Illinois Journal of Mathematics
Volume 46, Number 3, Fall 2002, Pages 755–780
S 0019-2082

A NEW DECOMPOSITION THEOREM FOR 3-MANIFOLDS

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Abstract. Let M be a (possibly non-orientable) compact 3-manifold with (possibly empty) boundary consisting of tori and Klein bottles. Let $X \subset \partial M$ be a trivalent graph such that $\partial M \setminus X$ is a union of one disc for each component of ∂M. Building on previous work of Matveev, we define for the pair (M, X) a complexity $c(M, X) \in \mathbb{N}$ and show that, when M is closed, irreducible and $\mathbb{P}^2$-irreducible, $c(M, \emptyset)$ is the minimal number of tetrahedra in a triangulation of M. Moreover c is additive under connected sum, and, given any $n \geqslant 0$, there are only finitely many irreducible and $\mathbb{P}^2$-irreducible closed manifolds having complexity up to n. We prove that every irreducible and $\mathbb{P}^2$-irreducible pair (M, X) has a finite splitting along tori and Klein bottles into pairs having the same properties, and complexity is additive on this splitting. As opposed to the JSJ decomposition, our splitting is not canonical, but it involves much easier blocks than all Seifert and simple manifolds. In particular, most Seifert and hyperbolic manifolds appear to have non-trivial splitting. In addition, a given set of blocks can be combined to give only a finite number of pairs (M, X). Our splitting theorem provides the theoretical background for an algorithm which classifies 3-manifolds of any given complexity. This algorithm has been already implemented and proved effective in the orientable case for complexity up to 9.

We develop in this paper a theory of complexity for pairs (M, X), where M is a compact 3-manifold such that $\chi(M) = 0$, and X is a collection of trivalent graphs, each graph τ being embedded in one component C of ∂M so that $C \setminus \tau$ is one disc. In the special case where M is closed, so that $X = \emptyset$, our complexity coincides with that introduced by Matveev [6]. Extending his results we show that complexity of pairs is additive under connected sum and that, when M is closed, irreducible, $\mathbb{P}^2$-irreducible and different from $S^3, L_{3,1}, \mathbb{P}^3$, its complexity is precisely the minimal number of tetrahedra in a triangulation. These two facts show that complexity is indeed a very natural measure of how complicated a manifold or pair is. The former fact was known to Matveev in the closed case, the latter in the orientable case.

Received October 2, 2001; received in final form November 11, 2002.
2000 *Mathematics Subject Classification.* Primary 57M50. Secondary 57M25.

The most relevant feature of our theory is that it leads to a splitting theorem along tori and Klein bottles for irreducible and $\mathbb{P}^2$-irreducible pairs (so, in particular, for irreducible and $\mathbb{P}^2$-irreducible closed manifolds). The blocks of the splitting are themselves pairs, and the complexity of the original pair is the sum of the complexities of the blocks. Recalling that in [6] a complexity $c(M)$ was defined also for the case when $\partial M \neq \emptyset$, we emphasize here that our complexity $c(M, X)$ is typically different from $c(M)$. So the splitting theorem crucially depends on the extension of c from manifolds to pairs.

Our splitting differs from the JSJ decomposition [2][3] since it is not unique (see below for a further discussion of this point), but it has the great advantage that the blocks it involves, which we call *bricks*, are much easier than all Seifert and simple manifolds. As a matter of fact, our splitting is non-trivial on almost all Seifert and hyperbolic manifolds it has been tested on. Another advantage is that the graphs in the boundary reduce the flexibility of possible gluings of bricks. As a consequence, a given set of bricks can only be combined in a finite number of ways. This property is of course crucial for computations, and our theory actually leads to very effective algorithms for the enumeration of closed manifolds having small complexity.

Returning to the relation between our splitting and the JSJ decomposition, we mention that all the bricks found so far [4] are geometrically atoroidal, which suggests that our splitting is actually always a refinement of the JSJ decomposition (and we know it is in the orientable case for complexity up to 9; see [4]). Moreover, non-uniqueness for a Seifert manifold typically corresponds to non-uniqueness of its realization as a graph-manifold. We have also found some non-uniqueness instances in the hyperbolic case [5].

The orientable version of the theory developed in this paper, culminating in the splitting theorem, was established in [4]. In the same paper we have proved several strong restrictions on the topology of bricks and, using a computer program, we have been able to classify all orientable bricks of complexity up to 9. Using the bricks we have then listed all closed irreducible orientable 3-manifolds up to complexity 9, showing in particular that the only four hyperbolic ones are precisely those of least known volume. The splitting theorem proved below is the main theoretical tool needed to extend our program of enumerating 3-manifolds of small complexity from the orientable to the general case. We are planning to realize this program in the close future. This will allow us to provide information on the smallest non-orientable hyperbolic manifolds and on the density, in each given complexity, of orientable manifolds among all 3-manifolds.

We have decided to devote the present paper to the general theory and the splitting theorem, leaving computer implementation for a subsequent paper, because the non-orientable case displays certain remarkable phenomena which do not appear in the orientable case. To begin with, toric boundary components restrict the shape of the trivalent graph they contain to only one possibility, while Klein bottles allow two. Next, the assumption of $\mathbb{P}^2$-irreducibility has to be added to irreducibility to get the finiteness of closed manifolds of a given complexity. More surprisingly, these assumptions do not suffice when a non-empty boundary is allowed, because the drilling of a boundary-parallel orientation-reversing loop never changes complexity. Because of these facts, the intrinsic definition of *brick* given below is somewhat subtler than that given in [4], and the proof of some of the key results (including additivity under connected sum) is considerably harder.

1. Manifolds with marked boundary

If C is a connected surface, we call *spine* of C a trivalent graph τ embedded in C in such a way that $C \setminus \tau$ is an open disc. If C is disconnected then a spine of C is a collection of spines for all its components. We denote by $\mathcal{X}$ the set of all pairs (M, X), where M is a connected and compact 3-manifold with (possibly empty) boundary made of tori and Klein bottles, and X is a spine of ∂M. Elements of $\mathcal{X}$ will be viewed up to homeomorphism of pairs.

REMARK 1.1. Since S^2 and $\mathbb{P}^2$ do not admit spines with vertices, a pair (M, X) with X a spine of ∂M belongs to $\mathcal{X}$ if and only if $\chi(M) = 0$ and all the elements of X have vertices.

Spines of the torus T and the Klein bottle K. A spine of T or K must be a trivalent graph with two vertices, and there are precisely two such graphs, namely the θ-curve and the frame σ of a pair of spectacles. Both θ and σ can serve as spines of K, as shown in Fig. 1, left and center. The following result will be shown in the appendix:

PROPOSITION 1.2.

(1) *Every spine of K is isotopic to one of the two graphs shown in Fig. 1.*
(2) *With notation as in Fig. 1, for both $\tau = \theta$ and $\tau = \sigma$ there exists $f \in \mathrm{Aut}(K)$ such that $f(\tau) = \tau$, $f(e') = e''$, and $f(e'') = e'$, but for all $f \in \mathrm{Aut}(K)$ such that $f(\tau) = \tau$ we have $f(e''') = e'''$.*

The situation for T is completely different. First of all, σ is not a spine of T. In addition, θ can be used as a spine of T in infinitely many non-isotopic ways, because the position of θ on T is determined by the triple of loops on T which are contained in θ. Note that any two of these loops generate $H_1(T; \mathbb{Z})$, and any such triple determines one spine θ. However we have the following result, which we leave to the reader to prove using the facts just stated.

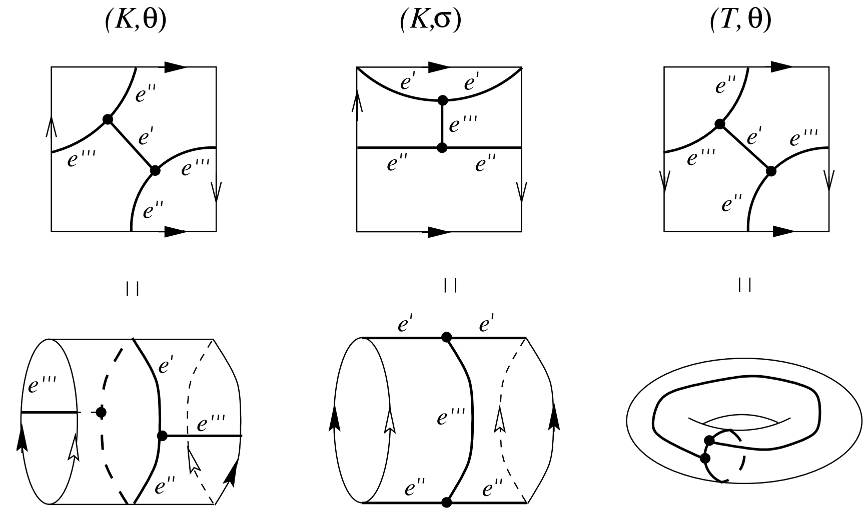

FIGURE 1. Spines of the Klein bottle and the torus.

