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TWO-SIDED ESTIMATES ON THE DENSITY OF BROWNIAN MOTION WITH SINGULAR DRIFT

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ABSTRACT. Let $\mu = (\mu^1, \dots, \mu^d)$ be such that each μ^i is a signed measure on $\mathbf{R}^d$ belonging to the Kato class $\mathbf{K}_{d,1}$. The existence and uniqueness of a continuous Markov process X on $\mathbf{R}^d$, called a Brownian motion with drift μ, was recently established by Bass and Chen. In this paper we study the potential theory of X. We show that X has a continuous density q^μ and that there exist positive constants c_i, $i = 1, \cdots, 9$, such that

$$c_1 e^{-c_2 t} t^{-\frac{d}{2}} e^{-\frac{c_3|x-y|^2}{2t}} \le q^\mu(t,x,y) \le c_4 e^{c_5 t} t^{-\frac{d}{2}} e^{-\frac{c_6|x-y|^2}{2t}}$$

and

$$|\nabla_x q^\mu(t,x,y)| \le c_7 e^{c_8 t} t^{-\frac{d+1}{2}} e^{-\frac{c_9|x-y|^2}{2t}}$$

for all $(t,x,y) \in (0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d$. We further show that, for any bounded $C^{1,1}$ domain D, the density $q^{\mu,D}$ of X^D, the process obtained by killing X upon exiting from D, has the following estimates: for any $T > 0$, there exist positive constants C_i, $i = 1, \cdots, 5$, such that

$$\begin{aligned} C_1(1 \wedge \frac{\rho(x)}{\sqrt{t}})(1 \wedge \frac{\rho(y)}{\sqrt{t}}) t^{-\frac{d}{2}} e^{-\frac{C_2|x-y|^2}{t}} &\le q^{\mu,D}(t,x,y) \\ &\le C_3(1 \wedge \frac{\rho(x)}{\sqrt{t}})(1 \wedge \frac{\rho(y)}{\sqrt{t}}) t^{-\frac{d}{2}} e^{-\frac{C_4|x-y|^2}{t}} \end{aligned}$$

and

$$|\nabla_x q^{\mu,D}(t,x,y)| \le C_5(1 \wedge \frac{\rho(y)}{\sqrt{t}}) t^{-\frac{d+1}{2}} e^{-\frac{C_4|x-y|^2}{t}}$$

for all $(t,x,y) \in (0,T] \times D \times D$, where $\rho(x)$ is the distance between x and ∂D. Using the above estimates, we then prove the parabolic Harnack principle for X and show that the boundary Harnack principle holds for the nonnegative harmonic functions of X. We also identify the Martin boundary of X^D.

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1. Introduction

In this paper we always assume that $d \geq 3$. Suppose $\mu = (\mu^1, \dots, \mu^d)$ is such that each μ^i is a signed measure on $\mathbf{R}^d$ belonging to the Kato class $\mathbf{K}_{d,1}$. (See Definition 2.1 for the precise definition of $\mathbf{K}_{d,1}$.) Informally, a Brownian motion in $\mathbf{R}^d$ with drift μ is a diffusion process in $\mathbf{R}^d$ with generator $\frac{1}{2}\Delta + \mu \cdot \nabla$. When each μ^i is given by $U^i(x)dx$ for some function U^i, a Brownian motion with drift μ is a diffusion in $\mathbf{R}^d$ with generator $\frac{1}{2}\Delta + U \cdot \nabla$ and it is a solution to the SDE

$$dX_t = dW_t + U(X_t) \cdot dt.$$

To give the precise definition of a Brownian motion with drift μ in $\mathbf{K}_{d,1}$, we fix a nonnegative smooth radial function $\varphi(x)$ in $\mathbf{R}^d$ with $\mathrm{supp}[\varphi] \subset B(0,1)$ and $\int \varphi(x)dx = 1$. Let $\varphi_n(x) = 2^{nd}\varphi(2^n x)$. For $1 \leq i \leq d$, define

$$U_n^i(x) = \int \varphi_n(x-y)\mu^i(dy).$$

Put $U_n(x) = (U_n^1(x), \dots, U_n^d(x))$. The following definition is taken from [4].

Definition 1.1. Suppose $\mu = (\mu^1, \dots, \mu^d)$ is such that each μ^i is a signed measure on $\mathbf{R}^d$ belonging to the Kato class $\mathbf{K}_{d,1}$. A Brownian motion with drift μ is a family of probability measures $\{\mathbf{P}_x : x \in \mathbf{R}^d\}$ on $C([0,\infty), \mathbf{R}^d)$, the space of continuous $\mathbf{R}^d$-valued functions on $[0,\infty)$, such that under each $\mathbf{P}_x$ we have

$$X_t = x + W_t + A_t,$$

where

(a) $A_t = \lim_{n\to\infty} \int_0^t U_n(X_s)ds$ uniformly over t in finite intervals, where the convergence is in probability;

(b) there exists a subsequence $\{n_k\}$ such that

$$\sup_k \int_0^t |U_{n_k}(X_s)|ds < \infty$$

almost surely for each $t > 0$;

(c) W_t is a standard Brownian motion in $\mathbf{R}^d$ starting from the origin.

In this paper we will fix a μ in $\mathbf{K}_{d,1}$ and use X to denote a Brownian motion with drift μ. The existence and uniqueness of X were established in [4] by Bass and Chen. In fact, they showed that it is a Feller process.

Bass and Chen raised the following question in [4]: Do the Harnack principle and the boundary Harnack principle hold for the positive harmonic functions of X?

In this paper we will try to answer the above question by studying the densities of X and X^D, the process obtained by killing X upon exiting from a bounded $C^{1,1}$ domain D. We show that X has a continuous density q^μ and that there exist positive constants $c_i = c_i(d,\mu)$, $i = 1, \cdots, 9$, such that

$$c_1 e^{-c_2 t} t^{-\frac{d}{2}} e^{-\frac{c_3|x-y|^2}{2t}} \le q^\mu(t,x,y) \le c_4 e^{c_5 t} t^{-\frac{d}{2}} e^{-\frac{c_6|x-y|^2}{2t}} \tag{1.1}$$

and

$$|\nabla_x q^\mu(t,x,y)| \le c_7 e^{c_8 t} t^{-\frac{d+1}{2}} e^{-\frac{c_9|x-y|^2}{2t}} \tag{1.2}$$

for all $(t,x,y) \in (0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d$. We further show that, for any bounded $C^{1,1}$ domain D, X^D has a continuous density $q^{\mu,D}$ which has the following estimates: for any $T > 0$, there exist positive constants $c_i = c_i(d,\mu,T,D), i = 10, \ldots, 14$, such that

