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VISCOSITY SOLUTIONS ON GRUSHIN-TYPE PLANES

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Abstract. This paper examines viscosity solutions to a class of fully nonlinear equations on Grushin-type planes. First, viscosity solutions are defined, using subelliptic second order superjets and subjets. Then, a Grushin maximum principle is proved, and as an application, comparison principles for certain types of nonlinear functions follow. This is accomplished by establishing a natural relationship between Euclidean and subelliptic jets, in order to use the viscosity solution technology of Crandall, Ishii, and Lions (1992). The particular example of infinite harmonic functions on certain Grushin-type planes is examined in further detail.

1. Background and main results

In [5], viscosity solutions to a class of non-linear differential equations are defined and Euclidean results are extended to the Heisenberg group, which is the most elementary subelliptic environment. However, the extension of the results still exploits the group structure. In this paper, we examine the same class of equations, but now consider a subelliptic environment that is not a group. In particular, we will extend the results to Grushin-type planes.

In order to construct Grushin-type planes, we begin by fixing an arbitrary polynomial $\rho\colon \mathbb{R} \mapsto \mathbb{R}$ with degree $n \geq 1$. Using this polynomial, we consider the vector fields on $\mathbb{R}^2$ given by

$$X_1 = \frac{\partial}{\partial x},\ X_2 = \rho(x)\frac{\partial}{\partial y},\ X_3 = \rho'(x)\frac{\partial}{\partial y},\ \ldots,\ X_{n+2} = \rho^{(n)}(x)\frac{\partial}{\partial y}.$$

Observe that applying the Lie Bracket yields

$$[X_1, X_j] = X_{j+1} \text{ for } j = 2, \ldots, n+1.$$

Clearly, at all points $\{X_1, X_2, \ldots, X_{n+2}\}$ generates $\mathbb{R}^2$. Endow $\mathbb{R}^2$ with an inner product (singular at points where $\rho(x) = 0$) so that X_1 and X_2 are orthonormal. This vector space, which we shall denote by g_n, is the underlying

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manifold of a Grushin-type plane, denoted by G_n. We shall also denote the coordinates of an arbitrary point p in G_n by $p = (x, y)$, the coordinates of a fixed point p_0 in G_n by $p_0 = (x_0, y_0)$, and use the notation $p - p_0$ for $(x - x_0, y - y_0)$. In addition, at each point p_0, there is a unique smallest integer $r_{p_0} \in \{0, 1, \ldots n\}$ so that $\rho^{(r_{p_0})}(x_0) \neq 0$. Note that $r_p = 0$ except on the vertical lines $x = x_0$ where $\rho(x_0) = 0$.

Even though G_n is not a group, it is a metric space with the natural metric being the Carnot-Carathéodory distance, which is defined for the points p and q by

$$d_C(p, q) = \inf_{\Gamma} \int_0^1 \|\gamma'(t)\| dt,$$

where the set Γ is the set of all curves γ such that $\gamma(0) = p, \gamma(1) = q$ and $\gamma'(t)$ is in $\operatorname{span}\{X_1(\gamma(t)), X_2(\gamma(t))\}$. By Chow's theorem (see, for example, [4]) any two points can be connected by such a curve, which means that $d_C(p, q)$ is an honest metric. Using this metric, we can define a Carnot-Carathéodory ball of radius r centered at a point p_0 by

$$B = B(p_0, r) = \{p \in G_n : d_C(p, p_0) < r\};$$

similarly, we shall denote a bounded domain in G_n by Ω.

The Carnot-Carathéodory metric behaves differently on lines on which $\rho(x)$ vanishes. In particular, the estimate for this distance changes on these lines. Using Theorem 7.34 from [4] we obtain the local estimate at p_0

$$d_C(p_0, p) \sim |x - x_0| + |y - y_0|^{1/(r_{p_0}+1)}. \tag{1.1}$$

Observe that if $\rho(x_0) \neq 0$, then the metric is locally Riemannian because $r_{p_0} = 0$.

Having established the basic structure on G_n, our attention turns to differentiation and calculus. Given a smooth function f on G_n, and a multi-index $I = (i_1, i_2, \ldots, i_{n+2})$, the derivative $X^I f$ is defined by

$$X^I f = X_1^{i_1} X_2^{i_2} \ldots X_{n+2}^{i_{n+2}} f.$$

The function f is C^k if $X^I f$ is continuous for all multi-indices I such that

$$d(I) \equiv i_1 + i_2 + 2i_3 + 3i_4 + \cdots + (n+1)i_{n+2} \leq k. \tag{1.2}$$

In light of the Carnot-Carathéodory metric, the important first and second order derivatives that we will consider are given by

$$\nabla_0 f(p) = (X_1 f(p), X_2 f(p))$$

and

$$\left(D^2 f(p)\right)^\star = \begin{pmatrix} X_1 X_1 f(p) & \frac{1}{2}\left(X_1 X_2 f(p) + X_2 X_1 f(p)\right) \\ \frac{1}{2}\left(X_1 X_2 f(p) + X_2 X_1 f(p)\right) & X_2 X_2 f(p) \end{pmatrix}.$$

It should also be noted that, for any open set $\mathcal{O} \subset G_n$, the function f is in the horizontal Sobolev space $HW^{1,q}(\mathcal{O})$ if f, $X_1 f$ and $X_2 f$ are in $L^q(\mathcal{O})$.

Replacing $L^q(\mathcal{O})$ by $L^q_{\text{loc}}(\mathcal{O})$, the space $HW^{1,q}_{\text{loc}}(\mathcal{O})$ is defined similarly. The space $HW^{1,q}_0(\mathcal{O})$ is the closure in $HW^{1,q}(\mathcal{O})$ of smooth functions with compact support.

Using these derivatives, the class of equations we consider is given by

$$F\left(p, u(p), \nabla_0 u(p), \left(D^2 u(p)\right)^\star\right) = 0,$$

where the continuous function

$$F\colon G_n \times \mathbb{R} \times g_n \times S^2 \mapsto \mathbb{R}$$

satisfies

$$F(p, r, \eta, X) \leq F(p, s, \eta, Y)$$

when $r \leq s$ and $Y \leq X$ (that is, F is proper; see [7]). Recall that S^2 is the set of 2×2 real symmetric matrices. An example of this type of equation is the quasilinear horizontal q-Laplacian

$$\begin{aligned} \operatorname{div}\left(\|\nabla_0 u\|^{q-2}\nabla_0 u\right) = X_1 &\left(\left(X_1^2 u + X_2^2 u\right)^{(q-2)/2} X_1 u\right) \\ &+ X_2\left(\left(X_1^2 u + X_2^2 u\right)^{(q-2)/2} X_2 u\right) \end{aligned}$$

for $2 < q < \infty$. Formally taking the limit as $q \to \infty$ yields the horizontal infinite Laplacian

$$\Delta_{0,\infty} f = \sum_{i,j=1}^{2} X_i f X_j f X_i X_j f = \left\langle \nabla_0 f, \left(D^2 f\right)^\star \nabla_0 f \right\rangle.$$

For a more complete discussion of the q-Laplacian and the infinite Laplacian see [1], [5], [9].

