Illinois Journal of Mathematics
Volume 46, Number 3, Fall 2002, Pages 695–721
S 0019-2082

MINIMAL LAGRANGIAN SUBMANIFOLDS IN THE COMPLEX HYPERBOLIC SPACE

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Abstract. In this paper we construct new examples of minimal Lagrangian submanifolds in the complex hyperbolic space with large symmetry groups, obtaining three 1-parameter families with cohomogeneity one. We characterize these submanifolds as the only minimal Lagrangian submanifolds in $\mathbb{CH}^n$ that are foliated by umbilical hypersurfaces of Lagrangian subspaces $\mathbb{RH}^n$ of $\mathbb{CH}^n$. By suitably generalizing this construction, we obtain new families of minimal Lagrangian submanifolds in $\mathbb{CH}^n$ from curves in $\mathbb{CH}^1$ and $(n-1)$-dimensional minimal Lagrangian submanifolds of the complex space forms $\mathbb{CP}^{n-1}$, $\mathbb{CH}^{n-1}$ and $\mathbb{C}^{n-1}$. We give similar constructions in the complex projective space $\mathbb{CP}^n$.

1. Introduction

Special Lagrangian submanifolds of the complex Euclidean space $\mathbb{C}^n$ (or of a Calabi-Yau manifold) have been studied widely over the last few years. For example, A. Strominger, S.T. Yau and E. Zaslow [SYZ] proposed an explanation of mirror symmetry of a Calabi-Yau manifold in terms of the moduli spaces of special Lagrangian submanifolds. These submanifolds are volume minimizing and, in particular, are minimal submanifolds. Furthermore, any oriented minimal Lagrangian submanifold of $\mathbb{C}^n$ (or a Calabi-Yau manifold) is a special Lagrangian submanifold with respect to one of the 1-parameter families of special Lagrangian calibrations which this kind of Kaehler manifolds has (see [HL, Proposition 2.17]).

A very important problem here is to find non-trivial examples of special Lagrangian submanifolds (i.e., oriented minimal Lagrangian submanifolds). In [HL] R. Harvey and H.B. Lawson constructed the first examples in $\mathbb{C}^n$, and, in particular, the Lagrangian catenoid ([HL, Example III.3.B]). More recently, D.D. Joyce ([J1], [J2], [J3], [J4]) and M. Haskins [H] have developed

Received June 4, 2001; received in final form October 29, 2001.
2000 *Mathematics Subject Classification.* Primary 53B25, 53C42, 53D12.
Research partially supported by a MCYT and FEDER grant BFM2001-2967.

methods for constructing important families of special Lagrangian submanifolds of $\mathbb{C}^n$. We are particularly interested in examples with large symmetry groups (see [J1]), i.e., submanifolds which are invariant under the action of certain subgroups of the isometries group of $\mathbb{C}^n$.

Using some ideas from the above-mentioned papers, we construct examples of minimal Lagrangian submanifolds of the complex hyperbolic space $\mathbb{CH}^n$ with large symmetry groups. In particular, we consider the groups of isometries of the sphere $\mathbb{S}^{n-1}$, the real hyperbolic space $\mathbb{RH}^{n-1}$, and the Euclidean space $\mathbb{R}^{n-1}$, which we denote by $\mathrm{SO}(n)$, $\mathrm{SO}^1_0(n)$ and $\mathrm{SO}(n-1) \propto \mathbb{R}^{n-1}$, respectively, acting on $\mathbb{CH}^n$ as holomorphic isometries (see Section 2.1). In Theorems 1, 2 and 3 we classify the minimal Lagrangian submanifolds of $\mathbb{CH}^n$ that are invariant under the groups $\mathrm{SO}(n)$, $\mathrm{SO}^1_0(n)$ and $\mathrm{SO}(n-1) \propto \mathbb{R}^{n-1}$, respectively. In each of these theorems we obtain a 1-parameter family of minimal Lagrangian submanifolds M in $\mathbb{CH}^n$ with cohomogeneity one, i.e., such that the orbits of the symmetry group are of codimension one in M. In particular, M is foliated by a 1-parameter family of orbits parameterized by $s \in \mathbb{R}$, which are, respectively, geodesic spheres, tubes over hyperplanes, and horospheres (i.e., umbilical hypersurfaces) of Lagrangian subspaces $\mathbb{RH}^n_s$ of $\mathbb{CH}^n$. In Theorem 4 we characterize the above examples as the only minimal Lagrangian submanifolds of $\mathbb{CH}^n$ foliated by umbilical hypersurfaces of Lagrangian subspaces $\mathbb{RH}^n$ of $\mathbb{CH}^n$. A similar result characterizing the Lagrangian catenoid in $\mathbb{C}^n$ was proved in [CU2].

Following an idea given independently in [H, Theorem A], [J1, Theorem 6.4] and [CU2, Remark 1] (see Remark 1 in the latter paper for details), we construct, in Propositions 3 and 4, families of minimal Lagrangian submanifolds of $\mathbb{CH}^n$ from curves in $\mathbb{CH}^1$ and $(n-1)$-dimensional minimal Lagrangian submanifolds of the complex space forms $\mathbb{CP}^{n-1}$, $\mathbb{CH}^{n-1}$ and $\mathbb{C}^{n-1}$. The examples described in Theorems 1, 2 and 3 are the simplest examples of this construction.

In Theorem 5 and Proposition 6 we obtain similar results in the complex projective space $\mathbb{CP}^n$. In this case, there are fewer families of minimal Lagrangian submanifolds since there is only one family of umbilical hypersurfaces of the Lagrangian subspaces $\mathbb{RP}^n$ of $\mathbb{CP}^n$, namely the geodesic spheres. We will focus on the case of $\mathbb{CH}^n$ in this paper since the results and proofs are far more difficult in this case.

2. Preliminaries

2.1. The complex hyperbolic space. In this paper we will consider the following model for the complex hyperbolic space. In $\mathbb{C}^{n+1}$ we define the Hermitian form $(,)$ by

$$(z,w) = \sum_{i=1}^{n} z_i \bar{w}_i - z_{n+1}\bar{w}_{n+1}$$

for $z, w \in \mathbb{C}^{n+1}$, where $\bar{z}$ stands for the conjugate of z. If

$$\mathbb{H}_1^{2n+1} = \left\{ z \in \mathbb{C}^{n+1} \mid (z,z) = -1 \right\}$$

is the anti-de Sitter space, then $\Re(,)$ (where $\Re$ denotes the real part) induces on $\mathbb{H}_1^{2n+1}$ a Lorentzian metric of constant curvature -1. Letting $(\mathbb{CH}^n = \mathbb{H}_1^{2n+1}/\mathbb{S}^1, \langle , \rangle)$ denote the complex hyperbolic space of constant holomorphic sectional curvature -4, we have

$$\mathbb{CH}^n = \left\{ \Pi(z) = [z] \mid z = (z_1, \dots, z_{n+1}) \in \mathbb{H}_1^{2n+1} \right\},$$

where $\Pi : \mathbb{H}_1^{2n+1} \to \mathbb{CH}^n$ is the Hopf projection. The metric $\Re(,)$ becomes Π in a pseudo-Riemannian submersion. The complex structure of $\mathbb{C}^{n+1}$ induces, via Π, the canonical complex structure J on $\mathbb{CH}^n$. The Kähler two–form Ω in $\mathbb{CH}^n$ is defined by $\Omega(u,v) = \langle Ju, v \rangle$. We recall that $\mathbb{CH}^n$ has a smooth compactification $\mathbb{CH}^n \cup \mathbb{S}^{2n-1}(\infty)$, where $\mathbb{S}^{2n-1}(\infty) = \pi(\mathcal{N})$,

$$\mathcal{N} = \{ z \in \mathbb{C}^{n+1} - \{0\} \mid (z,z) = 0 \},$$

and $\pi : \mathcal{N} \to \mathbb{S}^{2n-1}(\infty)$ is the projection given by the natural action of $\mathbb{C}^*$ over $\mathcal{N}$.

Moreover, in the paper we will denote by $\mathbb{CP}^n$ the n-dimensional complex projective space endowed with the Fubini-Study metric of constant holomorphic sectional curvature 4, and by $\Pi : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ the Hopf fibration from the $(2n{+}1)$-dimensional unit sphere $\mathbb{S}^{2n+1}$. We also denote the complex structure and the Kähler two-form in $\mathbb{CP}^n$ by J and Ω, respectively.

If $U^1(n{+}1)$ is the group preserving the Hermitian form $(,)$, then

$$U^1(n{+}1) = \{ A \in GL(n{+}1, \mathbb{C}) \mid \bar{A}^t S A = S \},$$

where

$$S = \left(\begin{array}{c|c} \mathrm{I}_n & \\ \hline & -1 \end{array} \right),$$

with I_n the identity matrix of order n. Then $PU^1(n+1) = U^1(n+1)/\mathbb{S}^1$ is the group of the holomorphic isometries of $(\mathbb{CH}^n, \langle , \rangle)$.

Throughout this paper we will work with the special orthogonal group $\mathrm{SO}(n)$, the identity component of the indefinite special orthogonal group $\mathrm{SO}_0^1(n)$, and the group of isometries of the Euclidean $(n{-}1)$-space $\mathrm{SO}(n{-}1) \propto \mathbb{R}^{n-1}$. These groups act on $\mathbb{CH}^n$ as subgroups of holomorphic isometries as follows:

$$A \in \mathrm{SO}(n) \longmapsto \left[\left(\begin{array}{c|c} A & \\ \hline & 1 \end{array} \right) \right] \in PU^1(n+1),$$

$$A \in \mathrm{SO}_0^1(n) \longmapsto \left[\left(\begin{array}{c|c} 1 & \\ \hline & A \end{array} \right) \right] \in PU^1(n+1),$$

$$(A,a)\in \mathrm{SO}(n-1)\propto \mathbb{R}^{n-1}\longmapsto \left[\left(\begin{array}{c|c|c} A & Aa^t & Aa^t \\ \hline -a & 1-|a|^2/2 & -|a|^2/2 \\ \hline a & |a|^2/2 & 1+|a|^2/2 \end{array}\right)\right] \in PU^1(n+1),$$

where $a=(a_1,\dots,a_{n-1})$. Here $[\ ]$ stands for the class in $U^1(n+1)/\mathbb{S}^1$.