$$\begin{aligned} c_{10}(1 \wedge \frac{\rho(x)}{\sqrt{t}})(1 \wedge \frac{\rho(y)}{\sqrt{t}}) t^{-\frac{d}{2}} e^{-\frac{c_{11}|x-y|^2}{t}} &\le q^{\mu,D}(t,x,y) \\ &\le c_{12}(1 \wedge \frac{\rho(x)}{\sqrt{t}})(1 \wedge \frac{\rho(y)}{\sqrt{t}}) t^{-\frac{d}{2}} e^{-\frac{c_{13}|x-y|^2}{t}} \end{aligned} \tag{1.3}$$

and

$$|\nabla_x q^{\mu,D}(t,x,y)| \le c_{14}(1 \wedge \frac{\rho(y)}{\sqrt{t}}) t^{-\frac{d+1}{2}} e^{-\frac{c_{13}|x-y|^2}{t}} \tag{1.4}$$

for all $(t,x,y) \in (0,T] \times D \times D$, where $\rho(x)$ is the distance between x and ∂D. Then we use these estimates to establish that the Harnack principle and the boundary Harnack principle hold for the positive harmonic functions of X.

The Gaussian estimates (1.1) were first established by Aronson in [2] under the assumption that each μ^i is given by $U^i(x)dx$ with U^i belonging to $L^p(B(0,R))$ for some $p > d$ and $R > 0$ and bounded outside $B(0,R)$. The estimates (1.1) and (1.2) in the case when each μ^i is given by $U^i(x)dx$ with $U^i \in \mathbf{K}_{d,1}$ were stated by Zhang in [28], although they were only proved under the assumption that each U^i is bounded and smooth. In [17], Kondratiev, Liskevich, Sobol and Us gave a proof of (1.1) and (1.2) in the general case when each μ^i is given by $U^i(x)dx$ with $U^i \in \mathbf{K}_{d,1}$, but it seems that the proof there is not quite complete; see Lemma 2.7 there and the argument right before the lemma. In [22], Riahi established the estimates (1.3) and (1.4) in the case when each μ^i is given by $U^i(x)dx$ with $U^i \in \mathbf{K}_{d,1}$. But the proof in [22] also seems to be not quite complete since the first display on page 389 of [22] (attributed to [28]), although it can be easily checked when the U^i's are bounded and smooth (see the beginning of the proof of Theorem 4.2 below), needs justification in the general case.

It is known that, for heat equations on manifolds, one can prove Gaussian heat kernel estimates by checking the volume doubling property and the Poicaré inequality (see [15] and [23]). However, it seems that Brownian motions with singular drifts do not fit into this framework.

Let X^n be a Brownian motion with drift U_n, where U_n is defined in terms of μ as in the beginning of this section. Our strategy for establishing (1.1) and (1.2) for $\mu \in \mathbf{K}_{d,1}$ is as follows. First we use the result of [28] to establish (1.1) and (1.2) for q_n, the density of X^n, with the constants c_i, $1 = 1, \dots, 9$, independent of n and depending on μ only via the rate at which the function

$$r \mapsto \max_{1\le i\le d} \sup_{x\in\mathbf{R}^d} \int_{|x-y|\le r} \frac{|\mu^i|(dy)}{|x-y|^{d-1}}$$

goes to zero as $r \downarrow 0$. Then we show that the densities q_n converge uniformly in each compact subset in $(0,\infty)\times\mathbf{R}^d\times\mathbf{R}^d$. The uniform convergence of the transition densities shows that the approximation scheme proposed above is well-suited for the purpose of this paper.

The strategy for establishing (1.3) and (1.4) for $\mu \in \mathbf{K}_{d,1}$ is similar. The difference is that we have to first establish them for q_n^D, the density of $X^{n,D}$, and then establish the uniform convergence of q_n^D on arbitrary compact subsets of $(0,\infty)\times D\times D$.

The uniform convergences of q_n and q_n^D are essential for our approach to establish (1.1), (1.2), (1.3) and (1.4) and they can be regarded as stability results for the fundamental solutions under perturbations. Most of Sections 3 and 4 are devoted to proving these uniform convergences.

The content of this paper is organized as follows. In Section 2, we first recall the definition of the Kato class and discuss some basic properties. In Section 3, we establish the two-sided estimates on the density of X. In Section 4, we deal with estimates of the density for X^D, the process obtained by killing X upon exiting from D. We establish two-sided estimates on the density of X^D when D is a bounded $C^{1,1}$ domain D. In Section 5, we prove that the parabolic Harnack principle is valid for positive harmonic functions of X by using estimates obtained in Section 4. In Section 6, we establish two-sided estimates on the Green function of X^D and show that a boundary Harnack principle is valid for positive harmonic functions of X in bounded $C^{1,1}$ domains. In the last section we show that, when D is a bounded $C^{1,1}$ domain, the Martin boundary and minimal Martin boundary of X^D coincide with the Euclidean boundary.

In this paper we will use the following convention. The values of the constants $M_1, M_2, \cdots$ will remain the same throughout this paper, while the values of the constants $C_1, C_2, \cdots$ might change from one appearance to another. The labeling of the constants $C_1, C_2, \cdots$ starts anew in the statement of each result.

Recall that a bounded domain D in $\mathbf{R}^d$ is said to be a $C^{1,1}$ domain if there is a localization radius $r_0 > 0$ and a constant $\Lambda > 0$ such that for every $Q \in \partial D$, there is a $C^{1,1}$-function $\phi = \phi_Q : \mathbf{R}^{d-1} \to \mathbf{R}$ satisfying $\phi(0) = \nabla\phi(0) = 0$, $\|\nabla\phi\|_\infty \le \Lambda$, $|\nabla\phi(x) - \nabla\phi(z)| \le \Lambda|x-z|$, and an orthonormal coordinate system $y = (y_1, \cdots, y_{d-1}, y_d) := (\tilde{y}, y_d)$ such that $B(Q, r_0) \cap D = B(Q, r_0) \cap$

$\{y : y_d > \phi(\tilde{y})\}$. The pair (r_0, Λ) is called the characteristics of the $C^{1,1}$ domain D.

2. Preliminaries

First we recall the definition of the Kato class $\mathbf{K}_{d,\alpha}$ for $\alpha \in (0, 2]$, although we will only use $\mathbf{K}_{d,1}$ in this paper.