Within this environment, we first will define solutions to the equation

$$F\left(p, u(p), \nabla_0 u(p), \left(D^2 u(p)\right)^\star\right) = 0$$

in the viscosity sense. In order to achieve this goal, we must define the subelliptic jets. (For a thorough discussion of jets, the interested reader is directed to [7].) Given a function $f\colon G_n \mapsto \mathbb{R}$, it is natural to consider inequalities based on the Taylor expansion. Namely, we consider the following inequalities:

$$\begin{aligned} f(p) \leq f(p_0) &+ (x - x_0)\eta_1 + \frac{1}{\rho(x_0)}(y - y_0)\eta_2 \\ &+ \frac{1}{2}(x - x_0)^2 X_{11} + \frac{1}{2\rho(x_0)^2}(y - y_0)^2 X_{22} \\ &+ (x - x_0)(y - y_0)\left(\frac{1}{\rho(x_0)} X_{12} - \frac{\rho'(x_0)}{2\rho(x_0)^2}\eta_2\right) \\ &+ o\left(d_C(p_0, p)^2\right) \text{ as } p \to p_0 \text{ when } r_{p_0} = 0, \end{aligned} \tag{1.3}$$

$$(1.4)\qquad f(p) \leq f(p_0) + (x-x_0)\eta_1 + \frac{2}{\rho'(x_0)}(y-y_0)X_{12} + \frac{1}{2}(x-x_0)^2 X_{11} + o\left(d_C(p_0,p)^2\right) \text{ as } p \to p_0 \text{ when } r_{p_0} > 0.$$

If $\rho'(x_0)=0$, we consider the term $\frac{2}{\rho'(x_0)}(y-y_0)X_{12}$ in Equation (1.4) to be zero.

Given an open set $\mathcal{O} \subset G_n$ and a function $f\colon \mathcal{O} \mapsto \mathbb{R}$, define the second order superjet of f at p_0, denoted $J^{2,+}f(p_0)$, as follows:

$$(\eta, X) \in J^{2,+}_{\mathcal{O}} f(p_0) \Longleftrightarrow p, p_0 \in \mathcal{O} \text{ and (1.3) holds} \quad (r_{p_0}=0)$$

or

$$(\eta, X) \in J^{2,+}_{\mathcal{O}} f(p_0) \Longleftrightarrow p, p_0 \in \mathcal{O} \text{ and (1.4) holds} \quad (r_{p_0}>0).$$

The second order subjet of u at p_0, denoted $J^{2,-}u(p_0)$, is defined by

$$J^{2,-}_{\mathcal{O}} f(p_0) = -J^{2,+}_{\mathcal{O}}(-f)(p_0).$$

Using these jets, we can define viscosity solutions to our class of functions.

Definition 1. Let $\mathcal{O}$ be an open set in G_n and let $u\colon \mathcal{O} \mapsto \mathbb{R}$. If u is upper semi-continuous and

$$F(p, u(p), \eta, X) \leq 0 \text{ for all } p \in \mathcal{O} \text{ and all } (\eta, X) \in J^{2,+}_{\mathcal{O}} u(p),$$

then u is a *viscosity subsolution* of $F(p, u(p), \nabla_0 u(p), (D^2 u(p))^\star) = 0$.

If u is lower semi-continuous and

$$F(p, u(p), \eta, X) \geq 0 \quad \text{for all } p \in \mathcal{O} \quad \text{and all } (\eta, X) \in J^{2,-}_{\mathcal{O}} u(p),$$

then u is a *viscosity supersolution* of $F(p, u(p), \nabla_0 u(p), (D^2 u(p))^\star) = 0$.

The function u is a *viscosity solution* if it is both a viscosity subsolution and a viscosity supersolution.

In order to use the machinery of [7] to prove comparison principles, a relationship between Euclidean and subelliptic jets must be established. This is accomplished through the following lemma.

Main Lemma. *Let the points $p, p_0 \in \mathbb{R}^2$ be denoted by $p=(x,y)$ and $p_0=(x_0,y_0)$. Let $\eta \in \mathbb{R}^2$ and $X \in S^2$. Also, let $\langle\cdot,\cdot\rangle_E$ denote the Euclidean inner product in $\mathbb{R}^2$. Then define the standard Euclidean superjet, denoted $J^{2,+}_\sharp$, by*

$$J^{2,+}_\sharp u(p_0) = \Big\{(\eta, X) : u(p) \leq u(p_0) + \langle \eta, p-p_0\rangle_E + \frac{1}{2}\langle X(p-p_0), p-p_0\rangle_E + o\left(\langle p-p_0, p-p_0\rangle_E\right) \ \text{as } p \to p_0\Big\}.$$

If η and X are defined by

$$\eta = (\eta_1, \eta_2) \quad \textit{and } X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix},$$

define the g_n vector

$$\tilde{\eta} = \eta_1 X_1 + \rho(x_0)\eta_2 X_2$$

and the symmetric matrix Y by

$$\begin{pmatrix} X_{11} & \rho(x_0)X_{12} + \frac{1}{2}\rho'(x_0)\eta_2 \\ \rho(x_0)X_{12} + \frac{1}{2}\rho'(x_0)\eta_2 & \rho(x_0)^2 X_{22} \end{pmatrix}.$$

Then, given $(\eta, X) \in \overline{J}^{2,+}_{\sharp} u(p_0)$, we have $(\tilde{\eta}, Y) \in \overline{J}^{2,+} u(p_0)$.

This lemma is the key to proving comparison principles. The first comparison principle involves strictly monotone elliptic equations. Such equations satisfy the following properties:

$$\begin{aligned} \sigma(r-s) &\leq F(p,r,\eta,X) - F(p,s,\eta,X), \\ |F(p,r,\eta,X) - F(q,r,\eta,X)| &\leq w_1(d_C(p,q)), \\ |F(p,r,\eta,X) - F(p,r,\eta,Y)| &\leq w_2(\|Y-X\|), \\ |F(p,r,\eta,X) - F(p,r,\nu,X)| &\leq w_3(|\|\eta\| - \|\nu\||), \end{aligned}$$

where $\sigma > 0$ is a constant and the functions $w_i \colon [0,\infty] \mapsto [0,\infty]$ satisfy $w_i(0^+) = 0$ for $i = 1,2,3$. The appropriate comparison principle is given below.

THEOREM 1.1. *Let F satisfy the above properties. Let u be an upper semi-continuous subsolution and v a lower semi-continuous supersolution to*

$$F\left(p, f(p), \nabla_0 f(p), \left(D^2 f(p)\right)^{\star}\right) = 0$$

in a domain Ω so that

$$\limsup_{q \to p} u(q) \leq \liminf_{q \to p} v(q)$$

when $p \in \partial\Omega$, where both sides are not ∞ or $-\infty$ simultaneously. Then

$$u(p) \leq v(p)$$

for all $p \in \Omega$.

The second comparison principle involves Jensen's auxiliary function used in the proof of uniqueness for infinite harmonic functions (see [9]). This function is defined by

$$F_\varepsilon(\eta, X) = \min\left\{\|\eta\|^2 - \varepsilon^2, -\langle X\eta, \eta\rangle\right\},$$

where ε is a positive real number.

THEOREM 1.2. *Let u be an upper semi-continuous subsolution and v a lower semi-continuous supersolution to*

$$F_\varepsilon\left(\nabla_0 f(p), \left(D^2 f(p)\right)^\star\right) = 0$$

in a domain Ω so that

$$\limsup_{q\to p} u(q) \leq \liminf_{q\to p} v(q)$$

when $p \in \partial\Omega$, where both sides are not ∞ or $-\infty$ simultaneously. Then

$$u(p) \leq v(p)$$

for all $p \in \Omega$.

