Illinois Journal of Mathematics
Volume 46, Number 3, Fall 2002, Pages 885–891
S 0019-2082

ON LOCALLY FINITE GROUPS IN WHICH EVERY ELEMENT HAS PRIME POWER ORDER

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ABSTRACT. A group is called a CP-group if every element of the group has prime power order. The complete classification of locally finite CP-groups is given in this article.

1. Introduction

DEFINITION. A group is called a CP-group if every element of the group has prime power order.

This definition is equivalent to the statement that the centralizer of every nontrivial element is a p-group, for some prime p which depends on the element. This is a generalization of groups of prime power order. Examples of CP-groups include p-groups, where p is a prime, and Tarski groups, which are simple groups whose proper subgroups have prime order. This shows how complicated the structure of infinite CP-groups can be.

Finite CP-groups were first studied by Higman [3] in 1957. He showed that a finite solvable CP-group is a split extension of its Fitting subgroup, which must clearly be a p-group, by a complement acting fixed-point-freely. Moreover, the order of a finite solvable CP-group is divisible by at most two primes. In the same article, Higman studied the structure of finite insolvable CP-groups and showed that such a group has a non-abelian simple section which largely determines its structure. Suzuki classified finite simple CP-groups in his celebrated work [7], finding that only eight finite simple CP-groups exist. Brandl continued this line of inquiry by classifying finite insolvable CP-groups in [2], but his work contained flaws. Finally, Bannuscher and Tiedt gave the complete classification of finite CP-groups in [1].

We can visualize this type of group by means of a graph as follows. The *prime graph* of a group G is the graph having the prime divisors of the orders of the elements of G as vertices and an edge between two vertices p and q if

Received January 10, 2002; received in final form April 18, 2002.
2000 *Mathematics Subject Classification.* 20F50.

G has an element of order pq. Then a group is a CP-group if and only if its prime graph is totally disconnected.

NOTATION. Throughout the paper, p and q are two distinct primes, $O_p(G)$ is the maximal normal p-subgroup of a group G, and $\pi(G)$ denotes the set of primes dividing orders of the elements of G.

We can now state our main result.

MAIN THEOREM. *Let G be a locally finite group. Then G is a CP-group with Fitting subgroup P if and only if one of the following holds:*

(1) *$G = P$, i.e., G is a p-group;*
(2) *$G = Q \ltimes P$ where Q acts on P fixed-point-freely and Q is either a subgroup of a locally quaternion group or of $\mathbb{Z}_{q^\infty}$ where $p \neq q$;*
(3) *$G = (H \ltimes Q) \ltimes P$ where H acts fixed-point-freely on Q, and Q acts fixed-point-freely on P; also HP is a Sylow p-subgroup of G, Q is a subgroup of $\mathbb{Z}_{q^\infty}$, and H is finite cyclic, where $p \mid q-1$;*
(4) *G is finite almost simple and is isomorphic to* $\mathrm{PSL}(2,q)$ $(q = 4, 7, 8, 9, 17)$, $\mathrm{PSL}(3,4)$, $\mathrm{Sz}(8)$, $\mathrm{Sz}(32)$, *or* M_{10};
(5) *$P = O_2(G) \neq 1$ and G/P is isomorphic to* $\mathrm{PSL}(2,4)$, $\mathrm{PSL}(2,8)$, $\mathrm{Sz}(8)$, *or* $\mathrm{Sz}(32)$. *Moreover, P is isomorphic to a direct sum of natural modules for G/P.*

2. Finite CP-groups

It is obvious that any subgroup of a CP-group is also a CP-group. It is only slightly less obvious that a factor group of a locally finite CP-group is a CP-group, since an element mapping to an element of non-prime power order would generate a cyclic group of non-prime power order. Therefore any section of a locally finite CP-group is also a CP-group.

THEOREM 1 ([3]). *Suppose G is a finite solvable CP-group with $O_p(G) = P \neq 1$. Then G has one of the following structures:*

(1) *G is a p-group;*
(2) *$G = Q \ltimes P$ where Q acts on P fixed-point-freely and Q is either generalized quaternion or cyclic;*
(3) *$G = (H \ltimes Q) \ltimes P$ where H acts fixed-point-freely on Q, and Q acts fixed-point-freely on P; also HP is a Sylow p-subgroup of G, and H and Q are cyclic.*

In each case, $|\pi(G)| \leq 2$.