For any function f on $\mathbf{R}^d$ and $r > 0$, we define

$$M_f^\alpha(r) = \sup_{x\in\mathbf{R}^d} \int_{|x-y|\le r} \frac{|f|(y)dy}{|x-y|^{d-\alpha}}, \qquad 0 < \alpha \le 2.$$

In this paper, by a signed measure we mean the difference of two nonnegative measures at most one of which can have infinite total mass. For any signed measure ν on $\mathbf{R}^d$, we use ν^+ and ν^- to denote its positive and negative parts, and $|\nu| = \nu^+ + \nu^-$ its total variation. For any signed measure ν on $\mathbf{R}^d$ and any $r > 0$, we define

$$M_\nu^\alpha(r) = \sup_{x\in\mathbf{R}^d} \int_{|x-y|\le r} \frac{|\nu|(dy)}{|x-y|^{d-\alpha}}, \qquad 0 < \alpha \le 2.$$

Definition 2.1. Let $0 < \alpha \le 2$. We say that a function f on $\mathbf{R}^d$ belongs to the Kato class $\mathbf{K}_{d,\alpha}$ if $\lim_{r\downarrow 0} M_f^\alpha(r) = 0$. We say that a signed Radon measure ν on $\mathbf{R}^d$ belongs to the Kato class $\mathbf{K}_{d,\alpha}$ if $\lim_{r\downarrow 0} M_\nu^\alpha(r) = 0$. We say that a d-dimensional vector valued function $V = (V^1, \cdots, V^d)$ on $\mathbf{R}^d$ belongs to the Kato class $\mathbf{K}_{d,\alpha}$ if each V^i belongs to the Kato class $\mathbf{K}_{d,\alpha}$. We say that a d-dimensional vector valued signed Radon measure $\mu = (\mu^1, \cdots, \mu^d)$ on $\mathbf{R}^d$ belongs to the Kato class $\mathbf{K}_{d,\alpha}$ if each μ^i belongs to the Kato class $\mathbf{K}_{d,\alpha}$.

Rigorously speaking a function f in $\mathbf{K}_{d,\alpha}$ may not give rise to a signed measure ν in $\mathbf{K}_{d,\alpha}$ since it may not give rise to a signed measure at all. However, for the sake of simplicity we use the convention that whenever we write that a signed measure ν belongs to $\mathbf{K}_{d,\alpha}$ we are implicitly assuming that we are covering the case of all the functions in $\mathbf{K}_{d,\alpha}$ as well.

It is easy to see that if $\nu \in \mathbf{K}_{d,\alpha}$, then for any $r > 0$ we have $M_\nu^\alpha(r) < \infty$. In fact, by definition we know that there exists $r_0 > 0$ such that $M_\nu^\alpha(r_0) < \infty$, thus

$$\sup_{x\in\mathbf{R}^d} |\nu|(\overline{B(x, r_0)}) \le \sup_{x\in\mathbf{R}^d} \int_{|x-y|\le r_0} \frac{r_0^{d-\alpha}|\nu|(dy)}{|x-y|^{d-\alpha}} = r_0^{d-\alpha} M_\nu^\alpha(r_0).$$

Using this one can easily get that for any $r > 0$,

$$\sup_{x\in\mathbf{R}^d} |\nu|(\overline{B(x, r)}) < \infty. \tag{2.1}$$

Thus

$$\sup_{x\in\mathbf{R}^d}\int_{|x-y|\le r}\frac{|\nu|(dy)}{|x-y|^{d-\alpha}}$$
$$\le \sup_{x\in\mathbf{R}^d}\int_{|x-y|\le r_0}\frac{|\nu|(dy)}{|x-y|^{d-\alpha}}+r_0^{\alpha-d}\sup_{x\in\mathbf{R}^d}|\nu|(\overline{B(x,r)})<\infty$$

and

$$\sup_{x\in\mathbf{R}^d}|\nu|(\overline{B(x,r)})\le \sup_{x\in\mathbf{R}^d}\int_{|x-y|\le r}\frac{r^{d-\alpha}|\nu|(dy)}{|x-y|^{d-\alpha}}=r^{d-\alpha}M_\nu^\alpha(r). \tag{2.2}$$

In this paper we will use $p(t,x,y)$ to denote the transition density of a standard Brownian motion in $\mathbf{R}^d$, that is,

$$p(t,x,y)=(2\pi t)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{2t}},\qquad (t,x,y)\in(0,\infty)\times\mathbf{R}^d\times\mathbf{R}^d.$$

For any function f on $\mathbf{R}^d$ and $t>0$, we define

$$N_f^\alpha(t)=\sup_{x\in\mathbf{R}^d}\int_0^t\int_{\mathbf{R}^d}s^{\frac{\alpha}{2}-1}p(s,x,y)|f(y)|dyds,\qquad 0<\alpha\le 2.$$

For any signed measure ν on $\mathbf{R}^d$ and $t>0$, we define

$$N_\nu^\alpha(t)=\sup_{x\in\mathbf{R}^d}\int_0^t\int_{\mathbf{R}^d}s^{\frac{\alpha}{2}-1}p(s,x,y)|\nu|(dy)ds,\qquad 0<\alpha\le 2.$$

The next two propositions are variations of Lemmas 2.1 and 2.2 in [5]. For the convenience of our readers we include the proofs of these results.

Proposition 2.2. *Suppose that ν is a signed measure on $\mathbf{R}^d$ and $0<\alpha\le 2$. If $\nu\in\mathbf{K}_{d,\alpha}$, then for any $t>0$,*

$$\sup_{x\in\mathbf{R}^d}\int_{\mathbf{R}^d}e^{-\frac{|x-y|^2}{2t}}|\nu|(dy)<\infty, \tag{2.3}$$

and

$$\lim_{R\uparrow\infty}\sup_{x\in\mathbf{R}^d}\int_{|x-y|>R}e^{-\frac{|x-y|^2}{2t}}|\nu|(dy)=0. \tag{2.4}$$

Moreover, there is a constant $L_1=L_1(d,\alpha)$ depending only on d and α with the following property: for every $r>0$, there exists a constant $L_2=L_2(d,\alpha,r)$ such that for any $t\in(0,1)$,

$$N_\nu^\alpha(t)\le (t\,L_2(d,\alpha,r)+L_1(d,\alpha))\,M_\nu^\alpha(r). \tag{2.5}$$

