TRANSFERRING FOURIER MULTIPLIERS FROM S^{2p-1} TO H^{p-1}

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ABSTRACT. We prove versions of de Leeuw's Theorems for the contraction of S^{2p-1} to H^{p-1}.

1. Introduction

In 1965, Karel de Leeuw proved two theorems relating Fourier multipliers of $L^q(\mathbb{R})$ to those of $L^q(\mathbb{T})$. Given a function ϕ on $\widehat{\mathbb{R}} = \mathbb{R}$, consider its restriction ϕ_1 to $\widehat{\mathbb{T}} = \mathbb{Z}$. He proved that if ϕ is uniformly continuous on $\mathbb{R}$, then:

(1) If T_ϕ is bounded on $L^q(\mathbb{R})$, then T_{ϕ_1} is bounded on $L^q(\mathbb{T})$ and $\|T_{\phi_1}\|_q \le \|T_\phi\|_q$. In fact, if we let ϕ_ε be the restriction of ϕ to $\varepsilon\mathbb{Z} \subseteq \mathbb{R}$, then $\|T_{\phi_\varepsilon}\|_q \le \|T_\phi\|_q$.

(2) If for every $\varepsilon \in \mathbb{R}_+$, T_{ϕ_ε} is bounded on $L^q(\mathbb{T})$ with $\lim_{\varepsilon \to 0} \sup \|T_{\phi_\varepsilon}\|_q = K < \infty$, then T_ϕ is bounded on $L^p(\mathbb{R})$ with $\|T_\phi\|_q \le K$.

In these statements, we have used the notion of Fourier multiplier: T_ϕ is a Fourier multiplier if for all $f \in L^q(\mathbb{R})$, $\widehat{(T_\phi f)} = \phi \widehat{f}$ is the Fourier transform of an $L^q(\mathbb{R})$ function, and $\|T_\phi\|_q = \sup_{f \in L^q(\mathbb{R})} \|T_\phi f\|_q / \|f\|_q$.

These two elegant theorems became the prototype for a number of "transference" results, where the L^q boundedness of a Fourier multiplier or a convolution kernel on a group may be checked on a different, hopefully simpler, group.

This theme was taken up in the context of non-commutative harmonic analysis, notably by Coifman and Weiss [CW1][CW2], who proved results of Marcinkiewicz type for $SU(2)$ and other Lie groups, replacing the quotient maps $\mathbb{R} \to \mathbb{T}$ by the mapping $X \to \exp \varepsilon X : \mathfrak{g} \to G$. Rubin [Ru] used the same circle of ideas in the context of $SO(3)$ and the Euclidean motion group, and this theme was taken up by Dooley, Gaudry and Rice [DRi][DG][D1] who showed that the notion of a contraction or continuous deformation of Lie groups was the key underlying idea.

We say that the Lie group G_2 is a *contraction*[1] of the Lie group G_1 if there is a family $(\pi_\varepsilon)_{\varepsilon>0}$ of local diffeomorphisms $\pi_\varepsilon : G_2 \to G_1$, which are approximate homomorphisms in the sense that

$$\pi_\varepsilon(x) \to e \;\; \text{as} \;\; \varepsilon \to 0$$

and

$$\pi_\varepsilon^{-1}(\pi_\varepsilon(x)\pi_\varepsilon(y)) \to xy \;\; \text{as} \;\; \varepsilon \to 0.$$

If (G, K) is a Riemannian symmetric pair of the compact or non-compact type, we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Letting $V = \mathfrak{p}$, the Cartan motion group $V \rtimes K$ is a contraction of G [DRi], given by the maps

$$\pi_\varepsilon : (X, k) \mapsto \exp_G(\varepsilon X) \cdot k$$

This generalises the homomorphism/dilation relationship between $\mathbb{R}$ and $\mathbb{T}$. In [DG], a version of (2) was proved in this setting. (See also [D1] for a discussion of the ideas in the framework of a more general contraction.)

Subsequently, one of the authors [D2] gave a version of (1), both for the Cartan motion group contraction, and for the Coifman-Weiss contraction of G to $\mathfrak{g}$. Unfortunately, the versions of (1) proved in [D2] no longer gave an exact converse of the version of (2) from [DG] and from [CW1]. We shall discuss this further below.

Now in the case of a rank one semisimple group with Iwasawa decomposition $G = KAN$, there is an obvious contraction from K to the semi-direct product NM (and which coincides with the above Cartan motion group contraction for $SO(n + 1)$ to $\mathbb{R}^n \rtimes SO(n)$ if $G = SO(n + 1, 1)$). One wished to prove de Leeuw theorems in this setting, or in the closely related setting of the contraction of K/M to N. The latter was done in [RRu] for the special case of $SU(2, 1)$—where K/M is $SU(2)$ and N is the Heisenberg group. Again, an analogue of (2) was proved.

Dooley and Ricci [DR] took the first step towards proving the result in generality, that is, they found an approximation for matrix coefficients—but were not able, in that generality, to describe an orthonormal basis for the representation space which leads to the full de Leeuw theorem. In [DGu], the authors strengthened the results of [DR] for the case of $G = SU(n + 1, 1)$.

In this article, we shall use the representation theory developed in [DGu] to prove de Leeuw theorems for M-invariant multipliers associated to $SU(n + 1, 1)$. More precisely, for the contraction of $K/M = S^{2n-1}$ to the Heisenberg group H^{n-1}, Theorem 4.1 below is an analogue of the main theorem of [RRu], constituting a version of (2) above. The essential novelty here is the description of an explicit orthonormal basis for the Fock space $\mathcal{F}_\lambda(N)$ and its relationship to bases for the representations of $SU(n + 1)$.

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[1]This terminology is derived from the physics literature, where one Lie algebra is contracted to another by setting some of its structure constants equal to zero. The contracted Lie algebra is in general a "more abelian" algebra than the original one.

The other main result of this paper is a version of (1) which is an exact converse to Theorem 4.1. This result has not previously been proved, even in the Cartan motion group case.

We would like to explain its relationship to the results of [D2], where a slightly different version of (1) was proved. The key to the original de Leeuw theorems, and to any generalization, lies in understanding the nature of the periodization map.

In the original version, the maps $\phi \mapsto \phi_\varepsilon$ can be realized as the "duals" of the family of homomorphisms $\pi_\varepsilon, r \mapsto e^{i\varepsilon r} : \mathbb{R} \to \mathbb{T}$. To be somewhat imprecise, for a sufficiently nice function ϕ on $\mathbb{R}$, the two operations of

(i) defining the multiplier ϕ_ε on $\mathbb{T}$ by

$$\langle \phi_\varepsilon f, g \rangle = \langle \phi f \circ \pi_\varepsilon, g \circ \pi_\varepsilon \rangle$$

for $f \in L^q$ and $g \in L^{q'}$,
(ii) restricting $\phi(\varepsilon\cdot)$ to $\mathbb{Z}$ to obtain ϕ_ε as a multiplier of Fourier series,

coincide.

(Checking all the details here, and making sure that the rather loose description given in (i) actually makes sense, amounts to a proof of (1).)

In [DG], as in the present paper, a definition of restriction inspired by (ii) is used to prove a version of de Leeuw's theorem (2) for Cartan motion groups. However in [D2], a different notion of restriction based on (i) and denoted by $i_\varepsilon\phi$ was used to prove an analogue of de Leeuw's theorem (1). The restriction $i_\varepsilon\phi$ however did not provide an exact converse to the de Leeuw's theorem of [DG].

Thus, a salient question is to find a suitable version of the "restriction" for which both versions of de Leeuw's theorem hold. In Theorem 5.1 below, we find a suitable version of restriction, denoted by ϕ_ε, for which both directions of de Leeuw's theorem hold in the case of the contraction of S^{2p-1} to H^{p-1}.

We believe that with further work these techniques can be extended to the contraction of K to $\bar{N}M$.

2. Notation and definitions

We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$, the natural numbers, integers, real numbers and complex numbers, respectively. Let $p \geq 2$ and $G = SU(p,1)$. Then the Lie algebra of G, denoted by $\mathfrak{g} = \mathfrak{su}(p,1)$, is equal to

$$\left\{ \begin{pmatrix} Z & \mathbf{z} \\ \mathbf{z}^* & -\mathrm{tr}Z \end{pmatrix} : Z \text{ skew-hermitian of order } p \text{ and } \mathbf{z} \in \mathbf{C}^p \right\}.$$

Let θ be the differential of the Cartan involution of G. Then $\theta Z = -Z^*$. We will denote by $B(,)$ the Killing form of $\mathfrak{g}$. A positive definite product on $\mathfrak{g}$ is then given by $(X,Y) = -B(X,\theta Y)$. $\|X\|$ denotes $(X,X)^{1/2}$. A Cartan

decomposition for $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{p} \bigoplus \mathfrak{k}$, where

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & \mathbf{z} \\ \mathbf{z}^* & 0 \end{pmatrix} : \mathbf{z} \in \mathbb{C}^p \right\},$$

and

$$\mathfrak{k} = \left\{ \begin{pmatrix} Z & \mathbf{0} \\ \mathbf{0}^* & -\mathrm{tr}Z \end{pmatrix} : Z \text{ skew-hermitian of order } p \right\}.$$

Also, $\mathfrak{p}$, $\mathfrak{k}$ have the following complexifications:

$$\mathfrak{p}^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & \mathbf{z} \\ \mathbf{w}^* & 0 \end{pmatrix} : \mathbf{z}, \mathbf{w} \in \mathbb{C}^p \right\},$$

$$\mathfrak{k}^{\mathbb{C}} = \left\{ \begin{pmatrix} Z & \mathbf{0} \\ \mathbf{0}^* & -\mathrm{tr}Z \end{pmatrix} : Z \text{ a matrix of order } p \right\}.$$

Let $\mathbf{e}_j$ be the vector in $\mathbb{C}^p$ with j-th coordinate 1 and other coordinates zero.