PROPOSITION 1.3. *If θ is a spine of T then all automorphisms of θ are induced by automorphisms of T. If θ, θ' are spines of T there exists $f \in$* $\mathrm{Aut}(T)$ *such that $f(\theta) = \theta'$.*

Examples of pairs. If M is a closed 3-manifold then $(M, \emptyset)$ is an element of $\mathcal{X}$. For simplicity we will often write only M instead of $(M, \emptyset)$. We list here several more elements of $\mathcal{X}$ needed below. Our notation will be consistent with that of [4]. The reader is invited to use Propositions 1.2 and 1.3 to make sure that all the pairs we introduce are well-defined up to homeomorphism. We start with the product pairs:

$$\begin{aligned} B_0 &= (T \times [0,1], \{\theta \times \{0\}, \theta \times \{1\}\}), \\ B_0' &= (K \times [0,1], \{\theta \times \{0\}, \theta \times \{1\}\}), \\ B_0'' &= (K \times [0,1], \{\sigma \times \{0\}, \sigma \times \{1\}\}). \end{aligned}$$

We next have two pairs B_1 and B_2 based on the solid torus $\boldsymbol{T}$ and shown in Fig. 2-left, and two on the solid Klein bottle $\boldsymbol{K}$, namely $B_1' = (\boldsymbol{K}, \theta)$ and $B_2' = (\boldsymbol{K}, \sigma)$.

For $k \geqslant 1$ we take now the 2-orbifold D^2 with k mirror segments on ∂D^2 and we define $Z_k \in \mathcal{X}$ as the Seifert manifold without singular fibers over this orbifold [10], with a spine σ in each of the k Klein bottles on the boundary. Note that Z_k can also be viewed as the complement of k disjoint orientation-reversing loops in $S^2 \mathbin{\tilde{\times}} S^1$. Another description of Z_k is given in Fig. 2-right.

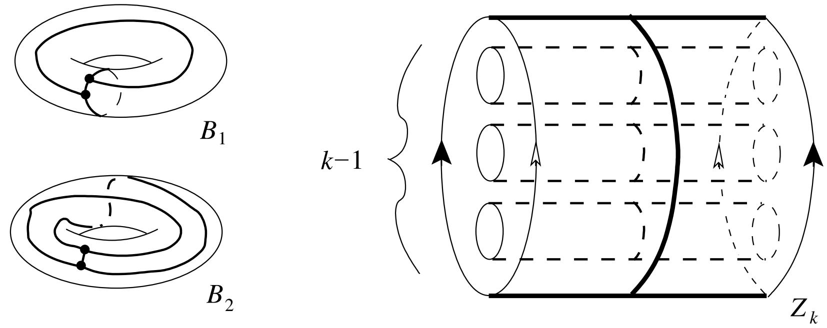

FIGURE 2. The pairs B_1, B_2, and Z_k for $k \geqslant 1$.

We also note that $Z_1 = B_2'$ and $Z_2 = B_0''$. We define now B_2'' to be Z_3, for a specific reason explained below.

We will now introduce three operations on pairs which allow us to construct new pairs from given ones. The ultimate goal is to show that all manifolds can be constructed via these operations using only certain building blocks.

Connected sum of pairs. The operation of connected sum obviously extends from manifolds to pairs. Namely, given (M, X) and (M', X') in $\mathcal{X}$, we define $(M, X)\#(M', X')$ as $(M\#M', X \cup X')$, where $M\#M'$ is one of the two possible connected sums of M and M'. Of course $S^3 = (S^3, \emptyset) \in \mathcal{X}$ is the identity element for operation $\#$. It is now natural to define (M, X) to be *prime* or *irreducible* if M is. Of course the only prime non-irreducible pairs are $S^2 \times S^1$ and $S^2 \mathbin{\widetilde{\times}} S^1$.

Assembling of pairs. Given (M, X) and (M', X') in $\mathcal{X}$, we pick spines $\tau \in X$ and $\tau' \in X'$ with $\tau \subset C \subset \partial M$ and $\tau' \subset C' \subset \partial M'$. If there is a homeomorphism $\psi : C \to C'$ such that $\psi(\tau) = \tau'$ we can construct the pair $(N, Y) = (M \cup_\psi M', (X \cup X') \setminus \{\tau, \tau'\})$. We call this an *assembling* of (M, X) and (M', X') and we write $(N, Y) = (M, X) \oplus (M', X')$. Of course two given elements of $\mathcal{X}$ can only be assembled in a finite number of inequivalent ways.

Considering the pairs B_i^* and Z_k introduced above, the reader may easily check as an exercise that $Z_k \oplus Z_h = Z_{h+k-2}$ and that the following holds:

REMARK 1.4.

1. $(M, X) \oplus B_0^* = (M, X)$ for any $(M, X) \in \mathcal{X}$.
2. For (i, j) equal to $(1, 1)$, $(1, 2)$, or $(2, 1)$, it is possible to assemble B_i and B_j along a certain map ψ in order to get S^3. So, for any (M, X), if we assemble $(M, X)\#B_i$ to B_j along ψ, we get the original (M, X) as a result.

3. The assembling of B_2'' with B_2' gives B_0'', so $((M,X) \oplus B_2'') \oplus B_2' = (M,X)$ provided B_2' is assembled to one of the free boundary components of B_2''.

This shows that we can discard various assemblings without impairing our capacity of constructing new manifolds. So we will call *trivial* an assembling $(M,X) \oplus (M',X')$ if, up to interchanging (M,X) and (M',X'), one of the following holds:

(1) (M',X') is of type B_0^*.
(2) $(M',X') = B_j$ for $j \in \{1,2\}$ and (M,X) can be expressed as $(N,Y)\#B_i$ for $i \in \{1,2\}$ with $(N,Y) \neq S^3$ in such a way that the assembling is performed along the boundary of B_i and $B_i \oplus B_j = S^3$.
(3) $(M',X') = B_2'$ and $(M,X) = (N,Y) \oplus B_2''$ with B_2' being assembled to B_2''.

Self-assembling. Given $(M,X) \in \mathcal{X}$, we pick two distinct $\tau, \tau' \in X$ with $\tau \subset C$ and $\tau' \subset C'$. If there is a homeomorphism $\psi : C \to C'$ we can choose one such that $\psi(\tau)$ and τ' intersect transversely in two points, and we define (N,Y) as $(M_\psi, X \setminus \{\tau,\tau'\})$. We call this a *self-assembling* of (M,X) and we write $(N,Y) = \odot(M,X)$. As above, only a finite number of self-assemblings of a given element of $\mathcal{X}$ are possible.

In the sequel it will be convenient to refer to a combination of assemblings and self-assemblings of pairs just as an *assembling*. Note that of course we can first do the assemblings and then the self-assemblings.

2. Complexity, bricks, and the decomposition theorem

In the following sections we will introduce and discuss a certain function $c : \mathcal{X} \to \mathbb{N}$ which we call *complexity*. In the present section we anticipate the definition of c very briefly and state several results about this function, which could also be taken as axiomatic properties. Then we show how to deduce the splitting theorem from these properties only. Proofs of the properties are given in Sections 3–5.

Given $(M,X) \in \mathcal{X}$ we denote by $c(M,X)$ the minimal number of vertices of a polyhedron P embedded in M such that $P \cup \partial M$ is simple, $P \cap \partial M = X$, and the complement of $P \cup \partial M$ is an open 3-ball. Here 'simple' means that the link of every point embeds in the 1-skeleton of the tetrahedron, and a point of P is a 'vertex' if its link is precisely the 1-skeleton of the tetrahedron. We obviously have $c(M,\emptyset) = c(M)$ if M is a closed 3-manifold and $c(M)$ is Matveev's complexity [6]. Note that $c(M)$ is also defined in [6] for $\partial M \neq \emptyset$, but typically $c(M,X) \neq c(M)$.

Axiomatic properties. We start with three theorems which suggest to restrict the study of $c(M,X)$ to pairs (M,X) which are irreducible and $\mathbb{P}^2$-irreducible. Recall that M is called $\mathbb{P}^2$*-irreducible* if it does not contain any two-sided embedded projective plane $\mathbb{P}^2$ (see [1] for generalities about this notion, and in particular for the proof that a connected sum is $\mathbb{P}^2$-irreducible if and only if the individual summands are). When M is closed, we call *singular* a triangulation of M with multiple and self-adjacencies between tetrahedra. The first and second theorems, respectively, extend results of Matveev [6] from the closed to the marked-boundary case, and from the orientable to the possibly-non-orientable case. The extension is easy for the second theorem, but less so for the first theorem. The third theorem shows that the non-orientable theory is far richer than the orientable one.

THEOREM 2.1 (Additivity under #). *For any* (M, X) *and* (M', X') *we have*

$$c((M, X)\#(M', X')) = c(M, X) + c(M, X').$$

Moreover $c(S^2 \times S^1) = c(S^2 \mathbin{\tilde{\times}} S^1) = 0$.

THEOREM 2.2 (Naturality). *If* M *is closed, irreducible,* $\mathbb{P}^2$*-irreducible, and different from* S^3, $\mathbb{P}^3$, $L_{3,1}$, *then* $c(M) = c(M, \emptyset)$ *is the minimal number of tetrahedra in a singular triangulation of* M.

