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THE LAPLACIAN-b RANDOM WALK AND THE SCHRAMM-LOEWNER EVOLUTION

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Dedicated to the memory of J. L. Doob

ABSTRACT. The Laplacian-b random walk is a measure on self-avoiding paths that at each step has translation probabilities weighted by the bth power of the probability that a simple random walk avoids the path up to that point. We give a heuristic argument as to what the scaling limit should be and call this process the Laplacian-b motion, LM_b. In simply connected domains, this process is the Schramm-Loewner evolution with parameter $\kappa = 6/(2b+1)$. In non-simply connected domains, it corresponds to the harmonic random Loewner chains as introduced by Zhan.

1. Introduction

In this paper we consider possible scaling limits of the Laplacian-b walk. Although this paper will focus primarily on two dimensions, we will start by describing the problem for $d \geq 2$.

If $b \in \mathbb{R}$ and $d \geq 3$, the *Laplacian-b walk* (from 0 to infinity) in $\mathbb{Z}^d$ is defined to be the non-Markovian process $\hat{S}_0, \hat{S}_1, \hat{S}_2, \ldots$ supported on self-avoiding paths that weights each step by the appropriate power of the escape probability of simple random walk. To be more precise, for any finite $K \subset \mathbb{Z}^d$, let $h_K(z)$ be the probability that simple random walk starting at z never visits K. By definition, $h_K(z) = 0$ if $z \in K$. Then h_K is the solution of the discrete Laplace equation

$$\Delta h_K(z) := \frac{1}{2d} \sum_{|y-z|=1} [h_K(y) - h_K(z)] = 0, \quad z \in \mathbb{Z}^d \setminus K,$$

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with boundary conditions $h_K \equiv 0$ on K and $h_K(z) \to 1$ as $z \to \infty$. The Laplacian-b walk starting at z_0 is the process with transition probabilities

$$\mathbf{P}\{\hat{S}_{n+1} = z_{n+1} \mid [\hat{S}_0, \ldots, \hat{S}_n] = [z_0, \ldots, z_n]\} = \frac{h_{K_n}(z_{n+1})^b}{\sum_{|y-z_n|=1} h_{K_n}(y)^b},$$

where $K_n = \{z_0, \ldots, z_n\}$ and we use the convention $0^b = 0$ even if $b \leq 0$. For $d = 2$, we can use a similar definition where we change the boundary conditions on h_K to be $h_K \equiv 0$ on K and $h_K(z) \sim \log|z|$ as $z \to \infty$. The definition can be modified (see Section 2) to give Laplacian-b walks in subsets of $\mathbb{Z}^d$ and Laplacian-b walks going to points $w \in \mathbb{Z}^d$.

The case $b = 1$ first arose in [11] where it was noted that this is the same as the *loop-erased random walk*. If S denotes a simple random walk in $\mathbb{Z}^d, d \geq 3$, and a self-avoiding path is obtained by erasing loops chronologically from the path, then the corresponding process has the transitions of the Laplacian-1 walk. A similar fact is true for $d = 2$ where the definition of the loop-erased walk needs to modified. The general Laplacian-b walk was introduced in [17].

If $d \geq 4$, it is known that the scaling limit of the loop-erased (Laplacian-1) walk is Brownian motion [11], [12]. Logarithmic corrections to scaling appear in the critical dimension $d = 4$. There is a $c = c_d > 0$ such that

$$\frac{\hat{S}_n}{\sqrt{c\,n}} \longrightarrow N, \quad d \geq 5,$$

$$\frac{\hat{S}_n}{\sqrt{c\,n\,\log^{1/3} n}} \longrightarrow N, \quad d = 4,$$

where N denotes a standard d-dimensional normal. The proofs make strong use of the loop-erased definition of the walk and the results have not been extended to other values of b. We conjecture that the results hold for a wide range of b although the logarithmic correction exponent in four dimensions ($1/3$ if $b = 1$) may depend on b.

For $d < 4$, it is not believed that the scaling limit is Brownian motion. As an approach to understanding the scaling limit, it is natural to first try to find a continuous object that is a candidate for the scaling limit and then try to prove convergence of the discrete process to this limit. We will call the candidate for the scaling limit the Laplacian-b motion, LM_b. For ease, let us consider the $b = 1$ case. Intuitively, such a process X_t should have the following behavior. Suppose $X[0, t]$ is known and we are interested in $X_s, s > t$. Let $\phi = \phi_{X[0,t]}$ be the positive harmonic function on $\mathbb{R}^d \setminus X[0, t]$ with boundary condition $\phi \equiv 0$ on $X[0, t]$ and as $z \to \infty$, $\phi(z) \sim 1$ (if $d = 3$) or $\phi(z) \sim \log|z|$ (if $d = 2$). Then locally the probabilities should be weighted by the normal derivative of ϕ at the point X_t. Unfortunately, it is not easy to make rigorous sense of this since the curve $X[0, t]$ will be fractal and not smooth near X_t.

Oded Schramm [18] overcame this difficulty in $d = 2$ by using Loewner theory from complex variables. By making the assumption that the limit was conformally invariant, he was able to map the nonsmooth curve $X[0, t]$ to a smooth curve (the real line), and thereby make things precise. This led to his definition of what is now called the *Schramm-Loewner evolution (SLE_κ)*. This is a one-parameter family of conformally invariant curves indexed by $\kappa > 0$. Schramm showed that if the limit of loop-erased walk existed and was conformally invariant then it must be SLE_κ for some particular value of κ. Particular known quantities for the loop-erased walk allowed him to determine the value $\kappa = 2$. Later papers [14], [22] proved that, in fact, the loop-erased walk converges to SLE_2.

There is strong reason to believe that the scaling limit of the Laplacian-b walk is conformally invariant for many other values of b. Under this assumption, it follows from Schramm's original argument that the process in simply connected domains must be SLE_κ for some value of κ. In this paper, we give a simple argument to show that we would expect $b = (6-\kappa)/(2\kappa)$ for $b > -1/2$. We call this process the two-dimensional Laplacian-b motion. We also show how this process is defined in non-simply connected domains. It turns out that there are many natural extensions of SLE_κ to non-simply connected domains. The LM_b is only one of these possibilities.

This conjectured relationship between b and κ is not new in this paper. Hastings [9] gave a nonrigorous argument phrased in terms of the multifractal spectrum for the Laplacian-b walk. This paper restricts to the $b \geq 1/4, \kappa \leq 4$ case where the appropriate scaling limit is expected to be supported on simple paths. In [9] η is used for b and $m = m(\eta)$ is used for a parameter which has the property that the dimension of the paths is $1 + (m/2)$. Hence, if the paths are to be SLE_κ paths, $m = \kappa/4$. The paper then relates the spectrum for a given m to that of central charge

$$c = 13 - 6m - \frac{6}{m} = 13 - \frac{3\kappa}{2} - \frac{24}{\kappa} = \frac{(3-4a)(3a-1)}{a}.$$

Here $a = 2/\kappa$ which is a parameter used in this paper. Although [9] gives arguments to compute the multifractal spectrum directly, one can derive the multifractal spectrum of SLE directly; these are essentially the "crossing" exponents or generalized Cardy's formula, see, e.g., [13, 6.9]. For a survey of the relationship between multifractal spectrum, conformal field theory, and quantum gravity from a physics perspective, see [7].

The process LM_b is also not new. After studying this process for a while, I realized that the process derived in this paper is essentially the same process introduced by Zhan in [23]. The notations and viewpoints of that paper is different from this and I will not try to relate the two. That paper also does not relate the process directly with the Laplacian-b walk except in the case $b = 1$ which is particular nice. I have chosen not to use Zhan's term harmonic random Loewner chain but rather to use the term Laplacian-b motion because it relates it more closely to the Laplacian walk and uses b as the parameter.

It is still an open question to define LM_b for $d = 3$. We will not try here.

1.1. h-processes. The Laplacian-b motion can be considered a generalization of Doob's construction of h-processes. In some sense, it is an h-process with moving boundary. In this section, we will review the facts about h-processes which serve to motivate the definition of LM_b.

Consider a one-dimensional Brownian motion B_t on the domain $D = (0, 1)$ and let τ_D be the first time that the process reaches $\partial D = \{0, 1\}$. The well known "gambler's ruin" estimate tells us that $\mathbf{P}^x\{B_{\tau_D} = 1\} = x$. If we condition the Brownian motion on the event $\{B_{\tau_D} = 1\}$ we impose a drift. The gambler's ruin estimate shows that we can obtain this conditioning by weighting paths at time t by the martingale $M_t = B_{t\wedge\tau_D}$, which satisfies the SDE

$$dM_t = \frac{1}{B_t}\, M_t\, dB_t, \quad t < \tau_D.$$

The Girsanov transformation tells us that this conditioned process has the same distribution as a Bessel process X_t satisfying

$$dX_t = \frac{1}{X_t}\, dt + dW_t,$$

where W_t is a standard Brownian motion with respect to the weighted measure.

If $b \in \mathbb{R}$, we can also consider processes obtained by weighting the paths by B_t^b. This is not a martingale, but since

$$d[B_t^b] = B_t^b \left[\frac{b(b-1)}{2\, B_t^2}\, dt + \frac{b}{B_t}\, dB_t\right],$$

the process

$$M_t = B_{t\wedge\tau_D}^b \exp\left\{-\frac{b(b-1)}{2} \int_0^{t\wedge\tau_D} \frac{ds}{B_s^2}\right\}$$

is a local martingale satisfying

$$dM_t = \frac{b}{B_t}\, M_t\, dB_t, \quad t < \tau_D.$$

In this case the weighted process satisfies the Bessel SDE,

$$dX_t = \frac{b}{X_t}\, dt + dW_t.$$

It is well known that this process does not reach 0 with probability one if and only if $b \geq 1/2$. One can still make sense of this for $b < 1/2$, but the fact that the weighted process reaches the origin gives technical difficulties.

Now suppose B_t is a d-dimensional Brownian motion in a domain $D \subset \mathbb{R}^d$ with smooth boundary and let $w \in \partial D$. We can define an excursion (h-process) in D by weighting the paths by the martingale $h(B_t)$ where h is the harmonic function $h(z) = H_D(z, w)$. Here H_D denotes the Poisson kernel, i.e., the density of the hitting measure of ∂D. The martingale $M_t = h(B_{t\wedge\tau_D})$ satisfies the SDE

$$dM_t = \frac{\nabla h(B_t)}{h(B_t)} M_t \, dB_t, \quad t < \tau_D.$$

Under this weighting the paths satisfy

$$dX_t = \frac{\nabla h(X_t)}{h(X_t)} \, dt + dW_t.$$

Here W_t is a standard d-dimensional Brownian motion with respect to the weighted measure. If we choose to weight the paths by $h(B_t)^b$, we first note that

$$dh(B_t)^b = h(B_t)^b \left[\frac{b(b-1) \, |\nabla h(B_t)|^2}{h(B_t)^2} \, dt + \frac{b \, \nabla h(B_t)}{h(B_t)} \, dB_t \right].$$

Hence, the local martingale

$$M_t = h(B_{t\wedge\tau_D})^b \exp\left\{ -b(b-1) \int_0^{t\wedge\tau_D} \frac{|\nabla h(B_s)|^2}{h(B_s)^2} \, ds \right\}$$

satisfies

$$dM_t = \frac{b \, \nabla h(B_t)}{h(B_t)} M_t \, dB_t, \quad t < \tau_D.$$

The weighted paths satisfy

$$dX_t = \frac{b \, \nabla h(X_t)}{h(X_t)} \, dt + dW_t, \quad t < \tau_D.$$

An argument similar to the one-dimensional argument shows that $b = 1/2$ is again critical.

This process is a continuous analogue of an easily defined process derived from simple random walk. Suppose S_n denotes a simple random walk in $\mathbb{Z}^d$ and V is a finite connected subset of $\mathbb{Z}^d$. Let $w \in \partial V = \{z : \mathrm{dist}(z, V) = 1\}$ and for $z \in V$, let $H_V(z, w)$ denote the probability that the walk leaves V first at w. Then the generalized h-process associated to the function $H_V(\cdot, w)^b$ is the Markov chain with absorbing state w with transition probabilities

$$P_V(z, z') = \mathbf{P}\{\hat{S}_{n+1} = z' \mid \hat{S}_n = z\} = \frac{H_V(z', w)^b}{Z_V(z)}$$

where

$$Z_V(z) = Z_V(z; w, b) = \sum_{|z-z_1|=1} H_V(z_1, w)^b.$$

With the discrete process there is no problem defining the h-process at each step with moving boundaries. This is more subtle at the continuous level. In order to understand the scaling limit, we will find some property of the discrete model that should hold in the scaling limit. We will then look for a continuous model that satisfies this property.

If $V_1 \subset V$ and $w \in \partial V_1$ as well, then the transition probabilities for the process in V_1 are

$$P_{V_1}(z,z') = \frac{H_{V_1}(z',w)^b}{Z_{V_1}(z)} = P_V(z,z')\,\left[\frac{H_{V_1}(z',w)}{H_V(z',w)}\right]^b\,\frac{Z_V(z)}{Z_{V_1}(z)}. \tag{1.1}$$

The quantity $H_{V_1}(z',w)/H_V(z',w)$ is the probability that a simple random walk starting at z' exits V_1 at w given that it exits V at w. We see that we can get the h-process in V_1 by starting with the process in V and then weighting the paths by the probability of staying in V_1 raised to the b power. This is the property that we will postulate holds on the continuous level as well and this will determine the value of κ.

Our definition of LM_b will essentially be an h-process defined on an SLE_κ which in turn involves a weighting on the driving Brownian motion. The basic idea is a generalization of a martingale introduced by Lawler, Schramm, and Werner in [15].

1.2. Chordal SLE_κ. If $a > 0$, the chordal Schramm-Loewner evolution with parameter $\kappa = 2/a$ ($SLE_{2/a}$) is the random curve $\gamma : [0,t] \to \overline{\mathbb{H}}$ with $\gamma(0) = 0$ satisfying the following. Let H_t denote the unbounded component of $\mathbb{H} \setminus \gamma(0,t]$ and let $g_t : H_t \to \mathbb{H}$ denote the unique conformal transformation with $g_t(z) - z = o(1)$ as $z \to \infty$. Then g_t satisfies

$$\dot{g}_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{1.2}$$

where $U_t = -B_t$ and B_t is a standard one-dimensional Brownian motion. The curve γ is parametrized such that g_t has expansion

$$g_t(z) = z + \frac{at}{z} + O(|z|^{-2})$$

at infinity. See [13], [20] for background on SLE. If $Y_t = Y_t^z = g_t(z) - U_t$, then Y_t satisfies the Bessel SDE

$$dY_t = \frac{a}{Y_t}\,dt + dB_t, \quad Y_0 = z, \tag{1.3}$$

and we see that there is a phase transition at $a = 1/2$. For $a \geq 1/2$ ($\kappa \leq 4$), SLE paths are simple (nonselfintersecting) with $\gamma(0,\infty) \subset \mathbb{H}$.

As mentioned above, chordal SLE was introduced by Schramm [18] as a candidate for the limit of the loop-erased (Laplacian-1) random walk. The definition was (up to a trivial time change) as in the previous paragraph for $\mathbb{H}$ and he defined SLE in simply connected domains by conformal transformation. However, this did not give a natural way to define the process in non-simply connected domains.

Let $b \in \mathbb{R}$, and suppose that the scaling limit of the Laplacian-b walk were $SLE_{2/a}$. Then if $D \subset \mathbb{H}$ with $\mathbb{H} \setminus D$ bounded and $\mathrm{dist}(0, \mathbb{H} \setminus D) > 0$, we should be able to define $SLE_{2/a}$ in D by weighting the paths in a way analogous to (1.1). If $x \in \mathbb{R}$, we define $Q(D; x)$ to be the probability that a Brownian excursion from x to infinity in $\mathbb{H}$ never hits $\mathbb{H} \setminus D$. Let $D_t = g_t(D)$. Then, at time t we will weight the path of $SLE_{2/a}$ using the weight $Q(D_t; U_t)^b$. We can do this for any choice of a, b, but the key fact is that if we choose

$$b = \frac{3a - 1}{2} = \frac{6 - \kappa}{2\kappa},$$

then the process defined is conformally invariant. In other words, if $f : D, \hat{D}$ are two such domains and γ is $SLE_{2/a}$ in D then the image path $f \circ \gamma$ is (a time change of) $SLE_{2/a}$ in $\hat{D}$. In particular, our definition agrees with that of Schramm if the domain is simply connected. It is important that this conformal invariance holds only under the assumption that f is a conformal transformation of D; we do not assume that f can be extended to a conformal transformation of $\mathbb{H}$.

1.3. Arbitrary domains. If we wish to consider $SLE_{2/a}$ as the scaling limit of Laplacian-b walks, then we can determine how to define the process in domains D that are not simply connected. We will give the definition here but leave the justification for this definition for later in the paper. Suppose for ease that our domain is of the form $D = \mathbb{H} \setminus (A_1 \cup \cdots \cup A_n)$ where $A_1, \ldots, A_n$ are "holes" which for the moment we will assume are disjoint, connected, compact subsets of $\mathbb{H}$, larger than a single point, with smooth boundary. Suppose $w \in \partial D \setminus \{0\}$. We will define chordal $SLE_{2/a}$ from 0 to w in D as follows (this definition is modulo a time reparametrization).