Set

$$H = \begin{pmatrix} 0 & \mathbf{e}_1 \\ \mathbf{e}_1^t & 0 \end{pmatrix}.$$

Then $\mathfrak{a} = \{aH : a \in \mathbb{R}\}$ is a maximal abelian subalgebra of $\mathfrak{p}$. The restricted positive roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$ are $\{\lambda, 2\lambda\}$, $\lambda(H) = 1$, and the corresponding root spaces are

$$\mathfrak{g}_\lambda = \left\{ \begin{pmatrix} 0 & \mathbf{z}^* & 0 \\ -\mathbf{z} & 0 & \mathbf{z} \\ 0 & \mathbf{z}^* & 0 \end{pmatrix} : \mathbf{z} \in \mathbb{C}^{p-1} \right\}$$

and

$$\mathfrak{g}_{2\lambda} = \{a(\theta X_0) : a \in \mathbb{R}\},$$

where X_0 is the matrix

$$X_0 = \begin{pmatrix} i/2 & \mathbf{0}^* & i/2 \\ \mathbf{0} & 0 & \mathbf{0} \\ -i/2 & \mathbf{0}^* & -i/2 \end{pmatrix}$$

of order $p + 1$. Set $\mathfrak{n} = \mathfrak{g}_\lambda \bigoplus \mathfrak{g}_{2\lambda}$. Let N, A, K denote the Lie groups corresponding to Lie algebras $\mathfrak{n}, \mathfrak{a}, \mathfrak{k}$, respectively. Then the Iwasawa decomposition of G is KAN, where $K = S(U(p) \times U(1)) \equiv U(p)$. Set $\bar{\mathfrak{n}} = \mathfrak{g}_{-\lambda} \bigoplus \mathfrak{g}_{-2\lambda}$. By $\bar{N}$, we shall denote the image of N under the Cartan involution $\theta X = (X^*)^{-1}$ of G. We will identify $\bar{N}$ with $\bar{\mathfrak{n}}$ via the exponential mapping between them. Then $\bar{N}$ is isomorphic to the Heisenberg group H^{p-1}. Let M be the centralizer of A in K. Then the Lie algebra of M is

$$\mathfrak{m} = \left\{ \begin{pmatrix} -\mathrm{tr}Z/2 & \mathbf{0}^* & 0 \\ \mathbf{0} & Z & \mathbf{0} \\ 0 & \mathbf{0}^* & -\mathrm{tr}Z/2 \end{pmatrix} : Z \text{ skew-hermitian of order } p-1 \right\}.$$

Next we define a family of contraction mappings π_ε between $\bar{N}M$ and K. Each element of $\bar{N}M$ can be expressed uniquely as $\exp(X + aX_0)m$, where $X \in \mathfrak{g}_{-\lambda}$, $a \in \mathbb{R}$, and $m \in M$, and we denote it by $(X, a)m$. For $X \in \mathfrak{g}$, X_θ denotes $X + \theta X$. Then for $\varepsilon > 0$, define

$$\pi_\varepsilon(\bar{n}m) = \exp(\varepsilon^{1/2}X_\theta + a\varepsilon(X_0)_\theta)m.$$

We close this section with a theorem (see [Va]):

THEOREM 2.1. *There exists a relatively compact open set O in $\bar{N}M$ such that $\{\exp(X_\theta + a(X_0)_\theta)m : (X, a)m \in O\}$ is open and dense in K and π_1 is a diffeomorphism on O.*

3. M-class-1 representations of K and representations of $\bar{N}$

We say that a unitary representation T of K is of M-class-1 if the carrier Hilbert space $\mathcal{H}$ of T has non-zero vectors a such that $T(m)a = a$ for all $m \in M$ and the dimension of the linear space spanned by such vectors is one. We describe a family of M-class-1 representations of K which we need here. For a complete description of M-class-1 representations of K see [VK]. For $\mathbf{n} \in \mathbb{N}^{p-1}$, let $|\mathbf{n}| = n_2 + \cdots + n_p$.

Let E_{jk} be the matrix of order $p + 1$ with (j, k)-entry equal to 1 and all other entries equal to zero. For $2 \le j \le p$, set

$$X_j = \frac{-1}{\sqrt{2}}(E_{1j} - E_{p+1,j} - E_{j1} - E_{j,p+1})$$

For $g \in K$, define $\Gamma(g) = 2(E_{1,p+1}, \mathrm{Ad}(g)E_{1,p+1})$ and for $2 \le j \le p$ set

$$\Gamma_j(g) = (X_j - \theta X_j, \quad \mathrm{Ad}(g)2E_{1,p+1}) = \frac{1}{\sqrt{2}}(E_{j,p+1}, \mathrm{Ad}(g)2E_{1,p+1}).$$

For $l \in \mathbb{N}$, we put

$$\Gamma_{l,\mathbf{n}} = \frac{\Gamma^{l-|\mathbf{n}|}\prod_{j=2}^{p}\Gamma_j^{n_j}}{\left\|\Gamma^{l-|\mathbf{n}|}\prod_{j=2}^{p}\Gamma_j^{n_j}\right\|_{L^2(K)}}.$$

Next, define

$$w = \begin{pmatrix} -1 & \mathbf{0}^t & 0 \\ \mathbf{0} & I_{p-1} & \mathbf{0} \\ 0 & \mathbf{0}^t & -1 \end{pmatrix}.$$

The following theorem (for a proof see [DGu]) describes a family of M-class-1 representations of K.

THEOREM 3.1. *Let τ denote the left regular representation of K. For $l \in \mathbb{N}$, define $\mathcal{H}_l$ to be the subspace of $L^2(K)$ spanned by the set*

$$\mathcal{B}_l = \left\{\Gamma_{l,\mathbf{n}} : \mathbf{n} = (n_2, \ldots, n_p) \in \mathbb{N}^{p-1}, |\mathbf{n}| \le l\right\}.$$

Then the following statements are true.

(i) $\mathcal{H}_l$ *is an invariant subspace of* $L^2(K)$ *under* τ *and* τ *acts irreducibly on* $\mathcal{H}_l$. *Also* $\mathcal{B}_l$ *is a complete set in* $\mathcal{H}_l$ *and* Γ^l *is* M-*invariant.*

(ii) *For* $l \in \mathbb{N}$ *and* $g \in K$, *define*

$$\tau^{-l}(g) = \tau^l(w(g^t)^{-1}w).$$

Then τ^{-l} *is a* M-*class*-1 *irreducible representation of* K *on* $\mathcal{H}_l$.

For $l \in \mathbb{Z}$, we denote by τ^l the representation τ of K on $\mathcal{H}_{|l|}$.

Define $J : \mathfrak{g} \to \mathfrak{g}$ by

$$JX = [X_0, \theta X].$$

Then $(J|_{\mathfrak{g}_{-\lambda}})^2 = -\operatorname{Id}$ (see [DR]). Thus $(\mathfrak{g}_{-\lambda}, J)$ is a complex space, i.e., J induces a complex structure on $\mathfrak{g}_{-\lambda}$. Indeed, $\mathfrak{g}_{-\lambda} = \mathcal{R} \oplus J\mathcal{R}$, where $\mathcal{R}$ is given by

$$\left\{ \begin{pmatrix} 0 & \mathbf{x}^t & 0 \\ -\mathbf{x} & 0 & -\mathbf{x} \\ 0 & -\mathbf{x}^t & 0 \end{pmatrix} : \mathbf{x} \in \mathbb{R}^{p-1} \right\}.$$

The complex inner product on $(\mathfrak{g}_{-\lambda}, J)$ is

$$(Z, X)_{\mathbb{C}} = (Z, X)_{\mathbb{R}} + i(Z, JX)_{\mathbb{R}}.$$

For $A, B \in \mathcal{R}$ and $X = A + JB \in \mathfrak{g}_{-\lambda}$, $\bar{X}$ is defined as $A - JB$. Let E_{jk} be the matrix of order $p+1$ with (j, k) entry 1 and other entries zero. For $2 \leq j \leq p$, set

$$X_j = -\frac{1}{\sqrt{2}}(E_{1j} - E_{p+1,j} - E_{j1} - E_{j,p+1}).$$

Then $\{X_j\}_{j=2}^p$ is an orthonormal basis for $\mathfrak{g}_{-\lambda}$ as a complex vector space. Let λ be a positive real number. Define $\mathcal{F}_\lambda$ to be the generalised Fock space consisting of holomorphic functions F on $\mathfrak{g}_{-\lambda}$ satisfying

$$\int_{\mathfrak{g}_{-\lambda}} |F(X)|^2 e^{-2\lambda\pi\|X\|^2} dX < \infty.$$

$\mathcal{F}_\lambda$ is a Hilbert space with the inner product given by

$$\langle F, G \rangle = \int_{\mathfrak{g}_{-\lambda}} F(X)\bar{G}(X)e^{-2\|X\|^2\lambda\pi} dX.$$

Now we define two representations of $\bar{N}$ on the Hilbert space $\mathcal{F}_\lambda$:

Let $F \in \mathcal{F}_\lambda$, $(X, a) \in \bar{N}$, $Z \in \mathfrak{g}_{-\lambda}$. Define

$$(\sigma^\lambda(X, a)F)(Z) = e^{2ia\lambda\pi}e^{-2\lambda\pi\left(\|X\|^2/2 + (Z,X)_{\mathbb{C}}\right)} F(Z + X),$$

$$(\sigma^{-\lambda}(X, a)F)(Z) = e^{-2ia\lambda\pi}e^{-2\lambda\pi\left(\|X\|^2/2 + (Z,-\bar{X})_{\mathbb{C}}\right)} F(Z - \bar{X}).$$

It is known that each σ^λ and each $\sigma^{-\lambda}$ is an irreducible representation of $\bar{N}$ and the set $\{\sigma^\lambda, \sigma^{-\lambda} : \lambda > 0\}$ provides a set of representations of full

Plancherel measure for $\bar{N}$. Next we construct σ^λ on a function space on $\bar{N}$. Given $F \in \mathcal{F}_\lambda$, let $\widetilde{\Gamma}^\lambda F$ be the function on $\bar{N}$ given by

$$(\widetilde{\Gamma}^\lambda F)(X, a) = F(-X)G_\lambda(X, a),$$

where

$$G_\lambda(X, a) = e^{-\pi\lambda(2ia + \|X\|^2)}.$$

The action of (X, a) on F by σ^λ is conjugate to the action of (X, a) by the left regular representation σ of $\bar{N}$ on $\widetilde{\Gamma}^\lambda F$, i.e.,

$$\sigma(X, a)\widetilde{\Gamma}^\lambda F = \widetilde{\Gamma}^\lambda \sigma^\lambda(X, a)F$$

on $\mathcal{F}_\lambda$. We denote by $\widetilde{\mathcal{F}_\lambda}$ the space $\widetilde{\Gamma}^\lambda \mathcal{F}_\lambda$ with the norm that makes $\widetilde{\Gamma}^\lambda$ an isometry and by $\tilde{\sigma}^\lambda$ the representation σ of $\bar{N}$ on $\widetilde{\mathcal{F}_\lambda}$. As to the representations $\sigma^{-\lambda}$, observe that $\sigma^{-\lambda}(X, a) = \sigma^\lambda(-\bar{X}, -a)$. Consequently, we can define the equivalent representations $\tilde{\sigma}^{-\lambda}$ of $\bar{N}$ on $\widetilde{\mathcal{F}_\lambda}$ by setting

$$\tilde{\sigma}^{-\lambda}(X, a) = \tilde{\sigma}^\lambda(-\bar{X}, -a).$$

The following is a description of an orthonormal basis of $\mathcal{F}_\lambda$: Let $Z \in \mathfrak{g}_{-\lambda}$ be given by

$$Z = \sum_{j=2}^{p} x_j X_j + \sum_{j=2}^{p} y_j \overline{JX_j}.$$

Then

$$(Z, X_j)_\mathbb{C} = (Z, X_j) + i(Z, JX_j) = x_j + iy_j.$$

We denote $(Z, X_j)_\mathbb{C}$ by z_j. The monomials given by $z^\mathbf{n} = z_2^{n_2} \dots z_p^{n_p}$ belong to the space $\mathcal{F}_\lambda$. Define

$$\widetilde{\Gamma}_\mathbf{n}^\lambda(z^\mathbf{n}) = \frac{\widetilde{\Gamma}^\lambda(z^\mathbf{n})}{\|\widetilde{\Gamma}^\lambda(z^\mathbf{n})\|_{L^2(\bar{N})}}.$$

The set $\mathcal{B}_\lambda = \{\widetilde{\Gamma}_\mathbf{n}^\lambda(z^\mathbf{n}) : \mathbf{n} \in \mathbb{N}^{p-1}\}$ is a complete set in $\widetilde{\mathcal{F}_\lambda}$.

We shall denote the matrix elements of τ^l with respect to basis $\mathcal{B}_l$ as follows: For $\mathbf{n}, \mathbf{m} \in \mathbb{N}^{p-1}$ and $l \in \mathbb{Z}$ set

$$\tau_{\mathbf{n},\mathbf{m}}^l(g) = \left(\tau^l(g)\Gamma_{|l|,\mathbf{n}}, \Gamma_{|l|,\mathbf{m}}\right).$$

We denote the matrix entries of $\tilde{\sigma}^\lambda$ with respect to $\mathcal{B}_\lambda$ as follows: For $\mathbf{n}, \mathbf{m} \in \mathbb{N}^{p-1}$ and $\lambda \in \mathbb{R}$ set

$$\tilde{\sigma}_{\mathbf{n},\mathbf{m}}^\lambda(X, a) = \left(\tilde{\sigma}^\lambda(X, a)\widetilde{\Gamma}_\mathbf{n}^\lambda, \widetilde{\Gamma}_\mathbf{m}^\lambda\right).$$

Next we put an ordering on $\mathbb{N}^{p-1}$. We say that $(n_2, \dots, n_p) \leq (m_2, \dots, m_p)$ if there exists a natural number k, $2 \leq k < p$, such that $n_j = m_j$ for $2 \leq j \leq k$ and $n_{k+1} < m_{k+1}$. For $f \in L^1(K)$, we denote by $\hat{f}(l, \mathbf{n}, \mathbf{m})$ the $d_{|l|} \times d_{|l|}$ matrix with $(\mathbf{n}, \mathbf{m})$-entry

$$\hat{f}(l, \mathbf{n}, \mathbf{m}) = \int_K f(g)\overline{\tau_{\mathbf{n},\mathbf{m}}^l}(g)dg.$$

Similarly, for $h \in L^1(\bar{N})$ and a non-zero real number λ, we call $\hat{h}(\lambda)$ the countably infinite matrix with $(\mathbf{n}, \mathbf{m})$-entry

$$\hat{h}(\lambda, \mathbf{n}, \mathbf{m}) = \int_{\bar{N}} h(g)\overline{\hat{\sigma}^{\lambda}_{\mathbf{n},\mathbf{m}}}(g)dg.$$

The following Plancherel formulas are well known.

Let $f \in L^1(K)$ be such that $\hat{f}$ is supported only on the irreducible unitary representations of K indexed by $\mathbb{Z}$. Let d_l denote the dimension of the Hilbert space $\mathcal{H}_l$. Then

$$\int_K |f(g)|^2 dg = \sum_{-\infty}^{\infty} d_{|l|} \sum_{|\mathbf{n}|,|\mathbf{m}|\leq l} \left|\hat{f}(l,\mathbf{n},\mathbf{m})\right|^2,$$

and for $f \in L^2(\bar{N})$,

$$\int_{\bar{N}} |f(\bar{n})|^2 d\bar{n} = \int_{-\infty}^{\infty} \sum_{\mathbf{n},\mathbf{m}\in\mathbf{N}^{p-1}} \left|\hat{f}(\lambda,\mathbf{n},\mathbf{m})\right|^2 |\lambda|^{p-1} d\lambda.$$

We define (right) multiplier transformations on the two groups as follows. On K, for each $l \in \mathbb{Z}$ we assign a $d_{|l|} \times d_{|l|}$ matrix $\phi(l)$ with entries $\phi(l,\mathbf{n},\mathbf{m})$. We say that this matrix-valued function ϕ induces a bounded multiplier T_ϕ on $L^q(K)$ if

$$\|T_\phi f\|_q \leq C\|f\|_q,$$

where $\widehat{(T_\phi f)}(l) = \hat{f}(l)\phi(l)$ with $f \in L^q(K)$. Similarly, on $\bar{N}$, for each $\lambda \neq 0$ we assign a countably infinite matrix $\phi(\lambda)$ with entries $\phi(\lambda, \mathbf{n}, \mathbf{m})$ that are measurable in λ for each $\mathbf{n}$ and $\mathbf{m}$, and say that it induces a bounded multiplier T_ϕ on $L^q(\bar{N})$ if

$$\|T_\phi f\|_q \leq C\|f\|_q,$$

where $\widehat{(T_\phi f)}(\lambda) = \hat{f}(\lambda)\phi(\lambda)$ for f in some dense subspace of $L^q(\bar{N})$.

4. The de Leeuw theorem

For $\varepsilon > 0$ and $\lambda > 0$, let $n_\lambda(\varepsilon) = [(\pi\lambda)/\varepsilon]$. Now we state our analogue of the de Leeuw theorem (2) in the setting of $SU(p,1)$ mentioned in the introduction.

THEOREM 4.1. *Let $q \in [1,\infty)$, $p \geq 2$, and $0 < \alpha < 1$. For each $\varepsilon > 0$, let ϕ_ε be a bounded L^q-multiplier on K such that ϕ_ε is supported on the family of representations indexed by $\mathbb{Z}$. Suppose that the corresponding multiplier transformations T_{ϕ_ε} have uniformly bounded norms and for almost every $\lambda > 0$, the limits*

$$\lim_{\varepsilon\to 0} \phi_\varepsilon(n_\lambda(\varepsilon), \mathbf{n}, \mathbf{m}) = \phi(\lambda, \mathbf{n}, \mathbf{m}),$$

$$\lim_{\varepsilon\to 0} (-1)^{|\mathbf{n}|+|\mathbf{m}|}\phi_\varepsilon(-n_\lambda(\varepsilon), \mathbf{n}, \mathbf{m}) = \phi(-\lambda, \mathbf{n}, \mathbf{m})$$

exist and define for every $\mathbf{n}$ and $\mathbf{m}$ measurable functions in λ. Then the multiplier transformation on $\bar{N}$ that is induced by the matrices $\phi(\lambda) = \{\phi(\lambda, \mathbf{n}, \mathbf{m}) : \mathbf{n}, \mathbf{m} \in \mathbb{N}^{p-1}\}$ is bounded on $L^q(\bar{N})$ and satisfies

$$\|T_\phi\|_q \leq C \sup \{\|T_{\phi_\varepsilon}\|_q : \varepsilon > 0\}.$$

REMARK. For $p = 2$, Theorem 4.1 is the main result in [RRu], namely, it is a statement of de Leeuw's theorem for the pair $(SU(2), H^1)$.