THEOREM 2.3 (Finiteness). *For all* $n \geqslant 0$ *the following holds:*

(1) *There exist finitely many irreducible and* $\mathbb{P}^2$*-irreducible pairs* (M, X) *such that* $c(M, X) = n$ *and* (M, X) *cannot be expressed as an assembling* $(N, Y) \oplus B_2''$.
(2) *If* $(N, Y) \in \mathcal{X}$ *is irreducible and* $\mathbb{P}^2$*-irreducible and* $c(N, Y) = n$ *then* (N, Y) *can be obtained from one of the pairs* (M, X) *described above by repeated assembling of copies of* B_2''. *Any such assembling has complexity* n.

The latter result is of course crucial for computational purposes. To better appreciate its "finiteness" content, note that whenever we assemble one copy of B_2'' the number of boundary components increases by one. Therefore the theorem implies that for all $n, k \geqslant 0$ the set

$$\mathcal{M}_{\leqslant n}^{\leqslant k} = \{(M, X) \in \mathcal{X} \text{ irred. and } \mathbb{P}^2\text{−irred., } c(M, X) \leqslant n,\ \#X \leqslant k\}$$

is finite. It should be emphasized that not only we can prove that $\mathcal{M}_{\leqslant n}^{\leqslant k}$ is finite, but the proof itself provides an explicit algorithm to produce a finite list of pairs from which $\mathcal{M}_{\leqslant n}^{\leqslant k}$ is obtained by removing duplicates. The theorem also implies that dropping the restriction $\#X \leqslant k$ we get infinitely many pairs, but only finitely many orientable ones. This fact, which is ultimately due to the existence of the Z_k series generated by B_2'' under assembling, is one of the key differences between the orientable and the general case (another important difference will arise in the proof of Theorem 2.1—see Proposition 5.2). Note also that an assembling with B_2'' geometrically corresponds to the drilling of a boundary-parallel orientation-reversing loop.

The following more specific version of the previous theorem for $n = 0$ is needed below.

PROPOSITION 2.4. *The only irreducible and* $\mathbb{P}^2$*-irreducible pairs having complexity* 0 *are* S^3*,* $L_{3,1}$*,* $\mathbb{P}^3$ *and all the pairs* B_i^* *and* Z_k *defined above.*

We turn now to the behaviour of complexity under assembling.

PROPOSITION 2.5 (Subadditivity). *For any* $(M,X),(M',X') \in \mathcal{X}$ *we have*

$$\begin{aligned} c((M,X)\oplus(M',X')) &\leqslant c(M,X)+c(M',X'),\\ c(\odot(M,X)) &\leqslant c(M,X)+6. \end{aligned}$$

We define now an assembling $(M,X)\oplus(M',X')$ to be *sharp* if it is nontrivial and $c((M,X)\oplus(M',X')) = c(M,X)+c(M',X')$. Similarly, a self-assembling $\odot(M,X)$ is *sharp* if $c(\odot(M,X)) = c(M,X)+6$. Proposition 2.5 readily implies the following facts.

REMARK 2.6.

1. If a combination of sharp (self-)assemblings is rearranged in a different order then it still consists of sharp (self-)assemblings.
2. Every assembling with B_2'' is sharp (unless it is trivial, which only happens when B_2'' is assembled to B_0'' or to B_2'). To see this, note again that $(M,X)\oplus B_2''\oplus B_2' = (M,X)$ and $c(B_2'') = c(B_2') = 0$.

THEOREM 2.7 (Sharp splitting). *Let* (N,Y) *be irreducible and* $\mathbb{P}^2$*-irreducible. If* (N,Y) *can be expressed as a sharp assembling* $(M,X)\oplus(M',X')$ *or as a self-assembling* $\odot(M'',X'')$ *then* (M,X)*,* (M',X')*, and* (M'',X'') *are irreducible and* $\mathbb{P}^2$*-irreducible.*

Proof. In both cases we are cutting N along a two-sided torus or Klein bottle, so $\mathbb{P}^2$-irreducibility is obvious. If $(N,Y) = \odot(M'',X'')$, this torus or Klein bottle is incompressible in N, and irreducibility of M'' is a general fact [1]. We are left to show that if $(N,Y) = (M,X)\oplus(M',X')$ sharply then M and M' are irreducible. Since these manifolds have boundary, it is enough to show that they are prime. Suppose they are not, and consider prime decompositions of (M,X) and (M',X') involving summands (M_i,X_i) and (M_j',X_j'). So one summand (M_i,X_i) is assembled to one (M_j',X_j'), and the other (M_i,X_i)'s and (M_j',X_j')'s survive in (N,Y). It follows that, up to permutation, (M,X) is prime, $(M',X') = (M_1',X_1')\#(M_2',X_2')$ with (M_1',X_1') and (M_2',X_2') prime, $(M,X)\oplus(M_1',X_1') = S^3$ and $(M_2',X_2') = (N,Y)$. Sharpness of the original assembling and additivity under # now imply that $c(M,X) = c(M_1',X_1') = 0$. So Proposition 2.4 applies to (M,X) and

(M'_1, X'_1). Knowing that $(M,X) \oplus (M'_1, X'_1) = S^3$ it is easy to deduce that (M,X) and (M'_1, X'_1) are either B_1 or B_2, and that the original assembling was a trivial one. This is a contradiction. $\square$

Bricks and decomposition. Taking the results stated above for granted, we define here the elementary building blocks and prove the decomposition theorem. Later we will comment on the actual relevance of this theorem.

A pair $(M,X) \in \mathcal{X}$ is called a *brick* if it is irreducible and $\mathbb{P}^2$-irreducible and cannot be expressed as a sharp assembling or self-assembling. Theorem 2.3 and Remark 2.6 easily imply that there are finitely many bricks of complexity n. From Proposition 2.4 it is easy to deduce that in complexity zero the only bricks are precisely the pairs B_i^* introduced above, which explains why we have given a special status to $Z_3 = B''_2$, and that the other irreducible and $\mathbb{P}^2$-irreducible pairs are assemblings of bricks. Now, we show more generally:

THEOREM 2.8 (Existence of splitting). *Every irreducible and $\mathbb{P}^2$-irreducible pair $(M,X) \in \mathcal{X}$ can be expressed as a sharp assembling of bricks.*

Proof. The result is true for $c(M,X) = 0$, so we proceed by induction on $c(M,X)$ and suppose $c(M,X) > 0$. By Theorem 2.3 we can assume that (M,X) cannot be split as $(N,Y) \oplus B''_2$, because every assembling with B''_2 is sharp, and we have seen that B''_2 is a brick. Now if (M,X) is a brick we are done. Otherwise (M,X) is either a sharp self-assembling $\odot(N,Y)$, in which case $c(N,Y) = c(M,X) - 6$ and we can conclude by induction using Theorem 2.7, or (M,X) is a sharp assembling $(N,Y) \oplus (N',Y')$. Theorem 2.7 states that (N,Y) and (N',Y') are irreducible and $\mathbb{P}^2$-irreducible. If both (N,Y) and (N',Y') have positive complexity we conclude by induction. Otherwise we can assume that $c(N',Y') = 0$ and apply Proposition 2.4. Since the assembling is non-trivial, (N',Y') is not of type B_0^*. It is also not B''_2 or Z_k for $k \geqslant 3$, by the property of (M,X) we are assuming. So (N',Y') is one of B_1, B'_1, B_2, B'_2. In particular, it is a brick.

Now we claim that (N,Y) cannot be split as $(N'',Y'') \oplus B''_2$. Assuming it can, we have two cases. In the first case the assembling of (N',Y') is performed along a free boundary component of B''_2, but then we must have $(N',Y') = B'_2$, and the assembling is trivial, which is absurd. In the second case (N',Y') is assembled to a free boundary component of (N'',Y''), and we have

$$(M,X) = \big((N'',Y'') \oplus (N',Y')\big) \oplus B''_2,$$

which is again absurd. Our claim is proved.

Now we know that (N,Y) again belongs to the finite list of irreducible and $\mathbb{P}^2$-irreducible pairs which have complexity n and cannot be split as an assembling with B''_2. However (N,Y) has one more boundary component than (M,X), which implies that by repeatedly applying this argument we must eventually end up with a brick. $\square$

Experimental facts. Theorem 2.8 shows that listing irreducible and $\mathbb{P}^2$-irreducible manifolds up to complexity n is easy once the bricks up to complexity n are classified. The finiteness features of our theory imply that there exists an algorithm which reduces such a classification to a recognition problem. Moreover it turns out experimentally that recognition needs to be carried out only on a comparatively short list of pairs. The complete list of *orientable* bricks up to complexity 9 was found in [4] (see also [11]), and it consists of 30 pairs, whereas there are 1901 closed, irreducible, and orientable 3-manifolds of complexity up to 9. As a matter of fact, only 7 bricks are already sufficient to obtain 1882 closed manifolds (the other 19 being themselves bricks). In addition, all bricks found are geometrically atoroidal, which makes it easy to recognize their assemblings; see also [5].

3. Skeleta

We introduce here the notion of *skeleton* of a pair (M, X), we define the complexity of (M, X) as the minimal number of vertices of a skeleton, and we discuss the first properties of minimal skeleta, deducing some of the results stated above. The other results require a deeper analysis and new techniques and will be proved later.

Simple skeleta and complexity. We recall that a compact polyhedron P is called *simple* if the link of every point of P embeds in the space given by a circle with three radii. The points having the whole of this space as a link are called *vertices*. They are isolated and therefore finite in number. Given a pair $(M, X) \in \mathcal{X}$, a polyhedron P embedded in M is called a *skeleton* of (M, X) if the following holds:

- $P \cup \partial M$ is simple;
- $M \setminus (P \cup \partial M)$ is an open ball;
- $P \cap \partial M = X$.