Consider the curve $\gamma : [0, t] \to \overline{\mathbb{H}}$ satisfying the Loewner equation (1.2) where U_t satisfies the SDE

$$dU_t = b\,[\log H_{\partial D_t}(U_t, g_t(w))]'\,dt + dB_t.$$

Here $U_t = g_t(\gamma(t))$, $D_t = g_t(D)$, $H_{\partial D}(z_1, z_2)$ denotes the "boundary Poisson kernel" (the normal derivative of the usual Poisson kernel), and the derivative is with respect to the first variable. In the case $w = \infty$, we replace $H_{\partial D_t}(U_t, g_t(w))$ with $Q(D_t; U_t)$, the probability that a Brownian excursion in $\mathbb{H}$ from 0 to infinity stays in D_t.

If A_1 is a singleton $\{w\}$, then chordal $SLE_{2/a}$ to w is called radial $SLE_{2/a}$ in $\mathbb{H} \setminus (A_2 \cup \cdots \cup A_n)$. In this case we replace the quantity $H_{\partial D_t}(U_t, g_t(w))$ with $H_{D_t}(g_t(w), U_t)$, where H_{D_t} denotes the Poisson kernel. (The derivative is then with respect to the second variable, i.e., with respect to the U_t variable.)

More generally, we can consider driving functions U_t satisfying

$$dU_t = b\,\Psi(U_t, D_t, g_t(w))\,dt + dB_t.$$

The solution of the Loewner equation (1.2) with this choice of driving function will give a conformally invariant measure if Ψ satisfies the conformal transformation rule

$$[\log \Psi(x, D, y)]' = f'(x)\,[\log \Psi(f(x), f(D), f(y)]' + [\log f'(x)]'$$

(where $'$ refers to x-derivatives). It is easy to check that our choice of Ψ satisfies this condition, but there are other choices as well.

We choose to label this process LM_b, rather than just referring to it as SLE_κ, in order to distinguish it from other extensions of SLE to non-simply connected domains.

1.4. Outline of the paper. We start by defining the discrete process precisely in Section 2. The definition is elementary, but we focus on the properties of the process that should hold in the scaling limit. Conformal invariance, the Markovian property, and a certain rule for perturbation of domains will characterize the candidate for the scaling limit.

Section 3 reviews a number of facts about Brownian motion and (Brownian) excursion measure. Excursion measure is a fundamental object in the study of conformally invariant processes, yet it does not seem to appear explicitly in many treatments of conformal maps. (However, the two main ingredients, the normal derivative of the Poisson kernel and the h-process associated to Brownian motions conditioned to leave at a particular point do arise separately.) Much of what is discussed here can be found in [13, Chapter 5] although mostly simply connected domains are treated there. Extensions to finitely connected domains are straightforward and discussed here. We also consider a classical question as to when two finitely connected domains are conformally invariant. In trying to understand the standard proof (see, e.g., [1], [5]), I found that it was easier to understand what is happening in terms of a process reflected off the holes. I call this conformally invariant process, *excursion reflected Brownian motion*. As it turns out, the process has been found before, see [8], although I have not seen the conformal invariance of this process exploited. I have included a proof of the result about conformal equivalence of domains in the final section of the paper. Although I do not explicitly use this in this paper, it may be useful for later understanding of Loewner equations in non-simply connected domains.

The relation between LM_b and SLE_κ is discussed in Section 4. We start by reviewing the Loewner equation and chordal SLE_κ. We then show that there is only one value of κ that is consistent with the domain perturbation rule and conformal invariance. We discuss the corresponding (local) martingale associated to LM_b in finitely connected domains; this is an extension of the martingale defined in [15] for simply connected domains. This gives a natural definition of LM_b in finitely connected domains. It turns out that this agrees with the harmonic random Loewner chain introduced by Zhan [23]. While this definition is natural in the case of Laplacian random walk, and particularly nice in the case $b = 1$, it is not the only way to extend SLE_κ to non-simply connected domains. For example, as we discuss in Section 4.5, it is not the natural definition for $SLE_{8/3}$ that would be obtained by restriction.

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2. Laplacian-b random walks in $\mathbb{Z}^2$

We will restrict our consideration to simple random walk in $\mathbb{Z}^2$ although everything in this section has immediate generalizations to Markov chains. Let S_j denote a simple nearest neighbor random walk in $\mathbb{Z}^2$ and if $V \subset \mathbb{Z}^2$, let

$$T_V = \min\{j \geq 0 : S_j \notin V\}.$$

We write $\partial V = \{z \in \mathbb{Z}^2 : \mathrm{dist}(z, V) = 1\}$ and $\overline{V} = V \cup \partial V$. A function $h : \overline{V} \to \mathbb{R}$ is (discrete) harmonic in V if

$$h(x) = \mathbf{E}^x[h(S_1)] = \frac{1}{4} \sum_{|y-x|=1} h(y), \quad x \in V.$$

If such a function is (strictly) positive on V, then the associated h-process is the Markov chain with transition probabilities

$$P_h(x, y) = \frac{1}{4} \frac{h(y)}{h(x)}, \quad x \in V, \, y \in \overline{V}, \, |x - y| = 1.$$

This process is stopped when it reaches ∂V. The n-step transitions for the chain are given for $x, y \in V$ by

$$P_h^n(x, y) = \mathbf{P}^x\{S_n = y; T_V > n\} \, \frac{h(y)}{h(x)}. \tag{2.1}$$

If V is a connected subset of $\mathbb{Z}^2$ and $y \in \partial V$, we let

$$H_V(x, y) = \mathbf{P}^x\{T_V < \infty; S_{T_V} = y\}.$$

Then $H_V(\cdot, y)$ is the bounded positive harmonic function on V with boundary value

$$H_V(z, y) = \delta(y - z), \quad z \in \partial V.$$

Let

$$P(x, x'; V, y) = P_{H_V(\cdot, y)}(x, x')$$

denote the transition probabilities for the associated h-process. We can extend this to a Markov chain on $\overline{V}$ by setting $P(y, y; V, y) = 1$ and

$$P(z, x; V, y) = \frac{H_V(x, y)}{\sum_{|x'-z|=1, x' \in V} H_V(x', y)},$$

for $z \in \partial V \setminus \{y\}, x \in V, |x - z| = 1$.

If V is an infinite connected subset of $\mathbb{Z}^2$ and h is an unbounded positive harmonic function on V with boundary value zero on ∂V, then the paths of the corresponding h-process have infinite lifetime. This process corresponds to random walk conditioned not to leave V (i.e., to reach infinity before any other boundary point of V). An important example is the *(random walk) half-plane excursion* in $V = \mathbb{Z}^2_+ := \{(z_1, z_2) \in \mathbb{Z}^2 : z_2 > 0\}$ with harmonic function $h(z) = z_2$. If the chain is at $(z_1, z_2) \in \mathbb{Z}^2_+$, then it moves to the right or left with probability $1/4$ each, moves up with probability $(z_2 + 1)/(4z_2)$ and moves down with probability $(z_2 - 1)/(4z_2)$. If the process starts at the boundary point $(z_1, 0)$, it immediately moves up.

The h-process can be considered as simple random walk weighted by h. Using this interpretation, we can define the h-process for any nonnegative function (not necessarily harmonic) on $\overline{V}$ that is positive on V. In this case the transition functions are given by

$$P_h(x, y) = \frac{1}{4}\, h(y)\; Z_h(x)^{-1}$$

where Z_h denotes the normalization constant

$$Z_h(x) = \frac{1}{4} \sum_{y' \in V; |x - y'| = 1} h(y').$$

This gives a Markov chain. If h is not harmonic, the n-step transition probabilities do not take the nice form (2.1).

The *Laplacian random walk with exponent* b *(Laplacian-*b *walk) in* V *starting at* x *ending at* y is a measure on self-avoiding walks $\omega = [\omega_0 = x, \omega_1, \ldots, \omega_\tau = y]$ with $\omega_1, \ldots, \omega_{\tau-1} \in V$. The lifetime τ will be finite but random. We can define the process as follows: let V be a connected subset of $\mathbb{Z}^2$ and $y \in \partial V$.

- If $x = y$, then the Laplacian-b walk gives probability one to the trivial walk of zero steps.
- If $x \neq y, |z - x| = 1, z \in V \cup \{y\}$, set $V_0 = V \setminus \{x\}$. The probability that the first step of the walk is to z is

$$\mathcal{Q}(x, z; V, y) := \frac{1}{4}\; H_{V_0}(z, y)^b\; Z_{V_0}(x, y)^{-1}, \tag{2.2}$$

  where

$$Z_{V_0}(x, y) = \frac{1}{4} \sum_{z' \in V \cup \{y\}, |x - z'| = 1} H_{V_0}(z', y)^b.$$

- If the first step is $z \neq y$, then set $V_1 = V_0 \setminus \{z\}$ and continue this process.

Remarks.

- The Laplacian-1 walk is known to be the same as the *loop-erased random walk*. The loop-erased random walk from x to y in V is obtained by taking simple random walk starting at x stopped upon exiting V; conditioning to exit V at y; and then erasing the loops chronologically from the path. It is a straightforward exercise to verify that loop-erasure gives the same measure as the Laplacian-1 walk.
- The definition of the Laplacian-b walk does not require b to be positive. However, if $b \leq 0$, we interpret $H_{V_0}(z,y)^b$ to be
$$H_{V_0}(z,y)^b \, 1\{H_{V_0}(z,y) > 0\}.$$
In other words, we are using the convention $0^b = 0$ for all $b \in \mathbb{R}$.
- The Laplacian-b walk going to y in V started at a boundary point $x \in V$ can be considered as a Markov chain $(V_n, \hat{S}_n)$ whose state space is subsets of $\mathbb{Z}^2$ with a marked boundary point. We start with $(V_0, \hat{S}_0) = (V, x)$ and the probability of the transition $(V, x) \longrightarrow (V \cup \{z\}, z)$ is given by (2.2).
- The Laplacian-b walk satisfies the following Markovian property. Suppose $[\hat{S}_0, \ldots, \hat{S}_k]$ are the first k steps of a Laplacian-b walk from x to y in V. Then the distribution of the remainder of the walk given $[\hat{S}_0, \ldots, \hat{S}_k]$ is the same as that of the Laplacian-b walk in $V \setminus \{\hat{S}_0, \ldots, \hat{S}_k\}$ from $\hat{S}_k$ to y.
- Suppose $V' \subset V$. Then
$$\mathcal{Q}(x,z;V',y) = \mathcal{Q}(x,z;V,y) \, \left[\frac{H_{V'}(z,y)}{H_V(z,y)}\right]^b \, \frac{Z_V(x,y)}{Z_{V'}(x,y)}.$$
Note that $[H_{V'}(z,y)/H_V(z,y)]$ is the conditional probability that a random starting at z exits V' at y given that it exits V at y. The term $[Z_V(x,y)/Z_{V'}(x,y)]$ is a normalization term that is independent of z. Hence we see that the Laplacian-b walk in V' can be obtained by running the Laplacian-b walk in V but then weighting the probabilities at each step by a certain probability to the b power. This probability is the probability that an excursion to y in V stays in V'.

We now consider the possible scaling limits for the Laplacian-b walk. Suppose $D \subset \mathbb{C}$ is a domain whose boundary consists of a finite number of components. Then a scaling limit of the Laplacian-b walk should be a family $\mu_D(z,w)$ of probability measures on curves $\gamma : [0, t_\gamma] \to \overline{D}$ with $\gamma(0) = z, \gamma(t_\gamma) = w, \gamma(0, t_\gamma) \subset D$ connecting distinct boundary points z, w of the domain D. We will consider curves modulo reparametrization. (One could also consider the scaling limit of the discrete parametrization but we will not consider this harder question in this paper.) From the discrete description we would expect the following properties to hold for the family of measures $\{\mu_D(z,w)\}$.

- **Markovian property.** Suppose the beginning of the curve $\gamma[0,t]$ is observed. Then the conditional distribution of $\gamma[t,t_\gamma]$ given $\gamma[0,t]$ is the same as $\mu_{D\setminus\gamma[0,t]}(\gamma(t),w)$.
- **Conformal invariance.** Suppose $f : D \to D'$ is a conformal transformation. Then
$$f \circ \mu_D(z,w) = \mu_{f(D)}(f(z),f(w)).$$
- **Perturbation of domains.** Suppose $D' \subset D$ and D', D agree in neighborhoods of z, w. Then $\mu_{D'}(z,w)$ can be obtained from $\mu_D(z,w)$ by weighting paths locally by the bth power of [probability that an excursion in D to w stays in D'].

It is well known that the first two conditions imply that for simply connected domains D, the measure $\mu_D(z,w)$ must be chordal SLE_κ for some κ. In fact, Schramm's first constructed SLE with the Laplacian-1 walk in mind. With this in mind we present the following conjecture.

Conjecture. *The scaling limit of the Laplacian-b walk in simply connected domains is chordal SLE_κ with*
$$b = \frac{6-\kappa}{2\kappa} = \frac{3a-1}{2}.$$

This conjecture is a theorem in the case $b = 1$. It has also been proved [19] in the case $b = 0$ on the triangular lattice because in this case the Laplacian-0 walk is the same as the "percolation exploration process". We will not try to prove the conjecture in this paper but rather discuss some rigorous properties of SLE that correspond to the conjectured properties of the limit.

3. Brownian measures

3.1. Domains. We let $\mathbb{D}$ denote the unit disk, $\mathbb{H}$ the upper half plane in $\mathbb{C}$, and $\mathbb{D}_+ = \mathbb{D} \cap \mathbb{H}$. Throughout this paper, we will consider proper subdomains of $\mathbb{C}$ of the form
$$D = \mathbb{C} \setminus [A_0 \cup A_1 \cup \cdots \cup A_n]$$
where $A_0, A_1, \ldots, A_n$ are closed, disjoint subsets of $\mathbb{C}$, larger than a single point, at most one is unbounded, and the bounded sets are connected. In other words, as a subset of the Riemann sphere, D is the sphere with $n+1$ disjoint closed sets removed. Such a domain is called n-connected. We write $\partial_j = [\partial D]_j = \partial D \cap A_j$. We let $\mathcal{D}$ be the set of such domains and $\mathcal{D}_n$ the set of $(n+1)$-connected domains in $\mathcal{D}$. Domains in $\mathcal{D}_0$ are called simply connected.

If B_t is a complex Brownian motion, we let
$$\tau_D = \inf\{t > 0 : B_t \notin D\}.$$

We let $h_0, h_1, \ldots, h_n$ be the unique bounded harmonic functions on D with boundary value $h_j(z) = 1$ if $z \in \partial_j$ and $h_j(z) = 0$ if $z \in \partial D \setminus \partial_j$. In other words, for $z \in D$,

$$h_j(z) = \mathbf{P}^z\{B_{\tau_D} \in A_j\}.$$

We let $\mathcal{D}_0^*$ be the set of simply connected subdomains D of $\mathbb{H}$ such that $\mathbb{H} \setminus D$ is bounded. If $D \in \mathcal{D}_0^*$, we let $g_D : D \to \mathbb{H}$ be the unique conformal transformation such that $g_D(z) - z = o(1)$ as $z \to \infty$. The Riemann mapping theorem can be used to show the existence and uniqueness of g_D and one can check that

$$\operatorname{Im}[g_D(z)] = v_D(z) := \operatorname{Im}(z) - \mathbf{E}^z[\operatorname{Im}(B_{\tau_D})]. \tag{3.1}$$

Note that v_D is well defined even if D is not simply connected. If $\mathbb{H} \setminus D$ is bounded, then $v_D(z) = \operatorname{Im}(z) + O(|z|^{-1})$ as $z \to \infty$. However, if D is not simply connected, we cannot write v_D as the imaginary part of an analytic function.

We will use η to denote smooth, simple, closed curves $\eta : [0, t_\eta] \to \mathbb{C}$ parametrized counterclockwise. We also write η for the set $\eta[0, t_\eta]$. Given η, we let U_η be the bounded domain with boundary η. If $D = \mathbb{C} \setminus [A_0 \cup A_1 \cup \cdots \cup A_n] \in \mathcal{D}$ we say that η *surrounds* A_j *in* D if $\eta \subset D$ and $A_j = U_\eta \cap (\mathbb{C} \setminus D)$. If it easy to see that if $A_1, \ldots, A_n$ are compact, then we can find disjoint curves $\eta_1, \ldots, \eta_n$ such that η_j surrounds A_j in D.

We let $\mathcal{Y}_n$ denote the set of domains $D \in \mathcal{D}_n$ with $A_0 = \mathbb{C} \setminus \mathbb{H}$. In other words, $D = \mathbb{H} \setminus (A_1 \cup \cdots \cup A_n)$ where $A_1, \ldots, A_n$ are disjoint, connected, compact subsets of $\mathbb{H}$, each larger than a single point. We will call such a domain a *canonical domain* if each A_j is a horizontal line segment. In this case, we will write $A_j = [\operatorname{Re}_-(A_j) + i \operatorname{Im}(A_j), \operatorname{Re}_+(A_j) + i \operatorname{Im}(A_j)]$, i.e.,

$$A_j = \{x + i \operatorname{Im}(A_j) : \operatorname{Re}_-(A_j) \le x \le \operatorname{Re}_+(A_j)\}.$$

We let $\mathcal{C}Y_n$ denote the set of canonical domains in $\mathcal{Y}_n$.