Proof. By using an argument similar to that of the proof of Lemma 1.1 in [24], one can easily prove (2.3) and (2.4). We skip the details. We now concentrate on proving the last assertion of this proposition. For any $r > 0$ and $s \in (0,1)$, we have

$$\begin{aligned}
&\int_0^s \int_{|x-y|>r} u^{\frac{\alpha}{2}-1} p(u,x,y)|\nu|(dy)du \\
&\quad \le s \sup_{u\in(0,s)} \int_{|x-y|>r} u^{\frac{\alpha}{2}-1} p(u,x,y)|\nu|(dy) \\
&\quad \le s\,(2\pi)^{-\frac{d}{2}} \sup_{u\in(0,s)} \left(u^{-\frac{d+2-\alpha}{2}} e^{-\frac{r^2}{4u}}\right) \sup_{x\in\mathbf{R}^d} \int_{\mathbf{R}^d} e^{-\frac{|x-y|^2}{4}} |\nu|(dy) \\
&\quad \le s\,(2\pi)^{-\frac{d}{2}} \sup_{u\in(0,1)} \left(u^{-\frac{d+2-\alpha}{2}} e^{-\frac{r^2}{4u}}\right) c_1(2,d,r,\alpha)\, M_\nu^\alpha(r) \\
&\quad := s\, L_2(d,\alpha,r)\, M_\nu^\alpha(r).
\end{aligned}$$

On the other hand, using

$$\int_0^t s^{\frac{\alpha}{2}-1} p(s,x,y)ds = 2^{\frac{d-\alpha}{2}} (2\pi)^{-\frac{d}{2}} |x-y|^{-d+\alpha} \int_{\frac{|x-y|^2}{2t}}^\infty u^{\frac{d-2-\alpha}{2}} e^{-u} du, \tag{2.6}$$

we get

$$\begin{aligned}
&\int_0^t \int_{|x-y|\le r} s^{\frac{\alpha}{2}-1} p(s,x,y)|\nu|(dy)ds \\
&\qquad = \int_{|x-y|\le r} \int_0^t s^{\frac{\alpha}{2}-1} p(s,x,y)ds|\nu|(dy) \\
&\qquad = \int_{|x-y|\le r} 2^{\frac{d-\alpha}{2}} (2\pi)^{-\frac{d}{2}} |x-y|^{-d+\alpha} \int_{\frac{|x-y|^2}{2t}}^\infty u^{\frac{d-2-\alpha}{2}} e^{-u} du|\nu|(dy) \\
&\qquad \le 2^{\frac{d-\alpha}{2}} (2\pi)^{-\frac{d}{2}} \int_0^\infty u^{\frac{d-2-\alpha}{2}} e^{-u} du \int_{|x-y|\le r} |x-y|^{-d+\alpha} |\nu|(dy) \\
&\qquad := L_1(d,\alpha) \int_{|x-y|\le r} |x-y|^{-d+\alpha} |\nu|(dy).
\end{aligned}$$

Therefore for every $s \in (0,1)$,

$$N_\nu^\alpha(s) \le (s\, L_2(d,\alpha,r) + L_1(d,\alpha))\, M_\nu^\alpha(r). \qquad \square$$

PROPOSITION 2.3. *Suppose that ν is a signed measure on $\mathbf{R}^d$. Then for any $0 < \alpha \le 2$, $\nu \in \mathbf{K}_{d,\alpha}$ if and only if $\lim_{t\to 0} N_\nu^\alpha(t) = 0$.*

Proof. By Proposition 2.2, if $\nu \in \mathbf{K}_{d,\alpha}$, then $\lim_{t\to 0} N^\alpha_\nu(t) = 0$. For the converse, by (2.6) we have

$$\begin{aligned}
&\int_0^{r^2}\int_{\mathbf{R}^d} s^{\frac{\alpha}{2}-1} p(s,x,y)|\nu|(dy)ds \\
&\qquad = \int_{\mathbf{R}^d}\int_0^{r^2} s^{\frac{\alpha}{2}-1} p(s,x,y)ds|\nu|(dy) \\
&\qquad \ge \int_{|x-y|\le r} 2^{\frac{d-\alpha}{2}}(2\pi)^{-\frac{d}{2}}|x-y|^{-d+\alpha}\int_{\frac{|x-y|^2}{2r^2}}^{\infty} u^{\frac{d-2-\alpha}{2}} e^{-u}du|\nu|(dy) \\
&\qquad \ge 2^{\frac{d-\alpha}{2}}(2\pi)^{-\frac{d}{2}}\int_{\frac{1}{2}}^{\infty} u^{\frac{d-2-\alpha}{2}} e^{-u}du \int_{|x-y|\le r}|x-y|^{-d+\alpha}|\nu|(dy). \qquad \square
\end{aligned}$$

By combining Proposition 2.3 above and Theorem A in [28] we get the following result.

THEOREM 2.4. *Suppose that $U(x) = (U^1(x), \ldots, U^d(x))$ is such that each component U^i is bounded. Then the Brownian motion with drift U has a transition density $q^U(t,x,y)$. q^U is the fundamental solution of the equation*

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta_x u(t,x) + U(x)\cdot\nabla_x u(t,x),$$

and is also called the heat kernel for $\frac{1}{2}\triangle + U\cdot\nabla$. There exist positive constants $C_j, 1 \le j \le 9$, depending on U only via the rate at which $\max_{1\le i\le d} M^1_{U^i}(r)$ goes to zero, such that

$$C_1 e^{-C_2 t} t^{-\frac{d}{2}} e^{-\frac{C_3|x-y|^2}{2t}} \;\le\; q^U(t,x,y) \le C_4 e^{C_5 t} t^{-\frac{d}{2}} e^{-\frac{C_6|x-y|^2}{2t}} \tag{2.7}$$

and

$$|\nabla_x q^U(t,x,y)| \;\le\; C_7 e^{C_8 t} t^{-\frac{d+1}{2}} e^{-\frac{C_9|x-y|^2}{2t}}$$

for all $(t,x,y) \in (0,\infty)\times\mathbf{R}^d\times\mathbf{R}^d$.

The meaning of the phrase "depending on μ only via the rate at which $\max_{1\le i\le d} M^1_{\mu^i}(r)$ goes to zero" is that if $w(r)$ is a decreasing function on $(0,\infty)$ with $\lim_{r\to 0} w(r) = 0$, then the statement is true for any signed measure μ with

$$\max_{1\le i\le d} M_{\mu^i}(r) \le w(r), \quad r > 0.$$

3. Two-sided estimates for the density of X

Throughout this paper, we assume that $\mu = (\mu^1, \cdots, \mu^d)$ is such that each μ^i belongs to $\mathbf{K}_{d,1}$ and that X is a Brownian motion with drift μ. In this section we shall establish two-sided estimates for the density of X.