Remark 3.1. If P is a skeleton of (M, X) then P is simple, and the vertices of P cannot lie on ∂M. When $\#X = 1$ then P is a *spine* of M (i.e., M collapses onto P), and when $\#X = 0$ (i.e., when M is closed) then P is a spine of $M \setminus \{\text{point}\}$. When $\#X \geqslant 2$ no such interpretation is possible.

The proof that every $(M, X) \in \mathcal{X}$ has a skeleton, already given in [4, Remark 2.1], extends *verbatim* to the non-orientable context. For a simple polyhedron P we denote by $v(P)$ the number of vertices of P, and we define the *complexity* $c(M, X)$ of a given $(M, X) \in \mathcal{X}$ as the minimum of $v(P)$ over all skeleta P of (M, X). So we have a function $c : \mathcal{X} \to \mathbb{N}$.

Some skeleta without vertices. If we remove one point from the closed manifolds S^3, $L_{3,1}$, $\mathbb{P}^3$, $S^2 \times S^1$, and $S^2 \mathbin{\tilde{\times}} S^1$ we can collapse the result respectively to a point, to the "triple hat," to $\mathbb{P}^2$, and to the join of S^2 and

S^1 (for both the last two cases). Here the triple hat is the space obtained by attaching D^2 to S^1 so that ∂D^2 runs three times around S^1. This shows that S^3, $L_{3,1}$, $\mathbb{P}^3$, $S^2 \times S^1$, and $S^2 \mathbin{\tilde{\times}} S^1$ all have complexity zero. It is a well-known fact, which we will prove again below, that these are the only prime and $\mathbb{P}^2$-irreducible manifolds having complexity zero.

Turning to the pairs B_i^* and Z_k defined in the previous section, we now show that they also have complexity 0. This is rather obvious for the product pairs B_0, B_0', and B_0'', because they have the product skeleta $P_0 = \theta \times [0,1] \subset T \times [0,1]$, $P_0' = \theta \times [0,1] \subset K \times [0,1]$, and $P_0'' = \sigma \times [0,1] \subset K \times [0,1]$.

For $B_1 = (\boldsymbol{T}, \{\theta\})$ we note that θ contains a meridian of the torus, so we can attach to X a meridional disc and get the skeleton P_1 shown in Fig. 3. The same construction applies to $B_1' = (\boldsymbol{K}, \{\theta\})$ and leads to the skeleton P_1' also shown in the figure. Of course P_1 and P_1' are isomorphic as abstract polyhedra (just as P_0 and P_0' are), but we use different names to keep track also of their embeddings.

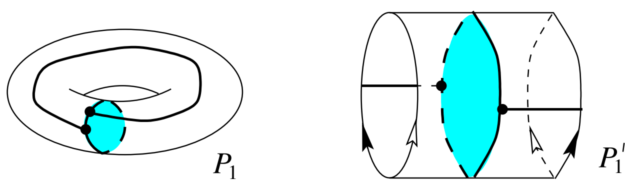

FIGURE 3. The skeleta P_1 and P_1' of B_1 and B_1'.

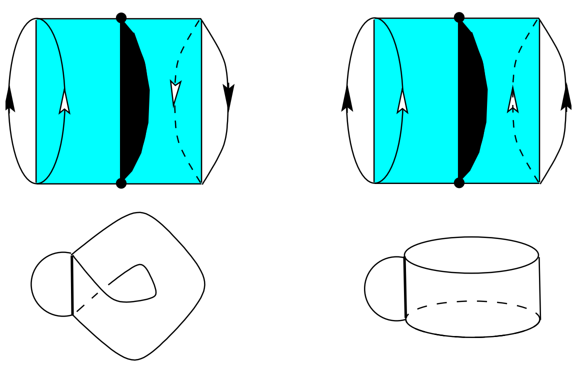

FIGURE 4. The skeleta P_2 and P_2' of B_2 and B_2'.

The skeleta P_2 and P_2' of B_2 and B_2', respectively, are shown in Fig. 4, both as abstract polyhedra, and as polyhedra embedded in $\boldsymbol{T}$ and $\boldsymbol{K}$. We conclude with the series Z_k for $k \geqslant 3$, for which a skeleton is shown in Fig. 5. Recalling that B_2'' was defined as Z_3, we denote this skeleton by P_2'' when $k = 3$.

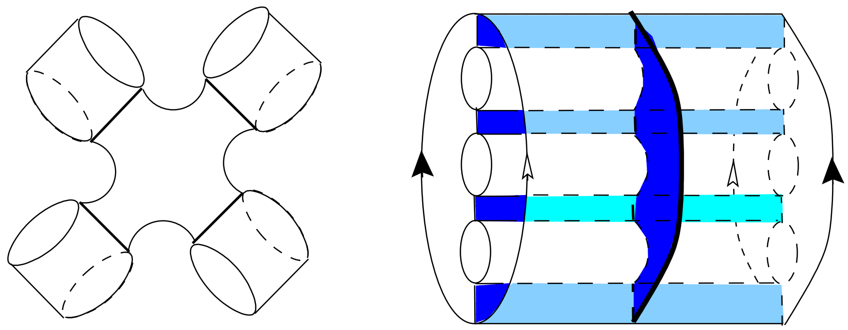

FIGURE 5. The skeleton of Z_k for $k = 4$.

Nuclear and standard skeleta. A skeleton of (M, X) is called *nuclear* if it does not collapse to a proper subpolyhedron which is also a skeleton of (M, X). A nuclear skeleton P of $(M, X) \in \mathcal{X}$ having $c(M, X)$ vertices is called *minimal*. Of course every (M, X) has minimal skeleta.

We will introduce now two more restricted classes of simple polyhedra. Later we will show that, under suitable assumptions, minimal polyhedra must belong to these classes. A simple polyhedron Q is called *quasi-standard with boundary* if every point has a neighborhood of one of the types (1)-(5) shown in Fig. 6. A point of type (3) was already defined above to be a *vertex* of Q. We denote now by $V(Q)$ the set of all vertices, and we define the *singular set* $S(Q)$ as the set of points of type (2), (3), or (5), and the *boundary* ∂Q as the set of points of type (4) or (5). Moreover we call 1-*components* of Q the connected components of $S(Q) \setminus V(Q)$ and 2-*components* of Q the connected components of $Q \setminus (S(Q) \cup \partial Q)$.

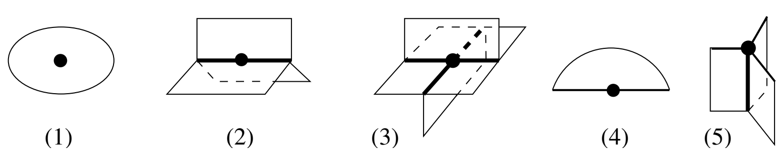

FIGURE 6. Typical neighborhoods of points in a quasi-standard polyhedron with boundary.

If the 2-components of Q are open discs (and hence are called just *faces*), and the 1-components are open segments (and hence called just *edges*), then we call Q a *standard polyhedron with boundary.* For short we will often just call Q a *standard* polyhedron, and possibly specify that ∂Q should be empty or non-empty. We prove now the first properties of nuclear skeleta.

LEMMA 3.2. *If P is a nuclear skeleton of a pair $(M, X) \in \mathcal{X}$, then $P = Q \cup s_1 \cup \ldots \cup s_m \cup G$, where:*

(1) *Q is a quasi-standard polyhedron with boundary $\partial Q \subset X$.*
(2) *For all components (C, τ) of $(\partial M, X)$, either $\partial Q \supset \tau$ or Q appears near C as in Fig. 7, so that $\partial Q \cap \tau$ is one or two circles, depending on the type of (C, τ).*
(3) *$s_1, \ldots, s_m$ are the edges of the τ's in X which do not already belong to Q.*
(4) *G is a graph with $G \cap (Q \cup s_1 \cup \ldots \cup s_m)$ finite and $G \cap V(Q \cup \partial M)$ empty.*

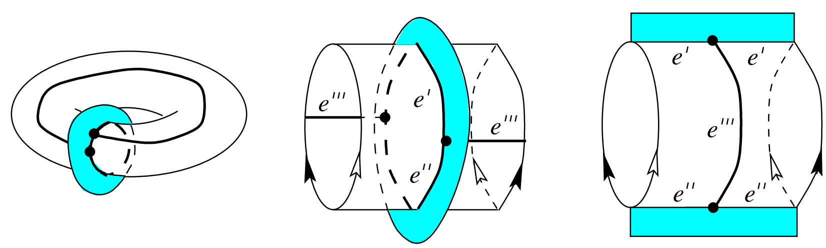

FIGURE 7. Local aspect of Q near C if $\partial Q \not\supset \tau$.