We will say that ∂D is *(locally) analytic* at z if in a neighborhood of z ∂D is an analytic curve. If D lies on both sides of the curve (as in the example of points in the interior of A_j for canonical domains), we will consider z as being two boundary points, say z_+ and z_-, representing approaches from the two sides. In this case we say that the boundary is analytic at z_+ and at z_-. If ∂D is analytic at z, then the *Poisson kernel* $H_D(w, z), w \in D$, is the hitting density for Brownian motion

$$\mathbf{P}^w\{B_{\tau_D} \in V\} = \int_V H_D(w, z')\, |dz'|,$$

at least for subarcs V on ∂D near z. Note that if z is "two-sided" we have two Poisson kernels $H_D(w, z_+)$ and $H_D(w, z_-)$. If $w \in \partial D$, and ∂D is analytic at w, then the *boundary Poisson kernel* is defined by

$$H_{\partial D}(w, z) = \frac{d}{dn} H_D(w, z).$$

Here n is the unit normal pointing into D. If w is two-sided, we again consider w as two different boundary points and we have separate values $H_{\partial D}(w_+, z), H_{\partial D}(w_-, z)$. If $f : D \to f(D)$ is a conformal transformation, $w \in D, z, z' \in \partial D$ and ∂D is analytic at z, z', then

$$H_D(w, z) = |f'(z)|\, H_{f(D)}(f(w), f(z)),$$
$$H_{\partial D}(z', z) = |f'(z')|\, |f'(z)|\, H_{\partial f(D)}(f(z'), f(z)).$$

Also if $\hat{D} \subset D$, D and $\hat{D}$ agree in a neighborhood of $z \in \partial D$, and ∂D is analytic at z, then

$$H_D(w, z) - H_{\hat{D}}(w, z) = \mathbf{E}^w[H_D(B_{\tau_{\hat{D}}}, z)] = \mathbf{E}^w[H_D(B_{\tau_{\hat{D}}}, z)\, 1\{\tau_{\hat{D}} < \tau_D\}].$$

If $w \in \partial D, w \neq z$,

$$H_{\partial D}(w, z) - H_{\partial \hat{D}}(w, z) = \frac{d}{dn} \mathbf{E}^w[H_D(B_{\tau_{\hat{D}}}, z)].$$

These formulas can be understood easily. The paths starting at w that exit D at z are those that exit $\hat{D}$ at z plus those that exit $\hat{D}$ at a point in $\partial \hat{D} \cap D$ and afterward exit D at z.

For future reference, suppose $D, \hat{D}$ are domains such that for some $\epsilon > 0$, $D \cap (\epsilon \mathbb{D}) = \hat{D} \cap (\epsilon \mathbb{D}) = \epsilon \mathbb{D}_+$, and suppose $f : D \to \hat{D}$ is a conformal transformation with $f(0) = 0$. Then if $w \in D, z \in \partial D$ and x is in a real neighborhood about the origin,

$$H_D(w, x) = f'(x)\, H_{\hat{D}}(f(w), f(x)),$$
$$H_{\partial D}(z, x) = f'(x)\, |f'(z)|\, H_{\partial \hat{D}}(f(z'), f(x)).$$

Hence,

$$[\log H_D(w, x)]' = f'(x)\, [\log H_{\hat{D}}(f(w), f(x))]' + [\log f'(x)]', \tag{3.2}$$

$$[\log H_{\partial D}(z, x)]' = f'(x)\, [\log H_{\partial \hat{D}}(f(z), f(x))]' + [\log f'(x)]'. \tag{3.3}$$

Here $[\log H_{\partial D}(\cdot, \cdot)]'$ denotes differentiation with respect to the second variable.

The Riemann mapping theorem implies that every domain $D \in \mathcal{D}$ is conformally equivalent to a domain $D' \in \mathcal{Y}$. Indeed, any conformal transformation $f : \mathbb{C} \setminus A_0 \to \mathbb{H}$ gives a transformation $f : D \to D'$ by restriction. We state below a well known proposition that every $D \in \mathcal{Y}_n$ is conformally equivalent to a canonical domain $D' \in \mathcal{C}Y_n$. In this case, there is a three parameter family of such transformations. This reduces to a two parameter family if we require infinity to be fixed. The other two parameters come from translation by a real number and scaling by a positive factor.

Corollary 3.1. *If $D \subset \mathcal{Y}_n$, there is a $D' \in \mathcal{C}Y_n$ and a conformal transformation $f : D \to D'$ with $f(\infty) = \infty$. If $f, \hat{f}$ are two such transformations, then*

$$\hat{f}(z) = r f(z) + x$$

for some $r > 0, x \in \mathbb{R}$.

For standard proofs, see [1], [5]. In Section 5.2 we will give a more probabilistic version of the proof. In particular, there is a unique conformal transformation, which we denote by f_D, from D onto a canonical domain $f_D(D)$ such that

$$f_D(z) = z + o(1), \qquad z \to \infty. \tag{3.4}$$

3.2. Useful estimates. Some basic estimates for Poisson kernels are used in proving Loewner equations. We summarize what we will need here. See [13, Section 2.3] for derivations.

Suppose $D \subset \mathbb{H}$ with $\operatorname{dist}(0, \mathbb{H} \setminus D) \geq 1$. For $0 < \epsilon < 1$, let σ_ϵ be the first time t such that $B_t \notin D \setminus \{|z| \leq \epsilon\}$. Let $p_\epsilon(\theta; D, z)$ denote the density of $\arg(B_{\sigma_\epsilon})$, assuming $B_0 = z$, conditioned on the event that $|B_{\sigma_\epsilon}| = \epsilon$. Then for $|z| > 1$,

$$p_\epsilon(\theta; D; z) = \frac{1}{2} \sin\theta \, [1 + O(\epsilon)]. \tag{3.5}$$

Also for $|z| < 1$,

$$\begin{aligned} H_D(z, 0) = H_{\mathbb{H}}(z, 0) - \mathbf{E}^z[H_{\mathbb{H}}(B_{\tau_D}, 0)] &= \frac{\operatorname{Im}(z)}{\pi\, |z|^2} - O(\operatorname{Im}(z)) \\ &= \frac{\sin(\arg(z))}{\pi\, |z|} - O(\operatorname{Im}(z)). \end{aligned}$$

Combining these estimates gives for $|z| > 1$,

$$\mathbf{E}^z[H_D(B_{\sigma_\epsilon}, 0) \mid |B_{\sigma_\epsilon}| = \epsilon] = \frac{1}{4\,\epsilon} + O(1),$$

which implies

$$H_D(z, 0) = \frac{1}{4\,\epsilon}\, \mathbf{P}^z\{|B_{\sigma_\epsilon}| = \epsilon\} + O(\epsilon/|z|).$$

3.3. Excursions. Suppose $D \in \mathcal{D}$, and ∂D is analytic at $w \in \partial D$. We say that X_t is a *(Brownian) excursion in D to w* if X_t is an h-process with harmonic function

$$h(z) = H_D(z, w).$$

In other words, X_t is a Brownian motion conditioned to leave D at w. It satisfies the SDE

$$dX_t = \frac{\nabla h(X_t)}{h(X_t)}\, dt + dB_t.$$

This process can be started at any $z \in \overline{D}$ and runs for a lifetime ρ_D that can be finite or infinite (if $z = w$ the lifetime is zero). We will be interested in this process modulo reparametrization, so whether or not the lifetime is finite is not important for us. It is not critical that ∂D be locally analytic at w; all that is necessary to define the process is a positive harmonic function $h(z)$

with boundary value 0 on $\partial D \setminus \{w\}$ and going to infinity as z approaches w (with appropriate modifications if w is "multi-sided").

If U is an open set with $\overline{U} \subset D$ and $z \in U$, then the distribution of the first visit of the h-process X_t to ∂U is absolutely continuous with respect to the distribution of the first visit by Brownian motion. The Radon-Nikodym derivative is $H_D(\cdot, w)/H_D(z, w)$. (Note that since $H_D(\cdot, w)$ is harmonic, this integrates to one.) A similar fact holds if we do not assume that $\overline{U} \subset D$ although now it is possible for the excursion to reach w without visiting $D \setminus \hat{U}$. In fact, if $\hat{D} \subset D$, $z \in D$, then

$$\mathbf{P}^z\{X(0, \rho_D) \not\subset \hat{D}\} = \frac{\mathbf{E}^z[H_D(B_{\tau_{\hat{D}}}, w)]}{H_D(z, w)}.$$

Note that with probability one, $H_D(B_{\tau_{\hat{D}}}, w)$ equals zero on the event $\{\tau_{D'} = \tau_D\}$. It is useful to think of the right hand side heuristically as follows. The denominator is "the probability that the Brownian motion leaves D at w" and the numerator is "the probability that the Brownian motion leaves D at w and leaves $\hat{D}$ before doing so." Hence the ratio gives a conditional probability of leaving $\hat{D}$ before leaving D given that the Brownian motion leaves D at w. For $z \in D$, we define

$$Q(\hat{D}; z \mid D, w) = \mathbf{P}^z\{X(0, \rho_D) \subset \hat{D}\} = 1 - \frac{\mathbf{E}^z[H_D(B_{\tau_{\hat{D}}}, w)]}{H_D(z, w)},$$

$$\Gamma^*(\hat{D}; z \mid D, w) = \pi\, \mathbf{E}^z[H_D(B_{\tau_{\hat{D}}}, w)].$$

The factor π is put in for convenience. For $z \in \partial D$, we define $Q(\hat{D}; z \mid D, w)$ as the appropriate limit. If ∂D is analytic at z we also define

$$\Gamma(\hat{D}; z \mid D, w) = \frac{d}{dn} \Gamma^*(\hat{D}; z \mid D, w) = \mathbf{E}^z[H_D(B_{\tau_{\hat{D}}}, w)],$$

where n is the inward unit normal. If z is two-sided, this can be defined on both sides. If $f : D \to f(D)$ is a conformal transformation, then it easy to check that

$$Q(\hat{D}; z \mid D, w) = Q(f(\hat{D}); f(z) \mid f(D), f(w)),$$

$$\Gamma(\hat{D}; z \mid D, w) = |f'(z)|\, |f'(w)|\, \Gamma(f(\hat{D}); f(z) \mid f(D), f(w)).$$

3.4. Excursion measure. The excursion measure $\mathcal{E}_D$ is a σ-finite measure on paths that begin and end at ∂D and otherwise stay in D. Roughly speaking, one starts at $z \in \partial D$, forces the process to immediately go into D, and then stops it when it leaves the domain. The measure is defined in a way so that it is conformally invariant.

There are a number of ways to define the measure. Assume first that the boundary components $\partial_0, \ldots, \partial_n$ are smooth curves. Then the excursion measure can be written as

$$\mathcal{E}_D(z, w) = H_{\partial D}(z, w)\, \mu^{\#}_D(z, w),$$

$$\mathcal{E}_D = \int_{\partial D} \mathcal{E}_D(z,w)\, |dz|\, |dw|,$$

where $\mu^{\#}_D(z,w)$ denotes the probability measure on paths given by an excursion from z to w in ∂D, and $H_{\partial D}(z,w)$ is the boundary Poisson kernel. Conformal invariance of Brownian motion implies that

$$f \circ \mu^{\#}_D(z,w) = \mu^{\#}_{f(D)}(f(z), f(w)).$$

Here we are considering $\mu^{\#}_D$ as a measure on paths modulo reparametrization. A similar formula holds for the measures on parametrized curves, but one needs to reparametrize the image curve appropriately. Also,

$$f \circ H_{\partial D}(z,w) = |f'(z)|\, |f'(w)|\, H_{\partial f(D)}(f(z), f(w)).$$

By combining these results, we see that $\mathcal{E}_D(z,w)$ is conformally covariant and $\mathcal{E}_D$ is conformally invariant, i.e.,

$$f \circ \mathcal{E}_D(z,w) = |f'(z)|\, |f'(w)|\, \mathcal{E}_{f(D)}(f(z), f(w)), \quad f \circ \mathcal{E}_D = \mathcal{E}_{f(D)}.$$

Since $\mathcal{E}_D$ is a conformal invariant there is no problem extending the definition of $\mathcal{E}_D$ to domains in $\mathcal{D}$ whose boundaries are not smooth. If V, V' are disjoint closed arcs in ∂D, then

$$|\mathcal{E}_D(V,V')| = \int_V \int_{V'} H_{\partial D}(z,w)\, |dz|\, |dw| < \infty.$$

Here $|\cdot|$ denotes total mass of the measure. The scalar quantity $|\mathcal{E}_D(V,V')|$ is a conformal invariant.

The excursion measure satisfies the *restriction property* which states that if $D' \subset D$, and $z, w \in \partial D \cap \partial D'$, then $\mathcal{E}_{D'}(z,w)$ is $\mathcal{E}_D(z,w)$ restricted to curves that stay in D'. The restriction property implies that it suffices to define the excursion measure in simply connected domains and conformal annuli for then it can be defined for domains of higher connectivity by restriction.

If $D = \mathbb{C} \setminus [A_0 \cup A_1]$ is a conformal annulus with ∂_0, ∂_1 smooth, and $z \in \partial_1$, we define

$$H_{\partial D}(A_0, z) = \int_{\partial D_0} H_{\partial D}(w,z)\, |dw|.$$

If $f : D \to D'$ is a conformal transformation, then

$$H_{\partial D}(A_0, z) = |f'(z)|\, H_{\partial D'}(f(A_0), f(z)).$$

Using this, we can see that $H_{\partial D}(A_0, z)$ is well defined for smooth ∂_1 even if ∂_0 is not smooth.

Suppose $D = \mathbb{C} \setminus [A_0 \cup \cdots \cup A_n] \in \mathcal{D}$, A_j is compact, and η_j surrounds A_j in D. Let $D_j = D \cap U_{\eta_j}$. Then if $V \subset \partial_j, V' \subset \partial D$,

$$|\mathcal{E}_D(V,V')| = |\mathcal{E}_{D_j}(V,V')| + \int_{\eta_j} \int_{V'} H_{\partial D_j}(z,w)\, H_D(w,w')\, |dw|\, |dw'|.$$

This can be considered as a combination of the restriction property and the strong Markov property of Brownian motion. If $V' \cap \partial_j = \emptyset$, then $|\mathcal{E}_{D_j}(V, V')| = 0$ and only the second term appears.

If u is a harmonic function in D and η, η' are smooth simple curves surrounding A_j in D, then Green's theorem shows that

$$\int_\eta \frac{d}{dn} u(z)\,|dz| = \int_{\eta'} \frac{d}{dn} u(z)\,|dz|, \tag{3.6}$$

where n denotes the outward normal. If ∂_j is analytic at $w \in \partial_j$ and $u \equiv 0$ on A_j, then

$$\frac{d}{dn} u(w) = \int_\eta H_{\partial D_j}(w, z)\, u(z)\,|dz|.$$

If $u \equiv 0$ on A_j, ∂_j is analytic, and u is continuous in a neighborhood of A_j, then integration of the last equality on ∂_j shows that the integrals in (3.6) equal

$$\int_\eta u(z)\, H_{\partial(D \cap U_\eta)}(A_0, z)\ |dz|.$$

In fact, this is true even for nonsmooth ∂_j; this can be verified by approximating ∂_j by analytic curves. More generally, if u is continuous in a neighborhood of A_j and takes the constant value $u(A_j)$ on A_j, then

$$\int_\eta u(z)\, H_{\partial(D \cap U_\eta)}(A_0, z)\ |dz| = u(A)\, |\mathcal{E}_{D_\eta}(A_j, \eta)| + \int_\eta \frac{d}{dn} u(z)\,|dz|. \tag{3.7}$$

3.5. $\mathbb{H}$-excursion. An $\mathbb{H}$-excursion B_t^* is an excursion to infinity in $\mathbb{H}$. This is the h-process in $\mathbb{H}$ associated to the harmonic function $h(z) = \mathrm{Im}(z)$. It corresponds to Brownian motion conditioned to stay in $\mathbb{H}$ for all time.