2.2. Lagrangian submanifolds in $\mathbb{CH}^n$. Let ϕ be an isometric immersion of a Riemannian n-manifold M in $\mathbb{CH}^n$ (resp. $\mathbb{CP}^n$). ϕ is called *Lagrangian* if $\phi^*\Omega\equiv 0$. We denote the Levi-Civita connection of M and the connection on the normal bundle by ∇ and $\nabla^\perp$, respectively. The second fundamental form is denoted by σ. If ϕ is Lagrangian, the formulas of Gauss and Weingarten lead to

$$\nabla^\perp_X JY = J\nabla_X Y,$$

and the trilinear form $\langle\sigma(X,Y),JZ\rangle$ is totally symmetric for any tangent vector fields X, Y and Z.

If $\phi: M\longrightarrow \mathbb{CH}^n$ (resp. $\mathbb{CP}^n$) is a Lagrangian immersion of a simply connected manifold M, then ϕ has a horizontal lift with respect to the Hopf fibration to $\mathbb{H}^{2n+1}_1$ (resp. $\mathbb{S}^{2n+1}$), which is unique up to isometries. We denote this horizontal lift by $\tilde\phi$. We note that only Lagrangian immersions in $\mathbb{CH}^n$ (resp. $\mathbb{CP}^n$) have (locally) horizontal lifts. Horizontal immersions from n-manifolds in $\mathbb{H}^{2n+1}_1$ (resp. $\mathbb{S}^{2n+1}$) are called *Legendrian immersions* (see [H]). Thus we can paraphrase the above reasoning as follows: *Lagrangian immersions in $\mathbb{CH}^n$ (resp. $\mathbb{CP}^n$) are locally projections of Legendrian immersions in $\mathbb{H}^{2n+1}_1$ (resp. $\mathbb{S}^{2n+1}$).*

If H is the mean curvature vector of the immersion $\phi: M\longrightarrow\mathbb{CH}^n$, then ϕ is called minimal if $H=0$. Minimality means that the submanifold is critical for compact supported variations of the volume functional. In [O], the second variation of the volume functional was studied for minimal Lagrangian submanifolds of Kaehler manifolds. Among other things, it was proved that *minimal Lagrangian submanifolds in $\mathbb{CH}^n$ are stable and without nullity.*

Let $\langle\langle,\rangle\rangle$ be the restriction of $(,)$ to $\mathbb{R}^{n+1}\equiv\Re\mathbb{C}^{n+1}$. The real hyperbolic space $\mathbb{RH}^n$ endowed with its canonical metric of constant sectional curvature -1 is defined as the following hypersurface of $(\mathbb{R}^{n+1},\langle\langle,\rangle\rangle)$:

$$\mathbb{RH}^n=\{x\in\mathbb{R}^{n+1}\mid \langle\langle x,x\rangle\rangle=-1,\ x_{n+1}\geq 1\}.$$

We recall that $\mathbb{RH}^n$ has also a smooth compactification $\mathbb{RH}^n\cup\mathbb{S}^{n-1}(\infty)$, where $\mathbb{S}^{n-1}(\infty)=\pi(\mathcal{N})$, with $\mathcal{N}=\{x\in\mathbb{R}^{n+1}-\{0\}\mid\langle\langle x,x\rangle\rangle=0\}$ the light cone and π the projection given by the natural action of $\mathbb{R}^*$ over $\mathcal{N}$. In addition, $\mathrm{SO}^1_0(n+1)$ is a group of isometries of $(\mathbb{RH}^n,\langle\langle,\rangle\rangle)$.

$\mathbb{RH}^n$ can be isometrically embedded in $\mathbb{CH}^n$ as a totally geodesic Lagrangian submanifold in the standard way, via the map

$$x \in \mathbb{RH}^n \mapsto [x] \in \mathbb{CH}^n.$$

Moreover, up to congruences, $\mathbb{RH}^n$ is the only totally geodesic Lagrangian submanifold of $\mathbb{CH}^n$. We also point out (for later use in Section 4) that the totally umbilical submanifolds of $\mathbb{CH}^n$ (which were classified in [ChO]) are either totally geodesic or umbilical submanifolds of totally geodesic Lagrangian submanifolds. Thus, *up to congruences, the* $(n-1)$*-dimensional totally umbilical (non-totally geodesic) submanifolds of* $\mathbb{CH}^n$ *are the umbilical hypersurfaces of* $\mathbb{RH}^n$ *embedded in* $\mathbb{CH}^n$ *in the above manner*. Up to congruences, the umbilical hypersurfaces of $\mathbb{RH}^n$ can be described as follows:

(1) *Geodesic spheres*. Given $r > 0$, let $\psi : \mathbb{S}^{n-1} \to \mathbb{RH}^n$ be the embedding given by

$$\psi(x) = (\sinh r\; x, \cosh r)\,.$$

Then $\psi(\mathbb{S}^{n-1})$ is the geodesic sphere of $\mathbb{RH}^n$ of center $(0, \dots, 0, 1)$ and radius r.

(2) *Tubes over hyperplanes*. Given $r > 0$, let $\psi : \mathbb{RH}^{n-1} \to \mathbb{RH}^n$ be the embedding given by

$$\psi(x) = (\sinh r, \cosh r\; x)\,.$$

Then $\psi(\mathbb{RH}^{n-1})$ is the tube to distance r over the hyperplane dual to $(1, 0, \dots, 0)$.

(3) *Horospheres*. Let $\psi : \mathbb{R}^{n-1} \to \mathbb{RH}^n$ be the embedding given by

$$\psi(x) = \left(x, \frac{|x|^2}{2}, \frac{|x|^2}{2} + 1\right).$$

Then $\psi(\mathbb{R}^{n-1})$ is a horosphere of $\mathbb{RH}^n$ with infinity point $\pi(0, \dots, 0, 1, 1)$.

We will refer to these examples as $(n-1)$-*geodesic spheres*, $(n-1)$-*tubes over hyperplanes* and $(n-1)$-*horospheres* of $\mathbb{CH}^n$.

3. Examples of minimal Lagrangian submanifolds with symmetries

In this section we describe the minimal Lagrangian submanifolds of $\mathbb{CH}^n$ that are invariant under the actions of $\mathrm{SO}(n)$, $\mathrm{SO}^1_0(n)$, and $\mathrm{SO}(n-1) \propto \mathbb{R}^{n-1}$, as subgroups of the isometries of $\mathbb{CH}^n$ given in Section 2.1. These manifolds may be regarded as the simplest examples of minimal Lagrangian submanifolds.

3.1. Examples invariant under $\mathrm{SO}(n)$.

THEOREM 1. *For any $\rho > 0$ there exists a minimal (non-totally geodesic) Lagrangian embedding*

$$\Phi_\rho : \mathbb{R} \times \mathbb{S}^{n-1} \longrightarrow \mathbb{CH}^n$$

defined by

$$\Phi_\rho(s,x) = \left[\left(\sinh r(s) \, \exp\left(ia \int_0^s \frac{dt}{\sinh^{n+1} r(t)} \right) x, \right.\right.$$
$$\left.\left. \cosh r(s) \, \exp\left(ia \int_0^s \frac{\tanh^2 r(t)\, dt}{\sinh^{n+1} r(t)} \right) \right) \right],$$

where $r(s)$, $s \in \mathbb{R}$, is the unique solution to

(1) $\quad r'' \sinh r \cosh r = (1-(r')^2)(\sinh^2 r + n \cosh^2 r),\ r(0) = \rho,\ r'(0) = 0,$

and $a = \cosh\rho \sinh^n \rho$.

The map Φ_ρ is invariant under the action of $\mathrm{SO}(n)$ and satisfies

$$\int_{\mathbb{R}\times\mathbb{S}^{n-1}} |\sigma|^n dv \, < \infty,$$

where dv is the canonical measure of the complete induced metric $ds^2 + \sinh^2 r(s) g_0$, with g_0 the canonical metric of the unit sphere $\mathbb{S}^{n-1}$.

Moreover, any minimal (non-totally geodesic) Lagrangian immersion in $\mathbb{CH}^n$ that is invariant under the action of $\mathrm{SO}(n)$, $n \geq 3$, is congruent to an open subset of one of the above submanifolds.

Proof. We begin with the analysis of the differential equation (1). The energy integral of (1) is given by

$$(1-(r')^2)\cosh^2 r \sinh^{2n} r = \text{constant},$$

which is equivalent to

$$(r')^2 + \frac{\mathrm{ch}_\rho^2 \, \mathrm{sh}_\rho^{2n}}{\cosh^2 r \sinh^{2n} r} = 1,$$

where $\mathrm{ch}_\rho = \cosh\rho$ and $\mathrm{sh}_\rho = \sinh\rho$.

By the qualitative theory of ordinary differential equations, for any initial condition $\rho = r(0) > 0$, there exists a unique solution $r(s)$ to (1), defined on the whole space $\mathbb{R}$. This will be the only absolute minimum of $r(s)$ and $r(-s) = r(s)$ for $s \in \mathbb{R}$.

It is easy to prove that Φ_ρ is a Lagrangian immersion that is invariant under the action of $\mathrm{SO}(n)$. The induced metric $ds^2 + \sinh^2 r(s) g_0$ is a complete metric, since $\sinh^2 r(s) \geq \sinh^2 \rho$. We consider the orthonormal frame for this metric given by

$$e_1 = \partial_s,\ e_j = \frac{v_j}{\sinh r}, \quad j = 2, \ldots, n,$$

where $\{v_2, \ldots, v_n\}$ is an orthonormal frame of $(\mathbb{S}^{n-1}, g_0)$, and we compute the second fundamental form σ of Φ_ρ:

$$\sigma(e_1, e_1) = -\frac{(n-1)Je_1}{\sinh^{n+1} r}, \quad \sigma(e_1, e_j) = \frac{Je_j}{\sinh^{n+1} r}, \quad \sigma(e_j, e_k) = \frac{\delta_{jk} Je_1}{\sinh^{n+1} r}.$$

Using this, it is not difficult to prove that Φ_ρ is minimal, and a computation shows that

$$\int_{\mathbb{R} \times \mathbb{S}^{n-1}} |\sigma|^n dv = 2((n+2)(n-1))^{n/2} c_{n-1} \int_0^{+\infty} \frac{ds}{\sinh^{n^2+1} r(s)},$$

where c_{n-1} denotes the volume of $(\mathbb{S}^{n-1}, g_0)$. Making the change of variable $t = \sinh r(s)$, we get

$$\int_0^{+\infty} \frac{ds}{\sinh^{n^2+1} r(s)} = \int_{\sinh \rho}^{+\infty} \frac{dt}{t^{n^2-n+1}\sqrt{t^{2n+2} + t^{2n} - \mathrm{ch}_\rho^2 \,\mathrm{sh}_\rho^{2n}}}.$$

The latter integral is a hyperelliptic integral, and we can establish its convergence using numerical methods.