Let $\varphi(x)$ be a nonnegative smooth radial function in $\mathbf{R}^d$ such that $\mathrm{supp}[\varphi] \subset B(0,1)$ and $\int \varphi(x)dx = 1$. We fix φ throughout this paper. Let $\varphi_n(x) := 2^{nd}\varphi(2^n x)$. For signed Radon measures μ_i on $\mathbf{R}^d$ with $1 \le i \le d$, define

$$U_n^i(x) = \int \varphi_n(x-y)\mu^i(dy) \tag{3.1}$$

and set

$$\mu_n^i(dx) = U_n^i(x)dx. \tag{3.2}$$

We write $U_n(x)$ for $(U_n^1(x), \cdots, U_n^d(x))$. It follows from (2.2) that when $\mu^i \in \mathbf{K}_{d,1}, i = 1, \dots, d$, each U_n^i is a bounded and smooth function on $\mathbf{R}^d$.

Lemma 3.1. *Each U_n^i belongs to the Kato class $\mathbf{K}_{d,1}$ and*

$$M^1_{U_n^i}(r) \le M^1_{\mu^i}(r) \qquad r > 0,\ 1 \le i \le d. \tag{3.3}$$

Proof. See the proof of Proposition 3.6 in [4]. □

Using Theorem 2.4 and Lemma 3.1 we can see that, for each n, the Brownian motion with drift U_n has a density q_n and there exist positive constants M_i, $i = 1, \dots, 9$, depending on μ only via the rate at which $\max_{1\le i\le d} M^1_{\mu^i}(t)$ goes to zero, such that

$$M_1 e^{-M_2 t} t^{-\frac{d}{2}} e^{-\frac{M_3|x-y|^2}{2t}} \ \le\ q_n(t,x,y)\ \le\ M_4 e^{M_5 t} t^{-\frac{d}{2}} e^{-\frac{M_6|x-y|^2}{2t}} \tag{3.4}$$

and

$$|\nabla_x q_n(t,x,y)| \ \le\ M_7\, e^{M_8 t}\, t^{-\frac{d+1}{2}} e^{-\frac{M_9|x-y|^2}{2t}} \tag{3.5}$$

for all $(t,x,y) \in (0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d$ and $n \ge 1$.

The next lemma is a consequence of (2.2), which can be proved easily by a covering argument.

Lemma 3.2. *For any bounded $(d-1)$-rectifiable subset A of $\mathbf{R}^d$, we have*

$$\sum_{i=1}^d \sup_{x\in\mathbf{R}^d} |\mu^i|(A) \ =\ 0.$$

Proof. Let $N(A,\varepsilon)$ be the smallest number of ε-balls needed to cover A, i.e.,

$$N(A,\varepsilon) := \min\left\{ k : A \subset \bigcup_{j=1}^k B(x_j,\varepsilon) \quad \text{for some } x_j \in \mathbf{R}^d \right\}.$$

So for each $\varepsilon > 0$, there exists a sequence $\{x_j\}_{1\le j\le N(A,\varepsilon)}$ such that

$$|\mu^i|(A) \ \le\ \sum_{j=1}^{N(A,\varepsilon)} |\mu^i|(B(x_j,\varepsilon)), \qquad 1 \le i \le d.$$

Using (2.2), we get

$$|\mu^i|(A) \;\le\; \varepsilon^{d-1}\, N(A,\varepsilon)\, M^1_{\mu_i}(\varepsilon), \qquad 1 \le i \le d. \tag{3.6}$$

Let

$$A(\varepsilon) := \left\{x \in \mathbf{R}^d : \operatorname{dist}(x, A) \le \varepsilon\right\}.$$

It is well-known (see, for instance, (5.4) and (5.6) in [20]) that there exists a positive number $c_1 = c_1(d)$ such that

$$\varepsilon^d N(A,\varepsilon) \;\le\; c_1\, \mathcal{L}^d(A(\varepsilon)), \tag{3.7}$$

where $\mathcal{L}^d$ is d-dimensional Lebesgue measure. Since A is $(d-1)$-rectifiable, by Theorem 3.2.39 in [11], there exists a nonnegative real number $c_2 = c_2(A)$ such that

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\mathcal{L}^d(A(\varepsilon)) = c_2 < \infty. \tag{3.8}$$

Thus combining (3.6)–(3.8), we have for any i

$$|\mu^i|(A) \;\le\; c_1 \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\mathcal{L}^d(A(\varepsilon)) M^1_{\mu_i}(\varepsilon) \;=\; c_1 c_2 \lim_{\varepsilon\downarrow 0} M^1_{\mu_i}(\varepsilon) = 0. \qquad \square$$

LEMMA 3.3. *Let $0 < T_0 < T_1 < \infty$. Suppose K_1 is a compact subset of $\mathbf{R}^d$ and D_1 is a bounded domain with smooth boundary ∂D_1. Then for any continuous function f on $[T_0, T_1] \times K_1 \times K_1 \times \overline{D_1}$, we have*

$$\lim_{n\uparrow\infty} \sup_{(t,x,y)\in[T_1,T_2]\times K_1\times K_1} \left| \int_{D_1} f(t,x,y,z)(\mu^i_n - \mu^i)(dz) \right| \;=\; 0, \qquad 1 \le i \le d.$$

Proof. Fix an i and extend $f(t,x,y,\,\cdot\,)$ to be zero off $\overline{D_1}$. Let

$$A_n \;:=\; \left\{w \in \mathbf{R}^d : \operatorname{dist}(\partial D_1, w) \le 2^{-n}\right\}.$$

By Lemma 3.2, we have

$$\lim_{n\to\infty} |\mu^i|(A_n) \;=\; |\mu^i|(\partial D_1) \;=\; 0.$$

Given $\varepsilon > 0$, choose a large positive integer n_1 such that for every $n \ge n_1$,

$$\left(\sup_{(t,x,y,z)\in[T_0,T_1]\times K_1\times K_1\times\overline{D_1}} |f(t,x,y,z)| \right) |\mu^i|(A_n) \;<\; \frac{\varepsilon}{4}. \tag{3.9}$$

By Fubini's theorem, we have for every $(t,x,y) \in [T_1,T_2] \times K_1 \times K_1$ and every $n \ge n_1$,

$$
\begin{aligned}
&\left|\int_{D_1} f(t,x,y,z)(\mu_n^i-\mu^i)(dz)\right| \\
&\quad = \left|\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} \varphi_n(w-z)\mu^i(dw)f(t,x,y,z)dz - \int_{\mathbf{R}^d} f(t,x,y,w)\mu^i(dw)\right| \\
&\quad = \left|\int_{\mathbf{R}^d} ((\varphi_n * f(t,x,y,\,\cdot\,))(w) - f(t,x,y,w))\mu^i(dw)\right| \\
&\quad = \left|\int_{D_1\cup A_n} ((\varphi_n * f(t,x,y,\,\cdot\,))(w) - f(t,x,y,w))\mu^i(dw)\right| \\
&\quad \le \left|\int_{D_1\setminus A_n} ((\varphi_n * f(t,x,y,\,\cdot\,))(w) - f(t,x,y,w))\mu^i(dw)\right| + \frac{\varepsilon}{2}.
\end{aligned}
$$