If $z \in \overline{\mathbb{D}}_+$, let $q(z, \theta)$ denote the density of the argument of the first visit to $\partial\mathbb{D}$ by an $\mathbb{H}$-excursion starting at z. By direct computation or using conformal invariance, we can see that

$$q(0, \theta) = \frac{2}{\pi}\ \sin^2\theta, \quad 0 < \theta < \pi. \tag{3.8}$$

If $\bar{q}(0, \theta)$ denotes the corresponding density for the *last* visit to $\partial\mathbb{D}$, then the map $z \mapsto -1/z$ can be used to see that $\bar{q}(0, \theta) = q(0, \theta)$. By estimating $\partial_x g(x, \theta)$, we can see that

$$q(z, \theta) = \frac{2}{\pi}\ \sin^2\theta\ [1 + O(|z|)], \quad |z| \le 1/2. \tag{3.9}$$

Suppose D is a subdomain of $\mathbb{H}$ such that $\mathbb{H} \setminus D$ is bounded. If $z \in D$, we define

$$Q(D; z) = Q(D; z \mid \mathbb{H}, \infty) = \mathbf{P}^z\{B^*(0, \infty) \subset D\}.$$

We can write

$$Q(D; z) = 1 - \frac{\mathbf{E}^z[\mathrm{Im}(B_{\tau_D})]}{\mathrm{Im}(z)}. \tag{3.10}$$

Note that the probability on the left-hand side is with respect to the $\mathbb{H}$-excursion but the expectation on the right is with respect to the Brownian motion. If ρ_R denotes the first time that a Brownian motion B_t reaches $\{\mathrm{Im}(w) = R\}$, then we can write

$$Q(D;z) = \lim_{R\to\infty} \frac{\mathbf{P}^z\{\tau_D > \rho_R\}}{\mathbf{P}^z\{\tau_{\mathbb{H}} > \rho_R\}} = \lim_{R\to\infty} \frac{R\,\mathbf{P}^z\{\tau_D > \rho_R\}}{\mathrm{Im}(z)}.$$

If $D, \hat{D}$ are two such domains and $f : D \to \hat{D}$ is a conformal transformation with $f(z) \sim z/r$ as $z \to \infty$, then $\mathbf{P}^z\{\tau_D > \rho_R\} \sim \mathbf{P}^{f(z)}\{\tau_{\hat{D}} > \rho_{R/r}\}$ as $z \to \infty$. Hence,

$$Q(D;z) = r\,Q(\hat{D}; f(z))\,\frac{\mathrm{Im}[f(z)]}{\mathrm{Im}(z)}, \quad z \in D,$$

$$Q(D;x) = f'(x)\,r\,Q(\hat{D}; f(x)), \quad x \in \mathbb{R},\ \mathrm{dist}(x, \mathbb{H}\setminus D) > 0. \tag{3.11}$$

In particular, if $x \in \mathbb{R},\ \mathrm{dist}(x, \mathbb{H}\setminus D) > 0$,

$$[\log Q(D;x)]' = f'(x)\,[\log Q(\hat{D}; f(x))]' + [\log f'(x)]'. \tag{3.12}$$

There are two important cases.

- If $D \in \mathcal{D}_0^*$, $x \in \mathbb{R}$, and $\mathrm{dist}(x, \mathbb{H}\setminus D) > 0$, then $Q(D;x) = g_D'(x)$. This follows from (3.11) since $Q(\mathbb{H};x) = 1$.
- If $D = \mathbb{H}\setminus(A_1 \cup \cdots \cup A_n) \in \mathcal{C}Y$ is a canonical domain, then (3.10) implies that

$$Q(D;x) = \lim_{\epsilon\to 0+} Q(D; x + \epsilon i) = 1 - \sum_{j=1}^n H_{\partial D}(x, A_j)\ \mathrm{Im}(A_j).$$

More generally, if $D = \mathbb{H}\setminus(A_1 \cup \cdots \cup A_n) \in \mathcal{Y}$ and $f_D : D \to D'$ is the map onto a canonical domain D' as in (3.4), then

$$\begin{aligned} Q(D;x) &= f_D'(x)\ \left[1 - \sum_{j=1}^n H_{\partial D'}(f_D(x), \partial f_D(A_j))\ \mathrm{Im}\, f_D(A_j)\right] \\ &= f_D'(x) - \sum_{j=1}^n H_{\partial D}(x, A_j)\ \mathrm{Im}\, f_D(A_j) = \hat{g}_D'(x), \end{aligned}$$

where

$$\hat{g}_D(x) = f_D(x) - \sum_{j=1}^n |\mathcal{E}_D((-\infty, x], A_j)|\ \mathrm{Im}\, f_D(A_j). \tag{3.13}$$

The *(Brownian boundary) bubble measure at x of (bubbles leaving) D* is given by

$$\begin{aligned}\Gamma(D;x) = \Gamma(D;x \mid \mathbb{H},x) &= \pi \int_{\partial D} H_{\mathbb{H}}(z,x)\, d\mathcal{E}_D(x,z)\\ &= \pi \int_{\partial D} H_{\partial D}(x,z)\, H_{\mathbb{H}}(z,x)\, |dz|.\end{aligned}$$

The last expression assumes that ∂D is smooth. This is the measure of the set of bubbles in $\mathbb{H}$ at x that leave the domain D. Although the bubble measure is infinite, the measure of the set of bubbles that leave D is finite. Another way of writing $\Gamma(D;x)$ is

$$\begin{aligned}\Gamma(D;x) &= \lim_{\epsilon \to 0+} \frac{\pi}{\epsilon}\, \mathbf{E}^{x+i\epsilon}[H_{\mathbb{H}}(B_{\tau_D},x)]\\ &= \lim_{\epsilon \to 0+} \frac{\pi}{\epsilon}\, [H_{\mathbb{H}}(x+\epsilon i,x) - H_D(x+\epsilon i,x)].\end{aligned}$$

Here,

$$H_{\mathbb{H}}(z,x) = \frac{1}{\pi}\,\mathrm{Im}\left[\frac{1}{x-z}\right] = \frac{\mathrm{Im}(z)}{\pi\,|z-x|^2},$$

is the usual Poisson kernel in $\mathbb{H}$. Under our normalization $\Gamma(\mathbb{D}_+;0) = 1$. If $f : D \to D'$ is a conformal transformation, then

$$\Gamma(D;x) = |f'(x)|^2\, \Gamma(f(D);f(x)).$$

As $z \to 0$ we have the expansion

$$\pi\, H_D(z,0) = \pi\, H_{\mathbb{H}}(z,0) - \mathrm{Im}(z)\, \Gamma(D;0)\, [1 + O(|z|)]. \tag{3.14}$$

If $D \in \mathcal{D}_0^*$ with $\mathrm{dist}(x, \mathbb{H} \setminus D) > 0$, then

$$H_D(x+i\epsilon,x) = g_D'(x)\, H_{\mathbb{H}}(g_D(x+i\epsilon), g_D(x)).$$

Using this it can be seen that

$$\Gamma(D;x) = -\frac{1}{6}\, Sg_D(x) = \frac{Q'(D;x)^2}{4\,Q(D;x)^2} - \frac{Q''(D;x)}{6\,Q(D;x)},$$

where S denotes the Schwarzian derivative,

$$Sf(z) = \frac{f'''(z)}{f'(z)} - \frac{3\,f''(z)^2}{2\,f'(z)^2}.$$

4. Schramm-Loewner evolution

4.1. Loewner equation in $\mathbb{H}$. We will review some facts about half-plane capacity and the chordal or half-plane Loewner equation; see [13], especially Sections 3.4 and 4.1, for more details. If K is a bounded subset of $\mathbb{H}$, then the *half-plane capacity* (in $\mathbb{H}$ from infinity) is defined by

$$\mathrm{hcap}(K) = \lim_{y \to \infty} y\, \mathbf{E}^{iy}[\mathrm{Im}(B_{\rho_K})], \tag{4.1}$$

where $\rho_K = \inf\{t : B_t \in \mathbb{R} \cup K\}$. It is not difficult to show that this is well defined for bounded K and satisfies the scaling rule $\operatorname{hcap}(rK) = r^2 \operatorname{hcap}(K)$. In fact, if $s > \sup\{\operatorname{Im}(z) : z \in K\}$,

$$\operatorname{hcap}(K) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathbf{E}^{x+ik}[\operatorname{Im}(B_{\rho_K})] \, dx,$$

$$\operatorname{hcap}(K) = \frac{2s}{\pi} \int_0^{\pi} \mathbf{E}^{se^{i\theta}}[\operatorname{Im}(B_{\rho_K})] \, \sin\theta \, d\theta. \tag{4.2}$$

These equations come from considering the density, with respect to length, of the hitting distribution of $\{\operatorname{Im}(z) = s\}$ and $\mathbb{R} \cup \{|z| = s\}$, respectively, by a Brownian motion started at iy. If we multiply by y and take the limit as $y \to \infty$, in the first case the density is $(1/\pi)$ and in the second case it is $(2/\pi)\sin\theta$. The hcap can also be defined in terms of $\mathbb{H}$-excursions,

$$\operatorname{hcap}(K) = \lim_{z\to\infty} |z|^2 \, \mathbf{P}^z\{B^*(0,\infty) \cap K \neq \emptyset\}.$$

It may be surprising at first that this limit is independent of the angle at which z goes to infinity, but it is. (This can be seen more easily after conformally mapping by the logarithm to a doubly infinite strip.) In fact, if $\operatorname{rad}(K) = \sup\{|z| : z \in K\}$, then if $|z| \geq 2$, $\operatorname{rad}(K) \leq 1$,

$$\mathbf{P}^z\{B^*(0,\infty) \cap K \neq \emptyset\} = \frac{\operatorname{hcap}(K)}{|z|^2} \left[1 + O(\operatorname{rad}(K))\right]. \tag{4.3}$$

If $\gamma : (0,\infty) \longrightarrow \mathbb{H}$ is a simple curve with $\gamma(0+) \in \mathbb{R}$, we say that γ is *parametrized by capacity* if $\operatorname{hcap}(\gamma(0,t]) = t$.[1] More generally, if $a(t) = \operatorname{hcap}(\gamma(0,t])$, then $a(t)$ is continuous and strictly increasing, and $\hat{\gamma}(t) := \gamma(a^{-1}(t))$ is parametrized by capacity. We will say that γ has a *smooth (increasing) capacity parametrization* if $a(t)$ is C^1 and $\dot{a}(t) > 0$.

We will call K a *(boundary) hull* if $\mathbb{H}\setminus K$ is simply connected. If K is a hull, we recall that g_K is the unique conformal transformation $g_K : \mathbb{H} \setminus K \to \mathbb{H}$ such that $g_K(z) - z = o(1)$ as $z \to \infty$. It is known that there is a c such that

$$\left| g_K(z) - z - \frac{\operatorname{hcap}(K)}{z} \right| \leq c \, \frac{\operatorname{hcap}(K) \, \operatorname{rad}(K)}{|z|^2}, \quad |z| \geq 2 \operatorname{rad}(K). \tag{4.4}$$

The proof of (4.4) starts by studying the imaginary part of g_k,

$$\operatorname{Im} g_k(z) = \operatorname{Im}(z) - \mathbf{E}^z[\operatorname{Im}(B_{\rho_K})].$$

Asymptotics for this can be taken for fixed K as $|z| \to \infty$ or for fixed z as $\operatorname{rad}(K) \to 0$. The estimate for $\operatorname{Im} g_k(z)$ is proved by writing $\mathbf{E}^z[\operatorname{Im}(B_{\rho_K})]$ as

$$\mathbf{P}^z\{B[0,\tau_{\mathbb{H}}] \cap \operatorname{rad}(K)\,\mathbb{D}_+ \neq \emptyset\} \, \mathbf{E}[\operatorname{Im}(B_{\rho_K}) \mid B[0,\tau_{\mathbb{H}}] \cap \operatorname{rad}(K)\,\mathbb{D}_+ \neq \emptyset].$$

[1] This differs from the usage in many papers where parametrization by capacity implies $\operatorname{hcap}(\gamma(0,t]) = 2t$.

The asymptotics for the first term can be given in terms of $H_{\mathbb{H}}(z,0)$ and the asymptotics for the second term are independent of z and determined using (4.2). We then get the asymptotics for g_K using the asymptotics for $\operatorname{Im} g_K$, the Cauchy-Riemann equations, and the condition $g_K(z) - z = o(1)$ which determines the additive constant in $\operatorname{Re} g_k$. An important fact that is used is

$$\operatorname{Im}\left[\frac{1}{z-x}\right] = -\pi\, H_{\mathbb{H}}(z,x).$$

We say that hulls K_t *shrink by capacity to* $x \in \mathbb{R}$ if $\operatorname{hcap}(K_t) = t$ and $r_t := \operatorname{rad}(K_t;x) \to 0$ as $t \to 0$. We do not need to assume that the sets are decreasing in t. If K_t decreases by capacity to x and $g_t = g_{K_t}$, then

$$g_t(z) = z + \frac{t}{z-x}\,\left[1 + O\left(\frac{r_t}{|z-x|}\right)\right], \quad t \to 0+\,. \tag{4.5}$$

An important example is $K_t = \sqrt{t}\,\mathbb{D}_+ = \{z \in \mathbb{H} : |z| < \sqrt{t}\}$. We write ϕ_t for the corresponding function

$$\phi_t(z) = g_{\sqrt{t}\,\mathbb{D}_+}(z) = z + \frac{t}{z}.$$

We say that K_t is an increasing collection of hulls parametrized by capacity if $t \mapsto K_t$ is increasing; $\operatorname{hcap}(K_t) = t$; and there exists a continuous $t \mapsto U_t$ such that for each t,

$$\lim_{\epsilon \to 0+} \operatorname{rad}(g_t(K_{t+\epsilon} \setminus K_t); U_t) = 0.$$

An example is $K_t = \gamma(0,t]$ where γ is a simple curve parametrized by capacity. In this case, (4.5) shows that g_t satisfies the Loewner equation

$$\dot{g}_t(z) = \xi(g_t(z), U_t), \tag{4.6}$$

where $\xi(w,x) = 1/(w-x)$ is the complex form of the Poisson kernel in $\mathbb{H}$, i.e., the analytic function with imaginary part $-\pi H_{\mathbb{H}}(w,x)$ satisfying $\xi(\infty,x) = 0$.

4.2. Calculating a time derivative. The Loewner equation shows that if K_t shrinks to 0 by capacity then the derivative $\dot{g}_0(z)$ is independent of the choice of K_t. In this section we consider some other quantities that have this property. The fact that the time derivative does not depend on the choice is critical in order to perform certain Itô formula calculations for random K_t.

If $\Psi : \mathcal{Y} \times \mathbb{R} \to \mathbb{R}$ is a function, we define the time derivative $\dot{\Psi}(D;x)$ of Ψ to be the (right) derivative with respect to time evaluated at $t = 0$ of the function

$$t \longmapsto \Psi(D_t;x),$$

where $D_t = g_t(D), g_t = g_{K_t}$, and K_t is a collection of hulls shrinking by capacity to x. Implicit in the definition is the fact that $\dot{\Psi}(D;x)$ exists only if the value is independent of the choice of hulls K_t. If the derivative exists, we can choose $g_t = \phi_t$ and hence

$$\dot\Psi(D;x) = \frac{d}{dt}\Psi(\phi_t(D);x)\ |_{t=0}\ .$$

More generally if $z \in \overline{D}\setminus\{x\}$, we define the time derivative $\dot\Psi(D;x;z)$ to be

$$\dot\Psi(D;x;z) = \frac{d}{dt}\Psi(D_t;x;g_t(z))\ |_{t=0},$$

assuming the limit is independent of the choice of K_t.

It follows from (4.4) that there is a c such that if K_t shrinks to zero by capacity,

$$|g_t(z)-\phi_t(z)| \le c\,\frac{t\,r_t}{|z|^2}, \qquad |z|\ge 2\,r_t.$$

By considering the function $g_t(w)-\phi_t(w)$ for $|w-z|\le |z|/4$, we can see that this implies that there is a constant c such that

$$|g_t'(z)-\phi_t'(z)| = \left|g_t'(z)-1+\frac{t}{z^2}\right| \le c\,\frac{t\,r_t}{|z|^3}, \qquad |z|\ge 3\,r_t.$$

Recall the definitions of $Q(D;x),\Gamma(D;x)$ from Section 3.5. The next proposition computes $\dot Q(D;x)$ in terms of $Q(D;x)$ and $\Gamma(D;x)$.

Corollary 4.1. *If $D\in\mathcal{Y}$ and $x\in\mathbb{R}$, then $\dot Q(D;x)$ exists and*

$$\dot Q(D;x) = Q(D;x)\,\Gamma(D;x) - \frac{1}{2}\,Q''(D;x). \tag{4.7}$$

Proof. We may assume $x=0$. Let K_s shrink to 0 by capacity, and write $Q(D_s),\Gamma(D_s)$ for $Q(D_s;0),\Gamma(D_s;0)$. We first show that it suffices to prove (4.7) for D with $\mathrm{dist}(0,\mathbb{H}\setminus D)>2$. Suppose $D\in\mathcal{Y}$ and choose r sufficiently large so that $\mathrm{dist}(0,\mathbb{H}\setminus rD)>2$. The scaling rule for hcap implies that $\mathrm{hcap}(rK_s)=r^2 s$. Hence $\hat K_s := rK_{s/r^2}$ shrinks to 0 by capacity with corresponding transformations $\tilde g_s(z) = r\,g_{s/r^2}(z/r)$ and domains $\tilde D_s = r\,D_{s/r^2}$. If (4.7) holds for $\mathrm{dist}(0,\mathbb{H}\setminus D)>2$, then

$$\begin{aligned}\frac{d}{ds}Q(D_s)\,|_{s=0} &= r^2\,\frac{d}{ds}Q(\tilde D_s;0)\,|_{s=0}\\ &= r^2\,[\Gamma(rD)\,Q(rD)-\frac{1}{2}\,Q''(rD;0)] = \Gamma(D)\,Q(D)-\frac{1}{2}\,Q''(D;0).\end{aligned}$$

Here we use the scaling rules $Q(rD;rx)=Q(D;x),\Gamma(rD)=r^{-2}\Gamma(D)$. Hence, for the remainder of the proof we assume that $\mathrm{dist}(0,\mathbb{H}\setminus D)>2$.

