=k}^{\ell-1}\left|f_{Q_{t,i(t)}}-f_{Q_{t+1,i(t+1)}}\right|\right)^2\\&\le\left(2C_5\sum_{t=k}^{\ell}\left\{V\left(f;Q_{t,i(t)}\right)\right\}^{1/2}\right)^2\\&\le 4C_5^2(1+\ell-k)\sum_{t=k}^{\ell}V\left(f;Q_{t,i(t)}\right),\end{aligned}$$

where the last inequality results from the Cauchy-Schwarz inequality. Let

$$\begin{aligned}G_{k,j;\ell,i}=\big\{(\nu,h):k\le\nu\le\ell,\ 1\le h\le m(\nu),\ \beta(u_{\nu,h},2^{-\nu})\cap\beta(u_{\ell,i},2^{-\ell})\ne\emptyset\\ \text{and } \beta(u_{\nu,h},2^{-\nu})\cap\beta(u_{k,j},9\cdot 2^{-k})\ne\emptyset\big\}.\end{aligned}$$

Then the choice of $(t,i(t))$ guarantees that $(t,i(t))\in G_{k,j;\ell,i}$ for all $k\le t\le\ell$. Therefore

$$\left|f_{Q_{\ell,i}}-f_{Q_{k,j}}\right|^2\le 4C_5^2(1+\ell-k)\sum_{(\nu,h)\in G_{k,j;\ell,i}}V\left(f;Q_{\nu,h}\right). \tag{3.5}$$

Substituting $2|f-f_{Q_{\ell,i}}|^2+2|f_{Q_{\ell,i}}-f_{Q_{k,j}}|^2$ for $|f-f_{Q_{k,j}}|^2$ in (3.4), it now follows from (3.5) that

$$V\left(f;S_{k,j}\right)\le V\left(f;Q_{k,j}\right)+C_6\left(A_{k,j}+B_{k,j}\right),$$

where

$$\begin{aligned}A_{k,j}&=\sum_{(\ell,i)\in F_{k,j}}2^{-(n+1)(\ell-k)}V\left(f;Q_{\ell,i}\right),\\ B_{k,j}&=\sum_{(\ell,i)\in F_{k,j}}2^{-(n+1)(\ell-k)}(1+\ell-k)\sum_{(\nu,h)\in G_{k,j;\ell,i}}V\left(f;Q_{\nu,h}\right).\end{aligned}$$

Since $(\ell,i)\in G_{k,j;\ell,i}$, we obviously have $A_{k,j}\le B_{k,j}$. Therefore

$$V\left(f;S_{k,j}\right)\le V\left(f;Q_{k,j}\right)+2C_6B_{k,j}. \tag{3.6}$$

Let us estimate $B_{k,j}$. First of all, if we set $C_7=\sup_{m\ge 0}2^{-m/2}(1+m)$, then

$$B_{k,j}\le C_7\sum_{(\ell,i)\in F_{k,j}}\sum_{(\nu,h)\in G_{k,j;\ell,i}}2^{-(n+1/2)(\ell-k)}V\left(f;Q_{\nu,h}\right). \tag{3.7}$$

Now, for each pair (ν,h) with $\nu\ge k$, if $\nu\le\ell$ and $\beta(u_{\nu,h},2^{-\nu})\cap\beta(u_{\ell,i},2^{-\ell})\ne\emptyset$, then $\beta(u_{\nu,h},9\cdot 2^{-\nu})\supset\beta(u_{\ell,i},2^{-\ell})$ by Lemma 3(i). Since the sets

$$\left\{\beta\left(u_{\ell,i},2^{-\ell}/9\right):1\le i\le m(\ell)\right\}$$

are pairwise disjoint, Lemma 4 tells us that, for each $\ell \geq \nu$, the cardinality of the set $\{i : 1 \leq i \leq m(\ell),\ \beta(u_{\ell,i}, 2^{-\ell}) \cap \beta(u_{\nu,h}, 2^{-\nu}) \neq \emptyset\}$ is at most $(\alpha_2/\alpha_1) \cdot 9^n \cdot ((9 \cdot 2^{-\nu})/(2^{-\ell}/9))^n = C_8 2^{n(\ell-\nu)}$. Hence, if we set