To prove Theorem 4.1, we first prove a sequence of lemmas.

LEMMA 4.2. *Let $A \in \mathfrak{g}_{-\lambda}$ and $a_0 \in \mathbb{R}$. For $\bar{n}_0 = \exp(A + a_0 X_0)$, we have:*

(i)
$$\lim_{\varepsilon \to 0} \left(\tau^{n_\lambda(\varepsilon)}(\pi_\varepsilon(\bar{n}_0)) \Gamma_{n_\lambda(\varepsilon), \mathbf{n}} , \Gamma_{n_\lambda(\varepsilon), \mathbf{m}} \right)$$
$$= \left(\tilde{\sigma}^\lambda(\bar{n}_0) \widetilde{\Gamma}_{\mathbf{n}}^\lambda(z^{\mathbf{n}}), \widetilde{\Gamma}_{\mathbf{m}}^\lambda(z^{\mathbf{m}}) \right),$$

(ii)
$$\lim_{\varepsilon \to 0} \left(\tau^{-n_\lambda(\varepsilon)}(\pi_\varepsilon(\bar{n}_0)) \Gamma_{n_\lambda(\varepsilon), \mathbf{n}} , \Gamma_{n_\lambda(\varepsilon), \mathbf{m}} \right)$$
$$= (-1)^{|\mathbf{n}|+|\mathbf{m}|} \left(\tilde{\sigma}^{-\lambda}(\bar{n}_0) \widetilde{\Gamma}_{\mathbf{n}}^\lambda(z^{\mathbf{n}}) , \widetilde{\Gamma}_{\mathbf{m}}^\lambda(z^{\mathbf{m}}) \right),$$

where in the above limit it is assumed that $n_\lambda(\varepsilon) > \max(|\mathbf{n}|, |\mathbf{m}|)$.

Proof. See [DGu]. $\square$

Let f be any function on $\bar{N}$. We will think of f as defined on $\bar{N}M$ by setting $f(\bar{n}m) = f(\bar{n})$, for $\bar{n} \in N$, $m \in M$. Let O be as in Section 2. For $\varepsilon > 0$, define

$$O_\varepsilon = \left\{ \exp \left(\varepsilon^{-1/2} X + a \varepsilon^{-1}(X_0) \right) m : (X, a)m \in O \right\}.$$

Note that as $\varepsilon \to 0$, $O_\varepsilon \uparrow \bar{N}M$. For f a measurable function on $\bar{N}$, define

$$f_\varepsilon(g) = \begin{cases} \varepsilon^{-p} f \circ \pi_\varepsilon^{-1}(g) & \text{on } \pi_\varepsilon(O_\varepsilon), \\ 0 & \text{elsewhere.} \end{cases}$$

LEMMA 4.3. *Let f be an integrable function on $\bar{N}$ with compact support. For $\lambda > 0$ we have*

$$\lim_{\varepsilon \to 0} \widehat{f_\varepsilon}([\pi\lambda/\varepsilon], \mathbf{n}, \mathbf{m}) = \widehat{f}(\lambda, \mathbf{n}, \mathbf{m})$$

and

$$\lim_{\varepsilon \to 0} \widehat{f_\varepsilon}(-[\pi\lambda/\varepsilon], \mathbf{n}, \mathbf{m}) = (-1)^{|\mathbf{n}|+|\mathbf{m}|} \widehat{f}(-\lambda, \mathbf{n}, \mathbf{m}).$$

Also, if $1 \leq q < \infty$ and $q' = q/(q-1)$, then

$$\varepsilon^{p/q'} \|f_\varepsilon\|_{L^q(K)} \leq \|f\|_{L^q(\bar{N})}$$

and

$$\lim_{\varepsilon \to 0} \varepsilon^{p/q'} \|f_\varepsilon\|_{L^q(K)} = \|f\|_{L^q(\bar{N})}.$$

Proof. The easy proof using Lemma 4.2 is left to the reader. □

In the proof of Theorem 4.1 we will consider only C^∞-functions on $\bar{N}$ with compact support whose Fourier transforms consist of matrices having nonzero entries only on a finite number of diagonals. Such functions form a dense subspace of every L^q-space, $1 \le q < \infty$, as follows from the next lemma.

LEMMA 4.4. *Let f be a C^∞-function on $\bar{N}$ with support contained in a compact subset F. There exists a sequence $\{f^T\}_{T=1}^\infty$ of C^∞-functions supported on F which converges to f uniformly and is such that for every $\lambda \ne 0$, $\widehat{f^T}(\lambda, \mathbf{n}, \mathbf{m}) = 0$ when $\max\{|n_j - m_j| : 2 \le j \le p\} > T$.*

Proof. Let K_T be the T-th Fejèr kernel on the torus. Define $f^T(\rho_2 e^{i\phi_2}, \ldots, \rho_p e^{i\phi_p}, t)$ to be

$$\frac{1}{(2\pi)^{(p-1)}} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\rho_2 e^{i\theta_2}, \ldots, \rho_p e^{i\theta_p}, t) \times$$
$$\times K_T(\phi_2 - \theta_2) \ldots K_T(\phi_p - \theta_p) d\theta_2 \ldots d\theta_p.$$

By passing to a larger set if necessary, it can be assumed that the radius of the compact set F depends only on the parameters $\rho_2, \ldots, \rho_p$ and t. Then it is clear that f^T is a C^∞-function supported on F. The fact that f^T converges uniformly to f follows from the classical Fejèr theorem and the fact that f is uniformly continuous. Also an easy computation shows that

$$\widetilde{\sigma}_{\mathbf{n},\mathbf{m}}^\lambda(\rho_2 e^{i\phi_2}, \ldots, \rho_p e^{i\phi_p}, t) = e^{i\sum_{j=2}^p (n_j - m_j)\phi_j} \, \widetilde{\sigma}_{\mathbf{n},\mathbf{m}}^\lambda(\rho_2, \cdots, \rho_p, t),$$

and using this relationship,

$$\widehat{f^T}(\lambda, \mathbf{n}, \mathbf{m}) = \widehat{f}(\lambda, \mathbf{n}, \mathbf{m}) \prod_{j=2}^p \widehat{K}_T(m_j - n_j),$$

which vanishes if $\max\{|n_j - m_j| : 2 \le j \le p\} > T$. □

We say that a function f on $\bar{N}$ is of type T if for every $\lambda \ne 0$, $\widehat{f}(\lambda, \mathbf{n}, \mathbf{m}) = \mathbf{0}$ when $\max\{|n_j - m_j| : 2 \le j \le p\} > T$. The same definition will be used for functions on K, with reference to the matrix coefficients $\widehat{f}(l, \mathbf{n}, \mathbf{m})$.

LEMMA 4.5. *Let f be a continuous function of type T supported on a compact subset F of $\bar{N}$. Then for sufficiently small ε, f_ε is of type T on K.*

Proof. Let D_N be the Dirichlet kernel on the torus and define $g(\rho_2 e^{i\phi_2}, \ldots, \rho_p e^{i\phi_p}, t)$ to be

$$\frac{1}{(2\pi)^{(p-1)}} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\rho_2 e^{i\theta_2}, \ldots, \rho_p e^{i\theta_p}, t) \times$$
$$\times D_T(\phi_2 - \theta_2) \ldots D_T(\phi_p - \theta_p) d\theta_2 \ldots d\theta_p.$$

As before, we see that

$$\widehat{g}(\lambda, \mathbf{n}, \mathbf{m}) = \widehat{f}(\lambda, \mathbf{n}, \mathbf{m}) \prod_{j=2}^{p} \widehat{D}_T(m_j - n_j).$$

Therefore, if $\max \{|m_j - n_j| : 2 \leq j \leq p\} > T$, we get $\widehat{g}(\lambda, \mathbf{n}, \mathbf{m}) = \mathbf{0}$, and for $\max \{|m_j - n_j| : 2 \leq j \leq p\} \leq T$ we have $\widehat{D}_T(m_j - n_j) = 1$. Hence

$$\widehat{g}(\lambda, \mathbf{n}, \mathbf{m}) = \widehat{f}(\lambda, \mathbf{n}, \mathbf{m}) \text{ for all } \lambda, \mathbf{n}, \mathbf{m}.$$

So, $g = f$.