Proof. Nuclearity is a property of local nature, and the result is trivial if $\partial M = \emptyset$. For $\partial M \neq \emptyset$, defining Q as the 2-dimensional portion of P and G as $P \setminus (Q \cup X)$, the only non-obvious point to show is (2). Of course $Q \cap C \subset \partial Q$ is either τ or a union of circles. To check that the only possibilities are those of Fig. 7 one recalls that $M \setminus (P \cup \partial M)$ is a ball, so $C \setminus (Q \cup G)$ is planar, and then $C \setminus Q$ is also planar. $\square$

REMARK 3.3. Every $(M, X) \in \mathcal{X}$ has a minimal skeleton $P = Q \cup s_1 \cup \ldots \cup s_m \cup G$ as above, where in addition $G \cap \partial M = \emptyset$. This is because, without changing $v(P)$, we can take the ends of G lying on ∂M and make them slide over $Q \cup s_1 \cup \ldots \cup s_m$ until they reach $\mathrm{int}(M)$. Note that the regular neighborhood of $\tau \in X$ in P is now either a product $\tau \times [0, 1]$ or as shown in Fig. 7.

Subadditivity. Some properties of complexity readily follow from the definition and from the first facts shown about minimal skeleta. To begin with, if P and P' are skeleta of (M,X) and (M',X') and we add to $P \sqcup P'$ a segment which joins $P \setminus V(P)$ to $P' \setminus V(P')$, we get a skeleton of $(M,X)\#(M',X')$ with $v(P)+v(P')$ vertices. Therefore $c((M,X)\#(M',X')) \leqslant c(M,X)+c(M',X')$. Turning to assembling, let P and P' be as in Remark 3.3, and let an assembling $(M,X) \oplus (M',X')$ be performed along a map $\psi : C \to C'$ with $\psi(\tau) = \tau'$. Then $P \cup_\psi P'$ is simple, and it is a skeleton of $(M,X) \oplus (M',X')$. We deduce that $c((M,X) \oplus (M',X')) \leqslant c(M,X) + c(M',X')$.

Now we consider a self-assembling $\odot(M,X)$. If P is a skeleton of (M,X) as in Remark 3.3 and the self-assembling is performed along a certain map $\psi : C \to C'$ such that $\tau' \cap \psi(\tau)$ consists of two points, then $(P \cup C \cup C')/_\psi$ is a skeleton of $\odot(M,X)$. It has the same vertices as (M,X) plus at most two from the vertices of τ, two from the vertices of τ', and two from $\tau' \cap \psi(\tau)$. This shows that $c(\odot(M,X)) \leqslant c(M,X) + 6$.

Surfaces determined by graphs. We will need very soon the idea of splitting a skeleton along a graph, so we spell out how the construction goes.

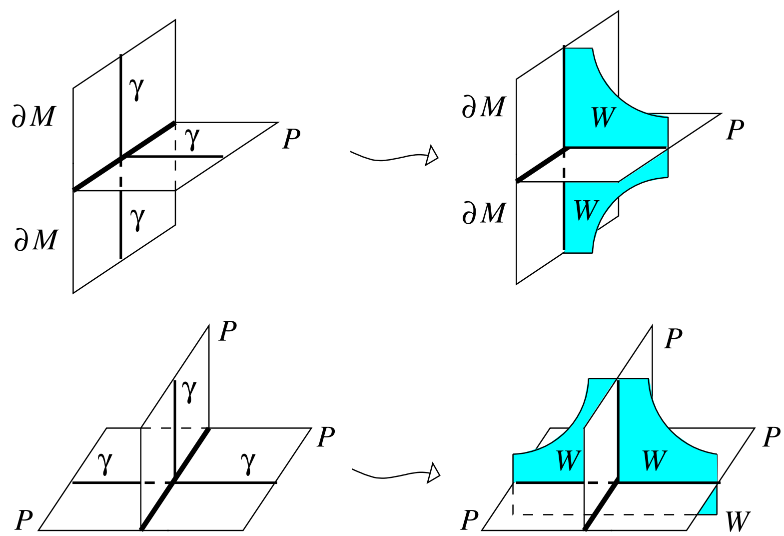

FIGURE 8. Surface determined by a trivalent graph.

LEMMA 3.4. *Let P be a quasi-standard skeleton of (M,X) and let γ be a trivalent graph contained in $P \cup \partial M$, locally embedded as in Fig. 8-left. Then there exists a surface S properly embedded in M such that $S \cap (P \cup \partial M) = \gamma$*

and $S \setminus \gamma$ is a union of discs. Moreover S is separating in M if and only if γ is separating in $P \cup \partial M$.

Proof. To construct S we first take a surface W with boundary as shown in Fig. 8-right, so that $W \cap (P \cup \partial M) = \gamma$. Then we attach disjoint discs to the components of ∂W lying in the interior of M. See [4, Remark 4.1] for details. □

REMARK 3.5. With the same notation as in the previous lemma, assume further that $\gamma \in \{\theta, \sigma\}$ is contained in P, that S is separating in M, and that $S \setminus \gamma$ is only one disc. Then cutting M along S and choosing γ as a spine for the two new boundary components we get a decomposition $(M, X) = (M_1, X_1) \oplus (M_2, X_2)$ which, at the level of skeleta, corresponds precisely to the splitting of P along γ.

Minimal skeleta are standard. We now prove a theorem on which most of our results are based. In particular, Proposition 2.4 readily follows from it, and Theorem 2.2 will be easily proved using it. We start with an easy remark.

REMARK 3.6. If P is a nuclear and standard skeleton of (M, X) then it is properly embedded, namely $\partial P = \partial M \cap P = X$, and $P \cup \partial M$ is standard without boundary. Moreover $P \cup \partial M$ is a spine of a manifold bounded by one sphere and some tori and Klein bottles, so $\chi(P \cup \partial M) = 1$. Knowing that $S(P \cup \partial M)$ is 4-valent, we then see that P has $\#X + v(P) + 1$ faces.

THEOREM 3.7. *Let $(M, X) \in \mathcal{X}$ be an irreducible and $\mathbb{P}^2$-irreducible pair, and let P be a minimal skeleton of (M, X). Then:*

1. *If $c(M, X) > 0$ then P is standard.*
2. *If $c(M, X) = 0$ and $X = \emptyset$ then $M \in \{S^3, L_{3,1}, \mathbb{P}^3\}$ and P is not standard.*
3. *If $c(M, X) = 0$ and $X \neq \emptyset$ then (M, X) is one of the B_i^* or Z_k, and P is precisely the skeleton described in Section 3, so P is standard unless (M, X) is B_1 or B_1'.*

Proof. We first show that if P is not standard then either $X = \emptyset$ and $M \in \{S^3, L_{3,1}, \mathbb{P}^3\}$, or $(M, X) \in \{B_1, B_1'\}$ and $P \in \{P_1, P_1'\}$. Later we will describe standard skeleta without vertices.

If P reduces to one point, then of course $M = S^3$. Let us first assume that P is not purely 2-dimensional, so there is segment e contained in the 1-dimensional part of P. We distinguish two cases depending on whether e lies in $\mathrm{int}(M)$ or on ∂M.

If $e \subset \mathrm{int}(M)$, we take a small disc Δ which intersects e transversely in one point. As in the proof of Lemma 3.4 we attach to $\partial\Delta$ a disc contained in the ball $M \setminus (P \cup \partial M)$, getting a sphere $S \subset M$ intersecting P in one point of e.

By irreducibility S bounds a ball B, and $P \cap B$ is easily seen to be a spine of B. Nuclearity now implies that $P \cap B$ contains vertices, so $P \setminus B$ is a skeleton of (M, X) with fewer vertices than P. This is a contradiction.

If $e \subset \partial M$, let C be the component of ∂M on which e lies. Since on C there is a circle which meets τ transversely in one point of e, looking at the ball $M \setminus (P \cup \partial M)$ again we see that in M there is a properly embedded disc D intersecting P in a point of e. We have now three cases depending on the type of the pair (C, τ).

- If $(C, \tau) = (T, \theta)$ then D is a compressing disc for T, so by irreducibility M is the solid torus. Knowing that ∂D meets P only in one point it is now easy to show also that $(M, X) = B_1$ and $P = P_1$.
- If $(C, \tau) = (K, \theta)$ then e must be contained in the edge e''' of θ by Lemma 3.2, and the same reasoning shows that $(M, X) = B_1'$ and $P = P_1'$.
- If $(C, \tau) = (K, \sigma)$ then e must be contained in the edge e''' of σ by Lemma 3.2. The complement in K of ∂D is now the union of two Möbius strips. If we choose any one of these strips and take its union with D, we get an embedded $\mathbb{P}^2$ in M. Since M is irreducible and $\mathbb{P}^2$-irreducible, it would then have to be $\mathbb{P}^3$, but $\partial M \neq \emptyset$, so we have obtained a contradiction.

We are left to deal with the case where P is purely two-dimensional, so that P is quasi-standard, but not standard. Let us first suppose that some 2-component F of P is not a disc. Then either F is a sphere, in which case P also reduces to only a sphere, which is clearly impossible because M would then be $S^2 \times [0,1]$, or there exists a loop γ in F such that one of the following holds:

(1) γ is orientation-reversing on F.
(2) γ separates F in two components none of which is a disc.

We consider now the closed surface S determined by γ as in Lemma 3.4, and note that S is either S^2 or $\mathbb{P}^2$. If $S = \mathbb{P}^2$ we deduce that $(M, X) = \mathbb{P}^3$. If $S = S^2$ irreducibility implies that S bounds a ball B in M. This is clearly impossible in case (1), so we are in case (2). Now we note that $P \cap B$ must be a nuclear spine of B. Knowing that $F \cap B$ is not a disc it is easy to deduce that $P \cap B$ must contain vertices. This contradicts the minimality because we could replace the whole of $P \cap B$ by only one disc, getting another skeleton of (M, X) with fewer vertices.

