Illinois Journal of Mathematics
Volume 46, Number 3, Fall 2002, Pages 837–839
S 0019-2082

ON FINITE APPROXIMATIONS OF GROUPS AND ALGEBRAS

MILOŠ ZIMAN

ABSTRACT. The notions of an LEF group and of an LEF algebra, introduced by Gordon and Vershik in [GV], have been formulated in terms of local embeddability into finite groups or finite dimensional algebras, respectively. We will prove that the group algebra $\mathbb{C}G$ of a group G is LEF if and only if G is. This solves a question raised in [GV].

In their paper [GV] E. I. Gordon and A. M. Vershik introduced the notion of a group locally embeddable into the class of finite groups (briefly, an LEF group), and the notion of an algebra locally embeddable into the class of finite dimensional algebras (briefly, an LEF algebra). (An algebra always means an associative linear algebra over the field $\mathbb{C}$.) This establishes the foundation of a new approach to approximations of groups by finite groups and approximations of algebras by finite dimensional algebras.

The following question was formulated in [GV]: *Let the group algebra of a countable group G be an LEF algebra. Is it true that G itself is an LEF group?*

In a subsequent paper [AGG], M. A. Alekseev, L. Yu. Glebskii and E. I. Gordon investigated, among other things, some connections between approximations of groups and Hopf algebras. In particular, they proved that the cocommutative Hopf algebra of a group G is approximable as a bialgebra by finite dimensional bialgebras if and only if G is an LEF group.

In this note we answer the original question by Gordon and Vershik affirmatively, even without the assumption that G be countable.

The following definition, which is better suited for our purpose, is easily seen to be equivalent to the original one from [GV].

DEFINITION 1. A group G is said to be *locally embeddable into the class of finite groups*, i.e., G is an *LEF group*, if for every finite subset $M \subseteq G$ there

Received December 6, 2001; received in final form February 25, 2002.

2000 *Mathematics Subject Classification.* Primary 20E25, 16U60. Secondary 16S34, 40L05.

Supported by the VEGA grant 1/7070/20.

exists a finite group H and an injective map $\psi : M \cup M^2 \to H$ such that for all $x, y \in M$ we have

$$\psi(x)\psi(y) = \psi(xy).$$

We define the corresponding notion for algebras similarly. For an algebra with a countable Hamel basis this definition coincides with that from [GV].

Definition 2. An algebra A is said to be *locally embeddable into the class of finite dimensional algebras*, i.e., A is an *LEF algebra*, if for every finite subset $M \subseteq A$ there exists a finite dimensional algebra B and an injective linear map $\psi : [M \cup M^2] \to B$ such that for all $x, y \in M$ we have

$$\psi(x)\psi(y) = \psi(xy),$$

where $[X]$ denotes the linear span of a set $X \subseteq A$.

Proposition 1. *Let A be an algebra. If A is an LEF algebra then the group of its invertible elements $G(A)$ is an LEF group.*

Proof. Let $M \subseteq G(A) \subseteq A$ be a finite subset; we can assume that $M = M^{-1}$. As A is an LEF algebra, there is a finite dimensional algebra B and a map $\psi : [M \cup M^2] \to B$ such that $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in M$.

The elements of B can be considered as linear operators on the finite dimensional vector space B. Thus B is a matrix algebra, and $G(B)$ is a matrix group. As Gordon and Vershik point out in [GV, Corollary 1.2], matrix groups, being locally residually finite, are LEF groups. As $\psi(M) \subseteq G(B)$, $G(A)$ is an LEF group as well. □

Corollary 1. *Let A be an algebra and G be a subgroup of $G(A)$. If A is an LEF algebra then G is an LEF group.*

Theorem 1. *Let $A = \mathbb{C}G$ be the group algebra of a group G. Then A is an LEF algebra if and only if G is an LEF group.*

Proof. If A is an LEF algebra then G is an LEF group by Corollary 1. So it remains to prove the reverse implication.

Let G be an LEF group and let $M \subseteq A$ be a finite set. Then there is a finite linearly independent set $L \subseteq G$ such that $M \cup M^2 \subseteq [L]$, and an injective map $\psi : L \cup L^2 \to H$ into a finite group H such that $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in L$.

Let $B = \mathbb{C}H$ denote the group algebra of H. As $L \cup L^2 \subseteq G$, it is a linearly independent subset of A. Hence the map ψ can be extended to an injective linear map $\lambda : [L \cup L^2] \to B$. Now it suffices to take the restriction $\varphi = \lambda \restriction [M \cup M^2] : [M \cup M^2] \to B$, which is an injective linear map satisfying $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in M$. □

REMARK. In the same vein an analogous result can be proved for the group algebra KG of a group G over an arbitrary field K.

The question under which conditions the implication of Corollary 1 can be reversed remains open. In view of Theorem 1 let us formulate the following conjecture.

CONJECTURE. *Let A be an algebra and G be a subgroup of $G(A)$ which generates A as a vector space. Then A is an LEF algebra if and only if G is an LEF group.*

REFERENCES

[AGG] M. A. Alekseev, L. Yu. Glebsky, and E. I. Gordon, *On approximations of groups, group actions and Hopf algebras*, J. Math. Sci. **107** (2001), 4305–4332.

[GV] E. I. Gordon and A. M. Vershik, *Groups that are locally embeddable in the class of finite groups*, St. Petersburg Math. J. **9** (1998), 49–67.

[S] S. Sakai, *C^*-algebras and W^*-algebras*, Springer-Verlag, New York, 1971.

DEPARTMENT OF ALGEBRA AND NUMBER THEORY, FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS, COMENIUS UNIVERSITY, 842 48 BRATISLAVA, SLOVAKIA

E-mail address: ziman@fmph.uniba.sk