A simple calculation now shows that if $|m| > T$ and $2 \leq j \leq p$, then

$$\int_0^{2\pi} f(\rho_2 e^{i\phi_2}, \ldots, \rho_p e^{i\phi_p}, t) e^{-im\phi_j} d\phi_j = 0.$$

It follows easily from the definitions that

$$\tau_{\mathbf{n},\mathbf{m}}^l \circ \pi_\varepsilon(\rho_2 e^{i\phi_2}, \ldots, \rho_p e^{i\phi_p}, t) = \prod_{j=2}^{p} e^{i(n_j - m_j)\phi_j} \, \tau_{\mathbf{n},\mathbf{m}}^l \circ \pi_\varepsilon(\rho_2, \ldots, \rho_p, t).$$

Let ε_0 be such that the support of f (considered f as a function on $\bar{N}M$) is contained in O_{ε_0}. For $\varepsilon < \varepsilon_0$,

$$
\begin{aligned}
\widehat{f_\varepsilon}(l, \mathbf{n}, \mathbf{m}) &= \int_K f_\varepsilon(g) \overline{\tau_{\mathbf{n},\mathbf{m}}^l}(g) \, dg \\
&= \int_{\bar{N}M} f_\varepsilon \circ \pi_\varepsilon(\bar{n}m) \overline{\tau_{\mathbf{n},\mathbf{m}}^l}(\pi_\varepsilon(\bar{n}m)) \, |\det d\pi_\varepsilon(\bar{n}m)| \, d\bar{n}dm \\
&= \varepsilon^{-p} \int_{O_\varepsilon} f(\bar{n}) \overline{\tau_{\mathbf{n},\mathbf{m}}^l} \pi_\varepsilon(\bar{n}) \, |\det d\pi_\varepsilon(\bar{n})| \, d\bar{n}dm.
\end{aligned}
$$

By a change of variable, the last integral is 0 if $\max \{|n_j - m_j| : 2 \leq j \leq p\}$ is greater than T.   □

Next we estimate the rate of decay of the Fourier transforms of C^∞-functions on K. For $2 \leq j \leq p$, set $P_j = (1/2)(E_{1j} - E_{j1})$, $Q_j = (i/2)(E_{1j} + E_{j1})$, $T_j = i(-E_{11} + E_{jj})$. Let $L = \sum_{j=2}^{p}(P_j^2 + Q_j^2)$ and $L_1 = \sum_{j=2}^{p} T_j^2$.

LEMMA 4.6. *Let f be a C^∞-function of compact support on $\bar{N}$ and r a positive number. Then for all sufficiently small ε and $l \geq 0$, $|\widehat{f_\varepsilon}(l, \mathbf{n}, \mathbf{m})|$ is bounded by both of the expressions*

(i) $C \left[1 + \dfrac{\varepsilon}{2}(|\mathbf{n}| + |\mathbf{m}|) + \varepsilon(|\mathbf{n}|(l - |\mathbf{n}|) + |\mathbf{m}|(l - |\mathbf{m}|)) \right.$

$$\left. + \frac{\varepsilon}{2}(p-1)(2l - (|\mathbf{n}| + |\mathbf{m}|)) \right]^{-r}$$

and

(ii)
$$C\left[1 + \varepsilon\left(\sum_{k=2}^{p}(l - |\mathbf{n}| - n_k)^2 + (l - |\mathbf{m}| - m_k)^2\right)\right]^{-r},$$

where C depends only on f and r, but not on ε.

Proof. It is easily seen that

$$d\tau(P_j)\Gamma = -\Gamma_j, \qquad d\tau(P_j)\Gamma_k = \delta_{jk}\Gamma, \qquad d\tau(Q_j)\Gamma = i\Gamma_j,$$
$$d\tau(Q_j)\Gamma_k = i\delta_{jk}\Gamma, \qquad d\tau(T_k)\Gamma = -i\Gamma, \qquad d\tau(T_k)\Gamma_j = i\delta_{jk}\Gamma.$$

Using these relations, we get that

$$d\tau^l(L)\Gamma_{l,\mathbf{n}} = -\{|\mathbf{n}|(l - |\mathbf{n}|) + (1/2)(l - |\mathbf{n}|)(p - 1) + (1/2)|\mathbf{n}|\}\Gamma_{l,\mathbf{n}}$$

and

$$d\tau^l(L_1)\Gamma_{l,\mathbf{n}} = -\sum_{k=2}^{p}(l - |\mathbf{n}| - n_k)^2\ \Gamma_{l,\mathbf{n}}.$$

Hence

$$\widehat{L}(l, \mathbf{n}, \mathbf{m}) = \begin{cases} -\{|\mathbf{n}|(l - |\mathbf{n}|) + (1/2)(l - |\mathbf{n}|)(p - 1) + (1/2)|\mathbf{n}|\} & \text{for } \mathbf{m} = \mathbf{n}, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\widehat{L_1}(l, \mathbf{n}, \mathbf{m}) = \begin{cases} -\{\sum_{k=2}^{p}(l - |\mathbf{n}| - n_k)^2\} & \text{for } \mathbf{m} = \mathbf{n}, \\ 0 & \text{otherwise}. \end{cases}$$

The remaining part of the proof of the Lemma 4.6 is similar to the proof of Lemma 5 in [RRu] and will be left to the reader.   □

Proof of Theorem 4.1. It will be enough to show that

$$\left|\int_{-\infty}^{\infty}\text{Tr}\left(\widehat{f}(\lambda)\phi(\lambda)\widehat{g}(\lambda)\right)|\lambda|^{p-1}d\lambda\right| \leq C\|f\|_{L^q(\bar{N})}\|g\|_{L^{q'}(\bar{N})}$$

when f and g are C^{∞}-functions on $\bar{N}$ with compact support and of type T for some T. Let ε_0 be such that the support of f and g is contained in O_{ε_0}. Consider $\varepsilon < \varepsilon_0$. By hypothesis and Lemma 4.3,

$$\left|\sum_{-\infty}^{\infty}d_{|l|}\text{Tr}\left(\widehat{f_\varepsilon}(l)\phi_\varepsilon(l)\widehat{g_\varepsilon}(l)\right)\right| \leq C\|f_\varepsilon\|_{L^q(K)}\|g_\varepsilon\|_{L^{q'}(K)}$$
$$\leq C\varepsilon^{-p}\|f\|_{L^q(\bar{N})}\|g\|_{L^q(\bar{N})}.$$

Therefore the theorem is proved if we show that

$$\lim_{\varepsilon\to 0}\varepsilon^p\sum_{-\infty}^{\infty}d_{|l|}\text{Tr}\left(\widehat{f_\varepsilon}(l)\phi_\varepsilon(l)\widehat{g_\varepsilon}(l)\right) = C\int_{-\infty}^{\infty}\text{Tr}\left(\widehat{f}(\lambda)\phi(\lambda)\widehat{g}(\lambda)\right)|\lambda|^{p-1}d\lambda.$$

Next,

$$\varepsilon^p \sum_{-\infty}^{\infty} d_{|l|} \operatorname{Tr}\left(\widehat{f}_\varepsilon(l)\phi_\varepsilon(l)\widehat{g}_\varepsilon(l)\right) = A_\varepsilon + B_\varepsilon$$

say, where

$$A_\varepsilon = \varepsilon^p \sum_{l=0}^{\infty} d_l \operatorname{Tr}\left(\widehat{f}_\varepsilon(l)\phi_\varepsilon(l)\widehat{g}_\varepsilon(l)\right)$$

and

$$B_\varepsilon = \varepsilon^p \sum_{l=1}^{\infty} d_{|l|} \operatorname{Tr}\left(\widehat{f}_\varepsilon(-l)\phi_\varepsilon(-l)\widehat{g}_\varepsilon(-l)\right).$$

Let

$$d(\varepsilon, p, \lambda) = \frac{\pi^p \ (p + [\varepsilon^{-1}\pi\lambda] - 1)!}{(p-1)![\varepsilon^{-1}\pi\lambda]! \{(([\varepsilon^{-1}\pi\lambda] + 1)^p - [\varepsilon^{-1}\pi\lambda]^p\}}.$$

Next, using the fact that

$$d_l = \left(\begin{array}{c} p + l - 1 \\ l \end{array}\right)$$

we get that

$$A_\epsilon = p \sum_{l=0}^{\infty} \int_{l\varepsilon/\pi}^{(l+1)\varepsilon/\pi} d(\varepsilon, p, \lambda) \operatorname{Tr}\left(\widehat{f}_\varepsilon([\varepsilon^{-1}\pi\lambda])\phi_\varepsilon([\varepsilon^{-1}\pi\lambda])\widehat{g}_\varepsilon([\varepsilon^{-1}\pi\lambda])\right) \lambda^{p-1}\,d\lambda$$

$$= \int_0^{\infty} d(\varepsilon, p, \lambda) \operatorname{Tr}\left(\widehat{f}_\varepsilon([\varepsilon^{-1}\pi\lambda])\phi_\varepsilon([\varepsilon^{-1}\pi\lambda])\widehat{g}_\varepsilon([\varepsilon^{-1}\pi\lambda])\right) \lambda^{p-1}\,d\lambda.$$

We denote $[\varepsilon^{-1}\pi\lambda]$ by $n_\lambda(\varepsilon)$. Now for ε sufficiently small,

$$\operatorname{Tr}\left(\widehat{f}_\varepsilon(n_\lambda(\varepsilon))\phi_\varepsilon(n_\lambda(\varepsilon))\widehat{g}_\varepsilon(n_\lambda(\varepsilon))\right)$$

$$= \sum_{|\mathbf{n}|=0}^{n_\lambda(\varepsilon)} \sum_{|\mathbf{m}|,|\mathbf{p}|=0}^{n_\lambda(\varepsilon)} \widehat{f}_\varepsilon(n_\lambda(\varepsilon), \mathbf{n}, \mathbf{m})\phi_\varepsilon(n_\lambda(\varepsilon), \mathbf{m}, \mathbf{p})\widehat{g}_\varepsilon(n_\lambda(\varepsilon), \mathbf{p}, \mathbf{n}).$$

Hence

$$A_\varepsilon = \int_0^{\infty} d(\varepsilon, p, \lambda) \sum_{|\mathbf{n}|=0}^{n_\lambda(\varepsilon)} \sum_{|\mathbf{m}|,|\mathbf{p}|=0}^{n_\lambda(\varepsilon)} \widehat{f}_\varepsilon(n_\lambda(\varepsilon), \mathbf{n}, \mathbf{m}) \times$$

$$\times \phi_\varepsilon(n_\lambda(\varepsilon), \mathbf{m}, \mathbf{p})\widehat{g}_\varepsilon(n_\lambda(\varepsilon), \mathbf{p}, \mathbf{n})\lambda^{p-1}\,d\lambda.$$

We note that the number of terms in the sum depends on ε. By our assumption, the sums are over the regions where both $\max\{|n_j - m_j| : 2 \le j \le p\} \le T$ and $\max\{|n_j - p_j| : 2 \le j \le p\} \le T$. For fixed $\mathbf{n}$, the number of $\mathbf{m}, \mathbf{p}$ satisfying this inequality is at most $(2T + 1)^{2(p-1)}$.