If P is quasi-standard and its 2-components are discs then either P is standard or $S(P)$ reduces to a single circle. Then it is easy to show that P must be the triple hat and $(M, X) = L_{3,1}$.

We are left to analyze the case where P is standard and $c(M, X) = 0$, so that $X \neq \emptyset$. Denoting $\#X$ by n, Remark 3.6 shows that P has $n + 1$ faces.

We consider first the case $n = 1$. Since P has one edge and two faces, it is easy to see that it must be homeomorphic to either P_2 or P_2' (see Fig. 4) as an abstract polyhedron. This does not quite imply that (M, X) is B_2 or B_2', because in general a skeleton P alone is not enough to determine a pair (M, X). However $P \cup \partial M$ certainly does determine (M, X), because it is a standard spine of M minus a ball, and $X = P \cap \partial M$. We are left to analyze all the polyhedra of the form $P_2 \cup_\psi T$ for $\psi : \partial P_2 \to \theta \subset T$, of the form $P_2 \cup_\psi K$ for $\psi : \partial P_2 \to \theta \subset K$, and of the form $P_2' \cup_\psi K$ for $\psi : \partial P_2' \to \sigma \subset K$. Among these polyhedra we must select those which can be thickened to manifolds with two boundary components (a sphere plus either a torus or a Klein bottle). The symmetries of (T, θ), (K, θ), and (K, σ) described in Propositions 1.3 and 1.2 imply that there are actually not many such polyhedra. More precisely, there is just one $P_2 \cup_\psi T$, which gives B_2. There are two $P_2 \cup_\psi T$, one of which is not thickenable (i.e., not the spine of any manifold), and the other can be thickened to a manifold with three boundary components (a sphere and two Klein bottles). Finally, there are two $P_2' \cup_\psi K$, one of which is not thickenable, and the other gives B_2'. This concludes the proof for $n = 1$.

Having worked out the case $n = 1$, we turn to the case $n \geqslant 2$, so that P has n edges and $n + 1$ faces. If a face of P meets ∂M in one arc only, then it meets $S(P)$ in one edge only, and this edge joins a component of ∂M to itself, which easily implies that $n = 1$, contradicting our present assumption $n \geqslant 2$. If a face of P is an embedded rectangle, with two opposite edges on ∂M and two in $S(P)$, then it readily follows that $n = 2$ and P is either $\theta \times [0, 1]$ or $\sigma \times [0, 1]$. As above, to conclude that $(M, X) \in \{B_0, B_0', B_0''\}$, we must consider the various polyhedra obtained by attaching (T, θ), (K, θ), and (K, σ) to the upper and lower bases of $\theta \times [0, 1]$ and $\sigma \times [0, 1]$. Using again Propositions 1.3 and 1.2 one sees that there are only six such polyhedra. Three of them are not thickenable, and the other three give B_0, B_0', B_0''.

Returning to the general case with $n \geqslant 2$, we note that there are a total of $3n$ edges on ∂M, so there are $3n$ germs of faces starting from ∂M. Knowing that there is a total of $n + 1$ faces and none of them uses only one germ, we see that at least one face uses only two germs, and so it is a rectangle R, possibly an immersed one. If $n = 2$ we have three rectangles, one of which must be embedded, and we are led back to a case already discussed. If $n \geqslant 3$ then R must be immersed, so in particular it joins a component (K_1, σ_1) of $(\partial M, X)$ to another (K_2, σ_2) component. A regular neighborhood in P of $R \cup \sigma_1 \cup \sigma_2$ is shown in Fig. 9. The boundary of this neighborhood is again a graph σ which determines a separating Klein bottle according to Remark 3.5. If we cut P along σ we get a disjoint union $P_2'' \sqcup P'$, which at the level of manifolds gives a splitting $(M, X) = B_2'' \oplus (M', X')$. Moreover P' is a nuclear skeleton of (M', X'), so $c(M', X') = 0$, P' is minimal, and $\#X' = n - 1$. Now either $(M', X') = B_0''$ and $P' = P_0''$ or we can proceed, eventually getting that $(M, X) = B_2'' \oplus \ldots \oplus B_2''$, so $(M, X) = Z_k$ for some

$k \geqslant 3$, and P is the corresponding skeleton constructed in Section 3. The proof is now complete. $\square$

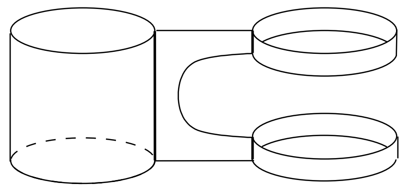

FIGURE 9. An immersed rectangle joins two (K, σ) components.

Proof of Theorem 2.2. By the previous result, a minimal spine of M is standard with vertices, and dual to it there is a singular triangulation with $c(M)$ tetrahedra (and one vertex). A singular triangulation of M with n tetrahedra and k vertices dually gives a standard polyhedron Q with n vertices whose complement is a union of k balls. If we puncture $k-1$ suitably chosen faces of Q we get a skeleton of $(M, \emptyset)$, whence the conclusion at once. $\square$

4. Finiteness

The proof of Theorem 2.3 will be based on the following result.

PROPOSITION 4.1. *Let (M, X) be an irreducible and $\mathbb{P}^2$-irreducible pair which does not split as $(M, X) = (N, Y) \oplus B_2''$. Assume that $c(M, X) > 0$ and let P be a standard skeleton of (M, X). Then every edge of P is incident to at least one vertex of P.*

Proof. Assume by contradiction that an edge e of P is not incident to any vertex of P, i.e., that both the ends of e lie on ∂M. If the ends of e lie on the same spine $\tau \in X$ then $\tau \cup e$ is a connected component of $S(P) \cup \partial M$. The standardness of P implies that P has no vertices, which contradicts the assumption that $c(M, X) > 0$. So the ends of e lie on distinct spines $\tau, \tau' \in X$. Let C and C' be the components of ∂M on which τ and τ' lie, and let R be a regular neighborhood in P of $C \cup C' \cup e$. By construction R is a quasi-standard polyhedron with boundary $\partial R = \tau \sqcup \tau' \sqcup \gamma$. Here γ is a trivalent graph with one component homeomorphic to θ or to σ, and possibly another component homeomorphic to the circle.

Let us first consider the case where γ has a circle component γ_0. This circle lies on P and is disjoint from $S(P)$. The standardness of P then implies that γ_0 bounds a disc D contained in P and disjoint from $S(P)$. In this case we set $\gamma' = \gamma \setminus \gamma_0$ and $R' = R \cup D$. In case γ is connected we just set $\gamma' = \gamma$ and $R' = R$. In both cases we have found a graph γ' homeomorphic to θ or to σ

which separates P. Moreover one component R' of $P \setminus \gamma'$ is standard without vertices and is bounded by $\tau \sqcup \tau' \sqcup \gamma'$.

According to Lemma 3.4, the graph γ' determines a separating surface S in M such that $S \cap P = \gamma'$. Since $\chi(\gamma') = -1$ and $S \setminus \gamma'$ consists of discs, we have $\chi(S) \geqslant 0$. Of course $\chi(S) \neq 1$, for otherwise S would be an embedded $\mathbb{P}^2$, but we are assuming that M is irreducible and $\mathbb{P}^2$-irreducible and has non-empty boundary. We will now show that if $\chi(S) = 2$ then $c(M, X) = 0$, and if $\chi(S) = 0$ then (M, X) splits as $(M, X) = (N, Y) \oplus B''_2$. This will imply the conclusion.

Assume that $\chi(S) = 2$, so S is a sphere. We denote by B the open 3-ball $M \setminus (P \cup \partial M)$ and note that $S \cap B = S \setminus \gamma'$ consists of three disjoint open 2-discs, which cut B into four open 3-balls. By irreducibility, S bounds a closed 3-ball D, and $B \setminus D$ is the union of some of the four open 3-balls just described. Viewing (D, γ') abstractly we can now easily construct a new simple polyhedron $Q \subset D$ without vertices such that $Q \cap S = \gamma'$ and $D \setminus Q$ consists of three distinct 3-balls, each incident to one of the three open 2-discs which constitute $S \setminus \gamma'$. Let us consider now the simple polyhedron $P' = R' \cup_{\gamma'} Q$ viewed as a subset of M. By construction $P' \cap \partial M = \tau \cup \tau' = X$. Moreover $M \setminus (P' \cup \partial M)$ is obtained from $B \setminus D$ (which consists of open 3-balls) by attaching each of the three 3-balls of $D \setminus Q$ along only one 2-disc (a component of $S \setminus \gamma'$). It follows that $M \setminus (P' \cup \partial M)$ still consists of open 3-balls. By puncturing some of the 2-components of P' we can then construct a skeleton of (M, X) without vertices, so indeed $c(M, X) = 0$.