This comparison principle produces a corollary, whose proof is similar to that of the theorem.

COROLLARY 1.3. *Let ε be a positive real number. Then a comparison principle for*

$$H_\varepsilon(\eta, X) = \min\left\{\varepsilon^2 - \|\eta\|^2, -\langle X\eta, \eta\rangle\right\}$$

holds as in the theorem.

Using the Theorem and the Lemma and letting $\varepsilon \to 0$, we obtain a comparison principle for infinite harmonic functions:

THEOREM 1.4. *Let u be an upper semicontinuous subsolution and v be a lower semicontinuous supersolution of*

$$\Delta_{0,\infty} u = 0$$

in a domain Ω such that if $p \in \partial\Omega$, then

$$\limsup_{q\to p} u(q) \leq \limsup_{q\to p} v(q),$$

where both sides are not $-\infty$ or $+\infty$ simultaneously. Then, for all $p \in \Omega$,

$$u(p) \leq v(p).$$

This paper is organized as follows. Section 2 is concerned with formulating Taylor's Theorem on Grushin-type planes. Section 3 defines second order jets on Grushin-type planes and proves needed properties. Section 4 establishes a Grushin maximum principle, and Section 5 proves various comparison principles. The paper ends with Section 6, which focuses on a specific class of Grushin-type planes and examines infinite harmonic functions there.

2. Taylor polynomials

In order to proceed, our attention must turn to Taylor polynomials. There are two forms of the Taylor polynomial on the Grushin plane, depending on the location of the base point. The following proposition formalizes this fact.

PROPOSITION 2.1. *Let $f\colon G_n \mapsto \mathbb{R}$ be a C^2 function. Let p_0 be denoted by (x_0, y_0). If $r_{p_0} = 0$ (that is, $\rho(x_0) \neq 0$), then*

$$\begin{aligned} f(p) &= f(p_0) + (x - x_0)X_1 f(p_0) + \frac{1}{\rho(x_0)}(y - y_0)X_2 f(p_0) \\ &\quad + \frac{1}{2}(x - x_0)^2 X_1^2 f(p_0) + \frac{1}{2\rho(x_0)^2}(y - y_0)^2 X_2^2 f(p_0) \\ &\quad + (x - x_0)(y - y_0)\frac{1}{2\rho(x_0)}\left(X_1 X_2 f(p_0) + X_2 X_1 f(p_0) - \frac{\rho'(x_0)}{\rho(x_0)} X_2 f(p_0)\right) \\ &\quad + o\left(d_C(p_0, p)^2\right). \end{aligned}$$

If $r_{p_0} \neq 0$ (that is, $\rho(x_0) = 0$), then

$$\begin{aligned} f(p) &= f(p_0) + (x - x_0)X_1 f(p_0) + (y - y_0)\frac{1}{\rho^{(r_{p_0})}(x_0)} X_{r_{p_0}+2} f(p_0) \\ &\quad + \frac{1}{2}(x - x_0)^2 X_1^2 f(p_0) + o\left(d_C(p_0, p)^2\right). \end{aligned}$$

Proof. Case 1: $r_{p_0} = 0$.

Define the polynomial $P(p)$ by

$$\begin{aligned} P(p) &= f(p_0) + (x - x_0)X_1 f(p_0) + \frac{1}{\rho(x_0)}(y - y_0)X_2 f(p_0) \\ &\quad + \frac{1}{2}(x - x_0)^2 X_1^2 f(p_0) + \frac{1}{2\rho(x_0)^2}(y - y_0)^2 X_2^2 f(p_0) \\ &\quad + (x - x_0)(y - y_0)\frac{1}{2\rho(x_0)}\left(X_1 X_2 f(p_0) + X_2 X_1 f(p_0) - \frac{\rho'(x_0)}{\rho(x_0)} X_2 f(p_0)\right). \end{aligned}$$

Then computation shows that the following equations hold:

$$\begin{aligned} X_1 P(p) &= X_1 f(p_0) + (x - x_0)X_1^2 f(p_0) \\ &\quad + (y - y_0)\frac{1}{2\rho(x_0)}\left(X_1 X_2 f(p_0) + X_2 X_1 f(p_0) - \frac{\rho'(x_0)}{\rho(x_0)} X_2 f(p_0)\right), \\ X_2 P(p) &= \rho(x) \times \left(\frac{1}{\rho(x_0)} X_2 f(p_0) + \frac{1}{\rho(x_0)^2}(y - y_0) X_2^2 f(p_0) \right. \\ &\quad + (x - x_0)\frac{1}{2\rho(x_0)}\left(X_1 X_2 f(p_0) + X_2 X_1 f(p_0) \right. \\ &\quad \left.\left. - \frac{\rho'(x_0)}{\rho(x_0)} X_2 f(p_0)\right)\right), \end{aligned}$$

$$
\begin{aligned}
X_1X_2P(p) &= \frac{\rho(x)}{2\rho(x_0)}\left(X_1X_2f(p_0)+X_2X_1f(p_0)-\frac{\rho'(x_0)}{\rho(x_0)}X_2f(p_0)\right)\\
&\qquad +\frac{\rho'(x)}{\rho(x)}X_2P(p),\\
X_2X_1P(p) &= \frac{\rho(x)}{2\rho(x_0)}\left(X_1X_2f(p_0)+X_2X_1f(p_0)-\frac{\rho'(x_0)}{\rho(x_0)}X_2f(p_0)\right),\\
X_1X_1P(p) &= X_1^2f(p_0),\\
X_2X_2P(p) &= \rho(x)^2\frac{1}{\rho(x_0)^2}X_2^2f(p_0).
\end{aligned}
$$

Evaluation at p_0 and recalling the relation

$$X_3=[X_1,X_2]=\frac{\rho'(x_0)}{\rho(x_0)}X_2$$

gives $X^IP(p_0)=X^If(p_0)$ for $d(I)\le 2$. By Theorem 4.10 in [4], $f(p)-P(p)$ is $O(d_C(p_0,p)^3)$, and so it is $o(d_C(p_0,p)^2)$.

Case 2: $r_{p_0}\neq 0$. The proof is similar to the above case, except that now X_2 is the zero vector. □

We point out that by equation (1.1), $y-y_0$ is $O(d_C(p_0,p)^{r_{p_0}+1})$. Thus, in the case $r_{p_0}>1$ this term is technically part of the error. However, in order to maintain a connection with the cases $r_{p_0}\le 1$, this term must be included. This connection will be necessary in the next section. Before proceeding to the next section, we rewrite the Taylor polynomial for the case $r_{p_0}>0$ in a way that emphasizes the symmetry. Namely,

$$
\begin{aligned}
f(p) &= f(p_0)+(x-x_0)X_1f(p_0)+\frac{1}{2}(x-x_0)^2X_1^2f(p_0)\\
&\quad +\frac{2}{\rho'(x_0)}(y-y_0)\frac{1}{2}\left(X_1X_2f(p_0)+X_2X_1f(p_0)\right)+o\left(d_C(p_0,p)^2\right).
\end{aligned}
$$

We write this with the understanding that if $\rho'(x_0)=0$, then the $y-y_0$ term is to be treated as a zero term. This is consistent with the Taylor polynomial, for if $\rho(x_0)=\rho'(x_0)=0$ (that is, if $r_{p_0}\ge 2$), the vector fields X_2 and X_3 are zero, resulting in X_2X_1 and X_1X_2 also being zero.