We will first do the computation for $K_s^* = \sqrt{s}\,\mathbb{D}_+$ for which

$$g_s(z)=\phi_s(z) = z+\frac{s}{z}.$$

Let $\xi_s = \sup\{t : |B_t^*| = \sqrt{s}\}$. Then,

$$\begin{aligned} Q(D) &= \mathbf{P}\{B^*(0,\infty) \subset D\} \\ &= \mathbf{P}\{B^*[\xi_s,\infty) \subset D\} - \mathbf{P}\{B^*(0,\xi_s] \not\subset D, B^*[\xi_s,\infty) \subset D\}. \end{aligned}$$

We can establish (4.7) for K_t^* by deriving the following three estimates:

$$\mathbf{P}\{B^*(0,\xi_s] \not\subset D, B^*[\xi_s,\infty) \subset D\} = Q(D)\,\Gamma(D)\,s\,[1+o(1)], \tag{4.8}$$

$$\mathbf{P}\{B^*[\xi_s,\infty) \subset D\} = Q(D_s) + \frac{1}{2}\,Q''(D_s;0)\,s + o(s), \tag{4.9}$$

$$Q''(D_s;0) = Q''(D_0;0) + o(1). \tag{4.10}$$

Let τ_D^* be the first time $t > 0$ that an $\mathbb{H}$-excursion B_t^* leaves D. Any path with $B^*(0,\xi_s^*] \not\subset D$ can be split into the first time that it reaches ∂D and then the first time after that that it enters $s\mathbb{D}_+$. Given that its first visit to ∂D is at z, the probability that it will enter $s\mathbb{D}_+$ again is $1 - Q(\mathbb{H} \setminus s\mathbb{D}_+; z)$. Hence,

$$\mathbf{P}\{B^*(0,\xi_s] \not\subset D\} = \mathbf{E}\left[\,1\{\tau_D^* < \infty\}\,[1 - Q(\mathbb{H} \setminus \sqrt{s}\,\mathbb{D}_+; B^*_{\tau_D^*})]\,\right].$$

By (4.3) we know that

$$1 - Q(\mathbb{H} \setminus \sqrt{s}\,\mathbb{D}_+; B^*_{\tau_D^*}) = \frac{s}{|B^*_{\tau_D^*}|^2}\,[1+o(1)]\,.$$

Also,

$$\begin{aligned} \mathbf{E}\left[|B^*_{\tau_D^*}|^{-2}\,1\{\tau_D^* < \infty\}\right] &= \lim_{\epsilon\to 0+} \mathbf{E}^{i\epsilon}\left[\,|B^*_{\tau_D^*}|^{-2}\,1\{\tau_D^* < \infty\}\,\right] \\ &= \lim_{\epsilon\to 0+} \epsilon^{-1}\mathbf{E}^{i\epsilon}\left[\mathrm{Im}(B_{\tau_D})\,|B_{\tau_D^*}|^{-2}\,\right] \\ &= \Gamma(D). \end{aligned}$$

Therefore,

$$\mathbf{P}\{B^*(0,\xi_s] \not\subset D\} = s\,\Gamma(D)\,[1+o(1)].$$

Since $\mathrm{dist}(0, \mathbb{H} \setminus D) > 1$, (3.9) implies that

$$Q(D;z) = Q(D;0)\,[1 + O(|z|)], \qquad |z| \le 1/2,$$

and hence

$$\mathbf{P}\{B^*[\xi_s,\infty) \subset D \mid B^*(0,\xi_s] \not\subset D\} = Q(D)\,[1+O(s^{1/2})] = Q(D)\,[1+o(1)].$$

Combining these gives (4.8).

Conformal invariance gives

$$\mathbf{P}\{B^*[\xi_s,\infty) \subset D\} = \int_0^\pi Q(D_s; \phi_s(\sqrt{s}\,e^{i\theta}))\;\bar{q}(\theta)\;d\theta,$$

where $\bar{q}$ denotes the density of $\arg(B_{\xi_s})$. Recall from (3.9) and the following sentence that $\bar{q}(\theta) = (2/\pi)\,\sin^2\theta$. Also, $\phi_s(\sqrt{s}\,e^{i\theta}) = 2\sqrt{s}\,\cos\theta$. Therefore,

$$\mathbf{P}\{B^*[\xi_s,\infty) \subset D\} = \int_0^\pi Q(D_s; 2\sqrt{s}\,\cos\theta)\,\frac{2}{\pi}\sin^2\theta\,d\theta.$$

But,

$$\begin{aligned}&Q(D_s; 2\sqrt{s}\,\cos\theta) + Q(D_s; -2\sqrt{s}\,\cos\theta)\\&\qquad = 2\,Q(D_s;0) + 4\,s\,\cos^2\theta\,Q''(D_s;0) + o(s).\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbf{P}\{B^*[\xi_s,\infty) \subset D\}\\&\quad = Q(D_s;0) + \frac{8}{\pi}\,Q''(D_s;0)\,s\int_0^{\pi/2}\cos^2\theta\,\sin^2\theta\,d\theta + o(s)\\&\quad = Q(D_s;0) + \frac{1}{2}\,Q''(D_s;0)\,s + o(s).\end{aligned}$$

This gives (4.9).

It is not difficult to show that for $z \in \overline{\mathbb{D}}_+$,

$$Q(D_s;z) = Q(D;z) + o(1), \quad s \to 0+.$$

Indeed, (4.4) shows that

$$Q(D_s;z) = Q(D;g_s^{-1}(z)) = Q(D; z - \frac{s}{z} + o(s)). \tag{4.11}$$

Using standard derivative estimates, we then get

$$Q''(D_s;0) = Q''(D;0)\,[1+o(1)].$$

This gives (4.10) and finishes the derivation of (4.7) in the case of $K_s = \sqrt{s}\,\mathbb{D}_+$.

We now let K_s be a any sequence of hulls shrinking to 0 by capacity and let $g_s = g_{K_s}$. Let $D_s = g_s(D)$, $D_s^* = \phi_s(D)$. It suffices to show that

$$Q(D_s;0) = Q(D_s^*;0) + o(s). \tag{4.12}$$

For $|z| \geq 1$,

$$\phi_s(z) = z + \frac{s}{z} + o(s), \quad g_s(z) = z + \frac{s}{z} + o(s).$$

Let $\psi_s = g_s \circ \phi_s^{-1}$, which is a conformal transformation of D_s^* onto D_s that satisfies

$$\psi_s(z) = z + o(s), \quad |z| \geq 1.$$

Let η denote a smooth simple closed curve in $D \cap \{|z| > 1\}$ such that $\mathbb{H} \setminus D$ is contained in D_η, the bounded domain with boundary η. For s sufficiently small, $\mathbb{H} \setminus D_s, \mathbb{H} \setminus D_s^*$ are also contained in D_η. Note that

$$Q(D_s;0) - Q(D_s^*;0) = \int_\eta [Q(D_s;z) - Q(D_s^*;z)]\,d\nu(z),$$

where $\nu = \nu_\eta$ denotes the subprobability measure associated to the first visit to η by an $\mathbb{H}$-excursion starting at 0. But conformal invariance implies that

$$Q(D_s^*; z) = Q(D_s; \psi_s(z)),$$

and standard derivative estimates show that

$$|Q(D_s; z) - Q(D_s; \psi_s(z))| \leq c\,|z - \psi_s(z)| = o(s),$$

which establishes (4.12). □

Corollary 4.2. *If $D \in \mathcal{Y}$, K_t is a sequence of hulls shrinking by capacity to $x \in \mathbb{R}$, $z \in D$, and ∂D is analytic at $w \in \partial D \setminus \{x\}$, then*

$$\frac{d}{dt} H_{D_t}(x, g_t(z)) \,\,|_{t=0} = \Gamma(D; x)\, H_D(x, z) - \frac{1}{2}\, H_D''(x, z), \tag{4.13}$$

$$\frac{d}{dt} H_{\partial D_t}(x, g_t(w)) \,\,|_{t=0} = \big(\Gamma(D; x) + \mathrm{Re}[(z-x)^{-2}]\big)\,\, H_{\partial D}(x, w) - \frac{1}{2}\, H_{\partial D}''(x, w). \tag{4.14}$$

Here $D_t = g_t(D)$ and the spatial derivative is with respect to the first variable.

Proof. For ease, we choose $x = 0$. We fix D, z, w. All error terms will be as $t \to 0+$ and may depend on D, z, w. We will do the computation for ϕ_t. The proof that the limit is independent of the choice of K_t is very similar to the previous proof and we omit it. Let $U_t = D \setminus (\sqrt{t}\,\mathbb{D}_+)$ and $r = r_t = \sqrt{t}$. Then,

$$\begin{aligned} H_{\partial U_t}(re^{i\theta}, z) &= |\phi_t'(re^{i\theta})|\, |\phi_t'(z)|\, H_{\partial D_t}(\phi_t(re^{i\theta}), \phi_t(z)) \\ &= 2\, \sin\theta\, |\phi_t'(z)|\, H_{\partial D_t}(2r\, \cos\theta, \phi_t(z)). \end{aligned} \tag{4.15}$$

Using the Loewner equation, we see that

$$|\phi_t'(z)| = |\, 1 - \frac{t}{z^2} + o(t)\, | = 1 - t\, \mathrm{Re}[z^{-2}] + o(t).$$

By the strong Markov property, for r sufficiently small,

$$H_D(0, z) = \int_0^\pi H_{\partial U_t}(z, re^{i\theta})\, H_D(re^{i\theta}, 0)\, r\, d\theta.$$

Also,

$$\begin{aligned} H_D(re^{i\theta}, 0) &= H_{\mathbb{H}}(re^{i\theta}, 0) - \mathbf{E}^{re^{i\theta}}[H_{\mathbb{H}}(B_{\tau_D}, 0)] \\ &= \frac{1}{\pi r}\, \sin\theta - \mathbf{E}^{re^{i\theta}}[H_{\mathbb{H}}(B_{\tau_D}, 0)] \\ &= \frac{1}{\pi r} \sin\theta\, [1 - r^2\, \Gamma(D; 0) + o(r^2)]. \end{aligned}$$

Therefore,

$$H_D(0, z) = |\phi_t'(z)|\, [1 - r^2\, \Gamma(D; 0)] \int_0^\pi (\frac{2}{\pi}\, \sin^2\theta)\, H_{\partial D_t}(2r\cos\theta, \phi_t(z))\, d\theta.$$

Expansion by Taylor series gives

$$\int_0^\pi \left(\frac{2}{\pi}\,\sin^2\theta\right) H_{\partial D_t}(2r\cos\theta, \phi_t(z))\,d\theta$$
$$= H_{\partial D_t}(0, \phi_t(z)) + \frac{r^2}{2}\,H''_{\partial D_t}(0, \phi_t(z)) + o(r^2).$$

Therefore, up to an error of $o(t)$, $H_D(0, z)$ equals

$$|\phi_t'(z)|\,\left(H_{\partial D_t}(0, \phi_t(z)) + t\,\left[-H_{\partial D_t}(0, \phi_t(z))\,\Gamma(D; 0) + \frac{1}{2}\,H''_{\partial D_t}(0, \phi_t(z))\right]\right).$$

Continuity gives

$$H_{\partial D_t}(0, \phi_t(z)) = H_{\partial D}(0, z) + o(1),$$
$$H''_{\partial D_t}(0, \phi_t(z)) = H''_{\partial D}(0, z) + o(1).$$

Combining all these estimates gives (4.14). A similar argument works for (4.13), but (4.15) is replaced with

$$H_{U_t}(z, re^{i\theta}) = 2\,\sin\theta\,H_{D_t}(\phi_t(z), 2r\,\cos\theta). \qquad \square$$

4.3. Chordal SLE**.** Suppose B_t is a standard Brownian motion. Chordal $SLE_{2/a}$ in $\mathbb{H}$ (from 0 to infinity parametrized so that $\operatorname{hcap}(\gamma(0, t]) = at$) is the random curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ satisfying the following. Let H_t denote the unbounded component of $\mathbb{H} \setminus \gamma(0, t]$ and let $g_t : H_t \to \mathbb{H}$ be the unique conformal transformation with $g_t(z) - z = o(1)$ as $z \to \infty$. Then,

$$\dot{g}_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{4.16}$$

where $U_t = -B_t$ is a standard Brownian motion. If $a \geq 1/2$, then the paths are simple. Recall that

$$b = b(a) = \frac{3a - 1}{2}.$$

REMARK. It is standard to define SLE_κ parametrized so that $\operatorname{hcap}(\gamma(0, t]) = 2t$. It is defined as the solution of

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z,$$

where $U_t = \sqrt{\kappa}\,B_t$ or $U_t = B_{\kappa t}$ and B denotes a standard Brownian motion. We have chosen our parametrization (and the negative sign on B_t) so that $g_t(z) - U_t$ satisfies the Bessel equation (1.3) without an additional change of variables.

More generally, if J_t is a continuous process adapted to the filtration of B_t, we define chordal $SLE_{2/a}$ with drift J_t if $\gamma(t), g_t$ satisfy (4.16) with the driving function U_t satisfying

$$dU_t = J_t\,dt - dB_t,$$

Girsanov's Theorem tells us that chordal $SLE_{2/a}$ with drift is absolutely continuous with respect to driftless $SLE_{2/a}$. When $SLE_{2/a}$ paths are mapped by a locally real conformal map, the distribution of the image is absolutely continuous with respect to that of $SLE_{2/a}$. The effect of the transformation is to change the drift term. The next lemma computes the drift term. Versions of this lemma have appeared in a number of places.

LEMMA 4.3. *Suppose γ is $SLE_{2/a}$ starting at $x \in \mathbb{R}$ with drift J_t. Suppose F is a conformal transformation of $\{z \in \mathbb{H} : |z - x| < \epsilon\}$ into $\mathbb{H}$ which maps $(x - \epsilon, x + \epsilon)$ into $\mathbb{R}$. For $t < \inf\{t' : |\gamma(t') - x| = \epsilon\}$, define*

$$\hat{\gamma}(t) = F \circ \gamma(t).$$

Let

$$F_t = g_{\hat{\gamma}(0,t]} \circ F \circ g_t^{-1}, \quad \hat{U}_t = F_t(U_t) = g_{\hat{\gamma}(0,t]}(\hat{\gamma}(t)).$$

Then there is a time change $\sigma(t)$ such that $g_t^* := \hat{g}_{\sigma(t)}$ satisfies (4.16) with driving function U_t^* satisfying

$$dU_t^* = \Big[\ \Psi_t'(U_t^*)\, J_{\sigma(t)} + b\, [\log \Psi_t'(U_t^*)]'\ \Big]\ dt - dW_t, \tag{4.17}$$

where $\Psi_t = F_{\sigma(t)}^{-1}$, $'$ denotes the spatial derivative, and W_t is a standard Brownian motion.

Proof. Itô's formula gives

$$d\hat{U}_t = d[f_t(U_t)] = \left[\dot{f}_t(U_t) + \frac{1}{2}\, f_t''(U_t)\right]\ dt + f_t'(U_t)\, dU_t.$$

A computation shows that $\dot{f}_t(U_t) = -(3a/2)\, f_t''(U_t)$. (See [13, (4.35)] where the computation is done assuming $\mathrm{hcap}(\gamma(0,t]) = 2t$.) Therefore,

$$d\hat{U}_t = [-b\, f_t''(U_t) + f'(U_t)\, J_t]\, dt - f_t'(U_t)\, dB_t.$$

If we define $\sigma(t), W_t$ by

$$\int_0^{\sigma(t)} f_s'(U_s)^2\, ds = t, \qquad W_t = \int_0^{\sigma(t)} f_s'(U_s)\, dB_s,$$

this becomes

$$dU_t^* = \left[-b\, \frac{f_{\sigma(t)}''(U_{\sigma(t)})}{f_{\sigma(t)}'(U_{\sigma(t)})^2} + \frac{J_{\sigma(t)}}{f_{\sigma(t)}'(U_{\sigma(t)})}\right]\ dt - dW_t.$$

If $\Psi(x) = f^{-1}(x)$, then

$$\Psi'(f(x)) = \frac{1}{f'(x)}, \qquad \Psi''(f(x)) = -\frac{f''(x)}{f'(x)^3} = -\,\Psi'(f(x))\, \frac{f''(x)}{f'(x)^2}.$$

This gives (4.17). □

REMARK. Suppose $D \subset \mathbb{H}$ is a simply connected domain with $\operatorname{dist}(x, \mathbb{H} \setminus D) > 0$. Let $w \in \partial D \setminus \{0\}$. Schramm defined $SLE_{2/a}$ in D from 0 to w (up to time change) as the image of $SLE_{2/a}$ in $\mathbb{H}$ from 0 to infinity under a conformal map $F : \mathbb{H} \to \mathbb{H} \setminus D$ with $F(0) = 0, F(\infty) = w$. By the lemma, we see that one can equivalently define $SLE_{2/a}$ in D as the solution to the Loewner equation (4.16) with driving function U_t satisfying

$$U_t = b\,[\log \Psi_t'(U_t)]'\,dt - dB_t.$$

Here Ψ_t is a conformal transformation of $g_t(D)$ onto $\mathbb{H}$ with $\Psi_t(g_t(w)) = \infty$. If $w \neq \infty$, the process under this parametrization has finite lifetime.