We next show that Φ_ρ is an embedding. Suppose $\Phi_\rho(s, x) = \Phi_\rho(\hat{s}, \hat{x})$. Then there exists $\theta \in \mathbb{R}$ such that the horizontal lift $\tilde{\Phi}_\rho$ of our immersion verifies

$$\tilde{\Phi}_\rho(\hat{s}, \hat{x}) = e^{i\theta} \tilde{\Phi}_\rho(s, x).$$

From the definition of Φ_ρ we deduce that $r(\hat{s}) = r(s)$, and so we have $\hat{s} = \pm s$. If $\hat{s} = s$, then necessarily $\hat{x} = x$. But if $\hat{s} = -s$, we get

$$\hat{x} = \exp\left(2i \int_0^s \frac{dt}{\cosh^2 r(t) \sinh^{n+1} r(t)}\right) x.$$

Using a similar reasoning as above, we can check that the increasing function

$$s \to 2 \int_0^s \frac{dt}{\cosh^2 r(t) \sinh^{n+1} r(t)}$$

is strictly less than π. Since the coordinates of x and $\hat{x}$ are real numbers, it follows that the case $\hat{s} = -s$ is impossible. Hence Φ_ρ must be an embedding.

Conversely, let $\phi : M \longrightarrow \mathbb{CH}^n$ be a non-totally geodesic minimal Lagrangian immersion that is invariant under the action of $\mathrm{SO}(n)$, and let $\tilde{\phi}$ be a local horizontal lift of ϕ to $\mathbb{H}_1^{2n+1}$. Let p be any point of M and let $z = (z_1, \ldots, z_{n+1}) = \tilde{\phi}(p)$. As ϕ is invariant under the action of $\mathrm{SO}(n)$, for any matrix A in the Lie algebra of $\mathrm{SO}(n)$ the curve $s \to [ze^{s\hat{A}}]$ with

$$\hat{A} = \left(\begin{array}{c|c} \mathrm{A} & \\ \hline & 0 \end{array}\right)$$

lies in the submanifold. Thus its tangent vector at $s = 0$ satisfies

$$\Pi_*(z\hat{A} + (z\hat{A}, z)z) \in \phi_*(T_pM).$$

Since ϕ is Lagrangian, this implies that

$$\Im(z\hat{A}\hat{B}\bar{z}^t) = 0$$

for any n-matrixes A and B in the Lie algebra of $\mathrm{SO}(n)$. Since $n \geq 3$ this easily implies that $\Re(z_1, \dots, z_n)$ and $\Im(z_1, \dots, z_n)$ must be linearly dependent. As $\mathrm{SO}(n)$ acts transitively on $\mathbb{S}^{n-1}$, we obtain that z is in the orbit (under the action of $\mathrm{SO}(n)$ described above) of the point $(a+ib, 0, \dots, 0, z_{n+1})$, with $a^2 + b^2 = |z_{n+1}|^2 - 1$. This implies that locally $\tilde{\phi}$ is the orbit under the action of $\mathrm{SO}(n)$ of a curve in $\mathbb{H}^3_1 \equiv \mathbb{H}^{2n+1}_1 \cap \{z_2 = \cdots = z_n = 0\}$. Therefore M is locally $I \times \mathbb{S}^{n-1}$, with I an interval in $\mathbb{R}$, and the lift $\tilde{\phi} : I \times \mathbb{S}^{n-1} \to \mathbb{H}^{2n+1}_1$ is given by

$$\tilde{\phi}(s, x) = (\gamma_1(s)x, \gamma_2(s)),$$

where $\gamma(s) = (\gamma_1(s), \gamma_2(s))$ is a horizontal curve in $\mathbb{H}^3_1$. Since the curve γ in $\mathbb{H}^3_1$ is horizontal, we can find real functions $r = r(s) > 0$ and $f = f(s)$, such that

$$\gamma(s) = \Bigg(\sinh r(s) \exp\left(i \int_{s_0}^s f(t)dt\right),\\ \cosh r(s) \exp\left(i \int_{s_0}^s f(t) \tanh^2 r(t) dt\right)\Bigg),$$

with $s_0 \in I$. Using the fact that ϕ is a minimal immersion, we can determine the functions f and r. After a long, but straightforward computation, one can prove that the immersion ϕ is minimal if and only if f and r satisfy

$$fr'' \tanh r = f^3 \tanh^2 r\,(n + \tanh^2 r) + (n+1)(r')^2 f + f' r' \tanh r. \tag{2}$$

If r is constant, then necessarily $f \equiv 0$ and γ degenerates into a point.

In order to analyze equation (2) in the remaining non-trivial case, we assume that γ is parameterized by the arc, i.e., $|\gamma'| = 1$. By computing $|\gamma'|$, we get

$$(r')^2 + f^2 \tanh^2 r = 1.$$

Differentiating this equation and applying it to (2), we can simplify equation (2) to

$$(n+1) f r' + f' \tanh r = 0.$$

If $f \equiv 0$, then $r(s)$ is a linear map, which leads to the totally geodesic case. The general solution to the above equation is

$$f(s) = \frac{a}{\sinh^{n+1} r(s)}, \quad a > 0.$$

It follows that $r(s)$ must satisfy the equation

$$(r')^2 + \frac{a^2}{\cosh^2 r \sinh^{2n} r} = 1.$$

The solutions to this differential equation are defined in whole space $\mathbb{R}$ and have only one critical point. Therefore we can take $s_0 = 0$ in the definition of γ and assume that $r'(0) = 0$, after a translation of the parameters. Thus $a^2 = \cosh^2 r(0) \sinh^{2n} r(0)$, and r is therefore a solution to (1). $\square$

We observe that for each $s \in \mathbb{R}$, $\Phi_\rho(\{s\} \times \mathbb{S}^{n-1})$ is a geodesic sphere of radius $r(s)$ and center $[(0, \dots, 0, 1)]$ in the Lagrangian subspace $\mathbb{R}\mathbb{H}^n_s$ of $\mathbb{C}\mathbb{H}^n$ defined by

$$\mathbb{R}\mathbb{H}^n_s = \Big\{[(x_1, \dots, x_{n+1})A(s)] \mid x_i \in \mathbb{R}\,, \quad \sum_{i=1}^n x_i^2 - x_{n+1}^2 = -1\,, \, x_{n+1} \geq 1\Big\},$$

where $A(s)$ is the matrix of $U^1(n+1)$ defined by

$$A(s) = \left(\begin{array}{c|c} e^{ia(s)} I_n & \\ \hline & e^{ib(s)} \end{array}\right),$$

with

$$a(s) = \int_0^s \frac{dt}{\sinh^{n+1} r(t)}, \quad b(s) = \int_0^s \frac{\tanh^2 r(t)\, dt}{\sinh^{n+1} r(t)}.$$

Moreover, if $s \neq s'$, then $\mathbb{R}\mathbb{H}^n_s \cap \mathbb{R}\mathbb{H}^n_{s'} = [(0, \dots, 0, 1)]$. Hence $\{\Phi(\{s\} \times \mathbb{S}^{n-1})\,, s \in \mathbb{R}\,\}$ defines a foliation on the minimal Lagrangian submanifold by $(n-1)$-geodesic spheres of $\mathbb{C}\mathbb{H}^n$.

In a more general context, we can classify pairs of Lagrangian subspaces of $\mathbb{C}\mathbb{H}^n$ intersecting only in a point as follows (compare with Proposition 6.2 in [J1]).

PROPOSITION 1. *Let $\mathbb{R}\mathbb{H}^n_a$ and $\mathbb{R}\mathbb{H}^n_b$ be two Lagrangian subspaces of $\mathbb{C}\mathbb{H}^n$ which intersect only at $[(0, \dots, 0, 1)]$. Then there exist $\theta_1, \dots, \theta_n \in (0, \pi)$ and $A \in U^1(n+1)$ such that*

$$\mathbb{R}\mathbb{H}^n_a = \Big\{[(x_1, \dots, x_{n+1})A] \mid x_i \in \mathbb{R}\,, \sum_{i=1}^n x_i^2 - x_{n+1}^2 = -1\,, \, x_{n+1} \geq 1\Big\}$$

and

$$\mathbb{R}\mathbb{H}^n_b = \Big\{[(e^{i\theta_1} x_1, \dots, e^{i\theta_n} x_n, x_{n+1})A] \mid x_i \in \mathbb{R}, \\ \sum_{i=1}^n x_i^2 - x_{n+1}^2 = -1\,, \, x_{n+1} \geq 1\Big\}.$$

Two Lagrangian subspaces $\mathbb{R}\mathbb{H}^n_a$ and $\mathbb{R}\mathbb{H}^n_b$ which intersect only at $[(0, \dots, 0, 1)]$ with $\theta_1 = \dots = \theta_n$ are said to be in *normal position*. In particular, in our family of Lagrangian subspaces $\{\mathbb{R}\mathbb{H}^n_s \mid s \in \mathbb{R}\}$ any two Lagrangian subspaces $\mathbb{R}\mathbb{H}^n_s$ and $\mathbb{R}\mathbb{H}^n_{s'}$ are in normal position.