We choose a number $\alpha > 0$ such that for all $\mathbf{n} \in \mathbb{N}^{p-1}$, $\sum_{k=2}^{p} n_k^2 \ge \alpha|\mathbf{n}|^2$. Next we choose $0 < \beta < 1$ such that $\Gamma = (\beta^2(\alpha + 1 + p) - (p + 1))$ is positive. We split the sum over $\mathbf{n}$ into two parts, Σ_1 and Σ_2: Σ_1 is the sum over the

terms with $|\mathbf{n}| \leq \beta n_\lambda(\varepsilon)$ and Σ_2 is the sum over the terms with $|\mathbf{n}| > \beta n_\lambda(\varepsilon)$. Using Lemma 4.6 (i), we obtain, for any positive integer r,

$$|\widehat{f_\varepsilon}(n_\lambda(\varepsilon), \mathbf{n}, \mathbf{m})| \leq C\Big[1 + \varepsilon(|\mathbf{n}|(n_\lambda(\varepsilon) - |\mathbf{n}|) + |\mathbf{m}|(n_\lambda(\varepsilon) - |\mathbf{m}|))$$
$$+ \frac{\varepsilon}{2}(|\mathbf{n}| + |\mathbf{m}|)\frac{\varepsilon}{2}(p-1)(2n_\lambda(\varepsilon) - (|\mathbf{n}| + |\mathbf{m}|))\Big]^{-r},$$

and similarly for g_ε. For the terms in Σ_1 one has

$$|\widehat{f_\varepsilon}(n_\lambda(\varepsilon), \mathbf{n}, \mathbf{m})| \leq C[1 + a\lambda(1 + |\mathbf{n}|)]^{-r},$$

$$|\widehat{g_\varepsilon}(n_\lambda(\varepsilon), \mathbf{n}, \mathbf{m})| \leq C[1 + a\lambda(1 + |\mathbf{n}|)]^{-r},$$

where a is a constant independent of $\mathbf{n}, \lambda$ and ε.

Also, by a theorem of Herz [H], for $\mathbf{m}, \mathbf{p} \in \mathbb{N}$ we have

$$|\phi_\varepsilon(n_\lambda(\varepsilon), \mathbf{m}, \mathbf{p})| \leq C.$$

Hence Σ_1 is dominated by

$$C \sum_{|\mathbf{n}|=0}^{\infty} \frac{(2T+1)^{(2p-2)}}{[1 + a\lambda(1 + |\mathbf{n}|)]^{2r}}.$$

Choosing $r \geq 2p$, we see that the sum converges.

Using Lemma 4.6 (ii), one has

$$|\widehat{f_\varepsilon}(n_\lambda(\varepsilon), \mathbf{n}, \mathbf{m})|$$
$$\leq C\left[1 + \varepsilon\left(\sum_{k=2}^{p}(n_\lambda(\varepsilon) - |\mathbf{n}| - n_k)^2 + (n_\lambda(\varepsilon) - |\mathbf{m}| - m_k)^2\right)\right]^{-r}.$$

By our choice of the constants α, β and Γ, for the terms in Σ_2 we have

$$\left(\sum_{k=2}^{p}(n_\lambda(\varepsilon) - |\mathbf{n}| - n_k)^2\right) \geq \Gamma n_\lambda^2(\varepsilon).$$

So,

$$|\widehat{f_\varepsilon}(n_\lambda(\varepsilon), \mathbf{n}, \mathbf{m})| \leq C[1 + \varepsilon\,\Gamma n_\lambda^2(\varepsilon)]^{-r}.$$

Let $0 < \varepsilon \leq \pi\lambda/2$. Then for the terms in Σ_2 we have

$$|\widehat{f_\varepsilon}(n_\lambda(\varepsilon), \mathbf{n}, \mathbf{m})| \leq C[1 + \pi\,\Gamma\,(\lambda/2)|\mathbf{n}|]^{-r}$$

and similarly for $|\widehat{g_\varepsilon}(n_\lambda(\varepsilon), \mathbf{n}, \mathbf{m})|$. Hence, Σ_2 is dominated by

$$C \sum_{|\mathbf{n}|=0}^{\infty} \frac{(2T+1)^{(2p-2)}}{[1 + \pi\,\Gamma\,(\lambda/2)\,|\mathbf{n}|]^{2r}},$$

which converges for $r \geq 2p$.

So we may apply the dominated convergence theorem to the double sum to see that

$$\lim_{\varepsilon \to 0} \mathrm{Tr}\left(\widehat{f}_\varepsilon(n_\lambda(\varepsilon))\phi_\varepsilon(n_\lambda(\varepsilon))\widehat{g}_\varepsilon(n_\lambda(\varepsilon))\right)$$

$$= \sum_{|\mathbf{n}|=0}^{\infty} \sum_{|\mathbf{m}|,|\mathbf{p}|=0}^{\infty} \widehat{f}(\lambda,\mathbf{n},\mathbf{m})\phi(\lambda,\mathbf{m},\mathbf{n})\widehat{g}(\lambda,\mathbf{n},\mathbf{p})$$

$$= \mathrm{Tr}\left(\widehat{f}(\lambda)\phi(\lambda)\widehat{g}(\lambda)\right).$$

Next, $\lim_{\varepsilon \to 0} d(\varepsilon,p,\lambda) = \pi^p/(p-1)!$ and $d(\varepsilon,p,\lambda) \le (p^{p-1}\pi^p)/(p-1)!$. Also, for $r = 2p$, the estimates on Σ_1 and Σ_2 allow us to use the dominated convergence theorem in the variable λ. Hence, we can take the $\lim_{\varepsilon \to 0}$ inside the integrands of A_ε. Therefore, we get

$$\lim_{\varepsilon \to 0} A_\varepsilon = \frac{\pi^p}{(p-1)!} \int_0^\infty \mathrm{Tr}\left(\widehat{f}(\lambda)\phi(\lambda)\widehat{g}(\lambda)\right) \lambda^{p-1} d\lambda.$$

We next consider B_ε. We have

$$B_\varepsilon = \varepsilon^p \sum_{l=1}^{\infty} d_{|l|} \mathrm{Tr}\left(\widehat{f}_\varepsilon(-l)\phi_\varepsilon(-l)\widehat{g}_\varepsilon(-l)\right)$$

$$= \sum_{l=1}^{\infty} \int_{l\varepsilon/\pi}^{(l+1)\varepsilon/\pi} d(\varepsilon,p,\lambda) \times$$

$$\times \mathrm{Tr}\left(\widehat{f}_\varepsilon(-[\varepsilon^{-1}\pi\lambda])\phi_\varepsilon(-[\varepsilon^{-1}\pi\lambda])\widehat{g}_\varepsilon(-[\varepsilon^{-1}\pi\lambda])\right) \lambda^{p-1} d\lambda$$

$$= \int_{\varepsilon/\pi}^{\infty} d(\varepsilon,p,\lambda)\mathrm{Tr}\left(\widehat{f}_\varepsilon(-[\varepsilon^{-1}\pi\lambda])\phi_\varepsilon(-[\varepsilon^{-1}\pi\lambda])\widehat{g}_\varepsilon(-[\varepsilon^{-1}\pi\lambda])\right) \lambda^{p-1} d\lambda$$

Using the same arguments as in calculating $\lim_{\varepsilon \to 0} A_\varepsilon$, we obtain

$$\lim_{\varepsilon \to 0} B_\varepsilon = \frac{\pi^p}{(p-1)!} \int_0^\infty \mathrm{Tr}\left(\widehat{f}(-\lambda)\phi(-\lambda)\widehat{g}(-\lambda)\right) \lambda^{p-1} d\lambda.$$

Hence

$$\lim_{\varepsilon \to 0} \varepsilon^p \sum_{-\infty}^{\infty} d_l \mathrm{Tr}\left(\widehat{f}_\varepsilon(l)\phi_\varepsilon(l)\widehat{g}_\varepsilon(l)\right)$$

$$= \frac{\pi^p}{(p-1)!} \int_{-\infty}^{\infty} \mathrm{Tr}\left(\widehat{f}(-\lambda)\phi(-\lambda)\widehat{g}(-\lambda)\right) |\lambda|^{p-1} d\lambda.$$

This completes the proof. $\square$

5. A converse of de Leeuw's theorem

It can be seen that the measures $\mu_t = \delta_{(0,t)}$ on H^1, when restricted to $SU(2)$ on the Fourier transforms side, are not bounded multipliers on any $L^q(SU(2))$. So a natural analogue of the classical converse of de Leeuw's

theorem is not true for the contraction of S^{2p-1} to H^{p-1}. Recently, using contractions, a converse of de Leeuw's theorem has been proved in a situation which is more general than ours (see [DGuR]). Using the ideas contained in [DGuR], we prove in Theorem 5.1 below a stronger form of the converse to de Leeuw's theorem in our set-up.

Fix a neighbourhood B of the identity in H^{p-1} such that $B = B^{-1}$ and $B^2 M$ is relatively compact in O.