Assume now that $\chi(S) = 0$, so S is a separating torus or Klein bottle. Remark 3.5 now shows that (M, X) is obtained by assembling some pair (N, Y) with a pair (N', Y') which has skeleton R'. By construction R' is standard without vertices and $\partial N'$ has three components, and it was shown in the proof of Theorem 3.7 that (N', Y') must then be B''_2. This completes the proof. □

Corollary 4.2. *Let (M, X) be irreducible and $\mathbb{P}^2$-irreducible. Assume $c(M, X) > 0$ and there is no splitting $(M, X) = (N, Y) \oplus B''_2$. Then $\#X \leqslant 2c(M, X)$.*

Proof. A minimal skeleton P of (M, X) is standard by Theorem 3.7, and we have just shown that each edge of P joins either $V(P)$ to itself or $V(P)$ to X. Since P has $c(M, X)$ quadrivalent vertices, there can be at most $4c(M, X)$ edges reaching X. Each component of X is reached by precisely two edges, so there are at most $2c(M, X)$ components. □

Proof of Theorem 2.3. The result is valid for $n = 0$ by the classification carried out in Theorem 3.7, so we assume $n > 0$. Let $\mathcal{F}_n$ be the set of all irreducible and $\mathbb{P}^2$-irreducible pairs (M, X) which cannot be split as $(M', X') \oplus B''_2$. By Theorem 3.7, each such (M, X) has a minimal standard spine P with

n vertices. By Corollary 4.2, we have that $S(P \cup \partial M)$ is a quadrivalent graph with at most $5n$ vertices. Since $P \cup \partial M$ is a standard polyhedron, there are only finitely many possibilities for $P \cup \partial M$ and hence for (M, X).

Given an irreducible and $\mathbb{P}^2$-irreducible pair (M, X) with $c(M, X) = n$, either $(M, X) \in \mathcal{F}_n$ or (M, X) splits along a Klein bottle K as $(M', X') \oplus B_2''$. The only case where K is compressible in M is when $(M', X') = B_2'$, but $B_2' \oplus B_2'' = B_0''$ and $c(B_0'') = 0$. So K is incompressible, whence M' is irreducible and $\mathbb{P}^2$-irreducible. Moreover $c(M', X') = n$ by Remark 2.6 (which depends on the now proved Propositions 2.4 and 2.5). Since (M', X') has one boundary component less than (M, X), we can iterate the process of splitting copies of B_2'' only a finite number of times, and then we get to an element of $\mathcal{F}_n$. □

5. Additivity

In this section we prove additivity under connected sum. This will require the theory of normal surfaces and more technical results on skeleta. We start with an easy general fact (see [4, Proposition 2.9] for a proof).

Proposition 5.1. *Given a pair $(M, X) \in \mathcal{X}$, let $Q \subset M$ be a quasi-standard polyhedron with $Q \cap \partial M = \partial Q \subset X$. Assume that $M \setminus Q$ has two components N' and N''. Then the 2-components of Q that separate N' from N'' form a closed surface $\Sigma(Q) \subset Q \subset \mathrm{int}(M)$ which cuts M into two components.*

Normal surfaces. Given a pair $(M, X) \in \mathcal{X}$, let P be a nuclear skeleton of (M, X). The simple polyhedron $P \cup \partial M$ is now a spine of M with a ball $B \subset M$ removed. Choose a triangulation of $P \cup \partial M$, and let ξ_P be the handle decomposition of $M \setminus B$ obtained by thickening the triangulation of $P \cup \partial M$, as in [7]. In this subsection we will study normal spheres in ξ_P. Note that there is an obvious example, namely the sphere parallel to ∂B and slightly pushed inside ξ_P. The following result deals with the other normal spheres. Its proof displays another remarkable difference between the orientable and the general case. Namely, it was shown in [4] that, when M is orientable, any normal surface reaching ∂M actually contains a component of ∂M. By contrast, when $(\partial M, X)$ contains some (K, σ) component, an arbitrary normal surface can reach K without containing it. As our proof shows, however, this cannot happen when the surface is a sphere.

Proposition 5.2. *Let P be a nuclear skeleton of $(M, X) \in \mathcal{X}$, and let S be a normal sphere in ξ_P. Then there exists a simple polyhedron Q such that $v(Q) \leqslant v(P)$, $Q \cap \partial M = X$ and $M \setminus (Q \cup \partial M)$ is a regular neighborhood of S. If in addition P is standard, $c(M, X) > 0$, and S is not the obvious sphere $\partial N(P \cup \partial M)$, then there exists Q as above with $v(Q) < v(P)$.*

Proof. Every region R of P carries a color $n \in \mathbb{N}$ given by the number of sheets of the local projection of S to R. Now we cut $P \cup \partial M$ open along S as explained in [7], i.e., we replace each R by its $(n+1)$-sheeted cover contained in the normal bundle of R in M. As a result we get a polyhedron $P' \subset M$ which contains ∂M, such that $M \setminus P'$ is the disjoint union of an open ball B and an open regular neighborhood N of S in M. By removing from each boundary component $C \subset \partial M$ the open disc $C \setminus \tau$ we get a polyhedron P'' intersecting ∂M in X. Now we puncture a 2-component which separates B from N and claim that the resulting polyhedron Q is as desired. Only the inequalities between $v(P)$ and $v(Q)$ are non-obvious.

We first prove that all the vertices of $P \cup \partial M$ which lie on ∂M disappear either when we cut P along S getting P' or later when we remove $\partial M \setminus X$ from P' to get P''. This of course implies the first assertion of the statement. We concentrate on one component (C, τ) of $(\partial M, X)$. By Lemma 3.2 either both vertices of τ are vertices of $P \cup \partial M$ or none of them is. In the latter case there is nothing to show, so we assume that there are three (possibly non-distinct) 2-components of P incident to τ. Let v and v' be the vertices of τ. Looking first at v, we denote by (n, n, n, p, q, r) the colors of the six germs at v of 2-component of $P \cup \partial M$. Here n corresponds to $C \setminus \tau$, which is triply incident to v.

The compatibility equations of normal surfaces now readily imply that (up to permutation) r is even, $p = q \geqslant r$, and that $n \geqslant p/2$ when $p = q = r$. Moreover:

- v disappears in P' if $p = q > r$;
- v survives in P' and remains on ∂M, so it disappears in P'', if $p = q = r$ and $n = p/2$;
- v survives in P' and moves to $\mathrm{int}(M)$ if $p = q = r$ and $n > p/2$.

Now if $\tau = \theta$ then the same coefficients appear at v'. The only case where v and v' do not both disappear in P'' is when $p = q = r$ and $n > p/2$. But in this case S would then contain $n - p/2$ parallel copies of C, which is impossible. The case $\tau = \sigma$ is easier, because if v survives in P'' the situation is as in Fig. 10. This is absurd because S would contain Möbius strips.

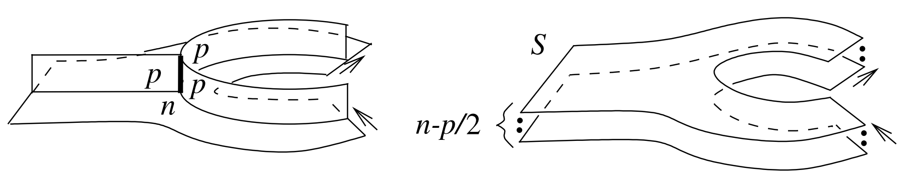

FIGURE 10. Möbius strips in a normal surface.

Now we turn to the second assertion. If $v(P'') < v(P)$ the conclusion is obvious, so we proceed assuming $v(P) = v(P'')$. It is now sufficient to show that some face of P'' which separates B from N contains vertices of P'', because we can then puncture such a face and collapse the resulting polyhedron until it becomes nuclear, getting fewer vertices. Assume by contradiction that there is no such face.

We note that P'' is the union of a quasi-standard polyhedron P''' and some arcs in X. The 2-components of P'' which separate B from N are the same as those of P''', so they give a closed surface $\Sigma \subset P''$ by Proposition 5.1. From the fact that $v(P'') = v(P)$ we deduce that near a vertex of P the transformation of P into P'' can be described as in Fig. 11, namely P'' can

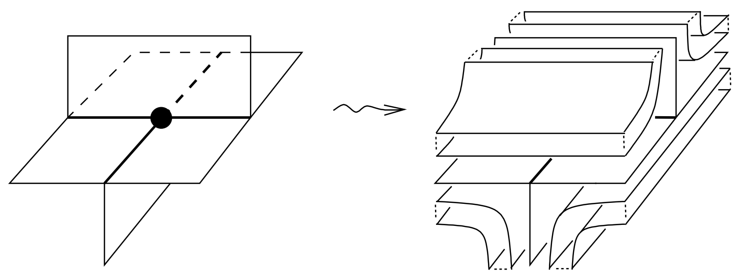

FIGURE 11. Transformation of P into P'' near a vertex of P.

be identified near the vertex with $P \cup S$. Of course this does not imply that globally $P'' = P \cup S$, because the components of P'' playing the role of P near vertices may not match across faces.

The closed surface Σ cannot be disjoint from $S(P'')$, because otherwise S would be the obvious sphere ∂B. On the other hand we are supposing $\Sigma \cap V(P'') = \emptyset$, so $\Sigma \cap S(P'')$ must be a non-empty union of loops. In particular, $S(P'')$ contains a loop γ disjoint from $V(P'')$.