PROPOSITION 2. *Let $\phi : M \to \mathbb{CH}^n$, $n \geq 3$, be a minimal Lagrangian immersion of a compact manifold with boundary ∂M. If $\phi(\partial M)$ is the union of two geodesic spheres centered at $[(0,\dots,0,1)]$ in two Lagrangian subspaces in normal position, then ϕ is congruent to one of the examples given in Theorem 1.*

Proof. It is clear that, up to a holomorphic isometry of $\mathbb{CH}^n$, the Lagrangian subspaces in normal position can be taken as

$$\mathbb{RH}^n_1 = \left\{ [(x_1,\dots,x_{n+1})] \mid x_i \in \mathbb{R}\,,\ \sum_{i=1}^n x_i^2 - x_{n+1}^2 = -1\,,\ x_{n+1} \geq 1 \right\}$$

and

$$\mathbb{RH}^n_2 = \left\{ [(e^{i\theta}x_1,\dots,e^{i\theta}x_n,x_{n+1})] \mid x_i \in \mathbb{R}, \sum_{i=1}^n x_i^2 - x_{n+1}^2 = -1\,,\ x_{n+1} \geq 1 \right\}.$$

Now, these Lagrangian subspaces and their corresponding geodesic spheres centered at $[(0,\dots,0,1)]$ are invariant under the action of the group $\mathrm{SO}(n)$ on $\mathbb{CH}^n$ (see Section 2). Hence, if X is a Killing vector field in the Lie algebra of $\mathrm{SO}(n)$, then its restriction to the submanifold is a Jacobi field on M vanishing on ∂M. As the nullity of the submanifold is zero, X also vanishes along the submanifold M. This means that the submanifold is invariant under the action of $\mathrm{SO}(n)$. The result now follows from Theorem 1. $\square$

3.2. Examples invariant under $\mathrm{SO}^1_0(n)$.

THEOREM 2. *For any $\rho > 0$ there exists a minimal (non-totally geodesic) Lagrangian embedding*

$$\Psi_\rho : \mathbb{R} \times \mathbb{RH}^{n-1} \longrightarrow \mathbb{CH}^n$$

defined by

$$\Psi_\rho(s,x) = \left[\left(\sinh r(s)\, \exp\left(ia \int_0^s \frac{\coth^2 r(t)\, dt}{\cosh^{n+1} r(t)} \right), \cosh r(s)\, \exp\left(ia \int_0^s \frac{dt}{\cosh^{n+1} r(t)} \right) x \right) \right],$$

where $r(s)$, $s \in \mathbb{R}$, is the unique solution to

$$r'' \sinh r \cosh r = (1-(r')^2)(\cosh^2 r + n \sinh^2 r),\ r(0) = \rho,\ r'(0) = 0, \tag{3}$$

and $a = \sinh \rho \cosh^n \rho$.

The map Ψ_ρ is invariant under the action of $\mathrm{SO}^1_0(n)$ and satisfies

$$\int_{\mathbb{R}\times\mathbb{RH}^{n-1}} |\sigma|^n dv < \infty,$$

where dv *is the canonical measure of the complete induced metric* $ds^2 + \cosh^2 r(s)\ \langle\langle,\rangle\rangle$.

Moreover, any minimal (non-totally geodesic) Lagrangian immersion in $\mathbb{CH}^n$ *that is invariant under the action of* $\mathrm{SO}^1_0(n)$, $n \geq 3$, *is congruent to an open subset of one of the above submanifolds.*

We omit the proof of this result, as it is similar to the proof of Theorem 1.

3.3. Examples invariant under $\mathrm{SO}(n{-}1) \propto \mathbb{R}^{n-1}$.

THEOREM 3. *For any* $\rho > 0$ *there exists a minimal Lagrangian embedding*

$$\Upsilon_\rho : \mathbb{R} \times \mathbb{R}^{n-1} \longrightarrow \mathbb{CH}^n$$

defined by

$$\Upsilon_\rho(s,x) = \Bigg[e^{iA_{n+1}(s)}\Bigg(r(s)x, \frac{1+r(s)^2(|x|^2-1-2iA_{n+3}(s))}{2r(s)}, \frac{1+r(s)^2(|x|^2+1-2iA_{n+3}(s))}{2r(s)}\Bigg)\Bigg],$$

where $A_n(s) = \rho^n \int_0^s dt/r(t)^n$ *and* $r(s) = \rho \cosh^{\frac{1}{n+1}}((n+1)s)$.

The map Υ_ρ *is invariant under the action of* $\mathrm{SO}(n{-}1) \propto \mathbb{R}^{n-1}$ *and satisfies*

$$\int_{\mathbb{R}\times\mathbb{R}^{n-1}} |\sigma|^n dv < \infty,$$

where dv *denotes the canonical measure of the complete induced metric* $ds^2 + r(s)^2\langle,\rangle$, *and* $\langle,\rangle$ *is the canonical metric of the Euclidean space* $\mathbb{R}^{n-1}$.

Moreover, any minimal (non-totally geodesic) Lagrangian immersion in $\mathbb{CH}^n$ *that is invariant under the action of* $\mathrm{SO}(n-1) \propto \mathbb{R}^{n-1}$, $n \geq 3$, *is congruent to an open subset of one of the above submanifolds.*

Proof. As in Section 3.1, the geometric properties of Υ_ρ can be verified using the explicit formulas given in the statement of the theorem.

Conversely, let $\phi : M \to \mathbb{CH}^n$ be a non-totally geodesic minimal Lagrangian immersion that is invariant under the action of $\mathrm{SO}(n{-}1) \propto \mathbb{R}^{n-1}$, and let $\tilde\phi$ be a local horizontal lift of ϕ to $\mathbb{H}^{2n+1}_1$. Let p be a point of M and let $z = (z_1, \ldots, z_{n+1}) = \tilde\phi(p)$. As ϕ is invariant under the action of $\mathrm{SO}(n{-}1) \propto \mathbb{R}^{n-1}$, for any (A,a) in the Lie algebra of $\mathrm{SO}(n{-}1) \propto \mathbb{R}^{n-1}$, the curve $s \to [ze^{s\hat A}]$ with

$$\hat A = \left(\begin{array}{c|c|c} A & a^t & a^t \\ \hline -a & 0 & 0 \\ \hline a & 0 & 0 \end{array}\right)$$

lies in the submanifold. Hence,

$$\pi_*(z\hat A + (z\hat A, z)z) \in \phi_*(T_pM).$$

Since ϕ is Lagrangian, we deduce from this that

$$\Im(z\hat{A}\hat{B}^t\bar{z}^t + (z\hat{A}, z)(z, z\hat{B})) = 0,$$

for any elements $\hat{A}, \hat{B}$ in the Lie algebra of $\mathrm{SO}(n{-}1) \propto \mathbb{R}^{n-1}$. Since $n \geq 3$, it is easy to see from this that $(z_1, \dots, z_{n-1}) = (z_{n+1}{-}z_n)(x_1, \dots, x_{n-1})$, with $x = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. As $\mathrm{SO}(n{-}1) \propto \mathbb{R}^{n-1}$ acts transitively on $\mathbb{R}^{n-1}$, we conclude that z is in the orbit under the action of $\mathrm{SO}(n{-}1) \propto \mathbb{R}^{n-1}$ described above of the point

$$\left(0, \dots, 0, z_n - (z_{n+1}{-}z_n)\frac{|x|^2}{2}, z_{n+1} - (z_{n+1}{-}z_n)\frac{|x|^2}{2}\right).$$

This implies that locally $\tilde{\phi}$ is the orbit under the action of $\mathrm{SO}(n{-}1) \propto \mathbb{R}^{n-1}$ of a curve in $\mathbb{H}^3_1 \equiv \mathbb{H}^{2n+1}_1 \cap \{z_1 = \cdots = z_{n-1} = 0\}$. Therefore M is locally $I \times \mathbb{R}^{n-1}$, with I an interval in $\mathbb{R}$, and the lift $\tilde{\phi} : I \times \mathbb{R}^{n-1} \to \mathbb{H}^{2n+1}_1$ is given by

$$\tilde{\phi}(s, x) = (\gamma_2(s){-}\gamma_1(s))\left(x, \frac{|x|^2}{2}, \frac{|x|^2}{2}\right) + (0, \gamma_1(s), \gamma_2(s)),$$

where $\gamma(s) = (\gamma_1(s), \gamma_2(s))$ is a horizontal curve in $\mathbb{H}^3_1$. Writing

$$(\gamma_2 - \gamma_1)(s) = r(s)\exp\left(i\int_{s_0}^{s} f(t)dt\right)$$

with real functions $r = r(s) > 0$ and $f = f(s)$ and using the assumption that γ is horizontal, it follows that

$$\begin{aligned}\gamma(s) = \exp\left(i\int_{s_0}^{s} f(t)dt\right)\Bigg(&\frac{1 - r(s)^2}{2r(s)} - ir(s)\int_{s_0}^{s}\frac{f(t)}{r(t)^2}dt,\\ &\qquad\frac{1 + r(s)^2}{2r(s)} - ir(s)\int_{s_0}^{s}\frac{f(t)}{r(t)^2}dt\Bigg),\end{aligned}$$

with $s_0 \in I$. Arguing as in Section 3.1, we see that the minimality of the immersion ϕ translates into the equation

$$(n+1)f((r')^2 + r^2f^2) - frr'' + f'rr' + f(r')^2 = 0. \tag{4}$$

If r is constant, then necessarily $f \equiv 0$ and γ degenerates into a point.

To analyze the equation (4), we assume that γ is parameterized by the arc, i.e., $|\gamma'| = 1$. By computing $|\gamma'|$ we get

$$(r'/r)^2 + f^2 = 1.$$

Differentiating this equation and using it again in (4), we obtain

$$(n+1)fr' + f'r = 0.$$

If $f \equiv 0$, then $r(s) = \mu e^{\pm s}$ and the immersion ϕ is totally geodesic. The general solution is given by

$$f(s) = \frac{a}{r(s)^{n+1}}, \quad a > 0.$$

It follows that $r(s)$ must satisfy the equation

$$(r')^2 + \frac{a^2}{r^{2n}} = r^2.$$

The general solution of this equation is, up to a translation of parameters, the solution given the statement of the Theorem, with $a = \rho^{n+1}$. $\square$

4. More examples of minimal Lagrangian submanifolds

The examples given in Theorems 1, 2 and 3 have been constructed in a common way. By analyzing this construction, we will obtain new examples of minimal Lagrangian submanifolds in $\mathbb{CH}^n$. In fact, the examples in Theorems 1, 2 and 3 are obtained, respectively, as follows:

$$\begin{aligned}
(s,x) \in \mathbb{R} \times \mathbb{S}^{n-1} &\mapsto [(\gamma_1(s)\, x, \gamma_2(s))] \in \mathbb{CH}^n, \\
(s,x) \in \mathbb{R} \times \mathbb{CH}^{n-1} &\mapsto [(\gamma_1(s), \gamma_2(s)\, x)] \in \mathbb{CH}^n, \\
(s,x) \in \mathbb{R} \times \mathbb{R}^{n-1} &\mapsto \Big[(\gamma_2(s) - \gamma_1(s)) \left(x, \frac{|x|^2}{2}, \frac{|x|^2}{2} \right) \\
&\qquad + (0, \gamma_1(s), \gamma_2(s))\Big] \in \mathbb{CH}^n.
\end{aligned}$$

Here $[(\gamma_1(s), \gamma_2(s))]$ denote certain curves in $\mathbb{CH}^1$ and

$$\begin{aligned}
x \in \mathbb{S}^{n-1} &\mapsto [x] \in \mathbb{CP}^{n-1}, \\
x \in \mathbb{RH}^{n-1} &\mapsto [x] \in \mathbb{CH}^{n-1}, \\
x \in \mathbb{R}^{n-1} &\mapsto x \in \mathbb{C}^{n-1}
\end{aligned}$$

are the totally geodesic Lagrangian submanifolds in the $(n-1)$-dimensional complex models.