Suppose D is a domain with $\operatorname{dist}(0, \mathbb{H} \setminus D) > 0$ and suppose $z \in \overline{D} \setminus \{0\}$ ($z = \infty$ is allowed). Suppose that we are interested in extending the definition of $SLE_{2/a}$ so that we can discuss "$SLE_{2/a}$ in D from 0 to z". There is no unique way of defining this. Any reasonable candidate should agree with the definition in simply connected domains and be absolutely continuous with respect to $SLE_{2/a}$ in $\mathbb{H}$ from 0 to infinity. For this reason, it is natural to define the process to be the solution of (4.16) where U_t satisfies

$$dU_t = \Psi_a(D_t; U_t; z_t)\,dt - dB_t,$$

where $D_t = g_t(D), z_t = g_t(z)$, and $\Psi_a(D; x; z)$ is a function which is C^1 in t and C^2 in x (where C^1 in t means existence of $\dot{\Psi}_a(D; x; z)$ as defined in the previous section). If we want the process to be conformally invariant, the previous lemma shows that we need the following:

- Suppose $f : D \to f(D)$ is a conformal transformation. Then

$$\Psi_a(D; x; z) = f'(x)\,\Psi_a(f(D); f(x); f(z)) + b\,[\log f'(x)]'.$$

In particular we can choose

$$\Psi_a(D; x; z) = b\,\Psi(D; x; z)$$

where Ψ satisfies the conformal transformation rule

$$\Psi(D; x; z) = f'(x)\,\Psi(f(D); f(x); f(z)) + [\log f'(x)]'.$$

Equivalently, we can choose $\Psi_a(D; x; z) = [\log \phi_a(D'; x; z)]'$, where ϕ_a satisfies

$$\phi_a(D; x; z) = f'(x)^b\,\phi_a(D'; f(x); f(z))\,u(z).$$

4.4. Definition of LM_b. We will use the Laplacian-b walk motivation to define LM_b from 0 to z in subdomains D of $\mathbb{H}$ where $z \in \overline{D} \setminus \{0\}$. If $z \in D$, then this will be a form of radial SLE and if $z \in \partial D$ this will be a form of chordal SLE. We will restrict our attention to domains D of the form

$$D = D^* \setminus (A_1 \cup \cdots \cup A_n)$$

where $D^* \subset \mathbb{H}$ is simply connected; $A_1, \ldots, A_n$ are disjoint, connected compact subsets of $\mathbb{H}$; and $\operatorname{dist}(0, \mathbb{H} \setminus D) > 0$. Also, if $z \in \partial D \setminus \{\infty\}$ we will assume enough regularity (very little is needed) so that there exists a strictly positive harmonic function $H_D(\cdot, z)$ on D with zero boundary conditions on $\partial D \setminus \{z\}$ (and hence blows up at z). In the case that z is many-sided we choose one of the sides (prime ends). If ∂D is analytic at z, then $H_D(\cdot, z)$ is the Poisson kernel. In what follows below we let

$$q(D; x; z) = \begin{cases} H_D(z, x) & \text{if } z \in D \\ H_{\partial D}(x, z) & \text{if } z \in \partial D \\ Q(D; x) & \text{if } z = \infty. \end{cases}$$

Let γ denote an $SLE_{2/a}$ path from 0 to infinity in $\mathbb{H}$ with driving Brownian motion $U_t = -B_t$. Let $T_D = \inf\{t > 0 : \gamma(t) \notin D\}$. Let $D_t = g_t(D)$ and $z_t = g_t(z)$. Note that if $t < T_D$, then $\operatorname{dist}(U_t, \mathbb{H} \setminus D_t) > 0$. Let $C_t = q(D_t; U_t; z_t)^b$. Using (4.7), (4.13), (4.14), and Itô's formula, we see that

$$dC_t = b\, C_t\ (\ J_t\ dt + [\log q(D_t; U_t; z_t)]'\ dU_t\ ),$$

where

$$J_t = a\, \Gamma(D_t; U_t) + \frac{1}{2}\,(1-a)\ \left[\frac{q''(D_t; U_t)}{q(D_t; U_t; z_t)} - \frac{3\, q'(D_t; U_t; z_t)^2}{2\, q(D_t; U_t; z_t)^2}\right]$$

if $z \in D$ or $z = \infty$, and

$$J_t = a\, \Gamma(D_t; U_t) + \frac{1}{2}\,(1-a)\ \left[\frac{q''(D_t; U_t; z_t)}{q(D_t; U_t; z_t)} - \frac{3\, q'(D_t; U_t; z_t)^2}{2\, q(D_t; U_t; z_t)^2}\right] + a\ \mathrm{Re}\left[\frac{1}{(g_t(z) - U_t)^2}\right]$$

if $z \in \partial D$. Let

$$M_t = C_t\ \exp\left\{-b \int_0^t J_s\, ds\right\}. \tag{4.18}$$

Then M_t is a local martingale satisfying

$$dM_t = b\, [\log q(D_t; U_t; z_t)]'\, M_t\, dU_t.$$

If $T_R = \inf\{t : M_t \in \{1/R, R\}\}$, then $M_t^{(R)} = M_{t \wedge T_R}$ is a continuous martingale. If we use the Girsanov transformation and consider the new measure $\mathbf{Q}^{(R)}$ defined by $d\mathbf{Q}_t^{(R)} = M_t^{(R)}\, d\mathbf{P}$, then under the measure $\mathbf{Q}^{(R)}$, U_t satisfies

$$dU_t = b\, [\log q(D_t; U_t; z_t)]'\, dt + d\hat{B}_t, \quad t < T_R,$$

where $\hat{B}_t$ is a standard Brownian motion under the new measure.

Using this motivation, we *define* LM_b in D to be the solution of the Loewner equation with driving function U_t satisfying

$$dU_t = b\, [\log q(D_t; U_t; z_t)]'\, dt + dB_t,$$

where B_t is a standard Brownian motion and $' = (d/dx)$. For the moment, we will only make this definition valid up to the first visit to $\mathbb{H} \setminus D$. From (3.2), (3.3), and (3.12), we see that this definition is conformally invariant. We state this as a proposition.

COROLLARY 4.4. *Suppose* $D, \hat{D}$ *are subdomains of* $\mathbb{H}$ *with* $\mathrm{dist}(0, \mathbb{H} \setminus D) > 0$ *and* $z \in (\overline{D} \cup \{\infty\}) \setminus \{0\}$. *Suppose* $f : D \to \hat{D}$ *is a conformal transformation with* $f(\infty) = \infty$. *Then if* γ *is* LM_b *from* 0 *to* z *in* D *(defined up to time* T_D*) then*

$$\hat{\gamma}(t) = f(\gamma(t)), \quad 0 < t < T_D,$$

has the distribution of (a time change of) LM_b *in* $\hat{D}$ *from* $f(0)$ *to* $f(z)$ *stopped at time* $T_{\hat{D}}$,

REMARK. This definition is clearly not changed if we replace q with cq for some constant c.

REMARK. As previously noted, if D is simply connected and $z \in \partial D$, this is the same as chordal $SLE_{2/a}$.

REMARK. If $D = \mathbb{H}$ and $z \in \mathbb{H}$ this gives radial $SLE_{2/a}$ from 0 to z. In this case, $D_t = \mathbb{H}$ for all t and we can choose

$$q(\mathbb{H}; x; z) = \pi\, H_{\mathbb{H}}(x, z) = \frac{\mathrm{Im}(z)}{|z - x|^2},$$

for which

$$[\log q(\mathbb{H}; x; z)]' = \frac{2\,(\mathrm{Re}(z) - x)}{|z - x|^2}.$$

Therefore radial $SLE_{2/a}$ from 0 to z in $\mathbb{H}$ is the solution of the Loewner equation (4.16) with

$$dU_t = \frac{2b\,(\mathrm{Re}(z_t) - U_t)}{|z_t - U_t|^2}\, dt - dB_t.$$

This is not the usual definition of radial $SLE_{2/a}$. As originally defined, radial $SLE_{2/a}$ from 1 to 0 in the unit disk $\mathbb{D}$ was obtained as the solution to the radial Loewner equation

$$\dot{\tilde{g}}(z) = \frac{a}{2}\, \tilde{g}_t(z)\, \frac{e^{iB_t} + \tilde{g}_t(z)}{e^{iB_t} - \tilde{g}_t(z)},$$

where B_t is a standard Brownian motion with $B_0 = 0$, This equation defines a random curve $\gamma : [0, \infty) \to \mathbb{D}$ with $\gamma(0) = 1$ and $\tilde{g}_t$ is the unique conformal transformation of $\mathbb{D} \setminus \tilde{\gamma}(0, t]$ onto $\mathbb{D}$ with $\tilde{g}_t(0) = 0, \tilde{g}_t'(0) > 0$. The curve has been parametrized so that $\tilde{g}_t'(0) = e^{at}$. Radial $SLE_{2/a}$ from 0 to z in $\mathbb{H}$ is then defined (modulo time reparametrization) as the image of this measure under a linear fractional transformation of $\mathbb{D}$ onto $\mathbb{H}$ taking 1 to 0 and 0 to z. We leave it to the reader to check directly that these definitions agree. Without calculation, one could also see that our definition has to be correct since it is correct for purely imaginary z by symmetry and it is conformally invariant.

REMARK. If D is simply connected, then

$$Q(D_t;U_t) = g'_{D_t}(U_t), \quad \Gamma(D_t;U_t) = -(1/6)\, Sg_{D_t}(U_t),$$

and hence

$$J_t = \left(\frac{1}{2} - \frac{2a}{3}\right) Sg_{D_t}(U_t),$$

$$M_t = C_t \exp\left\{\frac{a\lambda}{12}\int_0^t Sg_{D_s}(U_s)\, ds\right\} = C_t \exp\left\{-\frac{a\lambda}{2}\int_0^t \Gamma(D_s;U_s)\, ds\right\},$$

where

$$\lambda = \frac{(3a-1)\,(4a-3)}{2a}.$$

This is the local martingale considered in [15]. It is a bounded martingale if $\lambda \geq 0, b \geq 0$, i.e., if $a \geq 3/4$. It is a martingale for $a \geq 1/2$ (see the remark below).

REMARK. If $\gamma : (0,\infty) \to \mathbb{H}$ is a simple curve with $\mathrm{hcap}(\gamma[0,t]) = at$, then

$$\exp\left\{-\frac{a}{2}\int_0^\infty \Gamma(D_t,U_t)\, dt\right\}$$

denotes the probability that there is no loop in the Brownian loop soup (see [16]) in $\mathbb{H}$ that intersects both γ and D. This fact was used to construct restriction measures in [15].

REMARK. Note that if $D = \mathbb{H} \setminus (A_1 \cup \cdots \cup A_n) \in \mathcal{Y}$, then we can write

$$J_t = a\,\Gamma(D_t;U_t) + \frac{1}{2}\,(1-a)\, S\hat{g}_{D_t}(U_t), \tag{4.19}$$

where S denotes the Schwarzian derivative and $\hat{g}_{D_t}$ is as defined in (3.13). In contrast with the simply connected case, if $D \in \mathcal{Y}$, $\Gamma(D;x)$ is not a simple multiple of $S\hat{g}_D(x)$.

REMARK. If $a = 1$, the second term drops out and we get just $J_t = \Gamma(D_t;U_t)$. This has a natural interpretation in terms of loop-erased walk. The term

$$\exp\left\{\frac{1}{2}\int_0^\infty \Gamma(D_t,U_t)\, dt\right\}$$

gives a measure of the number of ways to "add loops back" to the process to retrieve a Brownian motion. This interpretation holds even in the case of non-simply connected domains.

REMARK. It is interesting to ask when M_t as defined in (4.18) is not just a local martingale but actually a martingale. To determine this we can consider $M_t^{(R)}$, which is a martingale and let $\mathbf{Q}^{(R)}$ be the probability measure on paths given by Girsanov,

$$d\mathbf{Q}_t^{(R)} = M_t^{(R)}\, d\mathbf{P}.$$

Suppose we know that for every $\epsilon > 0$, there is an $N = N_\epsilon < \infty$ such that for every R,

$$\mathbf{Q}^{(R)}\{T_N < \infty\} < \epsilon. \tag{4.20}$$

Then it is easy to verify that the collection of random variables $\{M_t^{(R)}\}$ is uniformly integrable and hence M_t is a uniformly integrable martingale. One example is when D is simply connected, $\mathbb{H}\setminus D$ is bounded, and $a \geq 1/2, \kappa \leq 4$. In this case the measure $\mathbf{Q}^{(R)}$ corresponds to $SLE_{2/a}$ from 0 to infinity in the domain D. We know that with probability one, the paths under this measure do not leave D and hence it is not difficult to verify (4.20). If $a \geq 3/4, \kappa \leq 8/3$, this is a bounded martingale while for $1/2 \leq a < 3/4$, the martingale is unbounded.

REMARK. Another case when M_t is a bounded martingale is $a = 1, \kappa = 2$ and D is any domain.

4.5. $SLE_{8/3}$ **versus** $LE_{5/8}$**.** The restriction property for $SLE_{8/3}$ gives a natural way to extend this process to non-simply connected domains so that the extension still satisfies the restriction property. This extension is not the same as $LE_{5/8}$. We will discuss this extension briefly in this section.

Throughout this section we will let γ denote a chordal $SLE_{8/3}$ from $x \in \mathbb{R}$ to ∞. If $D \subset \mathbb{H}$ is a domain, we let

$$Q_{SLE}(D; x) = \mathbf{P}^x\{\gamma(0, \infty) \subset D\}.$$

Here we use the superscript x to denote the starting point of the SLE curve. The restriction property for $SLE_{8/3}$ (see [13, Section 6.4]) states that if D is a simply connected domain with $\mathbb{H} \setminus D$ bounded and $\operatorname{dist}(x, \mathbb{H} \setminus D) > 0$, then

$$Q_{SLE}(D; x) = Q(D; x)^{5/8} = g_D'(x)^{5/8}.$$

However, the first equality does not hold if D is not simply connected. For example, if $D_\epsilon = \mathbb{H} \setminus \{z : |z - i| \leq \epsilon\}$, then as $\epsilon \to 0+$,

$$1 - Q_{SLE}(D; 0) \asymp \epsilon^{2/3}, \qquad 1 - Q(D; 0) \asymp \epsilon.$$

The second estimate is a straightforward estimate for Brownian motion. The first is closely related to the fact that the Hausdorff dimension of $\gamma(0, \infty)$ is $4/3$, see, e.g., [13, Theorem 7.9].

The restriction extension of $SLE_{8/3}$ to non-simply connected domains is simple: if $D \subset \mathbb{H}$, then $SLE_{8/3}$ in D starting at x is defined to be the conditional distribution of $SLE_{8/3}$ from x to infinity in $\mathbb{H}$ conditioned on the event $\{\gamma(0, \infty) \subset D\}$. Of course, we could define this quantity for SLE_κ for any κ; the key fact about $\kappa = 8/3$ is that this definition gives a conformally invariant family of curves. The restriction property proves this fact for simply connected domains; the following simple proposition extends this to $D \in \mathcal{Y}$. We will write $\nu_{D,x}$ for the subprobability measure on paths (modulo reparametrization) obtained by restricting $SLE_{8/3}$ from x to infinity to curves with $\gamma(0,\infty) \subset D$ and we write $\nu_D = \nu_{D,0}$ The restriction property tells us that if D is simply connected, then

$$g_D \circ \nu_D = g_D'(0)^{5/8}\, \nu_{\mathbb{H},g_D(0)}.$$

COROLLARY 4.5. *Suppose $D, D' \subset \mathcal{Y}$ and $f : D \to D'$ is a conformal transformation with $f(0) = 0$ and $f(z) \sim z$ as $z \to \infty$. Then,*

$$f \circ \nu_D = f'(0)^{5/8}\, \nu_{D'}.$$

In particular,

$$Q_{SLE}(D; 0) = f'(0)^{5/8}\, Q_{SLE}(D'; 0).$$

Proof. In order to determine the distribution of a simple curve (modulo reparametrization) it suffices to give the probability of being in U for every domain U. What we claim is that it suffices to give these probabilities for all simply connected domains U with $\mathbb{H} \setminus U$ bounded and $\mathrm{dist}(0, \mathbb{H} \setminus D) > 0$. Indeed, the events $\{\gamma(0,\infty) \subset U\}$, indexed by such simply connected U, form a π-system (closed under intersections). Hence if the σ-algebra generated by such U contains all such D, then we are finished (see [4, Theorem 3.4]).

Suppose $D = \mathbb{H} \setminus (A_1 \cup \cdots \cup A_n)$. Suppose $\gamma : (0,\infty) \to \mathbb{H}$ is a simple curve with $\gamma(0+) = 0$ and $\gamma(t) \to \infty$ as $t \to \infty$. Suppose also that $\gamma(0,\infty) \subset D$. A simple topological argument shows that there exist simple disjoint curves $\eta_j : [0,1) \to \mathbb{H}$ with $\eta_j(0) \in A_j$; $\eta(0,1) \subset D$; $\eta_j(1-) \in \mathbb{R} \setminus \{0\}$ and such that $\eta[0,1) \cap \gamma(0,\infty) = \emptyset$. Let $U = \mathbb{H} \setminus (A_1 \cup \cdots A_n \cup \eta_1 \cup \cdots \eta_n)$. Then U is a simply connected domain with $\mathbb{H} \setminus U$ bounded, $\mathrm{dist}(0, \mathbb{H} \setminus U) > 0$ and such that $\gamma(0,\infty) \subset U$.