In the last inequality above, we used (3.9). Note that from (2.1), we have $|\mu^i|(D_1) < c_1 < \infty$. So by taking the supremum over $(t,x,y) \in [T_0,T_1] \times K_1 \times K_1$, we get

$$
\begin{aligned}
&\sup_{\substack{t\in[T_0,T_1],\\(x,y)\in K_1\times K_1}} \left|\int_{D_1} f(t,x,y,z)(\mu_n^i-\mu^i)(dz)\right| \\
&\quad \le |\mu^i|(D_1) \sup_{\substack{t\in[T_0,T_1],\\(x,y)\in K_1\times K_1,\\ w\in D_1\setminus A_n}} |(\varphi_n * f(t,x,y,\,\cdot\,))(w) - f(t,x,y,w)| + \frac{\varepsilon}{2} \\
&\quad \le c_1 \sup_{\substack{t\in[T_0,T_1],\\(x,y)\in K_1\times K_1,\\ w\in D_1\setminus A_n}} \left|\int_{B(0,1)} \varphi(z)(f(t,x,y,w+2^{-n}z) - f(t,x,y,w))dz\right| + \frac{\varepsilon}{2} \\
&\quad \le c_1 \sup_{\substack{t\in[T_0,T_1],\\(x,y)\in K_1\times K_1,\\ w\in D_1\setminus A_n, |z|<2^{-n}}} |(f(t,x,y,z+w) - f(t,x,y,w))| + \frac{\varepsilon}{2}.
\end{aligned}
$$

The first term in the last line above goes to zero as $n \to \infty$ by the uniform continuity of f. $\square$

LEMMA 3.4. *Suppose that R is a positive number. Then for any $a > 0$, there exist positive constants C_1 and C_2 depending only on a and d such that for any measure ν on $\mathbf{R}^d$ and $t > 0$,*

$$
\begin{aligned}
\text{(3.10)}\qquad &\sup_{|x|,|y|<R/2} \int_0^t \int_{|x-z|\ge 4R} s^{-\frac{d}{2}} e^{-\frac{a|x-z|^2}{2s}} (t-s)^{-\frac{d+1}{2}} e^{-\frac{a|z-y|^2}{t-s}} \nu(dz)ds \\
&\qquad \le C_1 t^{-\frac{d}{2}} \sup_{|u|<R/2} \int_0^t \int_{|u-z|\ge 3R} s^{-\frac{d+1}{2}} e^{-\frac{a|u-z|^2}{4s}} \nu(dz)ds
\end{aligned}
$$

and

$$\sup_{|x|,|y|<R/2}\int_0^t\int_{|x-z|\geq 4R} s^{-\frac{d+1}{2}} e^{-\frac{a|x-z|^2}{2s}} (t-s)^{-\frac{d+1}{2}} e^{-\frac{a|z-y|^2}{t-s}} \nu(dz)ds \tag{3.11}$$
$$\leq C_2 t^{-\frac{d+1}{2}} \sup_{|u|<R/2}\int_0^t\int_{|u-z|\geq 3R} s^{-\frac{d+1}{2}} e^{-\frac{a|u-z|^2}{4s}} \nu(dz)ds.$$

Proof. One can follow the proof of Lemma 3.1 of [28] and show that for any $x, y \in \mathbf{R}^d$,

$$\int_0^t\int_{|x-z|\geq 4R} s^{-\frac{d}{2}} e^{-\frac{a|x-z|^2}{2s}} (t-s)^{-\frac{d+1}{2}} e^{-\frac{a|z-y|^2}{t-s}} \nu(dz)ds \tag{3.12}$$
$$\leq C_0\, t^{-\frac{d}{2}} e^{-\frac{a|x-y|^2}{2t}} \left(\int_0^t\int_{|x-z|\geq 4R} s^{-\frac{d+1}{2}} e^{-\frac{a|x-z|^2}{4s}} \nu(dz)ds \right.$$
$$\left. + \int_0^t\int_{|x-z|\geq 4R} s^{-\frac{d+1}{2}} e^{-\frac{a|y-z|^2}{4s}} \nu(dz)ds \right).$$

For $x, y \in B(0, R/2)$, if z satisfies $|x-z| > 4R$, we have $|y-z| > |x-z| - |x-y| > 4R - |x-y| > 3R$. Therefore

$$\int_0^t\int_{|x-z|\geq 4R} s^{-\frac{d+1}{2}} e^{-\frac{a|y-z|^2}{4s}} \nu(dz)ds$$
$$\leq \int_0^t\int_{|y-z|\geq 3R} s^{-\frac{d+1}{2}} e^{-\frac{a|y-z|^2}{4s}} \nu(dz)ds.$$

Now (3.10) follows by taking the supremum over $|x|, |y| < \frac{1}{2}R$. (3.11) can be proved similarly. □

LEMMA 3.5. *For any $\delta > 0$, there exists a constant $C_1 = C_1(d, \delta)$ depending only on d and δ such that, for every $t > \delta$, $x \in \mathbf{R}^d$, $R > 1$, $n \geq 1$, $M > 0$ and $1 \leq i \leq d$*

$$\int_\delta^t\int_{|y-x|\geq 3R} s^{-\frac{d+1}{2}} e^{-\frac{M|y-z|^2}{4s}} (|U_n^i(y)|dy + |\mu^i|(dz))ds$$
$$\leq C_1\, t \int_{|z-x|\geq R} e^{-\frac{M|x-z|^2}{4t}} |\mu^i|(dz).$$

Proof. For any $x \in \mathbf{R}^d$,

$$\int_\delta^t\int_{|y-x|\geq 3R} s^{-\frac{d+1}{2}} e^{-\frac{M|y-z|^2}{4s}} (|U_n^i(y)|dy + |\mu^i|(dz))ds$$
$$\leq \delta^{-\frac{d+1}{2}}\, t \int_{|y-x|\geq 3R} e^{-\frac{M|x-y|^2}{4t}} (|U_n^i(y)|dy + |\mu^i|(dz))dy.$$