We now construct new minimal Lagrangian submanifolds in $\mathbb{CH}^n$ using the same curves as above, but taking an arbitrary minimal Lagrangian submanifold in the $(n-1)$-dimensional complex models instead of the above totally geodesic Lagrangian submanifolds. In fact, it is straightforward to prove the following result.

Proposition 3.

(a) *Given a solution* $r(s)$ *of equation* (1) *in Theorem 1 and a minimal Lagrangian immersion* $\phi : N^{n-1} \to \mathbb{CP}^{n-1}$ *of a simply connected manifold* N*, the map* $\Phi : \mathbb{R} \times N \to \mathbb{CH}^n$ *defined by*

$$\Phi(s,x) = \left[\left(\sinh r(s)\, \exp\left(ia \int_0^s \frac{dt}{\sinh^{n+1} r(t)}\right) \tilde{\phi}(x), \right.\right. \\ \left.\left. \cosh r(s)\, \exp\left(ia \int_0^s \frac{\tanh^2 r(t)\, dt}{\sinh^{n+1} r(t)}\right)\right)\right]$$

is a minimal Lagrangian immersion in $\mathbb{CH}^n$*, where* $\tilde{\phi} : N \to \mathbb{S}^{2n-1}$ *is the horizontal lift of* ϕ *with respect to the Hopf fibration* $\Pi : \mathbb{S}^{2n-1} \to \mathbb{CP}^{n-1}$.

(b) *Given a solution* $r(s)$ *of equation* (3) *in Theorem 2 and a minimal Lagrangian immersion* $\psi : N^{n-1} \to \mathbb{CH}^{n-1}$ *of a simply connected manifold* N*, the map* $\Psi : \mathbb{R} \times N \to \mathbb{CH}^n$ *defined by*

$$\Psi(s,x) = \left[\left(\sinh r(s)\, \exp\left(ia \int_0^s \frac{\coth^2 r(t)\, dt}{\cosh^{n+1} r(t)}\right), \right.\right. \\ \left.\left. \cosh r(s)\, \exp\left(ia \int_0^s \frac{dt}{\cosh^{n+1} r(t)}\right) \tilde{\psi}(x)\right)\right]$$

is a minimal Lagrangian immersion in $\mathbb{CH}^n$*, where* $\tilde{\psi} : N \to \mathbb{H}_1^{2n-1}$ *is the horizontal lift of* ϕ *with respect to the Hopf fibration* $\Pi : \mathbb{H}^{2n-1} \to \mathbb{CH}^{n-1}$.

(c) *Given* $\rho > 0$*, a minimal Lagrangian immersion* $\eta : N^{n-1} \to \mathbb{C}^{n-1}$ *of a simply connected manifold* N *and a function* $f : N \longrightarrow \mathbb{C}$ *satisfying* $\Re f = |\eta|^2$ *and* $v(\Im f) = 2\langle \eta_* v, J\eta\rangle$ *for any vector* v *tangent to* N*, the map* $\Upsilon : \mathbb{R} \times N \to \mathbb{CH}^n$ *defined by*

$$\Upsilon(s,x) = \left[e^{iA_{n+1}(s)} \left(r(s)\eta(x), \frac{1 + r(s)^2(f(x) - 1 - 2iA_{n+3}(s))}{2r(s)}, \right.\right. \\ \left.\left. \frac{1 + r(s)^2(f(x) + 1 - 2iA_{n+3}(s))}{2r(s)}\right)\right],$$

where $A_n(s) = \rho^n \int_0^s dt/r(t)^n$ *and* $r(s) = \rho \cosh^{1/(n+1)}((n+1)s)$*, is a minimal Lagrangian immersion in* $\mathbb{CH}^n$.

We observe that if we take the map $\eta : \mathbb{R}^{n-1} \to \mathbb{C}^{n-1}$ in Proposition 3(c) to be the totally geodesic immersion $\eta(x) = x$, then $\Im f$ is constant and it is easy to prove that the corresponding immersion is congruent to that given in Theorem 3.

It is interesting to note that the totally geodesic Lagrangian submanifolds of $\mathbb{CH}^n$ can also be described in a form similar to the examples given in

Theorems 1, 2 and 3. In fact, we can give three different descriptions of the totally geodesic Lagrangian submanifolds of $\mathbb{CH}^n$:

$$(s,x) \in \mathbb{R}^+ \times \mathbb{S}^{n-1} \mapsto [(\sinh s\, x, \cosh s)] \in \mathbb{CH}^n,$$

$$(s,x) \in \mathbb{R} \times \mathbb{RH}^{n-1} \mapsto [(\sinh s, \cosh s\, x)] \in \mathbb{CH}^n,$$

and

$$(s,x) \in \mathbb{R} \times \mathbb{R}^{n-1} \mapsto \left[e^s \left(x, \frac{|x|^2}{2}, \frac{|x|^2}{2} \right) + (0, -\sinh s, \cosh s) \right] \in \mathbb{CH}^n.$$

In all three cases, the curve used is the geodesic $[(\sinh\, s, \cosh\, s)]$ of $\mathbb{CH}^1$ passing through the point $[(0,1)]$. As in Proposition 3, this idea leads to new examples of minimal Lagrangian submanifolds of $\mathbb{CH}^n$.

PROPOSITION 4.

(a) *Given a minimal Lagrangian immersion $\phi : N^{n-1} \to \mathbb{CP}^{n-1}$ of a simply connected manifold N, the map*

$$\begin{aligned} \Phi : \mathbb{R}^+ \times N &\to \mathbb{CH}^n \\ (s,x) &\mapsto \left[\left(\sinh s\, \tilde{\phi}(x)\, ,\, \cosh s \right) \right] \end{aligned}$$

is a minimal Lagrangian immersion in $\mathbb{CH}^n$, where $\tilde{\phi} : N \to \mathbb{S}^{2n-1}$ is the horizontal lift of ϕ with respect to the Hopf fibration $\Pi : \mathbb{S}^{2n-1} \to \mathbb{CP}^{n-1}$.

(b) *Given a minimal Lagrangian immersion $\psi : N^{n-1} \to \mathbb{CH}^{n-1}$, the map*

$$\begin{aligned} \Psi : \mathbb{R} \times N &\to \mathbb{CH}^n \\ (s,x) &\mapsto \left[\left(\sinh s\, ,\, \cosh s\, \tilde{\psi}(x) \right) \right], \end{aligned}$$

is a minimal Lagrangian immersion in $\mathbb{CH}^n$, where $\tilde{\psi} : N \to \mathbb{H}^{2n-1}_1$ is the horizontal lift of ϕ with respect to the Hopf fibration $\Pi : \mathbb{H}^{2n-1} \to \mathbb{CH}^{n-1}$.

(c) *Given a minimal Lagrangian immersion $\eta : N^{n-1} \to \mathbb{C}^{n-1}$ of a simply connected manifold N and a function $f : N \longrightarrow \mathbb{C}$ satisfying $\Re f = |\eta|^2$ and $v(\Im f) = 2\langle \eta_* v, J\eta \rangle$ for any vector v tangent to N, the map*

$$\begin{aligned} \Upsilon : \mathbb{R} \times N &\to \mathbb{CH}^n \\ (s,x) &\mapsto [e^s\,(\eta(x), f(x), f(x)) + (0, -\sinh s, \cosh s)] \end{aligned}$$

is a minimal Lagrangian immersion in $\mathbb{CH}^n$.

The examples described in Propositions 3 and 4 are unique in the following sense.

Proposition 5. *Let $\gamma = (\gamma_1, \gamma_2) : I \to \mathbb{H}^3_1$ be a Legendre curve.*

(a) *Given a Lagrangian immersion $\phi : N^{n-1} \to \mathbb{CP}^{n-1}$ of a simply connected manifold N, the map $\Phi : I \times N \to \mathbb{CH}^n$ defined by*

$$\Phi(s, x) = \left[\Big(\gamma_1(s) \tilde{\phi}(x), \gamma_2(s) \Big) \right],$$

where $\tilde{\phi} : N \to \mathbb{S}^{2n-1}$ is a horizontal lift of ϕ with respect to the Hopf fibration, is a minimal Lagrangian immersion if and only if Φ is congruent to one of the examples given in Propositions 3(a) *and* 4(a).

(b) *Given a Lagrangian immersion $\psi : N^{n-1} \to \mathbb{CH}^{n-1}$ of a simply connected manifold N, the map $\Psi : I \times N \to \mathbb{CH}^n$ defined by*

$$\Psi(s, x) = \left[\Big(\gamma_1(s), \gamma_2(s) \tilde{\psi}(x) \Big) \right],$$

where $\tilde{\psi} : N \to \mathbb{H}^{2n-1}_1$ is a horizontal lift of ψ with respect to the Hopf fibration, is a minimal Lagrangian immersion if and only if Ψ is congruent to one of the examples given in Propositions 3(b) *and* 4(b).

(c) *Given a Lagrangian immersion $\eta : N^{n-1} \to \mathbb{C}^{n-1}$ of a simply connected manifold N and a function $f : N \longrightarrow \mathbb{C}$ satisfying $\Re f = |\eta|^2$ and $v(\Im f) = 2\langle \eta_* v, J\eta \rangle$ for any vector v tangent to N, the map $\Upsilon : I \times N \to \mathbb{CH}^n$ defined by*

$$\Upsilon(s, x) = \left[(\gamma_2(s) - \gamma_1(s)) \left(\eta(x), \frac{f(x)}{2}, \frac{f(x)}{2} \right) + (0, \gamma_1(s), \gamma_2(s)) \right]$$

is a minimal Lagrangian immersion if and only if Υ is congruent to one of the examples given in Propositions 3(c) *and* 4(c).