This argument shows that it suffices to consider the measures ν_U where U ranges over all simply connected subdomains of $\mathbb{H}$ with $U \subset D$; $\mathbb{H}\setminus U$ bounded; and $\mathrm{dist}(0, \mathbb{H} \setminus U) > 0$. Since $f : U \to f(U)$ is a conformal transformation, we have

$$f \circ \nu_U = f'(0)^{5/8}\, \nu_{f(D)}. \qquad \square$$

REMARK. If $D = \mathbb{H} \setminus (A_1 \cup \cdots \cup A_n) \in \mathcal{Y}$, the event $\{\gamma(0,\infty) \subset D\}$ can written as the disjoint union of 2^n nontrivial events E_K indexed by subsets K of $\{1, \dots, n\}$. The event E_K is the event that A_j lies to the left of $\gamma(0,\infty)$ for $j \in K$ and A_j lies to the right of γ for $j \notin K$. We can write in a natural way

$$Q_{SLE}(D; x) = \sum_K Q_{SLE}(D; x; K).$$

The quantity $Q(D; x)$ does not admit such a simple decomposition since $\mathbb{H}$-excursions are not simple curves.

4.6. Boundary conditions. We will consider the case of boundary conditions for LM_b in D from 0 to infinity in the domain $D = \mathbb{H}\setminus(A_1\cup\cdots\cup A_n) \in \mathcal{Y}$. The other cases are handled similarly. We conjecture, but have not proved at this point, that if $a \geq 1/2, \kappa \leq 4$, then with probability one the paths do not hit $A_1 \cup \cdots \cup A_n$. However, we certainly expect this to happen for $\kappa > 4$.

Let $\xi = \xi_D$ denote the first time t such that $\gamma(t) \in A_1 \cup \cdots \cup A_n$. Assume, for ease, that $\gamma(\xi) \in A_1$. There are two possibilities: (1) the entire hole A_1 is swallowed up ($A_1 \cap \partial D_\xi = \emptyset$) in which case there is no problem defining the process for times slightly larger than ξ, or (2) $\partial_1 \subset \partial D$. In the latter case, the "hole" A_1 is connected to the real line by the path, or, in other words, it is no longer a hole. The point $\gamma(\xi)$ corresponds to two points, say $x_{\xi,1}, x_{\xi,2}$ after we map by g_ξ. We need to choose which one. The definition of the Laplacian walk shows that the natural thing to do is to choose randomly using the probability distribution proportional to $Q(D_\xi; x_{\xi,j})^b$. This choice is natural for the LM_b. However, other extensions of $SLE_{2/a}$, e.g., percolation in non-simply connected domains, would make other choices.

5. Non-simply connected domains

5.1. Excursion-reflected random walk. In order to motivate the definition of excursion-reflected Brownian motion, we will describe the discrete analogue. Suppose $A_1, \ldots, A_n$ are disjoint, finite, connected subsets of $\mathbb{Z}^2$ and let $V = \mathbb{Z}^2 \setminus (A_1 \cup \cdots \cup A_n)$. Then *excursion-reflected simple random walk (ERRW)* in V is the Markov chain with state space $V \cup \{A_1, \ldots, A_n\}$ obtained by identifying the vertices of a hole. The edges of the new graph are the edges from $\mathbb{Z}^2$ except that we do not include those edges that go from A_j to A_j. In this new graph, there may be multiple edges connecting two vertices. The excursion-reflected random walk is the simple random walk on this graph.

There is another way of constructing this process. Let $A_1, \ldots, A_n$ be as above and let X_m denote a Markov chain with state space $\mathbb{Z}^2$ that moves like simple random walk except whenever it reaches a point in A_j it instantaneously moves to a point in A_j chosen from the uniform distribution. This process has transition probabilities

$$p(x,y) = 1/4, \quad |x-y| = 1,\ y \in V,$$

$$p(x,y) = \frac{1}{4}\,\frac{\#\{y' \in A_j : |x-y'| = 1\}}{\#(A_j)}, \quad y \in A_j.$$

For any $x \in \mathbb{Z}^2$ and any $m > 0$, the function

$$y \longmapsto \mathbf{P}\{X_m = y \mid X_0 = x\},$$

is constant on each A_j. This Markov chain induces another Markov chain $\hat{X}_m$ on $V \cup \{A_1, \ldots, A_n\}$ by identifying the points. Here we associate the initial condition $\hat{X}_0 = A_j$ with the initial distribution of X_0 as the uniform on A_j. (Strictly speaking, we only allow initial conditions on X_0 that are uniform on each A_j.) From $\hat{X}_m$ we can get another induced chain by not allowing the process to go from A_j directly to A_j. This is the same as the ERRW of the previous paragraph.

Let us now consider the special case where the imaginary part is constant on each A_j, i.e., $A_j \subset \{\mathrm{Im}(z) = y_j\}$ with $y_1, \ldots, y_j > 0$. Suppose y is an integer greater than $\max\{y_1, \ldots, y_n\}$ and suppose we start independent walkers following the chain X_m at each point $k + iy, k \in \mathbb{Z}$. For each non-negative integer $l < y$, consider the measure on $\{\mathrm{Im}(z) = l\}$ given by the first visit of each of these walkers. Then it is easy to see that this measure is just the counting measure on $\{k + il : k \in \mathbb{Z}\}$. This implies a similar result about the induced chain $\tilde{X}_m$. We state this as a lemma.

LEMMA 5.1. *Let $A_1, \ldots, A_n$ be disjoint subsets of $\mathbb{Z}^2$ with $A_j \subset \{\mathrm{Im}(z) = y_j\}$ with $y_1, \ldots, y_n > 0$. Let τ be the first time that the ERRW reaches $\{\mathrm{Im}(z) = 0\}$ and for $z \in \mathbb{Z} \times i\mathbb{Z}_+$ let*

$$H(z, x) = \mathbf{P}^z\{\tilde{X}_\tau = x\}.$$

Then if $y > \max\{y_1, \ldots, y_n\}$, $x \in \mathbb{Z}$,

$$\sum_{k=-\infty}^{\infty} H(k + iy, x) = 1.$$

5.2. Excursion-reflected Brownian motion. In our proof of Proposition 3.1, we will use a process that we call *(excursion)-reflected Brownian motion (ERBM) in D.* Suppose $D = \mathbb{C} \setminus [A_0 \cup \cdots \cup A_n] \in \mathcal{D}$ with $A_1, \ldots, A_n$ compact. Let $\eta_1, \ldots, \eta_n$ be disjoint curves as in Section 3.1 with η_j surrounding A_j in D. We let $D_j = D \cap U_{\eta_j}$. ERBM in D acts like Brownian motion in D; is killed when it reaches A_0; and is reflected off the holes $A_1, \ldots, A_n$ is a way that makes the process a strong Markov process on $D \cup \mathbb{R} \cup \{A_1, \ldots, A_n\}$ with absorbing set $\mathbb{R}$. In other words, each hole A_j is identified to a single point.

We will now describe ERBM in $U = \{|z| > 1\}$. Let X_t, Y_t be independent one-dimensional processes where Y_t is a standard one-dimensional Brownian motion and X_t is one-dimensional Brownian motion reflected at the origin. Let

$$B_t = \exp\{X_{\sigma(t)} + iY_{\sigma(t)}\}$$

where the time change $\sigma(t)$ is chosen in the standard way so that B_t looks like a standard complex Brownian motion in U. Then B_t is Brownian motion with usual orthogonal reflection off the unit disk. To get excursion-reflected Brownian motion, we consider the excursions of B_t, i.e., the intervals (r_j, s_j) such that $|B_{r_j}| = |B_{s_j}| = 1$ and $|B_t| > 1$ for $r_j < t < s_j$. For each such excursion we choose an angle θ_j uniformly in $[0, 2\pi)$. The θ_j are chosen independently for each excursion. We then set

$$\tilde{B}_t = \begin{cases} \overline{\mathbb{D}} & \text{if } |B_t| = 1, \\ e^{i\theta_j}\, B_t & \text{otherwise,} \end{cases}$$

where θ_j denotes the angle associated to the excursion that B_t is in. It is easy to check that this process acts like Brownian motion on $\{|z| > 1\}$ and that the radial part acts like the radial part of a reflected Brownian motion. Roughly speaking, whenever the process reaches $\overline{\mathbb{D}}$ it chooses a new angle at random and goes out that angle. In particular, if $|\tilde{B}_0| > 1$ and T denotes the first time that $\tilde{B}_T = \overline{\mathbb{D}}$, then $\tilde{B}_{t+T} - \tilde{B}_T$ is independent of $\tilde{B}_t, 0 \le t \le T$, i.e., the process is Markovian on the state space $\{|z| > 1\} \cup \overline{\mathbb{D}}$.

This definition is a little confusing and it is useful to consider the hitting distribution of the circles $\mathcal{C}_r = \{|z| = r\}$. If $r > 1$, let $\tilde{\rho}_r$ denote the first time that the ERBM $\tilde{B}_t$ in U hits $\mathcal{C}_r$. If $\tilde{B}_0 = \overline{\mathbb{D}}$, then the distribution of $\tilde{B}_{\rho_r}$ is uniform on $\mathcal{C}_r$. If $1 < r < s$ and $\tilde{B}_0 = \overline{\mathbb{D}}$, we can consider the joint distribution of $(\tilde{B}_{\rho_r}, \tilde{B}_{\rho_s})$. Given $\arg(\tilde{B}_{\rho_r}) = \theta$ the density of $\arg(\tilde{B}_{\rho_s})$ is

$$s\, H_D(re^{i\theta}, se^{i\theta'}) + \frac{1}{2\pi}\left[1 - \frac{\log r}{\log s}\right],$$

where $D = \{1 < |w| < s\}$ and H_D denotes the Poisson kernel. The first term in the sum gives the density restricted to paths that reach $\mathcal{C}_s$ before $\mathcal{C}_1$ and the second term handles the paths that hit $\mathcal{C}_1$ first. Note that the probability of the first event is $(\log r/\log s)$. If s is fixed and $r \to 1+$, then $\tilde{B}_{\tilde{\rho}_r}$ and $\tilde{B}_{\tilde{\rho}_s}$ are asymptotically independent.

If A_j is a compact, connected set larger than a single point, then ERBM in $D = \mathbb{C} \setminus A_j$ can be defined from the process in the previous paragraph by conformal invariance. In this case, the process looks like Brownian motion on $\mathbb{C} \setminus A_j$ and when it hits A_j it chooses a new starting point based on harmonic measure from infinity on A_j which is the same as the image of the uniform measure of the circle under a conformal transformation of $\{|z| > 1\}$ onto $C \setminus A_j$ fixing infinity.

We can define ERBM in D by doing the reflection at the holes $A_1, \ldots, A_n$ and killing when it reaches A_0. If η_j is a curve surrounding A_j as in Section 3.1 and $\tilde{\tau}_{\eta_j}$ denotes the first visit to η_j, then

$$\mathbf{P}^{A_j}\{\tilde{B}_{\tilde{\tau}_{\eta_j}} \in V\} = \frac{|\mathcal{E}_{D_j}(A_j, V)|}{|\mathcal{E}_{D_j}(A_j, \eta_j)|}.$$

It is useful to consider this motion in steps: start at a point $z \in D$; do Brownian motion until we hit an A_j; if $j = 0$, stop; otherwise, do ERBM starting at A_j until we reach the curve η_j; then continue this procedure until we reach A_0. Note that the distribution of ERBM from A_j until it hits η_j depends only on A_j and η_j and not on the rest of the domain. Also it is important to realize that the definition of ERBM is independent of the choice of $\eta_1, \ldots, \eta_n$.

There is an induced discrete time Markov chain X_j with state space $\{A_0, A_1, \ldots, A_n\}$ whose distribution depends on the choice of $\eta_1, \ldots, \eta_n$. Let $X_0 = A_j, j \geq 1$ and assume that $\tilde{B}_{\tau_0} = A_j$. Let t be the first time after τ_{η_j} that $\tilde{B}_t \in A_0 \cup \{A_1, \ldots, A_n\}$ and set $X_t = \tilde{B}_t$. By iterating this, we get a Markov chain with transition probabilities

$$p_{jk} = \int_{\eta_j} h_k(z) \, \frac{H_{\partial D}(A_j, z)}{|\mathcal{E}_{D_j}(A_j, \eta_j)|} \, |dz|, \qquad j \geq 1,$$

and $p_{00} = 1$. From this Markov chain, we also have another induced Markov chain which corresponds to this chain when it changes to a new site. This chain also has absorbing state A_0 and has transition probabilities independent of the choice of $\eta_1, \ldots, \eta_n$,

$$q_{jk} = \frac{p_{jk}}{1 - p_{jj}} = \frac{|\mathcal{E}_D(A_j, A_k)|}{\sum_{l \geq 0, l \neq j} |\mathcal{E}_D(A_j, A_l)|}, \qquad j \geq 1, k \neq j.$$

Let Q denote the $n \times n$ matrix $Q = [q_{jk}]_{1 \leq j,k \leq n}$. Since $q_{j0} > 0$ for each j, all the eigenvalues of Q have absolute value strictly less than one, and the Green's matrix

$$M := (I - Q)^{-1} = I + Q + Q^2 + \cdots$$

is well defined. The entry M_{jk} represents the expected number of visits to A_k by the chain assuming that it starts in A_j. Note that

$$M_{jk} = \delta_{jk} + \sum_{l=1}^{n} q_{jl} \, M_{lk}.$$

Recall that if ERBM in D starts in state A_j, then the density of the hitting measure of the first visit to η_j is given by

$$\frac{H_{\partial D_j}(A_j, \cdot)}{|\mathcal{E}_{D_j}(A_j, \eta_j)|}.$$

We call a function $v : D \cup \mathbb{R} \cup \{A_1, \ldots, A_n\} \to \mathbb{R}$ *ER-harmonic* on $D \cup \{A_1, \ldots, A_n\}$ if v is harmonic on D and for each j,

$$v(A_j) = \int_{\eta_j} v(z) \, \frac{H_{\partial D_j}(A_j, z)}{|\mathcal{E}_{D_j}(A_j, \eta_j)|} \, |dz| \tag{5.1}$$

Using (3.7), we see that this definition is independent of the choice of $\eta_1, \ldots, \eta_n$ and that v is ER-harmonic on $D \cup \{A_1, \ldots, A_n\}$ if and only if it is harmonic in D and

$$\int_{\eta_j} \frac{d}{dn} v(z) \, |dz| = 0, \qquad j = 1, \ldots, n. \tag{5.2}$$

It is well known that this is a necessary and sufficient condition for there to exist an analytic function $f(z) = u(z) + iv(z)$ whose imaginary part is v. In fact, u is defined up to a real additive constant by

$$u(z) = u(z_0) + \int_\gamma \frac{d}{dn} v(z)\,|dz|,$$

where γ is a curve from z_0 to z and the normal derivative is chosen with the correct sign. The condition (5.2) shows that this is a well-defined single-valued function and a simple calculation shows that u, v satisfy the Cauchy-Riemann equations.

Note that ERBM satisfies the following:

- **Conformal invariance.** If $f : D \to D'$ is a conformal transformation and $\tilde{B}_t$ is ERBM in D, then $f(\tilde{B}_t)$ is (a time change of) ERBM in D'.