Having constructed Taylor polynomials in the proper form, we define subelliptic jets in G_n.

3. Subelliptic jets

Let S^2 be the set of all real 2×2 symmetric matrices. Let $\eta\in g_n$ and $X\in S^2$ be given by

$$\eta=\eta_1X_1+\eta_2X_2\quad\text{and}\quad X=\begin{pmatrix}X_{11} & X_{12}\\ X_{12} & X_{22}\end{pmatrix}.$$

Given a function $u\colon G_n \mapsto \mathbb{R}$, consider the following inequalities:

$$\begin{aligned}
u(p) \leq{}& u(p_0) + (x-x_0)\eta_1 + \frac{1}{\rho(x_0)}(y-y_0)\eta_2 \\
&+ \frac{1}{2}(x-x_0)^2 X_{11} + \frac{1}{2\rho(x_0)^2}(y-y_0)^2 X_{22} \\
&+ (x-x_0)(y-y_0)\left(\frac{1}{\rho(x_0)}X_{12} - \frac{\rho'(x_0)}{2\rho(x_0)^2}\eta_2\right) \\
&+ o\left(d_C(p_0,p)^2\right) \text{ as } p \to p_0 \text{ when } r_{p_0} = 0.
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
u(p) \leq{}& u(p_0) + (x-x_0)\eta_1 + \frac{2}{\rho'(x_0)}(y-y_0)X_{12} \\
&+ \frac{1}{2}(x-x_0)^2 X_{11} + o\left(d_C(p_0,p)^2\right) \text{ as } p \to p_0 \text{ when } r_{p_0} > 0.
\end{aligned} \tag{3.2}$$

Again, if $\rho'(x_0) = 0$, we consider the term $\frac{2}{\rho'(x_0)}(y-y_0)X_{12}$ in equation (3.2) to be zero.

Given an open set $\mathcal{O} \subset G_n$ and a function $u\colon \mathcal{O} \mapsto \mathbb{R}$, define the second order superjet of u at p_0, denoted $J^{2,+}u(p_0)$, as follows:

$$(\eta, X) \in J^{2,+}_{\mathcal{O}} u(p_0) \Longleftrightarrow p, p_0 \in \mathcal{O} \text{ and (3.1) holds} \quad (r_{p_0} = 0)$$

or

$$(\eta, X) \in J^{2,+}_{\mathcal{O}} u(p_0) \Longleftrightarrow p, p_0 \in \mathcal{O} \text{ and (3.2) holds} \quad (r_{p_0} > 0).$$

The second order subjet of u at p_0, denoted $J^{2,-}u(p_0)$, is defined by

$$J^{2,-}_{\mathcal{O}} u(p_0) = -J^{2,+}_{\mathcal{O}}(-u)(p_0).$$

Following [7], we define the closure of a jet by

$$\begin{aligned}
\overline{J}^{2,+}u(p_0) = \{(\eta, X) : \exists (p_n, \eta_n, X_n) \text{ so that } (\eta_n, X_n) \in J^{2,+}u(p_n) \\
\text{and } (p_n, u(p_n), \eta_n, X_n) \to (p_0, u(p_0), \eta, X)\}.
\end{aligned}$$

Having formally defined the concept of subelliptic jet on the Grushin plane, the following proposition characterizes the jets in terms of test functions that touch from above or below. This proposition and its proof are an extension of Crandall [6].

Proposition 3.1. *Let u, h, and $\mathcal{O}$ be as above. Define the set*

$$K^{u,p_0} = \left\{\left(\nabla_0\phi(p_0), \left(D^2\phi(p_0)\right)^\star\right) : u - \phi \textit{ has a local max at } p_0\right\}.$$

Then we have the equality

$$J^{2,+}_{\mathcal{O}} u(p_0) = K^{u,p_0}.$$

Proof. Let p_0 be the local maximum of $u - \phi$. Then, for p near p_0,

$$u(p) - \phi(p) \leq u(p_0) - \phi(p_0),$$

and so

$$u(p) \leq u(p_0) + \phi(p) - \phi(p_0).$$

Then Proposition 2.1 yields

$$K^{u,p_0} \subset J^{2,+}_{\mathcal{O}} u(p).$$

In order to show the reverse inclusion, a function ϕ with a strict maximum at p_0 that has the appropriate derivatives will be constructed. Define the function $a\colon G_n \mapsto \mathbb{R}$ by

$$a(p) = x^4 + y^4$$

for $p = (x, y)$. This function is C^2 and satisfies $a(p_0 - p) = O(d_C(p_0, p)^4)$ at every point p_0.

We will first consider the case when $r_{p_0} > 0$. We define the function $z(r)$ using a pair $(\eta, X) \in J^{2,+}_{\mathcal{O}} u(p_0)$ by setting $z(r)$ equal to

$$\sup \left(u(p) - u(p_0) - (x - x_0)\eta_1 - \frac{2}{\rho'(x_0)}(y - y_0)X_{12} - \frac{1}{2}(x - x_0)^2 X_{11} \right)^+,$$

where the sup is taken over all $p \in \mathcal{O}$ such that $a(p - p_0) \leq r$.

We proceed as in [5] to construct a C^2 function $\zeta\colon G_n \mapsto \mathbb{R}$ so that

$$X^I(\zeta(p - p_0))(p_0) = 0$$

for all multi-indices I with $d(I) \leq 2$. Define the function $\phi\colon G_n \mapsto \mathbb{R}$ by

$$\begin{aligned}\phi(p) = {} & \zeta(p - p_0) + a(p - p_0) + (x - x_0)\eta_1 \\ & + \frac{2}{\rho'(x_0)}(y - y_0)X_{12} + \frac{1}{2}(x - x_0)^2 X_{11}.\end{aligned}$$

With this definition, $\phi(p_0) = 0$, and so

$$\begin{aligned}u(p) - \phi(p) - u(p_0) + \phi(p_0) + s = {} & s + u(p) - \zeta(p - p_0) - a(p - p_0) \\ & - (x - x_0)\eta_1 - \frac{2}{\rho'(x_0)}(y - y_0)X_{12} \\ & - \frac{1}{2}(x - x_0)^2 X_{11} - u(p_0).\end{aligned}$$

By the construction of ζ, this gives

$$u(p) - \phi(p) - u(p_0) + \phi(p_0) + s \leq 0$$

in the region $s \leq a(p - p_0)$. Thus, $u - \phi$ has a strict local maximum at p_0. In addition, computation of the derivatives gives $\nabla_0 \phi(p_0) = \eta$ and $D^2(\phi(p_0))^\star = X$, so that

$$J^{2,+}_{\mathcal{O}} u(p) \subset K^{u,p_0}.$$

The case where $r_{p_0} = 0$ is similar and omitted. □

Due to the subelliptic structure of the Grushin plane on lines, where we have $\rho(x) = 0$, the maximum principle of Crandall, Ishii, and Lions [7] is not readily available. The next lemma shows explicitly how any traditional Euclidean superjet gives rise to a subelliptic superjet, so that the machinery of [7] may be employed.