Proof. In order to illustrate the idea, we will prove (a); (b) and (c) can be proved similarly.

By the properties of γ and ϕ, Φ is always a Lagrangian immersion. After a very long but straightforward computation, we arrive at the horizontal lift H^* of the mean curvature H of our Lagrangian immersion Φ, given by $nH^* = a(s) J\tilde{\phi}_s + (n-1)(\gamma_1 H^*_\phi, 0)/|\gamma_1|^2$, where

$$a = \frac{\langle \gamma'', J\gamma' \rangle}{|\gamma'|^4} + (n-1) \frac{\langle \gamma_1', J\gamma_1 \rangle}{|\gamma_1|^2 |\gamma'|^2}.$$

If we suppose that Φ is minimal, then necessarily ϕ is also minimal since H_ϕ must be zero and, in addition, $a \equiv 0$. We use this last equation to obtain an explicit expression for γ as in the proof of Theorem 1. A similar reasoning then shows that the only possible solutions for $r(s)$ are the solution of equation (1) and the trivial solution $r(s) = s$. $\square$

Remark 1. It is interesting to note the analogy between the above constructions of minimal Lagrangian submanifolds of $\mathbb{CH}^n$ and those given in the papers [CU2], [H] and [J1], where the ambient space is the complex Euclidean space $\mathbb{C}^n$. In fact, we can summarize some of the results in these papers as follows:

Proposition A ([CU2], [H], [J1]). *Let $\gamma : I \to \mathbb{C}^*$ be a regular curve and $\phi : N^{n-1} \to \mathbb{CP}^{n-1}$ a Lagrangian immersion of a simply connected manifold. Then the map $\Phi : I \times N \to \mathbb{C}^n$ defined by*

$$\Phi(s,x) = \gamma(s)\tilde{\phi}(x),$$

where $\tilde{\phi} : N \to \mathbb{S}^{2n-1}$ is a horizontal lift of ϕ with respect to the Hopf fibration, is a minimal Lagrangian submanifold if and only if ϕ is minimal and γ^n has curvature zero.

Thus, up to rotations in $\mathbb{C}^n$, the curve γ^n can be taken as $\gamma^n(s) = (s,c)$ with $c \geq 0$, i.e., $\Im \gamma^n = c$, where $\Im$ denotes the imaginary part. Hence, up to dilations, there are only two possibilities: $c = 0$ and $c = 1$. In the first case, the examples constructed in this way are cones with links $\tilde{\phi}$, and in the second case they are the examples given in [CU2, Remark 1], [H, Theorem A] and [J1, Theorem 6.4].

The ideas developed in Proposition 5 and Proposition A allow us to construct a wide family of Lagrangian submanifolds that are not necessarily minimal. This class of Lagrangian submanifolds has been thoroughly investigated in [RU] for the case when the ambient space is $\mathbb{C}^n$, and in [CMU] for the case when the ambient space is $\mathbb{CP}^n$ or $\mathbb{CH}^n$. Among other results, the class can be characterized by the existence of a closed and conformal vector field X on the Lagrangian submanifold satisfying $\sigma(X,X) = \rho\, JX$, for a certain function ρ.

5. A geometric characterization

As we pointed out in Section 3.1, the examples of minimal Lagrangian submanifolds of $\mathbb{CH}^n$ given in Theorem 1 are foliated by $(n-1)$-geodesic spheres of $\mathbb{CH}^n$ centered at the point $[(0,\dots,0,1)]$. In a similar way it can be checked that $\{\Psi_\rho(\{s\} \times \mathbb{RH}^{n-1}), s \in \mathbb{R}\}$ defines a foliation on the minimal Lagrangian submanifolds given in Theorem 2 by $(n-1)$-tubes over hyperplanes. Finally, $\{\Upsilon_\rho(\{s\} \times \mathbb{R}^{n-1}), s \in \mathbb{R}\}$ defines a foliation on the minimal Lagrangian submanifolds given in Theorem 3 by $(n-1)$-horospheres. In the following result we show that the examples described in Theorems 1, 2 and 3 are the only ones admitting such foliations.

Theorem 4. *Let $\phi : M \to \mathbb{CH}^n$, $n \geq 3$, be a minimal Lagrangian immersion in $\mathbb{CH}^n$.*

(a) *If ϕ is foliated by $(n-1)$-geodesic spheres of $\mathbb{CH}^n$, then ϕ is either totally geodesic or congruent to an open subset of one of the examples described in Theorem 1.*
(b) *If ϕ is foliated by $(n-1)$-tubes over hyperplanes of $\mathbb{CH}^n$, then ϕ is either totally geodesic or congruent to an open subset of one of the examples described in Theorem 2.*
(c) *If ϕ is foliated by $(n-1)$-horospheres of $\mathbb{CH}^n$, then ϕ is either totally geodesic or congruent to an open subset of one of the examples described in Theorem 3.*

Proof of (a). Our submanifold M is locally $I\times\mathbb{S}^{n-1}$, where I is an interval of $\mathbb{R}$ with $0\in I$, and for each $s\in I$, $\phi\left(\{s\}\times\mathbb{S}^{n-1}\right)$ is an $(n-1)$-geodesic sphere of $\mathbb{CH}^n$. It follows (see Section 2.2) that there exists a Lagrangian subspace

$$\mathbb{RH}^n_s=\{[zX(s)]\in\mathbb{CH}^n, z\in\mathbb{C}^{n+1}, z=\bar z\},$$

where $X(s)\in U^1(n+1)$, and there exists $Y(s)\in \mathrm{SO}^1_0(n+1)$ such that

$$\phi(s,x)=[(\sinh r(s)\,x,\cosh r(s))\,Y(s)X(s)]\,.$$

Setting $A(s)=X(s)Y(s)$, we obtain

$$\phi(s,x)=[(\sinh r(s)\,x,\cosh r(s))\,A(s)]\,,$$

with $A(s)\in U^1(n+1)$. Thus, $[(0,\dots,0,1)A(s)]$ and $r(s)$ are the center and the radius of the $(n-1)$-geodesic sphere $\phi(\{s\}\times\mathbb{S}^{n-1})$.

If we set

$$\hat\phi(s,x)=(\sinh r(s)\,x,\cosh r(s))\,A(s),$$

then $\hat\phi$ is a lift (not necessarily horizontal) of ϕ to $\mathbb{H}^{2n+1}_1$. But since (locally) Lagrangian immersions in $\mathbb{CH}^n$ have horizontal lifts to $\mathbb{H}^{2n+1}_1$, there exists a smooth function $\theta(s,x)$ such that $\tilde\phi=e^{i\theta}\hat\phi$ is a horizontal lift of ϕ to $\mathbb{H}^{2n+1}$. In particular, $(d\tilde\phi_{(s,x)}(0,v),\tilde\phi(s,x))=0$ for any $v\in T_x\mathbb{S}^{n-1}$, which means that $d\theta(v)=0$, and so $\theta(s,x)=\theta(s)$. Thus our horizontal lift is given by

$$\tilde\phi(s,x)=(\sinh r(s)\,x,\cosh r(s))\,B(s),$$

where $B(s)=e^{i\theta(s)}A(s)$. Moreover, as $(\tilde\phi_s,\tilde\phi)=0$ and $B(s)\in U^1(n+1)$, we obtain

$$(\sinh r(s)\,x,\,\cosh r(s))B'(s)S\bar B^t(s)(\sinh r(s)x,\cosh r(s))^t=0$$

for any $x\in\mathbb{S}^{n-1}$.

From the relation $B(s)S\bar B^t(s)=S$ we deduce $B'(s)S\bar B^t(s)+B(s)S{\bar B'}^t(s)=0$. Thus, $B'(s)S\bar B^t(s)=V(s)+iU(s)$, where $V(s)$ and $U(s)$ are real matrixes with $V(s)+V(s)^t=0$ and $U(s)=U(s)^t$. Hence the last equation becomes

$$(\sinh r(s)\,x,\cosh r(s))U(s)(\sinh r(s)x,\cosh r(s))^t=0,$$

for any $x \in \mathbb{S}^{n-1}$. From this equation it is easy to see that there exists a smooth function $a(s)$ such that the matrix $U(s)$ can be written as

$$U(s) = a(s) \left(\begin{array}{c|c} I_n & \\ \hline & -\tanh^2 r(s) \end{array} \right),$$

for any $s \in I$.

Now, we write $V(s)$ as follows:

$$V(s) = \left(\begin{array}{c|c} V_0(s) & -v^t(s) \\ \hline v(s) & 0 \end{array} \right).$$

Let $Z(s)$ be the solution to the equation

$$Z'(s) + Z(s)V_0(s) = 0, \quad Z(0) = I_n.$$

Since $V_0(s) + V_0^t(s) = 0$, we have $(Z(s)Z^t(s))' = 0$ and so $Z(s)Z^t(s) = Z(0)Z^t(0) = I_n$. Thus $Z(s)$ is a curve in $O(n)$, and reparametrizing our immersion by

$$(s, x) \in I \times \mathbb{S}^{n-1} \mapsto (s, xZ(s)) \in I \times \mathbb{S}^{n-1},$$

we obtain

$$\tilde{\phi}(s, x) = (\sinh r(s)\, x, \cosh r(s))\, C(s),$$

where

$$C(s) = \left(\begin{array}{c|c} Z(s) & \\ \hline & 1 \end{array} \right) B(s).$$

Now, $C'(s)S\bar{C}^t(s) = W(s) + iU(s)$, where

$$W(s) = \left(\begin{array}{c|c} 0 & -w(s)^t \\ \hline w(s) & 0 \end{array} \right),$$

with $w(s) = v(s)Z^t(s)$.