$$G_{k,j} = \left\{(\nu, h) : \nu \geq k, 1 \leq h \leq m(\nu), \beta\left(u_{\nu,h}, 2^{-\nu}\right) \cap \beta\left(u_{k,j}, 9 \cdot 2^{-k}\right) \neq \emptyset\right\},$$

then a change of the order of summation in (3.7) yields

$$\begin{aligned}(3.8)\quad B_{k,j} &\leq C_7 \sum_{(\nu,h)\in G_{k,j}} V\left(f; Q_{\nu,h}\right) \sum_{\ell=\nu}^{\infty} 2^{-(n+1/2)(\ell-k)} \\ &\qquad \times \operatorname{card}\left\{i : \beta\left(u_{\ell,i}, 2^{-\ell}\right) \cap \beta\left(u_{\nu,h}, 2^{-\nu}\right) \neq \emptyset\right\} \\ &\leq C_7 C_8 \sum_{(\nu,h)\in G_{k,j}} V\left(f; Q_{\nu,h}\right) \sum_{\ell=\nu}^{\infty} 2^{-(1/2)(\ell-k)} \cdot 2^{-n(\ell-k)} \cdot 2^{n(\ell-\nu)} \\ &= C_9 \sum_{(\nu,h)\in G_{k,j}} V\left(f; Q_{\nu,h}\right) 2^{-n(\nu-k)} \sum_{\ell=\nu}^{\infty} 2^{-(1/2)(\ell-k)} \\ &\leq C_{10} \sum_{(\nu,h)\in G_{k,j}} V\left(f; Q_{\nu,h}\right) 2^{-n(\nu-k)}.\end{aligned}$$

It is elementary that if $0 < r \leq 1$ and $a_m \geq 0$, then $(\sum a_m)^r \leq \sum a_m^r$. Since $p/2 \leq 1$, applying this to (3.8), we obtain

$$B_{k,j}^{p/2} \leq C_{10}^{p/2} \sum_{(\nu,h)\in G_{k,j}} \left\{V\left(f; Q_{\nu,h}\right)\right\}^{p/2} 2^{-n(\nu-k)p/2}.$$

Thus

$$\begin{aligned}\sum_{k,j} B_{k,j}^{p/2} &\leq C_{10}^{p/2} \sum_{k,j} \sum_{(\nu,h)\in G_{k,j}} \left\{V\left(f; Q_{\nu,h}\right)\right\}^{p/2} 2^{-n(\nu-k)p/2} \\ &= C_{10}^{p/2} \sum_{\nu,h} \left\{V\left(f; Q_{\nu,h}\right)\right\}^{p/2} \sum_{20\leq k\leq\nu} 2^{-n(\nu-k)p/2} \operatorname{card}(H_{\nu,h;k}),\end{aligned}$$

where, for any $20 \leq k \leq \nu$,

$$H_{\nu,h;k} = \left\{j : 1 \leq j \leq m(k),\ \beta(u_{k,j}, 9 \cdot 2^{-k}) \cap \beta(u_{\nu,h}, 2^{-\nu}) \neq \emptyset\right\}.$$

If $j \in H_{\nu,h;k}$, then Lemma 3(i) tells us that $\beta(u_{k,j}, 9^2 \cdot 2^{-k}) \supset \beta(u_{\nu,h}, 2^{-\nu})$ and, therefore, $\beta(u_{k,j}, 9^3 \cdot 2^{-k}) \supset \beta(u_{k,j'}, 9 \cdot 2^{-k}) \supset \beta(u_{k,j'}, 2^{-k}/9)$ for any other $j' \in H_{\nu,h;k}$. It follows from (2.2) and Lemma 4 that $\operatorname{card}(H_{\nu,h;k}) \leq C_{11}$. Hence

$$\begin{aligned}(3.9)\quad \sum_{k,j} B_{k,j}^{p/2} &\leq C_{12} \sum_{\nu,h} \left\{V\left(f; Q_{\nu,h}\right)\right\}^{p/2} \sum_{20\leq k\leq\nu} 2^{-n(\nu-k)p/2} \\ &\leq C_{13} \sum_{\nu,h} \left\{V\left(f; Q_{\nu,h}\right)\right\}^{p/2}.\end{aligned}$$