MAIN LEMMA. *Let the points $p, p_0 \in \mathbb{R}^2$ be denoted by $p = (x, y)$ and $p_0 = (x_0, y_0)$. Let $\eta \in \mathbb{R}^2$ and $X \in S^2$. Also, let $\langle \cdot, \cdot \rangle_E$ denote the Euclidean inner product in $\mathbb{R}^2$. Define the standard Euclidean superjet, denoted $J_\sharp^{2,+}$, by*

(3.3)

$$J_\sharp^{2,+}u(p_0) = \Big\{(\eta, X) : u(p) \leq u(p_0) + \langle \eta, p - p_0\rangle_E + \frac{1}{2}\langle X(p-p_0), p-p_0\rangle_E + o\left(\langle p-p_0, p-p_0\rangle_E\right) \text{ as } p \to p_0\Big\}.$$

If η and X are defined by

$$\eta = (\eta_1, \eta_2) \text{ and } X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix},$$

define the g_n vector by

$$\tilde{\eta} = \eta_1 X_1 + \rho(x_0)\eta_2 X_2$$

and the symmetric matrix Y by

$$\begin{pmatrix} X_{11} & \rho(x_0)X_{12} + \frac{1}{2}\rho'(x_0)\eta_2 \\ \rho(x_0)X_{12} + \frac{1}{2}\rho'(x_0)\eta_2 & \rho(x_0)^2 X_{22} \end{pmatrix}.$$

Then, given $(\eta, X) \in \overline{J}_\sharp^{2,+}u(p_0)$, we have $(\tilde{\eta}, Y) \in \overline{J}^{2,+}u(p_0)$.

Proof. Case 1: $(\eta, X) \in J_\sharp^{2,+}u(p_0)$.

We will consider the case where $r_{p_0} > 0$. The other case is similar and omitted. First, we observe that if β is $o(\langle p - p_0, p - p_0\rangle_E)$ then equation (1.1) leads to

$$\frac{\beta}{d_C(p_0, p)^2} \sim \frac{\beta}{\langle p - p_0, p - p_0\rangle_E} \times \frac{|x - x_0|^2 + |y - y_0|^2}{|x - x_0|^2 + |y - y_0|^{2/(r_{p_0}+1)}},$$

and so β is $o(d_C(p, p_0)^2)$. Also by equation (1.1), $y - y_0$ is $O(d_C(p, p_0)^{r_{p_0}+1})$. In particular, $(x - x_0)(y - y_0)$ and $(y - y_0)^2$ are $o(d_C(p, p_0)^2)$.

Using these estimates, we expand equation (3.3) to obtain

$$u(p) \leq u(p_0) + (x - x_0)\eta_1 + (y - y_0)\eta_2 + \frac{1}{2}(x - x_0)^2 X_{11} + o\left(d_C(p, p_0)^2\right).$$

The result then follows from equation (3.2) and the fact that $\rho(x_0) = 0$.

Case 2: $(\eta, X) \in \overline{J}_\sharp^{2,+}u(p_0)$.

Given $(\eta, X) \in \overline{J}_{\sharp}^{2,+}u(p_0)$, there is a sequence $\{p_n, \eta_n, X_n\} \in \Omega \times g_n \times S^2$ so that $(\eta_n, X_n) \in J_{\sharp}^{2,+}u(p_n)$ and $\{p_n, u(p_n), \eta_n, X_n\} \to (p_0, u(p_0), \eta, X)$. Now, (η_n, X_n) can be identified with $(\tilde{\eta_n}, Y_n) \in J^{2,+}u(p_n)$. By construction, $\tilde{\eta_n} \to \tilde{\eta}$ and $Y_n \to Y$. Thus, $(p_n, u(p_n), \tilde{\eta_n}, Y_n) \to (p_0, u(p_0), \tilde{\eta}, Y)$ and so we have $(\tilde{\eta}, Y) \in \overline{J}^{2,+}u(p_0)$. □

4. Maximum principle

We begin by stating a lemma analogous to Lemma 3.1 of [7]. The proof is similar and omitted.

LEMMA 4.1. *Let u be an upper-semicontinuous function in Ω and v a lower-semicontinuous function in Ω. For $\tau > 0$, $p = (x_1, y_1)$ and $q = (x_2, y_2)$ let the function $\varphi(p, q)$ be defined by*

$$\varphi(p, q) \equiv \frac{1}{2}(x_1 - x_2)^2 + \frac{1}{4}(y_1 - y_2)^4$$

and let the function M_τ be defined by

$$M_\tau = \sup_{\overline{\Omega} \times \overline{\Omega}} \left(u(p) - v(q) - \tau\varphi(p, q)\right).$$

Let $p_\tau = (x_1^\tau, y_1^\tau)$ and $q_\tau = (x_2^\tau, y_2^\tau)$ be so that

$$\lim_{\tau \to \infty} \left(M_\tau - \left(u(p_\tau) - v(q_\tau) - \tau\varphi(p_\tau, q_\tau)\right)\right) = 0.$$

Then,

$$\lim_{\tau \to \infty} \tau\varphi(p_\tau, q_\tau) = 0 \tag{4.1}$$

and

$$\lim_{\tau \to \infty} M_\tau = u\left(p^*\right) - v\left(p^*\right) = \sup_{\overline{\Omega}}(u(p) - v(p)) \tag{4.2}$$

whenever p^* is a limit point of p_τ as $\tau \mapsto \infty$.

Using the function $\varphi(p_\tau, q_\tau)$, we compute some important vectors and matrices that are dependent upon the Euclidean derivatives. We begin by defining the vectors Υ_{p_τ} and Υ_{q_τ} by

$$\Upsilon_{p_\tau} \equiv \begin{pmatrix} (x_1^\tau - x_2^\tau) \\ \rho\left(x_1^\tau\right)\left(y_1^\tau - y_2^\tau\right)^3 \end{pmatrix} \quad \text{and} \quad \Upsilon_{q_\tau} \equiv \begin{pmatrix} (x_1^\tau - x_2^\tau) \\ \rho\left(x_2^\tau\right)\left(y_1^\tau - y_2^\tau\right)^3 \end{pmatrix}.$$

Note that Υ_{p_τ} is the Euclidean derivative of $\varphi(p_\tau, q_\tau)$ with respect to p_τ twisted at p_τ using the Main Lemma and that Υ_{q_τ} is the negative of the Euclidean derivative of $\varphi(p_\tau, q_\tau)$ with respect to q_τ twisted at q_τ using the Main Lemma. Next, we consider the matrix $D^2\varphi(p_\tau, q_\tau)$ of second order Euclidean derivatives. A straightforward computation shows that $(D^2\varphi(p_\tau, q_\tau))^2 + D^2\varphi(p_\tau, q_\tau)$ equals

$$\begin{pmatrix} 3 & 0 & -3 & 0 \\ 0 & 3\left(y_1^\tau - y_2^\tau\right)^2 + 18\left(y_1^\tau - y_2^\tau\right)^4 & 0 & -3\left(y_1^\tau - y_2^\tau\right)^2 - 18\left(y_1^\tau - y_2^\tau\right)^4 \\ -3 & 0 & 3 & 0 \\ 0 & -3\left(y_1^\tau - y_2^\tau\right)^2 - 18\left(y_1^\tau - y_2^\tau\right)^4 & 0 & 3\left(y_1^\tau - y_2^\tau\right)^2 + 18\left(y_1^\tau - y_2^\tau\right)^4 \end{pmatrix}$$

and we shall denote this matrix by $\mathcal{C}$.