We now use the minimality of our immersion. To this end we first construct an orthonormal basis for our submanifold. For any $x \in \mathbb{S}^{n-1}$, the vectors

$$z(s) = \tanh^{-1} r(s)\, w(s) - (\tanh^{-1} r(s)\, w(s)x^t)x \in \mathbb{R}^n$$

are in $T_x\mathbb{S}^{n-1}$. It is easy to check that, for any $s \in I$, the vector $(1, -z(s))$ is a tangent vector to M and orthogonal to $(0, v)$ for any $v \in T_x\mathbb{S}^{n-1}$. Thus, an orthonormal basis of the submanifold $M = I \times \mathbb{S}^{n-1}$ at the point (s, x) is given by

$$e_1 = \frac{(1, -z(s))}{|(1, -z(s))|}\,,\; e_i = \frac{(0, v_i)}{\sinh r(s)}\,,\; i = 2, \dots, n,$$

where $\{v_2, \dots, v_n\}$ is an orthonormal basis of $T_x\mathbb{S}^{n-1}$. As $H = 0$, we have, in particular,

$$\langle \sum_{i=1}^{n} \sigma(e_i, e_i), J\tilde{\phi}_*(0, v) \rangle = 0$$

for any $v \in T_x\mathbb{S}^{n-1}$. But it is easy to check that

$$\langle \sigma(e_i, e_i), J\tilde{\phi}_*(0, v)\rangle = 0$$

for $i = 2, \dots, n$. Thus the above equation becomes

$$\langle \sigma(e_1, e_1), J\tilde{\phi}_*(0, v)\rangle = 0.$$

Using the definition of e_1, we obtain

$$\langle \tilde{\phi}_{ss}, J\tilde{\phi}_*(0, v)\rangle = 2\langle (\tilde{\phi}_s)_*(0, z(s)), J\tilde{\phi}_*(0, v)\rangle$$

for any $v \in T_x\mathbb{S}^{n-1}$. Using the properties of the second fundamental form of Lagrangian submanifolds, the definition of $z(s)$, and the fact that $C'(s) = (W(s)+iU(s))SC(s)$, it is now straightforward to prove that the last equation becomes

$$\frac{a(s)}{\cosh^2 r(s)} w(s) v^t = 0,$$

for any $s \in I$, $v \in T_x\mathbb{S}^{n-1}$ and $x \in \mathbb{S}^{n-1}$. Thus we have $a(s)w(s) = 0$ for any $s \in I$. Hence, setting

$$I_1 = \{s \in I \mid a(s) = 0\}, \quad I_2 = \{s \in I \mid w(s) = 0\},$$

we have $I_1 \cup I_2 = I$.

We first consider the open set $I - I_2$, where $U(s) = 0$ and thus $C'(s) = W(s)SC(s)$. This implies that the matrices $C(s)$ are real and hence are in $O^1(n+1)$. Therefore $\phi((I - I_2) \times \mathbb{S}^{n-1})$ lies in $\mathbb{RH}^n$, and so ϕ is totally geodesic on this open subset.

We next consider the open set $I - I_1$. On this set we have $W(s) = 0$ and thus $C'(s) = iU(s)SC(s)$. Using the definition of $U(s)$, we integrate this differential equation to obtain

$$C(s) = \left(\begin{array}{c|c} \exp\left(i \int_{s_0}^s a(r)dr\right) I_n & \\ \hline & \exp\left(-i \int_{s_0}^s a(r)\tanh^2 r dr\right) \end{array} \right).$$

This shows that, in this case, our immersion is invariant under the action of $\mathrm{SO}(n)$. Hence the map ϕ, defined on the open set $(I - I_1) \times \mathbb{S}^{n-1}$, is one of the examples described in Theorem 1.

Finally, since the second fundamental forms of the examples given in Theorem 1 are non-trivial and the set I is connected, the case when both $\mathrm{Int}(I_1) \neq \emptyset$ and $\mathrm{Int}(I_2) \neq \emptyset$ is impossible. This completes the proof of part (a).

We omit the proof of (b) because it is quite similar to that of (a).

Proof of (c). In this case our submanifold M is locally $I \times \mathbb{R}^{n-1}$, where I is an interval of $\mathbb{R}$ with $0 \in I$, and for each $s \in I$, $\phi(\{s\} \times \mathbb{R}^{n-1})$ is an

$(n-1)$-horosphere of some manifold $\mathbb{RH}^n_s$ embedded in $\mathbb{CH}^n$ as a totally geodesic Lagrangian submanifold. As in the proof of (a), we obtain

$$\phi(s,x) = \left[\hat{\phi}(s,x)\right] = [f(x)A(s)]\,,$$

where

$$f(x) = \left(x, \frac{|x|^2}{2}, \frac{|x|^2}{2} + 1\right).$$

Thus $\hat{\phi}$ is a lift (not necessarily horizontal) of ϕ to $\mathbb{H}^{2n+1}_1$. But since (locally) Lagrangian immersions in $\mathbb{CH}^n$ have horizontal lifts to $\mathbb{H}^{2n+1}_1$, there exists a smooth function $\theta(s,x)$ such that $\tilde{\phi} = e^{i\theta}\hat{\phi}$ is a horizontal lift of ϕ to $\mathbb{H}^{2n+1}$. In particular, $(d\tilde{\phi}_{(s,x)}(0,v), \tilde{\phi}(s,x)) = 0$ for any $v \in T_x\mathbb{S}^{n-1}$, which means that $d\theta(v) = 0$, and so $\theta(s,x) = \theta(s)$. Thus, our horizontal lift is given by

$$\tilde{\phi}(s,x) = f(x)B(s),$$

where $B(s) = e^{i\theta(s)}A(s)$. Moreover, as $(\tilde{\phi}_s, \tilde{\phi}) = 0$, we obtain

$$f(x)B'(s)S\bar{B}(s)^t f(x)^t = 0,$$

for any $s \in I$ and any $x \in \mathbb{R}^{n-1}$.

As in the proof of (a) we conclude from this that $B'(s)S\bar{B}(s)^t = V(s) + iU(s)$, where $V(s)$ and $U(s)$ are real matrixes with $V(s) + V(s)^t = 0$ and $U(s) = U(s)^t$. The last equation therefore becomes

$$f(x)U(s)f(x)^t = 0,$$

for any $s \in I$ and any $x \in \mathbb{R}^{n-1}$. From this equation it is easy to see that the matrix $U(s)$ has the form

$$U(s) = a(s)\left(\begin{array}{c|c|c} I_{n-1} & & \\ \hline & 2 & -1 \\ \hline & -1 & \end{array}\right),$$

for certain smooth function $a(s)$.

Now, define

$$V(s) = \left(\begin{array}{c|c|c} V_0(s) & -v_1(s)^t & -v_2(s)^t \\ \hline v_1(s) & 0 & -\rho(s) \\ \hline v_2(s) & \rho(s) & 0 \end{array}\right)$$

and let $Z(s)$ be the solution to the differential equation

$$Z'(s) + Z(s)V_0(s) = 0, \quad Z(0) = I_n.$$

Since $V_0(s) + V_0^t(s) = 0$, we obtain $(ZZ^t)'(s) = 0$, and so $Z(s)Z^t(s) = Z(0)Z^t(0) = I_n$. Thus $Z(s)$ is a curve in $O(n-1)$. Reparametrizing our immersion by

$$(s,x) \in I \times \mathbb{R}^{n-1} \mapsto (s, xZ(s)) \in I \times \mathbb{R}^{n-1},$$

we obtain

$$\tilde{\phi}(s,x) = f(x)C(s),$$

where

$$C(s) = \left(\begin{array}{c|c} Z(s) & \\ \hline & I_2 \end{array}\right) B(s).$$

It is easy to check that $C'(s)S\bar{C}^t(s) = W(s) + iU(s)$, where

$$W(s) = \left(\begin{array}{c|c|c} & -w_1^t(s) & -w_2^t(s) \\ \hline w_1(s) & & -\rho(s) \\ \hline w_2(s) & \rho(s) & \end{array}\right)$$

with $w_i(s) = v_i(s)Z^t(s)$ for $i = 1, 2$.

We now use the minimality of our immersion. To this end, we first construct an orthonormal basis in our submanifold at $(s,0)$. It is easy to check that the vector $(1, -w_2(s))$ is a tangent vector to M in $(s,0)$, and orthogonal to $(0,v)$ for any $v \in \mathbb{R}^{n-1}$. Thus, an orthonormal basis of the submanifold $M = I \times \mathbb{R}^{n-1}$ at the point $(s,0)$ is given by

$$e_1 = \frac{(1, -w_2(s))}{|(1, -w_2(s))|}, \; e_i = (0, v_i), \; i = 2, \dots, n,$$

where $\{v_2, \dots, v_n\}$ is an orthonormal basis of $\mathbb{R}^{n-1}$. As ϕ is a minimal immersion, we have, in particular,

$$\langle H(s,0), J\tilde{\phi}_*(0,v)\rangle = 0$$

for any $v \in \mathbb{R}^{n-1}$. But it is easy to check that

$$\langle \sigma(e_i, e_i), J\tilde{\phi}_*(0,v)\rangle = 0$$

for $i = 2, \dots, n$. Thus the above equation becomes

$$\langle \sigma(e_1, e_1), J\tilde{\phi}_*(0,v)\rangle = 0.$$

Using the definition of e_1, we obtain

$$\langle \tilde{\phi}_{ss}, J\tilde{\phi}_*(0,v)\rangle = 2\langle (\tilde{\phi}_s)_*(0, w_2(s)), J\tilde{\phi}_*(0,v)\rangle$$

for any $v \in \mathbb{R}^{n-1}$. Using the properties of the second fundamental form of Lagrangian submanifolds and the fact that $C'(s) = (W(s) + iU(s))SC(s)$, it is now straightforward to prove that the last equation becomes

$$a(s)(w_1(s) + w_2(s))v^t = 0,$$

for all $s \in I$ and $v \in \mathbb{R}^{n-1}$. Thus $a(s)(w_1(s) + w_2(s)) = 0$ for any $s \in I$. Hence, setting

$$I_1 = \{s \in I \mid a(s) = 0\}, \quad I_2 = \{s \in I \mid w_1(s) + w_2(s) = 0\},$$

we have $I_1 \cup I_2 = I$.

We first consider the open set $I - I_2$ where $U(s) = 0$ and so $C'(s) = W(s)SC(s)$. This implies that the matrices $C(s)$ are real and hence are in $O^1(n+1)$. Therefore, $\phi((I - I_2) \times \mathbb{R}^{n-1})$ lies in $\mathbb{RH}^n$, and so ϕ is totally geodesic on this open subset.