Since $p/2 \le 1$, it follows from (3.6) that

$$\{V(f; S_{k,j})\}^{p/2} \le \{V(f; Q_{k,j})\}^{p/2} + (2C_6)^{p/2} B_{k,j}^{p/2}.$$

Combining this with (3.9), the lemma is established. $\square$

What we have done thus far is valid for all $1 < p \le 2$; in fact, it can even be extended to the case $p = 1$ with minor changes only in the proof of Lemma 5. But for our next lemma, the requirement $p > 2n/(n+1)$ is absolutely indispensable.

LEMMA 7. *Suppose that* $2n/(n+1) < p \le 2$. *Suppose that* $f \in L^2(B_n, dV)$ *and that*

$$\sum_{k=20}^{\infty} \sum_{j=1}^{m(k)} \{V(f; S_{k,j})\}^{p/2} < \infty.$$

Then $\mathrm{MO}(f) \in L^p(B_n, d\lambda)$.

Proof. By (2.6) and (2.7), to prove that $\mathrm{MO}(f) \in L^p(B_n, d\lambda)$, it suffices to show that

$$\sum_{k,j} \sup_{z \in T_{k,j}} \{\mathrm{MO}(f)(z)\}^p < \infty. \tag{3.10}$$

Fix a pair of k, j and a $z \in T_{k,j}$ for the moment. We have $|k_z(\zeta)|^2 \le (1-|z|^2)^{n+1}/(1-|z|^2)^{2n+2} \le 2^{(k+1)(n+1)} \le C_1|S_{k,j}|^{-1}$ for all $\zeta \in B_n$, where the last inequality is due to the fact that $S_{k,j} \subset \{ru : 1 - 2^{-k} \le r < 1,\ u \in \beta(u_{k,j}, 9^3 \cdot 2^{-k})\}$. For any $20 \le \ell \le k$, there is an $i(\ell) \in \{1, \dots, m(\ell)\}$ such that $z/|z| \in \beta(u_{\ell,i(\ell)}, 2^{-\ell})$. We stipulate that $i(k) = j$, which is allowed because $z \in T_{k,j}$. For any $\ell < k$, it follows from (3.2) and the fact $z/|z| \in \beta(u_{\ell+1,i(\ell+1)}, 2^{-(\ell+1)})$ that, if $\zeta \in S_{\ell,i(\ell)} \backslash S_{\ell+1,i(\ell+1)}$, then

$$\begin{aligned}
|k_z(\zeta)|^2 &\le 2^{(\ell+2)(2n+2)} \left(1-|z|^2\right)^{n+1} \\
&\le 2^{(\ell+2)(2n+2)} \cdot 2^{(n+1)} \cdot 2^{-(n+1)k} \\
&= 2^{5n+5} \cdot 2^{(n+1)\ell} \cdot 2^{-(n+1)(k-\ell)} \\
&\le C_2 \left|S_{\ell,i(\ell)}\right|^{-1} 2^{-(k-\ell)(n+1)}.
\end{aligned}$$

Recall that we have defined $m(\ell)$ and $S_{\ell,i}$ only for $\ell \ge 20$ so far. We now set $m(19) = 1$, $S_{19,1} = B_n$, and $i(19) = 1$. It follows from the above analysis that

$$|k_z|^2 \le C_3 \sum_{\ell=19}^{k} \left|S_{\ell,i(\ell)}\right|^{-1} 2^{-(k-\ell)(n+1)} \chi_{S_{\ell,i(\ell)}}.$$

Applying this in the inequality $\{\mathrm{MO}(f)(z)\}^2 \le \int |f - f_{S_{k,j}}|^2 |k_z|^2 dV$, we obtain

$$\{\mathrm{MO}(f)(z)\}^2 \le C_3 \sum_{\ell=19}^{k} 2^{-(k-\ell)(n+1)} \frac{1}{\left|S_{\ell,i(\ell)}\right|} \int_{S_{\ell,i(\ell)}} \left|f - f_{S_{k,j}}\right|^2 dV. \tag{3.11}$$