NOTATION. A group as in (1) will be called a 1-*step group*; a group as in (2), a 2-*step group*; and a group as in (3), a 3-*step group*.

THEOREM 2 ([3]). *Let G be a finite insolvable CP-group. Then G has a normal series $G \geq N > P = O_p(G) \geq 1$, where*

(1) *G/N is cyclic or generalized quaternion, and, in fact, cyclic if $P > 1$;*
(2) *N/P is the unique minimal normal subgroup of G/P, N/P is nonabelian simple, and when $P > 1$, p divides $|N/P|$.*

THEOREM 3 ([7]). *A nonabelian simple CP-group is isomorphic to* $\mathrm{PSL}(2,q)$ $(q = 4,7,8,9,17)$, $\mathrm{PSL}(3,4)$, $\mathrm{Sz}(8)$, *or* $\mathrm{Sz}(32)$.

THEOREM 4 ([1]). *A group G is a finite CP-group if and only if one of the following holds:*

(1) *G is a* 1*-step group;*
(2) *G is a* 2*-step group;*
(3) *G is a* 3*-step group;*
(4) *G is isomorphic to* $\mathrm{PSL}(2,q)$ $(q = 4,7,8,9,17)$, $\mathrm{PSL}(3,4)$, $\mathrm{Sz}(8)$, $\mathrm{Sz}(32)$, *or* M_{10}*;*
(5) *$G/O_2(G)$ is isomorphic to* $\mathrm{PSL}(2,4)$, $\mathrm{PSL}(2,8)$, $\mathrm{Sz}(8)$, *or* $\mathrm{Sz}(32)$. *Moreover, $O_2(G)$ is isomorphic to a direct sum of natural modules for $G/O_2(G)$.*

3. Locally finite CP-groups

First of all, we show that there are no infinite locally finite simple CP-groups.

THEOREM 5. *Let G be a locally finite simple CP-group. Then G is finite.*

Proof. First, assume that G is countably infinite. Then by [5, 4.5], G is the union of a strictly ascending sequence $\{R_n : n \in \mathbb{N}\}$ of finite subgroups satisfying the following property: For each n there is a maximal normal subgroup M_{n+1} of R_{n+1} satisfying $M_{n+1} \cap R_n = 1$. Thus $R_n \simeq M_{n+1}R_n/M_{n+1}$, and so R_n is isomorphic to a subgroup of the simple group R_{n+1}/M_{n+1}.

If R_{n+1} is solvable for some n, then R_{n+1}/M_{n+1} has prime order and so does R_n. Thus the only possible solvable subgroups in $\{R_n\}$ are R_1 and R_2. Discarding these solvable subgroups from the set $\{R_n\}$, if necessary, we may assume that all R_n's are insolvable. Since R_n is isomorphic to a subgroup of a finite simple CP-group and there are only finitely many finite simple CP-groups (see Theorem 3), $\{R_n\}$ is a finite set and G is finite simple.

If G is not countable, then by [5, 4.4], G has a local system of countably infinite simple subgroups. This, however, was just shown to be impossible. □

HYPOTHESIS. From now until our main result, Theorem 10, we assume that G is an infinite locally finite CP-group.

We need to introduce the following group.

Definition. A group is called *locally quaternion* if it has a presentation

$$\langle X, y \mid X \simeq \mathbb{Z}_{2^\infty},\ x^y = x^{-1} \text{ for every } x \in X, \text{ and } y^2 \text{ is the involution of } X\rangle$$

In order to show our main result, we also need the following lemmas.

Notation. For $k = 1, 2, 3$, let $\mathbb{F}_k$ be the set of finite subgroups of G which are k-step groups, and let $\mathbb{F}_4$ be the set of finite insolvable subgroups of G with a non-trivial Fitting subgroup. Let $1 \leq k(G) \leq 4$ be maximal subject to $\mathbb{F}_{k(G)} \neq \emptyset$.

Note that $k(G)$ is well-defined by Theorem 4.