We next consider the open set $I - I_1$. On this set we have

$$W(s) = \left(\begin{array}{c|c|c} & -w_1^t(s) & w_1^t(s) \\ \hline w_1(s) & & -\rho(s) \\ \hline -w_1(s) & \rho(s) & \end{array}\right).$$

If $w(s)$ is a solution of $w'(s) + \rho(s)w(s) - w_1(s) = 0$, we can reparametrize our immersion as

$$(s, x) \in (I - I_1) \times \mathbb{R}^{n-1} \mapsto (s, x + w(s)) \in (I - I_1) \times \mathbb{R}^{n-1},$$

so that the immersion is given by

$$\tilde{\phi}(s, x) = f(x)D(s),$$

where

$$D(s) = \left(\begin{array}{c|c|c} I_{n-1} & w^t(s) & w^t(s) \\ \hline -w(s) & 1-\lambda & -\lambda \\ \hline w(s) & \lambda & 1+\lambda \end{array}\right) C(s)$$

with $\lambda = |w(s)|^2/2$. Now it is easy to check that

$$D'(s)S\bar{D}^t(s) = W^1(s) + iU(s)$$

with

$$W^1(s) = \left(\begin{array}{c|c|c} & & \\ \hline & & -\rho(s) \\ \hline & \rho(s) & \end{array}\right).$$

Now let

$$Y(s) = \left(\begin{array}{c|c|c} I_{n-1} & & \\ \hline & \cosh \int \rho(s) & -\sinh \int \rho(s) \\ \hline & -\sinh \int \rho(s) & \cosh \int \rho(s) \end{array}\right)$$

and set $F(s) = Y(s)D(s)$. Note that

$$Y(s)^{-1} = \left(\begin{array}{c|c|c} I_{n-1} & & \\ \hline & \cosh \int \rho(s) & \sinh \int \rho(s) \\ \hline & \sinh \int \rho(s) & \cosh \int \rho(s) \end{array}\right).$$

It is easy to check that $F'(s)S\bar{F}^t(s) = iF(s)U(s)\bar{F}^t(s)$. Thus, we arrive at the linear differential equation $F'(s) = G(s)F(s)$, where

$$G(s) = ia(s)\left(\begin{array}{c|c|c} I_{n-1} & & \\ \hline & 1+\cosh 2\int\rho(s) & 1+\sinh 2\int\rho(s) \\ \hline & -(1+\sinh 2\int\rho(s)) & 1-\cosh 2\int\rho(s) \end{array}\right).$$

The solution of this equation can be written as $F(s) = e^{\int G(s)}$. Therefore $D(s) = e^{\int G(s)}Y(s)^{-1}$, and it can be easily checked that the immersion is invariant under the action of $\mathrm{SO}(n–1) \propto \mathbb{R}^{n-1}$. Hence the map ϕ, defined on $(I - I_1) \times \mathbb{R}^{n-1}$, is one of the examples given in Theorem 3. Finally, since the second fundamental forms of the examples given in Theorem 3 are non-trivial and the set I is connected, the case when both $\mathrm{Int}(I_1) \neq \emptyset$ and $\mathrm{Int}(I_2) \neq \emptyset$ is impossible. $\square$

6. Minimal Lagrangian submanifolds in $\mathbb{CP}^n$

As we mentioned in the introduction, in this section we state (without proofs) the corresponding results when the ambient space is the complex projective space $\mathbb{CP}^n$.

If $U(n + 1)$ is the unitary group of order $n + 1$, then $PU(n + 1) = U(n + 1)/\mathbb{S}^1$ is the group of holomorphic isometries of $(\mathbb{CP}^n, \langle,\rangle)$. We consider the special orthogonal group $\mathrm{SO}(n)$ acting on $\mathbb{CP}^n$ as a subgroup of holomorphic isometries via the map

$$A \in \mathrm{SO}(n) \longmapsto \left[\left(\begin{array}{c|c} A & \\ \hline & 1 \end{array}\right)\right] \in PU(n+1),$$

where [ ] stands for the class in $U(n + 1)/\mathbb{S}^1$.

The unit sphere $\mathbb{S}^n$ can be isometrically immersed in $\mathbb{CP}^n$ as a totally geodesic Lagrangian submanifold in the standard way:

$$x \in \mathbb{S}^n \mapsto [x] \in \mathbb{CP}^n.$$

This immersion projects in the totally geodesic Lagrangian embedding of the real projective space $\mathbb{RP}^n$ in $\mathbb{CP}^n$. Moreover, up to congruences, $\mathbb{RP}^n$ is the only totally geodesic Lagrangian submanifold of $\mathbb{CP}^n$. It is interesting to note that the totally umbilical submanifolds of $\mathbb{CP}^n$ (which were classified in [ChO]) are either totally geodesic or umbilical submanifolds of totally geodesic Lagrangian submanifolds. Thus, *up to congruences, the $(n-1)$-dimensional totally umbilical (non-totally geodesic) submanifolds of $\mathbb{CP}^n$ are the umbilical hypersurfaces of $\mathbb{RP}^n$ embedded in $\mathbb{CP}^n$ in the above manner.* In this case, the umbilical hypersurfaces of $\mathbb{RP}^n$ are the geodesic spheres. We will refer to these examples as $(n-1)$-*geodesic spheres of* $\mathbb{CP}^n$.

THEOREM 5. *Let $\phi : M \to \mathbb{CP}^n$, $n \geq 3$, be a minimal (non-totally geodesic) Lagrangian immersion.*

(a) *ϕ is invariant under the action of* $\mathrm{SO}(n)$ *if and only if ϕ is locally congruent to one of the immersions in the 1-parameter family of minimal Lagrangian immersions $\{\Phi_\rho : \mathbb{R} \times \mathbb{S}^{n-1} \to \mathbb{CP}^n \mid \rho \in (0, \pi/2)\}$ given by*

$$\Phi_\rho(s,x) = \left[\left(\sin r(s)\, \exp\left(-ia\int_0^s \frac{dt}{\sin^{n+1} r(t)}\right) x, \right.\right. \\ \left.\left. \cos r(s)\, \exp\left(ia\int_0^s \frac{\tan^2 r(t)\, dt}{\sin^{n+1} r(t)}\right)\right)\right],$$

where $r(s)$, $s \in \mathbb{R}$, is the unique solution to

$$r'' \sin r \cos r = (1-(r')^2)(n\cos^2 r - \sin^2 r),\ r(0) = \rho,\ r'(0) = 0, \tag{5}$$

and $a = \cos\rho \sin^n \rho$.

(b) *ϕ is foliated by $(n-1)$-geodesic spheres of $\mathbb{CP}^n$ if and only if ϕ is locally congruent to one of the examples described in* (a)*.*

REMARK 2. In this case, $r(s) = \arctan\sqrt{n}$ gives a constant solution to equation (5). The corresponding minimal Lagrangian immersion $\Phi : \mathbb{R} \times \mathbb{S}^{n-1} \to \mathbb{CP}^n$ is given by

$$\Phi(s,x) = \left[\frac{1}{\sqrt{n+1}}\left(\sqrt{n}e^{-is/\sqrt{n}}\, x, e^{i\sqrt{n}s}\right)\right],$$

which provides a minimal Lagrangian immersion $\Phi : \mathbb{S}^1 \times \mathbb{S}^{n-1} \to \mathbb{CP}^n$ defined by

$$\Phi(e^{it},x) = \left[\frac{1}{\sqrt{n+1}}\left(\sqrt{n}\, e^{-it/(n+1)}\, x, e^{int/(n+1)}\right)\right].$$

If $h : \mathbb{S}^1 \times \mathbb{S}^{n-1} \to \mathbb{S}^1 \times \mathbb{S}^{n-1}$ is the diffeomorphism $h(e^{it},x) = (-e^{it}, -x)$, then Φ induces a minimal Lagrangian embedding $(\mathbb{S}^1 \times \mathbb{S}^{n-1})/h \to \mathbb{CP}^n$, which is a very well-known example studied by Naitoh (see [N, Lemma 6.2]).

REMARK 3. By studying the energy integral of equation (5) given by

$$(r')^2 + \frac{\sin^{2n}\rho \cos^2\rho}{\sin^{2n} r \cos^2 r} = 1,$$

it is easy to show that, in the case when r is not the constant solution, the orbits $s \mapsto (r(s), r'(s))$ are closed curves. Hence, all solutions of equation (5) are periodic functions. However, not all of the corresponding minimal Lagrangian submanifolds are embedded. In fact, in [CU1] minimal Lagrangian surfaces that are invariant by a 1-parameter group of holomorphic isometries of $\mathbb{CP}^2$ were classified, and as particular cases the versions of the examples given in Theorem 5 with $n = 2$ were obtained. Since the solutions of (5) for

$n = 2$ are elliptic functions (see [CU1]), it is not difficult to check that, except the Clifford torus, the examples given there do not provide embedded minimal Lagrangian tori. It may be interesting to point out here that Goldstein [G] recently constructed minimal Lagrangian tori in Einstein-Kaehler manifolds with positive scalar curvature.

We now give a method to produce examples of minimal Lagrangian submanifolds of $\mathbb{CP}^n$.

PROPOSITION 6. *Let $\phi : N^{n-1} \to \mathbb{CP}^{n-1}$ be a minimal Lagrangian immersion of a simply connected manifold N, and let $\tilde{\phi} : N \to \mathbb{S}^{2n-1}$ be the horizontal lift of ϕ with respect to the Hopf fibration $\Pi : \mathbb{S}^{2n-1} \to \mathbb{CP}^{n-1}$.*

(a) *Given a solution $r(s)$ of equation* (5) *in Theorem 5, the map Φ : $\mathbb{R} \times N \to \mathbb{CP}^n$ defined by*

$$\Phi(s,x) = \left[\left(\sin r(s)\, \exp\left(-ia\int_0^s \frac{dt}{\sin^{n+1} r(t)}\right) \tilde{\phi}(x),\right.\right.$$
$$\left.\left.\cos r(s)\, \exp\left(ia\int_0^s \frac{\tan^2 r(t)\, dt}{\sin^{n+1} r(t)}\right)\right)\right]$$

is a minimal Lagrangian immersion in $\mathbb{CP}^n$.

(b) *The map*

$$\Phi : (0,\pi/2) \times N \to \mathbb{CP}^n$$
$$(s,x) \mapsto \left[\left(\sin s\, \tilde{\phi}(x)\,,\, \cos s\right)\right]$$

is a minimal Lagrangian immersion in $\mathbb{CP}^n$.

(c) *Let $\gamma = (\gamma_1, \gamma_2) : I \to \mathbb{S}^3$ be a Legendre curve. The map $\Phi : I \times N \to \mathbb{CP}^n$ defined by*

$$\Phi(s,x) = \left[\left(\gamma_1(s)\tilde{\phi}(x), \gamma_2(s)\right)\right]$$

is a minimal Lagrangian immersion if and only if Φ is congruent to one of the examples given in (a) *and* (b).

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