We now proceed as in [7]. Let u be a viscosity subsolution and v a viscosity supersolution to $F(p, f(p), \nabla_0 f(p), (D^2 f(p))^\star) = 0$. Denote the points p and s by $p = (x_1, y_1)$ and $q = (x_2, y_2)$ and let $(p_\tau, q_\tau) = ((x_1^\tau, y_1^\tau), (x_2^\tau, y_2^\tau))$ be the maximum point of

$$u(p) - v(q) - \tau\varphi(p, q)$$

in $\overline{\Omega}\times\overline{\Omega}$. By the Euclidean maximum principle of semicontinuous functions [7], there are subsequences $p_{\tau_i} \to p_0$ and $q_{\tau_i} \to p_0$. Passing to the subsequence, we shall denote these points by p_τ and q_τ, respectively. In addition, there exist S^2 matrices X^τ and Y^τ, denoted by

$$X^\tau = \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} \quad \text{and} \quad Y^\tau = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12} & Y_{22} \end{pmatrix},$$

so that

$$(\tau D\varphi_p(p_\tau, q_\tau), X^\tau) \in \overline{J}^{2,+}_{\text{eucl}} u(p_\tau) \text{ and } (-\tau D\varphi_q(p_\tau, q_\tau), Y^\tau) \in \overline{J}^{2,-}_{\text{eucl}} v(q_\tau).$$

The matrices X^τ and Y^τ satisfy the estimate

$$\langle X^\tau \epsilon, \epsilon\rangle_E - \langle Y^\tau \kappa, \kappa\rangle_E \le \tau \langle \mathcal{C}\chi, \chi\rangle_E \tag{4.3}$$

for any vectors ϵ and κ in $\mathbb{R}^2$, where $\langle\cdot,\cdot\rangle_E$ is the standard Euclidean inner product and the vector χ is defined by $\chi = (\epsilon, \kappa)$. Using the Main Lemma, we obtain

$$(\tau\Upsilon_{p_\tau}, \mathcal{X}^\tau) \in \overline{J}^{2,+} u(p_\tau) \text{ and } (\tau\Upsilon_{q_\tau}, \mathcal{Y}^\tau) \in \overline{J}^{2,-} v(q_\tau),$$

where the matrices $\mathcal{X}^\tau$ and $\mathcal{Y}^\tau$ are defined by

$$\mathcal{X}^\tau = \begin{pmatrix} X_{11} & \rho(x_1^\tau)X_{12} + \frac{1}{2}\rho'(x_1^\tau)(y_1^\tau - y_2^\tau)^3 \\ \rho(x_1^\tau)X_{12} + \frac{1}{2}\rho'(x_1^\tau)(y_1^\tau - y_2^\tau)^3 & \rho(x_1^\tau)^2 X_{22} \end{pmatrix},$$

$$\mathcal{Y}^\tau = \begin{pmatrix} Y_{11} & \rho(x_2^\tau)Y_{12} + \frac{1}{2}\rho'(x_2^\tau)(y_1^\tau - y_2^\tau)^3 \\ \rho(x_2^\tau)Y_{12} + \frac{1}{2}\rho'(x_2^\tau)(y_1^\tau - y_2^\tau)^3 & \rho(x_2^\tau)^2 Y_{22} \end{pmatrix}.$$

The elements of the subelliptic jets also satisfy important estimates given by the following lemma.

LEMMA 4.2. *The vectors Υ_{p_τ} and Υ_{q_τ} satisfy*

$$\left|\|\Upsilon_{q_\tau}\|^2 - \|\Upsilon_{p_\tau}\|^2\right| = O(\varphi(p_\tau, q_\tau)^2). \tag{4.4}$$

In addition, with the usual ordering, the matrix $\mathcal{X}^\tau$ is smaller than the matrix $\mathcal{Y}^\tau$ with an error term. In particular, $\mathcal{X}^\tau \leq \mathcal{Y}^\tau + \mathcal{R}^\tau$, where $\mathcal{R}^\tau \to 0$ as $\tau \to \infty$.

Proof. A straightforward computation shows

$$\|\Upsilon_{q_\tau}\|^2 - \|\Upsilon_{p_\tau}\|^2 = (y_1^\tau - y_2^\tau)^6 \left(\rho\left(x_2^\tau\right)^2 - \rho\left(x_1^\tau\right)^2\right).$$

The vector difference estimate then follows from the definition of φ and the fact that $(\rho(x_2^\tau)^2 - \rho(x_1^\tau)^2)$ is $O(x_1^\tau - x_2^\tau)$.

We now focus on the matrix difference estimate. Using the definitions of $\mathcal{X}^\tau$ and $\mathcal{Y}^\tau$, we write $\langle \mathcal{X}^\tau \epsilon, \epsilon\rangle - \langle \mathcal{Y}^\tau \kappa, \kappa\rangle$ as

$$\left\langle X^\tau \begin{pmatrix} \epsilon_1 \\ \rho\left(x_1^\tau\right)\epsilon_2 \end{pmatrix}, \begin{pmatrix} \epsilon_1 \\ \rho\left(x_1^\tau\right)\epsilon_2 \end{pmatrix}\right\rangle - \left\langle Y^\tau \begin{pmatrix} \kappa_1 \\ \rho\left(x_2^\tau\right)\kappa_2 \end{pmatrix}, \begin{pmatrix} \kappa_1 \\ \rho\left(x_2^\tau\right)\kappa_2 \end{pmatrix}\right\rangle \\ + \frac{1}{2}\left(y_1^\tau - y_2^\tau\right)^3 \left(\rho'\left(x_1^\tau\right)\epsilon_1\epsilon_2 - \rho'\left(x_2^\tau\right)\kappa_1\kappa_2\right).$$

Using equation (4.3) we obtain

$$\text{(4.5)}\quad \langle \mathcal{X}^\tau \epsilon, \epsilon\rangle - \langle \mathcal{Y}^\tau \kappa, \kappa\rangle \leq \tau \langle \mathcal{C}\chi, \chi\rangle_E + \frac{1}{2}(y_1^\tau - y_2^\tau)^3(\rho'(x_1^\tau)\epsilon_1\epsilon_2 - \rho'(x_2^\tau)\kappa_1\kappa_2),$$

where $\chi = (\epsilon_1, \rho(x_1^\tau)\epsilon_2, \kappa_1, \rho(x_2^\tau)\kappa_2)$. Computing the inner product, we conclude

$$\text{(4.6)}\quad \begin{aligned} &\langle \mathcal{X}^\tau \epsilon, \epsilon\rangle - \langle \mathcal{Y}^\tau \kappa, \kappa\rangle \\ &\quad \leq \tau\left(3\left(y_1^\tau - y_2^\tau\right)^2 + 18\left(y_1^\tau - y_2^\tau\right)^4\right)\left(\rho\left(x_1^\tau\right)\epsilon_2 - \rho\left(x_2^\tau\right)\kappa_2\right)^2 \\ &\qquad + 3\tau(\epsilon_1 - \kappa_1)^2 + \frac{1}{2}\left(y_1^\tau - y_2^\tau\right)^3\left(\rho'\left(x_1^\tau\right)\epsilon_1\epsilon_2 - \rho'\left(x_2^\tau\right)\kappa_1\kappa_2\right). \end{aligned}$$