Lemma 6. *Suppose G is locally solvable and let $k = k(G)$. Let H_1 and H_2 be in $\mathbb{F}_k$. Then we have:*

(a) $\pi(\operatorname{Fit}(H_i)) = \pi(\operatorname{Fit}(\langle H_1, H_2\rangle)),\ i = 1, 2$;
(b) $\operatorname{Fit}(H_i) \leq \operatorname{Fit}(\langle H_1, H_2\rangle),\ i = 1, 2$.

Proof. First, note that $k \in \{1, 2, 3\}$ since G is locally solvable. Let $K = \langle H_1, H_2\rangle$. Then $K \in \mathbb{F}_k$. If $k = 1$, the claims are obvious. Put $\overline{K} = K/\operatorname{Fit}(K)$. Then $\overline{K}$ is a $(k-1)$-step group. If the claims do not hold, then $\overline{H_i} = H_i \operatorname{Fit}(K)/\operatorname{Fit}(K)$ is a k-step subgroup of $\overline{K}$, which is impossible. □

Recall that a group X is *almost simple* if $S \subseteq X \subseteq \operatorname{Aut}(S)$, for some simple group S.

We have a result identical to that of the previous lemma in the case that G is not locally solvable and $k = 4$.

Lemma 7. *Suppose G is not locally solvable. Then $\mathbb{F}_4 \neq \emptyset$, and for H_1 and H_2 in $\mathbb{F}_4$ we have:*

(a) $\pi(\operatorname{Fit}(H_i)) = \pi(\operatorname{Fit}(\langle H_1, H_2\rangle)),\ i = 1, 2$;
(b) $\operatorname{Fit}(H_i) \leq \operatorname{Fit}(\langle H_1, H_2\rangle),\ i = 1, 2$.

Proof. Since there are only finitely many types of finite CP-groups, $\mathbb{F}_4 \neq \emptyset$. Put $K = \langle H_1, H_2\rangle$. If K is not in $\mathbb{F}_4$, then K is almost simple and parts (4) and (5) of Theorem 4 show that $K \simeq \mathrm{PSL}(3, 4)$ and $H_1 \simeq H_2 \simeq 2^4 \cdot A_5$. This means that $G \geq \mathrm{PSL}(3, 4)$ and so $G = \mathrm{PSL}(3, 4)$ by Theorem 4 and Theorem 5, which is impossible.

Assume that $K \in \mathbb{F}_4$ and put $\overline{K} = K/\operatorname{Fit}(K)$. So $\overline{K}$ is almost simple. As $\operatorname{Fit}(K) \cap H_i \leq \operatorname{Fit}(H_i)$, we see that $H_i/(H_i \cap \operatorname{Fit}(K))$ is almost simple. Thus $\operatorname{Fit}(H_i) = \operatorname{Fit}(K) \cap H_i$ and the claims hold. □

Lemma 8. $\operatorname{Fit}(G) = O_p(G) \neq 1$ *for some unique prime p.*

Proof. Let $k = k(G)$, $H \in \mathbb{F}_k$, and $x \in G$. Let $K = \langle H, H^x \rangle$. By Lemmas 6 and 7, we have that $\mathrm{Fit}(H) \leq \mathrm{Fit}(K) = O_p(K)$ for some prime p. Thus $\langle O_p(H^y); y \in G \rangle$ is a p-group and $1 \neq O_p(H^G) \leq O_p(G)$, where H^G denotes the normal closure of H in G. □

Theorem 9. *Let G be a locally finite group and let G have a normal series $G \geq M > N > 1$. If the centralizer $C_M(n)$ lies in N for each nontrivial element n of N, then G splits over N.*

Proof. Clearly M is a locally finite Frobenius group with complement K, say, and kernel N. As all complements to N in M are conjugate in M, by [5, 1.J.2], a Frattini argument gives that $N_G(K)$ is a complement to N. □

Now we are able to prove our main result.

Proof of Main Theorem. By Theorem 4, we may assume that G is an infinite locally finite CP-group, and we put $k = k(G)$.

If $k = 1$, then (1) holds obviously.