5.3. Proof of Proposition 3.1

We will write B_t for a standard Brownian motion and $\tilde{B}_t$ for an ERBM. We let

$$\tau_D = \inf\{t : B_t \notin D\}, \quad \tilde{\tau}_D = \inf\{t : \tilde{B}_t \notin D\}, \quad \tilde{T}_D = \inf\{t : \tilde{B}_t \in \mathbb{R}\}.$$

The distributions of $B_t, 0 \le t < \tau_D$, and $\tilde{B}_t, 0 \le t < \tilde{\tau}_D$, are identical assuming they start at the same point $z \in D$. We define

$$v_D(z) = \lim_{r\to\infty} r\,\mathbf{P}^z\{\mathrm{Im}(B_t) = r \text{ for some } t < \tau_D\}, \quad z \in D,$$

$$\tilde{v}_D(z) = \lim_{r\to\infty} r\,\mathbf{P}^z\{\mathrm{Im}(\tilde{B}_t) = r \text{ for some } t < \tilde{T}_D\}, \quad z \in D \cup \{A_1, \ldots, A_n\}.$$

Recall, that the probability that a Brownian motion starting at z reaches $\{\mathrm{Im}(w) = r\}$ before hitting $\mathbb{R}$ is $\mathrm{Im}(z)/r$, assuming $r > \mathrm{Im}(z)$. Hence $v_{\mathbb{H}}(z) = \mathrm{Im}(z)$. The function v_D is the unique continuous function on $\overline{\mathbb{H}}$ that is harmonic in D, vanishes on $\overline{\mathbb{H}} \setminus D$, and satisfies

$$v_D(x + iy) \sim y, \qquad y \to \infty.$$

The function $\tilde{v}_D(z)$ can be considered as a function on $\overline{\mathbb{H}}$ with constant value $\tilde{v}_D(A_j)$ on A_j. It is the unique continuous function on $\overline{\mathbb{H}}$ that is harmonic in D, vanishes on $\mathbb{R}$, takes value $\tilde{v}_D(A_j)$ on A_j, and satisfies

$$\tilde{v}_D(x + iy) \sim y, \qquad y \to \infty.$$

This characterization does not determine the values $\tilde{v}_D(A_j)$. If $z \in D$, then the Markov property shows that

$$\tilde{v}_D(z) = v_D(z) + \sum_{j=1}^n \mathbf{P}^z\{\tilde{B}_{\tilde{\tau}_D} = A_j\}\,\tilde{v}_D(A_j). \tag{5.3}$$

(This last equation splits the event that $\tilde{B}_t$ reaches $\{\mathrm{Im}(w) = r\}$ into the events that it reaches before hitting $\{A_1, \ldots, A_n\}$ and that it reaches after hitting $\{A_1, \ldots, A_n\}$.) If we let $\nu_j = [H_{\partial D_j}(A_j, z)/|\mathcal{E}_{D_j}(A_j, \eta_j)|]\ |dz|$ denote the hitting distribution of η_j by $\tilde{B}$ assuming $\tilde{B}_0 = A_j$, we see that

$$\begin{aligned}\tilde{v}_D(A_j) &= \int_{\eta_j} \tilde{v}_D(z)\, d\nu_j(z) \\ &= \int_{\eta_j} v_D(z)\, d\nu_j(z) + \sum_{k=1}^{n} \int_{\eta_k} h_k(z)\, \tilde{v}_D(A_k)\, d\nu_j(z) \\ &= \int_{\eta_j} v_D(z)\, d\nu_j(z) + \sum_{k=1}^{n} p_{jk}\ \tilde{v}_D(A_k).\end{aligned}$$

If we define

$$v_D(A_j) = \frac{1}{1 - p_{jj}} \int_{\eta_j} v_D(z)\, d\nu_j(z),$$

this equation becomes

$$\tilde{v}_D(A_j) = v_D(A_j) + \sum_{k=1}^{n} q_{jk}\, \tilde{v}_D(A_k). \tag{5.4}$$

A little thought will show that $v_D(A_j)$ is independent of the choice of $\eta_1, \ldots, \eta_n$ (as are the transition probabilities q_{jk}, which we have already noted). This system of equations has a unique solution,

$$\tilde{v}_D(A_j) = \sum_{k=1}^{n} M_{jk}\, v_D(A_k).$$

We have defined $\tilde{v}_D$ so that it is an ER-harmonic function. As pointed out in Section 5.2, there is an analytic function f_D on D such that $\operatorname{Im} f_D = \tilde{v}_D$. This function is unique up to a real translation. We will specify it uniquely by adding the condition

$$f_D(z) = z + o(1), \qquad z \to \infty.$$

We also note that if $D, D' \in \mathcal{A}$ and $f : D \to D'$ is a conformal transformation with $f(\infty) = \infty, f'(\infty) = 1$, then conformal invariance of ERBM implies that

$$\tilde{v}_D(z) = \tilde{v}_{D'}(f(z)).$$

Let $\tilde{v}_D$ be as above and let f_D be an analytic function with $\operatorname{Im} f_D = \tilde{v}_D$. We claim that f_D is a conformal transformation and $D' := f_D(D)$ is a canonical domain. It is immediate that $D' \subset \mathbb{H}$. Let $\lambda_j = \tilde{v}_D(A_j)$. Let $\lambda \in (0, \infty) \setminus \{\lambda_1, \ldots, \lambda_n\}$ and let $V_\lambda = \{z : \tilde{v}_D(z) = \lambda\}$. Using standard facts about harmonic functions, we can see that V_λ is a doubly infinite curve in D that stays away from $A_1, \ldots, A_n$. This curve splits $\mathbb{H}$ into two sets; above the curve $\tilde{v}_D(z) > \lambda$ and below the curve $\tilde{v}_D(z) < \lambda$. Hence $(d/dn)\tilde{v}_D(z)$ has a consistent sign, and for this reason we see that the real part of f_D increases as we progress on this line in the "positive real direction". Also, the image is a doubly infinite curve. If $\lambda \in \{\lambda_1, \ldots, \lambda_n\}$, we use a similar argument except that the set V_λ contains at least one of the A_j. We still get that f_D is one-to-one, but there are intervals in $\mathbb{H}$ that are not hit. Hence we have that D' is a canonical domain and the y-component of $f_D(A_j)$ is $\tilde{v}_D(A_j)$. Since f_D is analytic and real in a neighborhood of infinity, we can expand f_D at infinity

$$f_D(z) = a_1 z + a_0 + a_{-1}\, z^{-1} + \cdots .$$

We know that $a_1 = 1$ since $v_D(iy) \sim y$. We can make $a_0 = 0$ by taking an appropriate real translation of f_D.

To show uniqueness, we use the conformal invariance of ERBM and the behavior at infinity to see that

$$\tilde{v}_D(z) = \tilde{v}_{D'}(f_D(z)).$$

This then gives uniqueness up to a real translation and exactly one of these translations gives $a_0 = 0$.

5.4. Poisson kernel for ERBM. Let $D \in \mathcal{Y}$. For $z \in D, x \in \mathbb{R}$, let $H_D(z,x)$ denote the Poisson kernel for Brownian motion killed upon leaving D, i.e.,

$$\mathbf{P}^z\{B_{\tau_D} \in [a,b]\} = \int_a^b H_D(z,x)\, dx.$$

The *Poisson kernel for ERBM in D*, $\tilde{H}_D(z,x)$, is defined to be the corresponding quantity for ERBM,

$$\mathbf{P}^z\{\tilde{B}_{T_D} \in [a,b]\} = \int_a^b \tilde{H}_D(z,x)\, dx.$$

COROLLARY 5.2. *For every $D \in \mathcal{Y}$, there exists a unique function $\zeta_D : D \times \mathbb{R} \to \mathbb{C}$ satisfying the following.*

- *For $x \in \mathbb{R}$, $z \mapsto \zeta_D(z,x)$ is analytic in D.*
- $\mathrm{Im}[\zeta_D(z,x)] = -\pi\, \tilde{H}_D(z,x)$.
- *For every $x \in \mathbb{R}$, there is a $c(D,x) > 0$ such that as $z \to \infty$,*

$$\zeta_D(z,x) = \frac{c(D,x)}{z-x} + O(|z|^{-2}).$$

- *For every $x \in \mathbb{R}$, there is a $r(D,x) \in \mathbb{R}$ such that as $z \to x$,*

$$\zeta_D(z,x) = \frac{1}{z-x} + r(D,x) + O(|z-x|).$$

Proof. We will assume $x = 0$; for $x \neq 0$, we can define $\zeta_D(z,x) = \zeta_{D-x}(z-x, 0)$. We write $H_D(z), \tilde{H}_D(z), \zeta_D(z)$ for $H_D(z,0), \tilde{H}_D(z,0), \zeta_D(z,0)$.

Similarly to the function $\tilde{v}_D$, we can write

$$\tilde{H}_D(z) = H_D(z) + \sum_{j=1}^{n} h_j(z)\, \tilde{H}_D(A_j), \tag{5.5}$$

$$\tilde{H}_D(A_j) = \int_{\eta_j} H_D(z)\, d\nu_j(z) + \sum_{k=1}^n p_{jk}\, \tilde{H}_D(A_k),$$

This shows that $\tilde{H}_D(z)$ is an ER-harmonic function on $D \cup \{A_1, \ldots, A_n\}$. As pointed out in Section 5.2, this implies that there is an analytic function $\zeta_D(z)$ in D with $\mathrm{Im}[\zeta_D(z)] = -\pi\, \tilde{H}_D(z)$. It is unique up to an additive real constant. By Schwarz reflection, the function $F(z) = \zeta_D(1/z)$ is defined and analytic in a neighborhood of the origin; we choose the additive constant for ζ_D so that $F(0) = 0$, i.e.,

$$\lim_{z\to\infty} \zeta_D(z) = 0.$$

For ease, we will assume that $A_1 \cup \cdots \cup A_n \subset \{|z| < 1\}$; other cases can be handled by scaling. Suppose $|z| \geq 2$ and we start an ERBM at z. By stopping the motion at the first time that it reaches $\mathbb{R} \cup \{|w| = 1\}$ (the process acts like a usual Brownian motion up to this time), (3.5) can be used to see that

$$\tilde{H}_D(z) = \frac{2}{\pi}\,[1 + O(|z|^{-1})]\, \frac{\mathrm{Im}(z)}{|z|^2} \int_0^\pi \tilde{H}_D(e^{i\theta})\, \sin\theta\, d\theta.$$

In particular,

$$\pi\, \tilde{H}_D(z) = c_D\, \frac{\mathrm{Im}(z)}{|z|^2}\, \left[1 + O(|z|^{-1})\right].$$

Let $f(z) = \zeta_D(z) - c_D\, z^{-1}$. Then, $|\,\mathrm{Im}[f(z)]| \leq O(|z|^{-2})$. By standard arguments, this implies $|f'(z)| \leq O(|z|^{-3})$. Since $f(\infty) = 0$, if $|x + iy| \geq 2$,

$$|f(x+iy)| = \left|\int_y^\infty f'(x+is)\, ds\right| \leq \int_y^\infty |f'(x+is)|\, ds = O(|x+iy|^{-2}).$$

Therefore,

$$\zeta_D(x) = \frac{c_D}{z} + O(|z|^{-2}).$$

Using (5.5), we see that as $z \to 0$,

$$\tilde{H}_D(z) = H_D(z) + \sum_{j=1}^n \mathrm{Im}(z)\, H_D(0, A_j)\; \tilde{H}_D(A_j)\; [1 + O(|z|)].$$

Also (3.14) gives

$$\pi\, H_D(z) = \pi\, H_{\mathbb{H}}(z) - \mathrm{Im}(z)\, \Gamma(D)\; [1 + O(|z|)].$$

Therefore,

$$\pi\, \tilde{H}_D(z) = \pi H_{\mathbb{H}}(z) - \mathrm{Im}(z)\, \tilde{\Gamma}(D)\; [1 + O(|z|)],$$

where

$$\tilde{\Gamma}(D) = \Gamma(D) - \sum_{j=1}^n \mathrm{Im}(z)\, H_D(0, A_j)\; \tilde{H}_D(A_j).$$

If we let $h(z) = \zeta(z) - (1/z)$, then these estimate show that $|h'(z)|$ is bounded in a neighborhood of the origin, and hence can be extended by Schwartz reflection to an analytic function near zero taking reals to reals. In particular, $h(z) = h(0) + O(|z|)$ with $h(0) \in \mathbb{R}$, giving

$$\zeta(z) = \frac{1}{z} + r_D + O(|z|), \quad z \to 0. \qquad \square$$

REMARKS.

- We call $\zeta_D(z, x)$ the complex Poisson kernel for EBRM in D. If $D = \mathbb{H}$, then
$$\zeta_D(z, x) = \frac{1}{z - x}.$$
- The function $z \mapsto \zeta_D(z, x)$ is the unique conformal transformation of D onto a canonical domain satisfying $\zeta_D(\infty, x) = 0$ and
$$\zeta_D(z, x) \sim \frac{1}{z - x}, \quad z \longrightarrow x.$$
- If $D, D' \in \mathcal{Y}$ and $f : D \to D'$ is a conformal transformation with $f(z) - z = o(1)$ as $z \to \infty$, then
$$\zeta_D(z, x) = f'(x)\, \zeta_{D'}(f(z), f(x)).$$
In particular, $c(D, x) = f'(x)\, c(D', f(x))$.
- We will see in the next section that if $D \in \mathcal{C}Y$, then $c(D, x) = 1$ for all $x \in \mathbb{R}$.

5.5. ERBM in canonical domains. The ERBM is conformally invariant and has particular nice properties in canonical domains. Suppose $D = \mathbb{H} \setminus (A_1 \cup \cdots \cup A_n) \in \mathcal{C}Y_n$ is a canonical domain so that

$$A_j = [\mathrm{Re}_-(A_j) + i\,\mathrm{Im}(A_j), \mathrm{Re}_+(A_j) + i\,\mathrm{Im}(A_j)]$$

is a line segment parallel to the real axis. Then the y-component of the ERBM acts exactly the same as a normal Brownian motion killed upon reaching the origin. In particular, if $D \in \mathcal{C}Y$,

$$\tilde{v}_D(z) = v_{\mathbb{H}}(z) = \mathrm{Im}(z).$$

When the ERBM hits an A_j, the x-component of the process changes. For this reason the Poisson kernel $\tilde{H}_D(z, x), \zeta_D(z, x)$ is *not* the same as for $\mathbb{H}$. However, the next lemma shows that $\tilde{H}_D(z, x)$ and $H_D(z, x)$ agree in the $z \to \infty$ limit. In the notation of Proposition 5.2, this lemma can be stated as $c(D, x) = 1$ if $D \in \mathcal{C}Y$.

LEMMA 5.3. *Suppose $D \in \mathcal{C}Y_n$. Then for every $x, x' \in \mathbb{R}$,*

$$\lim_{y \to \infty} \pi\, y\, \tilde{H}_D(x' + iy, x) = 1.$$

Proof. If $y' > \max\{\mathrm{Im}(A_1), \ldots, \mathrm{Im}(A_n)\}$, then it is easy to check that

$$\lim_{y\to\infty} \pi\, y\, \tilde{H}_D(x'+iy, x) = \int_{-\infty}^{\infty} H_D(x'+iy', x)\, dx'.$$

To show that the right-hand side equals one, we can approximate by random walk and take the limit. The random walk result is stated in Lemma 5.1. □

5.6. Loewner equation in canonical domains. Suppose $D = \mathbb{H}\setminus(A_1 \cup \cdots \cup A_n) \in \mathcal{C}Y$ and K_t is an increasing collection of hulls parametrized by capacity starting at $x \in \mathbb{R}$ with corresponding driving function U_t. Let us consider t sufficiently small so that $K_t \subset D$. As before, we let $g_t = g_{K_t}$ be the unique conformal transformation of $\mathbb{H} \setminus K_t$ onto $\mathbb{H}$ with $g_t(z) - z = o(1)$. Then g_t is a conformal transformation of D onto $g_t(D) \in \mathcal{Y}$; however, $g_t(D)$ need not be a canonical domain. By Proposition 3.1, we know that there is a conformal transformation $f_t = f_{g_t(D)}$ taking $g_t(D)$ onto a canonical domain such that $f_t(z) - z = o(1)$ as $z \to \infty$. Let $F_t = f_t \circ g_t$ which is the unique conformal transformation of $D \setminus K_t$ onto a canonical domain $D_s := F_t(D)$ such that $F_t(z) - z = o(1)$ as $z \to \infty$. In this section we give the Loewner equation for the evolution of F_t. The equation and derivation are similar to that used for the chordal equation; in fact, we will give the analogue of (4.6). The only difference is that we replace the Brownian motion with the ERBM. Let $V_t = f_t(U_t)$. This proposition is not new, see, e.g., [3, Theorem 3.1]; however, the identification of $\zeta_D(z,x)$ as the Poisson kernel for ERBM seems new.

COROLLARY 5.4. *F_t satisfies the equation*

$$\dot{F}_t(z) = \zeta_{D_t}(F_t(z), V_t), \qquad F_0(z) = z,$$

where $\zeta_D(z,x)$ denotes the complex Poisson kernel for ERBM as in Proposition 5.2.

Proof. We write

$$F_t(z) = u_t(z) + i\,\tilde{v}_t(z),$$

and we will first derive the equation for $\tilde{v}_t(z)$. For ease assume $t = 0$ and $V_0 = 0$. We have a characterization of $\tilde{v}_t(z)$ as the unique ER-harmonic function in $D \setminus K_t$ with boundary value 0 on $\mathbb{R} \cup K_t$ and such that

$$\tilde{v}_t(z) \sim \mathrm{Im}(z), \qquad z \to \infty.$$

Using this we can see that

$$\tilde{v}_t(z) = \mathrm{Im}(z) - \mathbf{E}^z[\mathrm{Im}(\tilde{B}_{\rho_t})],$$

where

$$\rho_t = \rho_{t,D,K_t} = \inf\{t : \tilde{B}_t \in \mathbb{R} \cup K_t\}.$$

We then prove as in (4.4) that

$$\mathbf{E}^z[\mathrm{Im}(\tilde{B}_{\rho_t})] = [\pi\, \tilde{H}_D(z,0)]\ \mathrm{hcap}(K_t) + o(t) = t\,[\pi\, \tilde{H}_D(z,0)]\ + o(t),$$

where the $o(t)$ term depends on z and the particular choice of K_t. Therefore,

$$\dot{\tilde{v}}_t(z) = -\pi\, \tilde{H}_{D_t}(z,0).$$

Note that this determines F_t up to a real additive constant r_t. By using the condition at infinity we get

$$\dot{F}_0(z) = \tilde{\zeta}_D(z,0). \qquad \square$$

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