Since φ is a nonnegative radial function supported by $B(0,1)$, we have for any $x \in \mathbf{R}^d$

$$\begin{aligned}
&\int_{|y-x|\geq 3R} e^{-\frac{M|x-y|^2}{4t}} |U_n^i(y)| dy \\
&\qquad \leq \int_{|y-x|\geq 3R} e^{-\frac{M|x-y|^2}{4t}} \int_{|y-z|\leq 1} \varphi_n(y-z)|\mu^i|(dz)dy \\
&\qquad \leq \int_{|z-x|\geq 2R} \int_{|y-z|\leq 1} \varphi_n(y-z) e^{-\frac{M|x-y|^2}{4t}} dy|\mu^i|(dz).
\end{aligned}$$

Using the change of variable $y = z - w$ in the inner integral we get that

$$\begin{aligned}
&\int_{|z-x|\geq 2R} \int_{|y-z|\leq 1} \varphi_n(y-z) e^{-\frac{M|x-y|^2}{4t}} dy|\mu^i|(dz) \\
&\quad = \int_{|z-x|\geq 2R} \int_{|w|\leq 1} \varphi_n(w) e^{-\frac{M|x-z+w|^2}{4t}} dw|\mu^i|(dz) \\
&\quad = \int_{|w|\leq 1} \varphi_n(w) \int_{|z-x|\geq 2R} e^{-\frac{M|x-z+w|^2}{4t}} |\mu^i|(dz)dw \\
&\quad \leq \sup_{|w|\leq 1} \int_{|z-x|\geq 2R} e^{-\frac{M|x-z+w|^2}{4t}} |\mu^i|(dz) \\
&\quad \leq \int_{|u-x|\geq R} e^{-\frac{M|x-u|^2}{4t}} |\mu^i|(du).
\end{aligned}$$

Therefore the lemma is valid with $C_1 = 2\delta^{-\frac{d+1}{2}}$. □

It is easy to check that, for any positive integer n, the function defined by

$$\tilde{q}_n(t,x,y) := p(t,x,y) + \int_0^t \int_{\mathbf{R}^d} q_n(s,x,z) U_n(z) \cdot \nabla_z p(t-s,z,y) dz ds$$

is a fundamental solution of the equation

$$\frac{\partial}{\partial t} u(t,x) = \frac{1}{2}\Delta_x u(t,x) + U_n(x) \cdot \nabla_x u(t,x).$$

Thus by Theorem 5 of [2] we get that for every $(t,x,y) \in (0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d$,

$$q_n(t,x,y) = p(t,x,y) + \int_0^t \int_{\mathbf{R}^d} q_n(s,x,z) U_n(z) \cdot \nabla_z p(t-s,z,y) dz ds. \tag{3.13}$$

We define $I_k^n(t,x,y)$ recursively for $k \geq 0$ and $(t,x,y) \in (0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d$:

$$\begin{aligned}
I_0^n(t,x,y) &:= p(t,x,y), \\
I_{k+1}^n(t,x,y) &:= \int_0^t \int_{\mathbf{R}^d} I_k^n(s,x,z) U_n(z) \cdot \nabla_z p(t-s,z,y) dz ds.
\end{aligned}$$

Then iterating (3.13) gives

$$q_n(t,x,y) = \sum_{k=0}^{\infty} I_k^n(t,x,y), \quad (t,x,y) \in (0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d. \tag{3.14}$$

It is easy to see that there exists $A > 0$ such that

$$|\nabla_x p(t,x,y)| \le At^{-\frac{d+1}{2}} e^{-\frac{|x-y|^2}{4t}}, \quad (t,x,y) \in (0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d. \tag{3.15}$$

By taking A larger if necessary, we may assume that

$$p(t,x,y) \le At^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{4t}}, \quad (t,x,y) \in (0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d. \tag{3.16}$$

We claim that there exist positive constants T_1 and M_0 such that for $k = 0, 1, \cdots$ and $(t,x,y) \in (0,T_1] \times \mathbf{R}^d \times \mathbf{R}^d$

$$|I_k^n(t,x,y)| \le 2^{-k-1} M_0 t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{8t}}. \tag{3.17}$$

In fact, by Lemma 3.1 in [28], (3.15) and (3.16), there exists a positive constant C_1 depending only on d such that

$$|I_1^n(t,x,y)| \le C_1 t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{8t}} \left(A \sum_{i=1}^{d} N^1_{U_n^i}(8t) \right).$$

Now suppose that

$$|I_k^n(t,x,y)| \le C_1 t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{8t}} \left(A \sum_{i=1}^{d} N^1_{U_n^i}(8t) \right)^k$$

is true. Then by Lemma 3.1 in [28] and (3.15), we have

$$\begin{aligned}
&|I_{k+1}^n(t,x,y)| \\
&\quad \le \int_0^t \int_{\mathbf{R}^d} |I_k^n(s,x,z)| |U_n(z) \cdot \nabla_z p(t-s,z,y)| dz ds \\
&\quad \le \int_0^t \int_{\mathbf{R}^d} C_1 s^{-\frac{d}{2}} e^{-\frac{|x-z|^2}{8s}} \left(A \sum_{i=1}^{d} N^1_{U_n^i}(8s) \right)^k \times \\
&\qquad\qquad \times \sum_{i=1}^{d} A(t-s)^{-\frac{d+1}{2}} e^{-\frac{|z-y|^2}{4(t-s)}} |U_n^i(z)| dz ds \\
&\quad \le C_1 t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{8t}} \left(A \sum_{i=1}^{d} N^1_{U_n^i}(8t) \right)^{k+1}.
\end{aligned}$$

Choose $T_1 > 0$ small so that

$$A \sup_{n \ge 1} \sum_{i=1}^{d} N^1_{U_n^i}(8T_1) < \frac{1}{2}. \tag{3.18}$$

By (2.5) and Lemma 3.1, T_1 depends on μ^i only via the rate at which $\max_{1\le i\le d} M^1_{\mu^i}(r)$ goes to zero. So the claim is proved. We will fix this constant T_1 until the end of this section.