Setting $\kappa = \epsilon$, we compute that

$$\text{(4.7)}\quad \begin{aligned} &\langle \mathcal{X}^\tau \epsilon, \epsilon\rangle - \langle \mathcal{Y}^\tau \epsilon, \epsilon\rangle \\ &\quad \leq \tau\epsilon_2^2\left(3\left(y_1^\tau - y_2^\tau\right)^2 + 18\left(y_1^\tau - y_2^\tau\right)^4\right)\left(\rho\left(x_1^\tau\right) - \rho\left(x_2^\tau\right)\right)^2 \\ &\qquad + \frac{1}{2}\epsilon_1\epsilon_2\left(y_1^\tau - y_2^\tau\right)^3\left(\rho'\left(x_1^\tau\right) - \rho'\left(x_2^\tau\right)\right). \end{aligned}$$

Now, $(y_1^\tau - y_2^\tau)^2 \leq (\varphi(p_\tau, q_\tau))^{1/2}$ and $(\rho(x_1^\tau) - \rho(x_2^\tau))^2$ is $O((\varphi(p_\tau, q_\tau))^{1/2})$, and so the right hand side of equation (4.7) goes to 0 as τ goes to infinity by equation (4.1). □

5. Comparison principles

Having established a maximum principle for this environment, we proceed to proving comparison principles for certain types of functions F. In our first example, we consider strictly monotone elliptic functions F. That is, we require F to satisfy the following properties:

$$\sigma(r-s) \leq F(p,r,\eta,X) - F(p,s,\eta,X), \tag{5.1}$$

$$|F(p,r,\eta,X) - F(q,r,\eta,X)| \leq w_1(d_C(p,q)), \tag{5.2}$$

$$|F(p,r,\eta,X) - F(p,r,\eta,Y)| \leq w_2(\|Y-X\|), \tag{5.3}$$

$$|F(p,r,\eta,X) - F(p,r,\nu,X)| \leq w_3(|\|\eta\| - \|\nu\||), \tag{5.4}$$

where $\sigma > 0$ is a constant and the functions $w_i \colon [0,\infty] \mapsto [0,\infty]$ satisfy $w_i(0^+) = 0$ for $i = 1,2,3$. We then formulate a comparison principle for such functions F.

Theorem 5.1. *Let F satisfy equations* (5.1), (5.2), (5.3), *and* (5.4). *Let u be an upper semi-continuous subsolution and v a lower semi-continuous supersolution to*

$$F\left(p, f(p), \nabla_0 f(p), \left(D^2 f(p)\right)^{\star}\right) = 0$$

in a domain Ω so that

$$\limsup_{q\to p} u(q) \leq \liminf_{q\to p} v(q)$$

when $p \in \partial\Omega$, where both sides are not ∞ or $-\infty$ simultaneously. Then

$$u(p) \leq v(p)$$

for all $p \in \Omega$.

Proof. Suppose $\sup_\Omega(u-v) > 0$. Using the Grushin maximum principle from the previous section, we obtain

$$\begin{aligned}
\sigma(u(p_\tau) - v(q_\tau)) &\leq F(p_\tau, u(p_\tau), \tau\Upsilon_{p_\tau}, \mathcal{X}^\tau) - F(p_\tau, v(q_\tau), \tau\Upsilon_{p_\tau}, \mathcal{X}^\tau) \\
&= F(p_\tau, u(p_\tau), \tau\Upsilon_{p_\tau}, \mathcal{X}^\tau) - F(q_\tau, v(q_\tau), \tau\Upsilon_{q_\tau}, \mathcal{Y}^\tau) \\
&\quad + F(q_\tau, v(q_\tau), \tau\Upsilon_{q_\tau}, \mathcal{Y}^\tau) - F(p_\tau, v(q_\tau), \tau\Upsilon_{q_\tau}, \mathcal{Y}^\tau) \\
&\quad + F(p_\tau, v(q_\tau), \tau\Upsilon_{q_\tau}, \mathcal{Y}^\tau) - F(p_\tau, v(q_\tau), \tau\Upsilon_{p_\tau}, \mathcal{Y}^\tau) \\
&\quad + F(p_\tau, v(q_\tau), \tau\Upsilon_{p_\tau}, \mathcal{Y}^\tau) - F(p_\tau, v(q_\tau), \tau\Upsilon_{p_\tau}, \mathcal{X}^\tau).
\end{aligned}$$

The first term is negative since u is a subsolution and v is a supersolution. Using equations (5.1), (5.2), (5.3), and (5.4), and Lemma 4.2, we obtain

$$0 < \sigma(u(p_\tau) - v(q_\tau)) \leq w_1(d_C(p_\tau, q_\tau)) + w_2(\|R_\tau\|) + w_3(\tau|\|\Upsilon_{q_\tau}\| - \|\Upsilon_{p_\tau}\||),$$

which goes to 0 as τ approaches ∞. □

In our second example, we consider a specific type of function F, namely,

$$F_\varepsilon(\eta, X) = \min\left\{\|\eta\|^2 - \varepsilon^2, -\langle X\eta, \eta\rangle\right\},$$

where ε is a positive real number. Before proving a comparison principle for such functions F, we state without proof a technical lemma from [11], which gives a function that approximates the identity and has useful properties.

LEMMA 5.2. *Let $A > 1$ and $\alpha > 1$ be given. Then the function $f : \mathbb{R} \mapsto \mathbb{R}$ given by*

$$f(t) = \frac{1}{\alpha} \log \left(1 + A\left(e^{\alpha t} - 1\right)\right)$$

satisfies $f(0) = 0$, $f'(t) > 1$ and $f''(t) < 0$ for all $t \geq 0$. In addition, f is invertible and $0 < f(t) - t < (A-1)/\alpha$ as $A \to 1^+$.

We now formulate the comparison principle.

THEOREM 5.3. *Let u be an upper semi-continuous subsolution and v a lower semi-continuous supersolution to*

$$F_\varepsilon \left(\nabla_0 f(p), (D^2 f(p))^\star\right) = 0$$

in a domain Ω so that

$$\limsup_{q \to p} u(q) \leq \liminf_{q \to p} v(q)$$

when $p \in \partial\Omega$, where both sides are not ∞ or $-\infty$ simultaneously. Then

$$u(p) \leq v(p)$$

for all $p \in \Omega$.

Proof. Suppose $\sup_\Omega (u - v) > 0$. We wish to replace v by w with $\|v - w\|_{L^\infty(\Omega)}$ small by using the previous lemma. Let $w = f(v)$ for A close to one, with f as in Lemma 5.2. Then let $\sup_{p \in \overline{\Omega}}(u(p) - w(p))$ occur at the (interior) point p_0. Let $\phi \in C^2(\Omega)$ so that $\phi(p_0) = w(p_0)$ and $\phi(p) < w(p)$ for $p \neq p_0$. Set $\Phi = f^{-1}(\phi)$, that is, $\phi = f(\Phi)$. Proceeding as in [5], we set

$$\mu(p) = \min\left\{\varepsilon^2 \left(f'(v(p))^2 - 1\right), -f''(v(p)) f'(v(p))^2 \varepsilon^4\right\},$$

and obtain

$$\min\left\{\nabla_0 \Phi(p_0) - \varepsilon^2, -\Delta_{0,\infty}\Phi(p_0)\right\} \geq \mu(p_0) > 0.$$

Thus, w is a strict supersolution of $F_\varepsilon = 0$.