Figure 11 now shows that $S(P'')$ coincides with $S(P)$ away from ∂M. Using the analysis of the transition from P to P'' near ∂M already carried out above, we also see that near a component (C, τ) of $(\partial M, X)$ either $S(P'')$ coincides with $S(P)$ or it is obtained from $S(P)$ by adding one edge of τ, and then slightly pushing the result inside M. When $(C, \tau) = (K, \sigma)$ the edge added is necessarily e'''. This implies that the loop γ described above can be viewed as a loop in $S(P \cup \partial M)$ such that $\gamma \cap V(P) = \emptyset$. In addition, if γ contains a vertex of $P \cup \partial M$ on a certain component of ∂M then it contains also the other vertex in that component. This readily implies that the union of γ with all the τ's in X touched by γ is a connected component of $S(P \cup \partial M)$. But $P \cup \partial M$ is standard, so $S(P \cup \partial M)$ is connected, and we deduce that P has no vertices. This is a contradiction. $\square$

Proof of Theorem 2.1. We have already shown that $c(S^2\times S^1) = c(S^2 \tilde{\times} S^1) = 0$ and that c is subadditive. Let us consider now a non-prime pair (M, X) and a minimal skeleton P of (M, X). Since (M, X) is not prime, there exists a normal sphere S in ξ_P which is essential in M, namely it either is non-separating or it separates M into two manifolds both different from B^3. Then we apply the first point of Proposition 5.2 to P and S, getting a polyhedron Q.

If S is separating and splits (M, X) as $(M_1, X_1)\#(M_2, X_2)$, we must have that Q is the disjoint union of polyhedra Q_1 and Q_2, where Q_i is a skeleton of (M_i, X_i). Since $v(Q_1) + v(Q_2) = v(Q) \leqslant v(P)$ we deduce that $c(M, X) \geqslant c(M_1, X_1) + c(M_2, X_2)$, so equality actually holds.

If S is not separating we identify a regular neighborhood of S in M with $S\times(-1, 1)$ and note that there must exist a face of Q having $S\times(-1, -1+\varepsilon)$ on one side and $S\times(1-\varepsilon, 1)$ on the other side. We puncture this face getting a polyhedron Q'. Now Q' is a skeleton of a pair (M', X) such that $(M, X) = (M', X)\#E$, where E is $S^2\times S^1$ or $S^2 \tilde{\times} S^1$. Moreover $v(Q') = v(Q) \leqslant v(P)$, and hence $c(M, X) \geqslant c(M', X)$, so equality actually holds.

We have shown so far that an essential normal sphere in (M, X) leads to a non-trivial decomposition $(M, X) = (M_1, X_1)\#(M_2, X_2)$ on which complexity is additive. If (M_1, X_1) and (M_2, X_2) are prime we stop; otherwise we iterate the procedure until we find a decomposition of (M, X) into primes on which complexity is additive. Since any other decomposition into primes actually consists of the same summands, we deduce that complexity is always additive on decompositions into primes. If we take the connected sum of two non-prime manifolds then a prime decomposition of the result is obtained from prime decompositions of the summands, so additivity holds also in general. $\square$

Appendix: Some facts about the Klein bottle

In this appendix, following Matveev [7], we classify all simple closed loops on the Klein bottle K and we deduce Proposition 1.2 from this classification. We also mention two more results on K which easily follow from the classification. These results are strictly speaking not necessary for the present paper, and they are probably well-known to experts, but we have decided to include them because they show a striking difference which exists between the orientable and non-orientable cases.

PROPOSITION A.1. *There exist on the Klein bottle only four non-trivial loops up to isotopy, as shown in Fig. 12. These loops are determined by their image in* $H_1(K;\mathbb{Z}) = \langle a, b | a+b = b+a,\ 2a = 0\rangle$*, as also shown in the picture. Moreover* a *and* $\pm 2b$ *are orientation-preserving on* K*, while* $\pm b$ *and* $a\pm b$ *are orientation-reversing.*

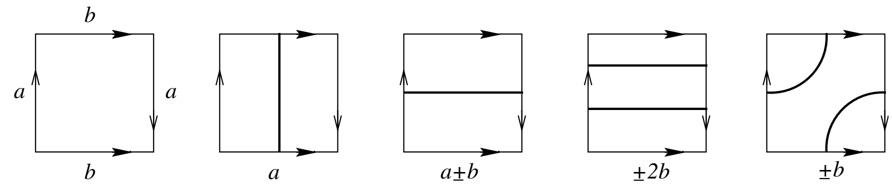

FIGURE 12. Non-trivial loops on the Klein bottle.

Proof. A non-trivial loop is isotopic to one which is normal with respect to a triangulation of K, i.e., it appears as in Fig. 13. We must have $n+m = n'+m'$,

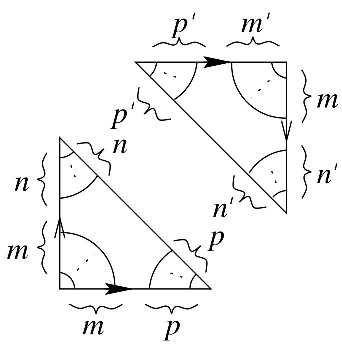

FIGURE 13. Normal loops in a triangulation of K.

$n + p = n' + p'$, $m + p = m' + p'$, so $n' = n$, $m' = m$, $p' = p$. If $p > m$, we further distinguish between the following cases: If $n < p$, since we look for a connected curve, we get $n = m = 0$ and $p = 1$, whence the loop a; if $n > p$ we do not get any solution; if $n = p$ we get $m = 0$ and $n = p \in \{1, 2\}$, whence the loops $\pm b$ and $\pm 2b$. If $m > p$ we must have $p = n = 0$ and $m \in \{1, 2\}$, whence the loops $\pm b$ and $\pm 2b$ again. If $m = p$, since the connected curve we look for is also non-trivial, we must have $m = p = 0$ and $n \in \{1, 2\}$, whence the loops $a \pm b$ and $\pm 2b$. □

Proof of Proposition 1.2. We start by showing that σ embeds uniquely as a spine of K. The closed edges e' and e'' of σ are disjoint simple loops in K, and they must be orientation-reversing. It easily follows that $\{e', e''\}$ must be $\{\pm b, a \pm b\}$. Now the ends of e''' can be isotopically slid over e' and e'' to reach the position of Fig. 1-centre, and uniqueness is proved.

Turning to the uniqueness of the embedding of θ, note that two of the three simple closed loops contained in θ must be orientation-reversing on K. Let e''' be the edge contained in both of these loops. If we perform the move shown in Fig. 14 along e''' we get a spine σ of K, and the newborn edge is the edge

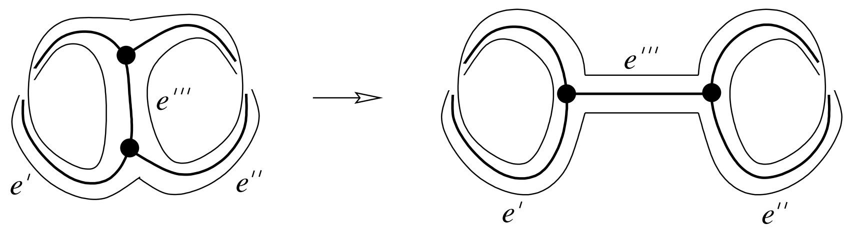

FIGURE 14. A move changing a spine θ of K into a spine σ.

e''' of σ. So θ is obtained from σ by the same move along $e''' \subset \sigma$. Since the embedding of σ is unique, we obtain the same conclusion for θ.

Having proved uniqueness, we must understand symmetries. Our description obviously implies that, in both σ and θ, the edges e' and e'' play symmetric roles, while the role of e''' is different, and the conclusion easily follows. The same conclusion could also be deduced from Proposition A.3 below. $\square$

PROPOSITION A.2. *If $\boldsymbol{K}$ is the solid Klein bottle and $K = \partial\boldsymbol{K}$ then every automorphism of K extends to $\boldsymbol{K}$. In particular, there is only one possible "Dehn filling" of a Klein bottle in the boundary of a given manifold.*

Proof. Proposition A.1 shows that the meridian a of $\boldsymbol{K}$ can be characterized in $K = \partial\boldsymbol{K}$ as the only orientation-preserving loop having connected complement. So every automorphism of K maps the meridian to itself and the conclusion follows. $\square$

PROPOSITION A.3. *The mapping class group of K is isomorphic to $\mathbb{Z}/_{2\mathbb{Z}} \times \mathbb{Z}/_{2\mathbb{Z}}$ and every automorphism of K is determined up to isotopy by its action on $H_1(K;\mathbb{Z})$.*

Proof. It is quite easy to construct commuting order-2 automorphisms ϕ and ψ of K such that their action on $H_1(K;\mathbb{Z})$ is given by

$$\phi(a) = a, \quad \phi(b) = -b, \qquad \psi(a) = a, \quad \psi(b) = a + b.$$

Given any other automorphism f, combining the geometric characterization of a with the observation that a is isotopic (not only homologous) to itself with opposite orientation, we deduce that (up to isotopy) f is the identity on a. Up to composing f with ϕ we can assume that f is actually the identity also near a, so f restricts to an automorphism of the annulus $K \setminus a$ which is the identity on the boundary. The mapping class group relative to the boundary of the annulus is now infinite cyclic generated by the restriction of ψ (but ψ has order 2 when viewed on K), and the conclusion follows. $\square$

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