Now, for each $20 \le \ell < k$, since

$$z/|z| \in \beta\left(u_{\ell,i(\ell)}, 2^{-\ell}\right) \cap \beta\left(u_{\ell+1,i(\ell+1)}, 2^{-\ell-1}\right),$$

we have

$$\beta\left(u_{\ell,i(\ell)}, 9 \cdot 2^{-\ell}\right) \supset \beta\left(u_{\ell+1,i(\ell+1)}, 2^{-\ell-1}\right)$$

and, therefore, $S_{\ell,i(\ell)} \supset Q_{\ell,i(\ell)} \supset T_{\ell+1,i(\ell+1)}$. Since $|S_{\ell,i}|$ is on the order of $2^{-(n+1)\ell}$ and since $|T_{\ell+1,i(\ell+1)}|$ is on the order of $2^{-(n+1)(\ell+1)} = 2^{-(n+1)} \cdot 2^{-(n+1)\ell}$, it follows that $|S_{\ell,i(\ell)}|/|S_{\ell,i(\ell)} \cap S_{\ell+1,i(\ell+1)}| \le C_4$. A similar bound holds for $|S_{\ell+1,i(\ell+1)}|/|S_{\ell,i(\ell)} \cap S_{\ell+1,i(\ell+1)}|$. Combining these bounds with (3.3), we now have

$$\left|f_{S_{\ell,i(\ell)}} - f_{S_{\ell+1,i(\ell+1)}}\right| \le C_5\left(\left\{V\left(f;S_{\ell,i(\ell)}\right)\right\}^{1/2} + \left\{V\left(f;S_{\ell+1,i(\ell+1)}\right)\right\}^{1/2}\right).$$

Thus, if $\ell < k$, then

$$\begin{aligned}\left|f - f_{S_{k,j}}\right|^2 &\le \left(\left|f - f_{S_{\ell,i(\ell)}}\right| + \sum_{t=\ell}^{k-1}\left|f_{S_{t,i(t)}} - f_{S_{t+1,i(t+1)}}\right|\right)^2 \\ &\le \left(\left|f - f_{S_{\ell,i(\ell)}}\right| + 2C_5\sum_{t=\ell}^{k}\left\{V\left(f;S_{t,i(t)}\right)\right\}^{1/2}\right)^2 \\ &\le (2+k-\ell)\left\{\left|f - f_{S_{\ell,i(\ell)}}\right|^2 + 4C_5^2\sum_{t=\ell}^{k}V\left(f;S_{t,i(t)}\right)\right\}.\end{aligned}$$

A substitution of this into (3.11) yields

$$\begin{aligned}&\{\mathrm{MO}(f)(z)\}^2 \\ &\quad\le C_3\left(1+4C_5^2\right)\sum_{\ell=19}^{k}2^{-(k-\ell)(n+1)}(2+k-\ell)\sum_{t=\ell}^{k}V\left(f;S_{t,i(t)}\right) \\ &\quad= C_3\left(1+4C_5^2\right)\sum_{t=19}^{k}V\left(f;S_{t,i(t)}\right)\sum_{\ell=19}^{t}2^{-(k-\ell)(n+1)}(2+k-\ell) \\ &\quad\le C_3\left(1+4C_5^2\right)\sum_{t=19}^{k}V\left(f;S_{t,i(t)}\right)\sum_{\nu=k-t}^{\infty}2^{-(n+1)\nu}(2+\nu).\end{aligned} \tag{3.12}$$

We now use the condition $p > 2n/(n+1)$. Because $(n+1)(p/2) - n > 0$, we can choose an $\epsilon > 0$ such that

$$(n+1-\epsilon)(p/2) - n > 0. \tag{3.13}$$

If we set

$$C_6 = \sum_{\nu=0}^{\infty}2^{-\epsilon\nu}(2+\nu),$$

then

$$\sum_{\nu=k-t}^{\infty} 2^{-(n+1)\nu}(2+\nu) \le C_6 2^{-(n+1-\epsilon)(k-t)}$$

in (3.12). Thus it follows from (3.12) that

$$\{\mathrm{MO}(f)(z)\}^2 \le C_7 \sum_{t=19}^{k} 2^{-(n+1-\epsilon)(k-t)} V\left(f; S_{t,i(t)}\right).$$