If $k = 2$, then, for every $H \in \mathbb{F}_2$, we have $O_p(H) \leq P \neq 1$, by Lemmas 6 and 8. Since $O_p(H)$ is the Sylow p-subgroup of H, elements of G not in P have order relatively prime to p. Thus elements of $G \backslash P$ act fixed-point-freely by conjugation on P. By Theorem 9, G splits over P and we may write $G = Q \ltimes P$. Put $\overline{G} = G/P$. Then $\overline{H}$ is isomorphic to either $\mathbb{Z}_{q^n}$ or to a generalized quaternion group. Therefore, Q is a subgroup of either $\mathbb{Z}_{q^\infty}$ or of a locally quaternion group.

If $k = 3$, then $P \neq 1$ by Lemma 6. Put $\overline{G} = G/P$, and $\overline{k} = k(\overline{G})$. Then $\overline{k} = 2$. Let $\mathbb{F}_{\overline{k}} = \{\overline{H} \leq \overline{G} \mid H \in \mathbb{F}_3\}$. By the result of the previous paragraph, $O_q(\overline{G}) \neq 1$, for some prime $q \neq p$, $\overline{G}$ splits over $O_q(\overline{G})$ and any complement to $O_q(\overline{G})$ acts fixed-point-freely on $O_q(\overline{G})$. Now G has a normal series $G > O_{p,q}(G) > P > 1$ and elements of $O_{p,q}(G)$ not in P act fixed-point-freely by conjugation on P. Therefore, by Theorem 9, G splits over P and G is a 3-step group.

Write $G = (H \ltimes Q) \ltimes P$. Since H acts fixed-point-freely on Q, and Q acts fixed-point-freely on P, it follows that Q and H are either subgroups of a locally quaternion group or a subgroup of $\mathbb{Z}_{q^\infty}$ and $\mathbb{Z}_{p^\infty}$, respectively. Moreover, if $\overline{Q}$ is locally quaternion, it has a characteristic subgroup of order 2, and so $\overline{H} = 1$; in the other case, $\overline{H}$ is a subgroup of the automorphism group of $\mathbb{Z}_{q^\infty}$ and hence is finite cyclic of order p^n.

If $k = 4$ and $H \in \mathbb{F}_4$, then $O_2(H) \leq \mathrm{Fit}(G)$ and so $p = 2$. By Lemma 7, $O_2(\overline{H}) = 1$ and $\overline{H}$ is isomorphic to $\mathrm{PSL}(2,4)$, $\mathrm{PSL}(2,8)$, $\mathrm{Sz}(8)$, or $\mathrm{Sz}(32)$. None of these simple groups contains any of the other ones, so $\overline{G} = \overline{H}$. By the results of Higman [4] and Martineau [6], (5) holds.

Conversely, assume that a group in the theorem has an element of order pq. Then the finite subgroup generated by that element must be contained in a finite group listed in Theorem 4, which is impossible. $\square$

4. Examples of infinite locally finite solvable CP-groups

EXAMPLE 1. Let p be an odd prime and V a 2-dimensional vector space over an infinite locally finite field F of characteristic p. Then a Sylow 2-subgroup, Q, of $\mathrm{SL}(2,F)$ is locally quaternion and acts fixed-point-freely on V. Thus $G = Q \ltimes V$ is a locally finite 2-step CP-group.

EXAMPLE 2. Let F be the locally finite field which is the direct limit of finite fields of order $2^{2\cdot 3^{k-1}}$ for all $k \geq 1$. By induction, it is easy to see that 3^k divides $2^{2\cdot 3^{k-1}} - 1$. Thus there is a subgroup H of F^* isomorphic to $\mathbb{Z}_{3^\infty}$. Let

$$H^0 = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in H \right\} \text{ and } z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $H^0 \simeq H$ and z inverts under conjugation each element of H^0. If V is a 2-dimensional vector space over F, then $G = (\langle z \rangle \ltimes H^0) \ltimes V$ is a locally finite 3-step CP-group.

It is worthwhile mentioning that the class of locally solvable CP-groups is contained in that of locally finite CP-groups since CP-groups are torsion. Moreover, it is known that a torsion group G has a unique maximal normal locally solvable subgroup R such that G/R has no non-trivial normal locally solvable subgroups (see [8]). R is called the locally solvable radical of G and G/R is said to be *locally solvably semisimple*. For instance, Tarski groups are locally solvably semisimple CP-groups. The structure of infinite locally solvably semisimple CP-groups remains to be settled.

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