THEOREM 5.1. *Let $k \in L^1(H^{p-1})$ with $\mathrm{supp}(k) \in B$. For $\varepsilon > 0$ the formula*

$$\langle K_\varepsilon, f \rangle = \int \int_{B \times B} k(z) f(\pi_\varepsilon(w)^{-1} \pi_\varepsilon(wz)) \, dw \, dz$$

defines a function $K_\varepsilon \in L^1(K)$ with support contained in $\pi_\varepsilon(B)\pi_\varepsilon(B^2)$. Furthermore, for every $q \in (1, \infty)$ there exists a constant C independent of k and ε such that:

(a) $K_\varepsilon\|_q \leq \|k\|_q,$

(b) $\lim_{\varepsilon \to 0} \widehat{K}_\varepsilon(n(\varepsilon), \mathbf{n}, \mathbf{m}) = \widehat{k}(\lambda, \mathbf{n}, \mathbf{m}), \ \lambda > 0,$

(c) $\lim_{\varepsilon \to 0} \widehat{K}_\varepsilon(-n(\varepsilon), \mathbf{n}, \mathbf{m}) = (-1)^{|\mathbf{n}|+|\mathbf{m}|} \widehat{k}(-\lambda, \mathbf{n}, \mathbf{m}), \ \lambda > 0,$

(d) $|k\|_q \leq \sup\{\|K_\varepsilon\|_q : \varepsilon > 0\}.$

To prove Theorem 5.1, we need a stronger form of Lemma 4.2, which we state below:

LEMMA 5.2.   *Let $w = (Y, b), \ z = (Z, c), \ Y, Z \in \mathfrak{g}_{-\lambda}$ and $b, c \in \mathbb{R}$. Then*

$$\lim_{\varepsilon \to 0} \tau_{\mathbf{n},\mathbf{m}}^{n(\varepsilon)}(\pi_\varepsilon(wz)^{-1}\pi_\varepsilon(w)) = \widetilde{\sigma}_{\mathbf{n},\mathbf{m}}^{\lambda}(z^{-1})$$

and

$$\lim_{\varepsilon \to 0} \tau_{\mathbf{n},\mathbf{m}}^{-n(\varepsilon)}(\pi_\varepsilon(wz)^{-1}\pi_\varepsilon(w)) = (-1)^{|\mathbf{n}|+|\mathbf{m}|} \widetilde{\sigma}_{\mathbf{n},\mathbf{m}}^{-\lambda}(z^{-1}).$$

We introduce some more notation: For any, $w = (Y, b) \in \bar{N}$, define

$$d_w = \sqrt{2\varepsilon}\|Y\|,$$

$$b_w = \sqrt{\varepsilon^2 b^2 + 4 \left(d_w\right)^2},$$

$$a_w = (2\sqrt{\varepsilon}/b_w) \sin(b_w/2)e^{3i\varepsilon b/2}.$$

For $\varepsilon > 0$, set $\bar{n}_\varepsilon = (X, \varepsilon^{-1}a), \ \phi_\varepsilon(\bar{n}) = \pi_\varepsilon(\bar{n}_\varepsilon)$ and

$$\Gamma_{j,\varepsilon} = \frac{1}{2\sqrt{\varepsilon}}\Gamma_j.$$

To prove Lemma 5.2, we need two lemmas.

LEMMA 5.3. *Let* $B = H - i(X_0 - \theta X_0) = 2E_{1,p+1}$, $w = (Y, b) \in \bar{N}$ *and* $Z \in \mathfrak{g}_{-\lambda}$. *Then the following statements are true.*

(1) $[Y_\theta, B] = (Y - \theta Y) - i(-JY + \theta JY)$,

(2) $\mathrm{Ad}(\pi_\varepsilon(w))B = \overline{\Gamma}(\pi_\varepsilon(w))B + a_w[Y_\theta, B]$,

(3) $\Gamma(\pi_\varepsilon(w)) = e^{-3i\varepsilon b/2}\left(\cos\left(b_w/2\right) - i\left(\varepsilon b/b_w\right)\sin\left(b_w/2\right)\right)$,

(4) $\mathrm{Ad}(\pi_\varepsilon(w))(Z - \theta Z)$
$$= (Z - \theta Z) - 2\varepsilon^{1/2}\left\{(X_0 - \theta X_0)(Y, JZ) + H(Y, Z)\right\} + o(\varepsilon).$$

Proof. See [DGu]. $\square$

LEMMA 5.4. *Let* $X, Y, Z \in \mathfrak{g}_{-\lambda}$ *and* $a, b, c \in \mathbb{R}$. *For* $\overline{n} = (X, a)$, $w = (Y, b)$, $z = (Z, c)$ *and* $X_a = (2/a)\sin(a/2)e^{ia/2}X$, *we have:*

(i) $\displaystyle \lim_{\varepsilon \to 0} \Gamma^{n(\varepsilon)}\left(\pi_\varepsilon(w)^{-1}\pi_\varepsilon(wz)\phi_\varepsilon(\overline{n})\right)\overline{\Gamma}^{n_\lambda(\varepsilon)}(\phi_\varepsilon(\overline{n}))$

$$= \lim_{\varepsilon \to 0} \Gamma^{n(\varepsilon)}(\pi_\varepsilon(z)\phi_\varepsilon(\overline{n}))\overline{\Gamma}^{n_\lambda(\varepsilon)}(\phi_\varepsilon(\overline{n}))$$

$$= e^{-2ic\lambda\pi}\, e^{-2\lambda\pi(\|Z\|^2/2 + (X_a, Z)_\mathbb{C})}\, e^{-2\lambda\pi\|X_a\|^2},$$

(ii) $\displaystyle \lim_{\varepsilon \to 0} \Gamma_{j,\varepsilon}(\pi_\varepsilon(w)^{-1}\pi_\varepsilon(wz)\phi_\varepsilon(\overline{n}))$

$$= \lim_{\varepsilon \to 0} \Gamma_{j,\varepsilon}(\pi_\varepsilon(z)\phi_\varepsilon(\overline{n}))$$

$$= e^{-2ia}(X_a + Z, X_j)_\mathbb{C}.$$

Proof. In the proof we will repeatedly use Lemma 5.3.

(i) By the definition of Γ,

$\Gamma(\pi_\varepsilon(w)^{-1}\pi_\varepsilon(wz)\phi_\varepsilon(\overline{n})) = \frac{1}{2}\left(\mathrm{Ad}(\pi_\varepsilon(w))B, \mathrm{Ad}(\pi_\varepsilon(wz))\,\mathrm{Ad}(\phi_\varepsilon(\overline{n}))B\right).$

Next,
$$\mathrm{Ad}(\pi_\varepsilon(w))B = \overline{\Gamma}(\pi_\varepsilon(w))B + a_w[Y_\theta, B],$$

so,

$\mathrm{Ad}(\pi_\varepsilon(wz))\,\mathrm{Ad}(\phi_\varepsilon(\overline{n}))B$

$= \mathrm{Ad}(\pi_\varepsilon(wz))\left\{\overline{\Gamma}(\phi_\varepsilon(\overline{n}))B + a_{\overline{n}_\varepsilon}[X_\theta, B]\right\}$

$= \overline{\Gamma}(\phi_\varepsilon(\overline{n}))\left\{\overline{\Gamma}(\pi_\varepsilon(wz))B + a_{wz}[Y_\theta + Z_\theta, B]\right\} + a_{\overline{n}_\varepsilon}\,\mathrm{Ad}(\pi_\varepsilon(wz))[X_\theta, B].$

Therefore

$\Gamma(\pi_\varepsilon(w)^{-1}\pi_\varepsilon(wz)\phi_\varepsilon(\overline{n}))$

$\qquad = \overline{\Gamma}(\pi_\varepsilon(w))\,\Gamma\left(\phi_\varepsilon(\bar{n})\right)\Gamma(\pi_\varepsilon(wz))$

$\qquad\qquad + (1/2)\overline{a}_{\overline{n}_\varepsilon}\,\overline{\Gamma}(\pi_\varepsilon(w))\left(B, \mathrm{Ad}(\pi_\varepsilon(wz))[X_\theta, B]\right)$

$\qquad\qquad + (1/2)a_w\overline{a}_{wz}\,\Gamma(\phi_\varepsilon(\overline{n}))\left([Y_\theta, B], [Y_\theta + Z_\theta, B]\right)$

$\qquad\qquad + (1/2)a_w\overline{a}_{\overline{n}_\varepsilon}\left([Y_\theta, B], \mathrm{Ad}(\pi_\varepsilon(wz))[X_\theta, B]\right).$

Next,

$$(B, \mathrm{Ad}(\pi_\varepsilon(wz))[X_\theta, B]) \;=\; -4\varepsilon^{1/2}(X, Y+Z)_{\mathbb{C}} + o(\varepsilon^{1/2}),$$
$$([Y_\theta, B], [Y_\theta + Z_\theta, B]) \;=\; 4\|Y\|^2 + 4(Z, Y)_{\mathbb{C}}$$

and

$$([Y_\theta, B], \mathrm{Ad}(\pi_\varepsilon(wz))[X_\theta, B]) \;=\; ([Y_\theta, B], [X_\theta, B]) + o(\varepsilon^{1/2})$$
$$=\; 4(X, Y)_{\mathbb{C}} + o(\varepsilon^{1/2}).$$