Since $p/2 \le 1$, we have $(\sum a_t)^{p/2} \le \sum a_t^{p/2}$ if $a_t \ge 0$. Hence

$$\{\mathrm{MO}(f)(z)\}^p \le C_7^{p/2} \sum_{t=19}^{k} 2^{-(n+1-\epsilon)(p/2)(k-t)} \left\{V\left(f; S_{t,i(t)}\right)\right\}^{p/2}.$$

Recall that each pair $(t, i(t))$ was chosen so that $z/|z| \in \beta(u_{t,i(t)}, 2^{-t})$. Thus, if we let

$$W_{k,j} = \big\{(t,h) : 20 \le t \le k,\ 1 \le h \le m(t),\\ \beta\left(u_{t,h}, 2^{-t}\right) \cap \beta\left(u_{k,j}, 2^{-k}\right) \ne \emptyset\big\} \cup \big\{(19,1)\big\},$$

then

$$\{\mathrm{MO}(f)(z)\}^p \le C_7^{p/2} \sum_{(t,h)\in W_{k,j}} 2^{-(n+1-\epsilon)(p/2)(k-t)} \left\{V\left(f; S_{t,h}\right)\right\}^{p/2}. \tag{3.14}$$

The set $W_{k,j}$ is, of course, independent of the choice of z in $T_{k,j}$. In other words, (3.14) holds for every $z \in T_{k,j}$. Therefore

(3.15)

$$\begin{aligned}\sum_{k,j} \sup_{z\in T_{k,j}} &\{\mathrm{MO}(f)(z)\}^p \\ &\le C_7^{p/2} \sum_{k,j} \sum_{(t,h)\in W_{k,j}} 2^{-(n+1-\epsilon)(p/2)(k-t)} \left\{V\left(f; S_{t,h}\right)\right\}^{p/2} \\ &= C_7^{p/2} \sum_{t,h} \left\{V\left(f; S_{t,h}\right)\right\}^{p/2} \sum_{k=\max\{t,20\}}^{\infty} 2^{-(n+1-\epsilon)(p/2)(k-t)} \operatorname{card}\left(U_{t,h;k}\right),\end{aligned}$$

where

$$U_{t,h;k} = \big\{j : 1 \le j \le m(k),\ \beta(u_{k,j}, 2^{-k}) \cap \beta(u_{t,h}, 2^{-t}) \ne \emptyset\big\}$$

for $t \ge 20$ and $U_{19,1;k} = \{1, \ldots, m(k)\}$. If $k \ge t \ge 20$ and $j \in U_{t,h;k}$, then $\beta(u_{t,h}, 9 \cdot 2^{-t}) \supset \beta(u_{k,j}, 2^{-k})$ by Lemma 3(i). Thus it follows from (2.2) and Lemma 4 that $\operatorname{card}(U_{t,h;k}) \le C_8(2^{-t}/2^{-k})^n = C_8 2^{n(k-t)}$ when $t \ge 20$.

Similarly, $\operatorname{card}(U_{19,1;k}) = m(k) \le C_9 2^{nk} = 2^{19n} C_9 2^{n(k-19)}$. An application of these bounds in (3.15) yields

$$\sum_{k,j} \sup_{z \in T_{k,j}} \{\mathrm{MO}(f)(z)\}^p \le C_{10} \sum_{t,h} \{V(f; S_{t,h})\}^{p/2} \sum_{k=t}^{\infty} 2^{-\{(n+1-\epsilon)(p/2)-n\}(k-t)}.$$

Because of (3.13), the above is finite whenever

$$\sum_{t,h} \{V(f; S_{t,h})\}^{p/2} < \infty.$$

This proves (3.10) and completes the proof of the lemma. □

Proof of Theorem 2. Under the condition $2n/(n+1) < p < 2$, it follows from Lemmas 5, 6 and 7 that, if H_f and $H_{\bar{f}}$ belong to $\mathcal{C}_p$, then $\mathrm{MO}(f) \in L^p(B_n, d\lambda)$. The converse of this was proved in the Introduction. □

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