Replacing v by w, we obtain

$$\begin{aligned} 0 < \mu(q_\tau) &\leq F_\varepsilon(\Upsilon_{q_\tau}, \mathcal{Y}^\tau) - F_\varepsilon(\Upsilon_{p_\tau}, \mathcal{X}^\tau) \\ &= \max\left\{\|\tau\Upsilon_{q_\tau}\|^2 - \|\tau\Upsilon_{p_\tau}\|^2, \langle \mathcal{X}^\tau \tau\Upsilon_{p_\tau}, \tau\Upsilon_{p_\tau}\rangle - \langle \mathcal{Y}^\tau \tau\Upsilon_{q_\tau}, \tau\Upsilon_{q_\tau}\rangle\right\}. \end{aligned}$$

Using Lemma 4.2, and in particular equation (4.6), we obtain the new inequality

$$0 < \mu(q_\tau) \lesssim C \max\left\{\tau^2 \varphi(p_\tau, q_\tau)^2, \tau^2\left(\tau\varphi(p_\tau, q_\tau)^3 + \varphi(p_\tau, q_\tau)^2\right)\right\}$$

for some finite constant C independent of τ. We arrive at a contradiction when we let τ approach infinity and apply Lemma 4.1. $\square$

This comparison principle produces a corollary whose proof is similar to that of the theorem.

Corollary 5.4. *Let ε be a positive real number. Then a comparison principle for*

$$H_\varepsilon(\eta, X) = \min\left\{\varepsilon^2 - \|\eta\|^2, -\langle X\eta, \eta\rangle\right\}$$

holds as in the theorem.

We now state a lemma whose proof is identical to the corresponding lemma found in [5]. The Euclidean version of this lemma was originally proved by Jensen [9]. It gives an estimate on the solutions as $\varepsilon \to 0$.

Lemma 5.5. *Let u^ε and u_ε be solutions to $F_\varepsilon = 0$ and $G_\varepsilon = 0$, respectively. Given $\delta > 0$, there exists an $\varepsilon > 0$ so that*

$$u_\varepsilon \leq u^\varepsilon \leq u_\varepsilon + \delta.$$

We then combine Theorem 5.3, Corollary 5.4, and Lemma 5.5 to obtain the following comparison principle for infinite harmonic functions:

Theorem 5.6. *Let u be an upper semicontinuous subsolution and v a lower semicontinuous supersolution of*

$$\Delta_{0,\infty} u = 0$$

in a domain Ω such that if $p \in \partial\Omega$, then

$$\limsup_{q \to p} u(q) \leq \limsup_{q \to p} v(q),$$

where both sides are not $-\infty$ or $+\infty$ simultaneously. Then, for all $p \in \Omega$,

$$u(p) \leq v(p).$$

6. Explicit calculations in a particular class of Grushin-type planes

In this section, we focus on infinite harmonic functions in the viscosity sense. Given a domain Ω and Lipschitz boundary data given by Θ, the existence proof of a viscosity infinite function u in Ω equal to Θ on the boundary follows that of [5]. By Theorem 5.6, such functions u are unique. We then desire to further examine the case when $\rho(x) = c(x-a)^n$ for any $n \in \mathbb{N}$ and $a, c \in \mathbb{R}$ with c non-zero. We shall not only exhibit a particular infinite harmonic function, but also relate it to the fundamental solution to the q-Laplacian. We begin by recalling that our choice of $\rho(x)$ above produces the following vector fields in g_n:

$$X_1 = \frac{\partial}{\partial x},\ X_2 = c(x-a)^n \frac{\partial}{\partial y},\ X_3 = cn(x-a)^{n-1}\frac{\partial}{\partial y},\ \ldots,\ X_{n+2} = cn!\frac{\partial}{\partial y}.$$

Motivated by [14] and [15], we define the function $r \colon G_n \mapsto \mathbb{R}$ by

$$r(x,y) = \left(c^2(x-a)^{2n+2} + (n+1)^2(y-b)^2\right)^{1/(2n+2)}$$

for any real number b. Using the vectors X_1 and X_2 above, an easy calculation shows that the function $r(x,y)$ is in $C^\infty(G_n \setminus (a,b)) \cap C(G_n)$. In addition, another routine calculation shows that in $G_n \setminus (a,b)$,

$$\Delta_{0,\infty} r(x,y) = 0.$$

Considering the domain $\Omega \equiv \{(x,y) \in G_n : 0 < r(x,y) < 1\}$, these facts make it clear that the unique (viscosity) solution to

$$\begin{cases} \Delta_{0,\infty} u = 0 & \text{in } \Omega, \\ u = r & \text{on } \partial\Omega \end{cases} \tag{6.1}$$

is the function $r(x,y)$ itself.

In addition to verification by computation, we can also use the limiting technique from [5] to show that $r(x,y)$ is the unique viscosity solution to equation (6.1). We begin first with the function $u_q \colon G_n \mapsto \mathbb{R}$ defined by

$$u_q(x,y) = (r(x,y))^{(2+n-q)/(1-q)}.$$

Using computations as in [14] and [15] we can show that u_2 is the fundamental solution to the Laplacian with singularity at (a,b). In particular, u_2 satisfies

$$\begin{cases} \Delta_{0,2} u = 0 & \text{in } \Omega, \\ u = r & \text{on } \partial\Omega. \end{cases}$$

The key to the proof of this fact is the use of "polar coordinates" in G_n (see [14], [15]). (For more recent results on polar coordinates in Carnot groups see [2].) A routine calculation as in [14] or [15] shows that if $\tau > 0$ and $\alpha = (2+n)(q-1)/(2+n-q)$, then $u_q \in L^{\alpha-\tau}_{\text{loc}}(G_n)$ but $u_q \notin L^{\alpha}_{\text{loc}}(G_n)$. Adapting Proposition 2.16 of [2], we obtain that $k_q u_q$ is the fundamental solution to the q-Laplacian for the appropriate constant k_q. In particular, u_q satisfies

$$\begin{cases} \Delta_{0,q} u = 0 & \text{in } \Omega, \\ u = r & \text{on } \partial\Omega, \end{cases}$$

so that formally taking $q \to \infty$ produces the desired result.

Concerning the fundamental solution, it should be noted that these results are true only when the singularity occurs at points (a,b). The results fail when the singularity is translated to points (z,b) for $z \neq a$, as an easy calculation shows that the function $\tilde{r} \colon G_n \mapsto \mathbb{R}$

$$\tilde{r}(x,y) = \left(c^2(x-z)^{2n+2} + (n+1)^2(y-b)^2\right)^{1/(2n+2)}$$

is not infinite harmonic when $z \neq a$ and

$$\Delta_{0,q} \tilde{r}(x,y)^{(2+n-q)/(1-q)} \neq 0$$

on $G_n \setminus (z,b)$. In fact, the Green's function at these points is much more complicated. The complete formulation and related Green's functions can be found in [3].

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