Set

$$A_\varepsilon = |\Gamma(\phi_\varepsilon(\overline{n}))|^2 \; \overline{\Gamma}(\pi_\varepsilon(w)) \; \Gamma(\pi_\varepsilon(wz))$$

and

$$B_\varepsilon = \varepsilon \frac{2\overline{a}_{\overline{n}_\varepsilon}}{\varepsilon^{1/2}} \frac{(X, Y+Z)_{\mathbb{C}}}{\Gamma(\phi_\varepsilon(\overline{n}))\Gamma(\pi_\varepsilon(wz))} - \frac{\varepsilon \frac{2a_w}{\varepsilon^{1/2}} \frac{\overline{a}_w}{\varepsilon^{1/2}} \{\|Y\|^2 + (Z,Y)_{\mathbb{C}}\}}{\overline{\Gamma}(\pi_\varepsilon(w))\Gamma(\pi_\varepsilon(wz))}$$
$$- \frac{\varepsilon \frac{2a_w}{\varepsilon^{1/2}} \frac{\overline{a}_{\overline{n}_\varepsilon}}{\varepsilon^{1/2}}(X,Y)_{\mathbb{C}}}{\Gamma(\phi_\varepsilon(\overline{n}))\overline{\Gamma}(\pi_\varepsilon(w))\Gamma(\pi_\varepsilon(wz))} - o(\varepsilon).$$

Then

$$\Gamma^{n(\varepsilon)}\left(\pi_\varepsilon^{-1}(w)\pi_\varepsilon(wz)\phi_\varepsilon(\overline{n})\right) \overline{\Gamma}^{n(\varepsilon)}(\phi_\varepsilon(\overline{n})) = A_\varepsilon^{n_\lambda(\varepsilon)}\{1 - B_\varepsilon\}^{n_\lambda(\varepsilon)}.$$

Note that

$$\frac{\overline{a}_{\overline{n}_\varepsilon}}{\varepsilon^{1/2}} \longrightarrow \frac{\sin(a/2)}{(a/2)} e^{-3ia/2} \quad \text{as } \varepsilon \longrightarrow 0,$$
$$\frac{a_w}{\varepsilon^{1/2}} \longrightarrow 1 \quad \text{as } \varepsilon \to 0,$$
$$\frac{a_{wz}}{\varepsilon^{1/2}} \longrightarrow 1 \quad \text{as } \varepsilon \longrightarrow 0.$$

Also, $\Gamma(\phi_\varepsilon(\overline{n})) \longrightarrow e^{-2ia}$ and $\Gamma^{n_\lambda(\varepsilon)}(\pi_\varepsilon(\overline{n})) \to e^{-2ia\lambda\pi} e^{-\pi\lambda\|X\|^2}$ as $\varepsilon \longrightarrow 0$. Therefore

$$\lim_{\varepsilon \to 0} \Gamma^{n(\varepsilon)}(\pi_\varepsilon(w)^{-1}\pi_\varepsilon(wz)\phi_\varepsilon(\overline{n}))\overline{\Gamma}^{n(\varepsilon)}(\phi_\varepsilon(\overline{n}))$$
$$= \; e^{-2\lambda\pi\|X_a\|^2} e^{2i\pi\lambda b} e^{-\pi\lambda\|Y\|^2} e^{-2i\lambda\pi(b+c+([Y,Z],X_0))} e^{-\pi\lambda\|Y+Z\|^2} \times$$
$$\times \, e^{-2\pi\lambda(X_a, Y+Z)_{\mathbb{C}}} e^{2\pi\lambda(\|Y\|^2 + (Z,Y)_{\mathbb{C}})} e^{2\pi\lambda(X_a, Y)_{\mathbb{C}}}$$
$$= \; e^{-2\lambda\pi\|X_a\|^2} e^{-2i\lambda\pi c} e^{-\pi\lambda\|Z\|^2} e^{-2\pi\lambda(X_a, Z)_{\mathbb{C}}}$$

as $-i([Y, Z], X_0) + (Z, Y)_{\mathbb{C}} - (Y, Z) = 0$. Hence (i) is proved.

(ii) By the definition of $\Gamma_{j,\varepsilon}$,

$$\Gamma_{j,\varepsilon}\left(\pi_\varepsilon(w)^{-1}\pi_\varepsilon(wz)\phi_\varepsilon(\overline{n})\right)$$
$$= \; \frac{1}{2\sqrt{\varepsilon}}(X_j - \theta X_j, \mathrm{Ad}(\pi_\varepsilon(w)^{-1}) \, \mathrm{Ad}(\pi_\varepsilon(wz)) Ad(\phi_\varepsilon(\overline{n}))B)$$
$$= \; \frac{1}{2\sqrt{\varepsilon}}(\mathrm{Ad}(\pi_\varepsilon(w))(X_j - \theta X_j), \, \mathrm{Ad}(\pi_\varepsilon(wz)) \, \mathrm{Ad}(\phi_\varepsilon(\overline{n}))B).$$

Next,

$$
\begin{aligned}
\mathrm{Ad}&(\pi_\varepsilon(w))(X_j - \theta X_j) \\
&= X_j - \theta X_j + \varepsilon^{1/2}\{-2(Y, JX_j)(X_0 - \theta X_0) - 2(Y, X_j)H\} + o(\varepsilon)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Ad}&(\pi_\varepsilon(wz))\,\mathrm{Ad}(\phi_\varepsilon(\overline{n}))B \\
&= \overline{\Gamma}(\phi_\varepsilon(\overline{n}))\{\overline{\Gamma}(\pi_\varepsilon(wz))B + a_{wz}[Y_\theta + Z_\theta, B]\} + a_{\overline{n}_\varepsilon}[X_\theta, B] + o(\varepsilon^{1/2}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Gamma_{j,\varepsilon}&(\pi_\varepsilon(w)^{-1}\pi_\varepsilon(wz)\phi_\varepsilon(\overline{n})) \\
&= \frac{1}{2\sqrt{\varepsilon}}\left\{\overline{a}_{wz}\Gamma(\phi_\varepsilon(\overline{n}))(X_j - \theta X_j, [Y_\theta + Z_\theta, B]) + \overline{a}_{\overline{n}_\varepsilon}(X_j - \theta X_j, [X_\theta, B])\right\} \\
&\qquad + \frac{1}{2}\left\{-2(Y, JX_j)(X_0 - \theta X_0, B)\right. \\
&\qquad\qquad \left. - 2(Y, X_j)(H, B)\right\}\Gamma(\phi_\varepsilon(\overline{n}))\Gamma(\phi_\varepsilon(wz)) + o(1).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lim_{\varepsilon\to 0}&\Gamma_{j,\varepsilon}(\pi_\varepsilon(w)^{-1}\pi_\varepsilon(wz)\phi_\varepsilon(\overline{n})) \\
&= e^{-2ia}(Y + Z, X_j)_{\mathbb{C}} + e^{-2ia}(X_a, X_j)_{\mathbb{C}} - e^{-2ia}(Y, X_j)_{\mathbb{C}} \\
&= e^{-2ia}(X_a + Z, X_j)_{\mathbb{C}},
\end{aligned}
$$

and (ii) is proved. $\qquad\square$

Proof of Lemma 5.2. The proof of Lemma 5.2 follows from Lemma 5.4. The details of this are given in [DGu]. $\qquad\square$

Proof of Theorem 5.1. The proof of the assertions $K_\varepsilon \in L^1(K)$, $\mathrm{supp}(K_\varepsilon) \subset \pi_\varepsilon(B)\pi_\varepsilon(B^2)$, and of part (a) is given in [DGuR]. We now prove (b), (c), and (d).

(b): By the definition of K_ε,

$$
\begin{aligned}
\widehat{K}_\varepsilon(n(\varepsilon), \mathbf{n}, \mathbf{m}) &= \langle K_\varepsilon, \tau_{\mathbf{n},\mathbf{m}}^{n(\varepsilon)}\rangle \\
&= \iint_{B\times B} k(z)\,\tau_{\mathbf{n},\mathbf{m}}^{n_\lambda(\varepsilon)}(\pi_\varepsilon(w)^{-1}\pi_\varepsilon(wz))dwdz \\
&= \iint_{B\times B} k(z)\,\tau_{\mathbf{n},\mathbf{m}}^{n_\lambda(\varepsilon)}(\pi_\varepsilon(wz)^{-1}\pi_\varepsilon(w))dwdz.
\end{aligned}
$$

An application of Lemma 5.2 and of the Dominated Convergence Theorem yields

$$
\begin{aligned}
\lim_{\varepsilon \to 0} \widehat{K}_\varepsilon(n(\varepsilon)\mathbf{n}, \mathbf{m}) &= \int\int_{B \times B} k(z)\widetilde{\sigma}^\lambda_{\mathbf{n},\mathbf{m}}(z^{-1})dwdz \\
&= |B| \int_B k(z)\widetilde{\sigma}^\lambda_{\mathbf{n},\mathbf{m}}(z^{-1})dz \\
&= |B|\widehat{k}(\lambda, \mathbf{n}, \mathbf{m}).
\end{aligned}
$$

By a suitable normalization of the Haar measure on H^{p-1} we get

$$
\lim_{\varepsilon \to 0} \widehat{K}_\varepsilon(n(\varepsilon), \mathbf{n}, \mathbf{m}) = \widehat{k}(\lambda, \mathbf{n}, \mathbf{m}).
$$

Part (c) follows exactly in the same way as (b).

Part (d) follows by de Leeuw's Theorem 4.1.

This completes the proof. $\square$

REMARK.   Using Theorem 5.1 (a) and (d), a conjecture of Herz regarding the asymmetry of L^P-multipliers on compact connected non-abelian Lie groups can be resolved. But since Theorem 5.1 (a) and (d) are proved in a more general set-up in [DGuR] and a result more general than the Herz conjecture has been proved in [DGuR], we do not provide any details of the proof of the Herz conjecture here.

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