Now we define $I_k(t,x,y)$ recursively for $k\ge 0$ and $(t,x,y)\in(0,T_1]\times\mathbf{R}^d\times\mathbf{R}^d$:

$$I_0(t,x,y) := p(t,x,y),$$
$$I_{k+1}(t,x,y) := \int_0^t\int_{\mathbf{R}^d} I_k(s,x,z)\nabla_z p(t-s,z,y)\cdot\mu(dz)ds.$$

Let

$$q(t,x,y) := \sum_{k=0}^{\infty} I_k(t,x,y), \quad (t,x,y)\in(0,T_1]\times\mathbf{R}^d\times\mathbf{R}^d. \tag{3.19}$$

Using a similar argument as in the previous paragraph, we also have that for $k=0,1,\cdots$ and $(t,x,y)\in(0,T_1]\times\mathbf{R}^d\times\mathbf{R}^d$

$$|I_k(t,x,y)| \le 2^{-k-1}M_0 t^{-\frac{d}{2}}\, e^{-\frac{|x-y|^2}{8t}}. \tag{3.20}$$

So $\sum_{k=0}^{\infty} I_k(t,x,y)$ converges absolutely on $(t,x,y)\in(0,T_1]\times\mathbf{R}^d\times\mathbf{R}^d$ and converges uniformly on $(t,x,y)\in[T_0,T_1]\times\mathbf{R}^d\times\mathbf{R}^d$ for every $0<T_0<T_1$, which implies that $q(t,x,y)$ is jointly continuous on $(0,T_1]\times\mathbf{R}^d\times\mathbf{R}^d$. Moreover

$$q(t,x,y) \le M_0 t^{-\frac{d}{2}}\, e^{-\frac{|x-y|^2}{8t}}, \qquad (t,x,y)\in(0,T_1]\times\mathbf{R}^d\times\mathbf{R}^d. \tag{3.21}$$

We will show that q_n converges uniformly on each compact subset of $(0,\infty)\times\mathbf{R}^d\times\mathbf{R}^d$ through several lemmas.

Lemma 3.6. *For any compact subsets K_1,K_2 of $\mathbf{R}^d$ and $T_0\in(0,T_1]$, we have*

$$\lim_{n\to\infty}\sup_{(t,x,y)\in[T_0,T_1]\times K_1\times K_2} |I_1^n(t,x,y)-I_1(t,x,y)| = 0.$$

Proof. Without loss of generality we may assume that $T_0<2$. We will prove this lemma for the case $K_1=K_2=\overline{B(0,r)}$ only. For any given $\varepsilon>0$, we first choose $r_1>0$ small such that

$$\sum_{i=1}^{d} M^1_{\mu^i}(r_1) < \frac{\varepsilon}{8\,L_1(d,1)}, \tag{3.22}$$

where $L_1(d,1)$ is the constant from Proposition 2.2. Then, by Proposition 2.2 and Lemma 3.1, we can choose $\delta = \delta(r_1, d) < \min(\frac{1}{2}T_0, 1)$ such that

$$\sum_{i=1}^{d} \sup_{x\in\mathbf{R}^d} \int_0^{\delta} \int_{\mathbf{R}^d} s^{-\frac{d+1}{2}} \exp\left(-\frac{|z-x|^2}{16s}\right) (|U_n^i|(z)dz + |\mu^i|(dz))ds \\ < \frac{T_0^{d/2}}{8A^2}(C_1^{-1} \wedge 2^{-\frac{d+1}{2}}T_0^{1/2})\varepsilon, \qquad n \geq 1, \tag{3.23}$$

where C_1 is the constant from Lemma 3.4 with $a = \frac{1}{4}$. Then for this δ, by Lemma 3.5, there exists a constant $c_1 = c_1(d,\delta)$ such that for any $x \in \mathbf{R}^d$ and $R > 1$,

$$\int_{\delta}^{T_1} \int_{|z-x|\geq 3R} s^{-\frac{d+1}{2}} \exp\left(-\frac{|z-x|^2}{16s}\right) (|U_n^i|(z)dz + |\mu^i|(dz))ds \\ \leq c_1 T_1 \int_{|z-x|\geq R} \exp\left(-\frac{|z-x|^2}{16T_1}\right) |\mu^i|(dz)ds.$$

By (2.4) we can choose $R > 2r$ large enough so that

$$\int_{\delta}^{T_1} \int_{|z-x|\geq 3R} s^{-\frac{d+1}{2}} \exp\left(-\frac{|z-x|^2}{16s}\right) (|U_n^i|(z)dz + |\mu^i|(dz))ds \\ < \frac{T_0^{d/2}\varepsilon}{8C_1A^2}, \qquad n \geq 1. \tag{3.24}$$

We split $|I_1^n(t,x,y) - I_1(t,x,y)|$ into four parts:

$$\begin{aligned}
&|I_1^n(t,x,y) - I_1(t,x,y)| \\
&\leq \sum_{i=1}^{d} \int_0^t \int_{|z|\geq 7R} p(s,x,z)|\nabla_z p(t-s,z,y)|(|U_n^i|(z)dz + |\mu^i|(dz))ds \\
&\quad + \sum_{i=1}^{d} \int_0^{\delta} \int_{|z|< 7R} p(s,x,z)|\nabla_z p(t-s,z,y)|(|U_n^i|(z)dz + |\mu^i|(dz))ds \\
&\quad + \sum_{i=1}^{d} \int_{t-\delta}^{t} \int_{|z|< 7R} p(s,x,z)|\nabla_z p(t-s,z,y)|(|U_n^i|(z)dz + |\mu^i|(dz))ds \\
&\quad + \sum_{i=1}^{d} \left| \int_{\delta}^{t-\delta} \int_{|z|<7R} p(s,x,z)\partial_{z_i} p(t-s,z,y) U_n^i(z)dzds \right. \\
&\qquad \left. - \int_{\delta}^{t-\delta} \int_{|z|<7R} p(s,x,z)\partial_{z_i} p(t-s,z,y)\mu^i(dz))ds \right| \\
&=: \mathrm{I}(n,t,x,y) + \mathrm{II}(n,t,x,y) + \mathrm{III}(n,t,x,y) + \mathrm{IV}(n,t,x,y).
\end{aligned}$$

Since $|x| \le r < R$, we have $|x-z| > |z| - |x| > 6R$ for $|z| \ge 7R$. So

$$\begin{aligned}
&\mathrm{I}(n,t,x,y) \\
&\qquad \le \sum_{i=1}^{d} \int_0^t \int_{|x-z|\ge 6R} p(s,x,z)|\nabla_z p(t-s,z,y)|(|U_n^i|(z)dz + |\mu^i|(dz))ds.
\end{aligned}$$

By Lemma 3.4, (3.16) and (3.15), we have

$$\begin{aligned}
&\sup_{|x|,|y|\le r, n\ge 1} \mathrm{I}(n,t,x,y) \\
&\le C_1 A^2 t^{-\frac{d}{2}} \sum_{i=1}^{d} \sup_{|u|<R/2} \int_0^t \int_{|u-z|\ge 3R} s^{-\frac{d+1}{2}} e^{-\frac{|u-z|^2}{16s}} (|U_n^i|(z)dz + |\mu^i|(dz))ds.
\end{aligned}$$

Therefore, by (3.23)–(3.24), we get

$$\begin{aligned}
&\sup_{\substack{|x|,|y|\le r, n\ge 1,\\ T_0\le t\le T_1}} \mathrm{I}(n,t,x,y) \\
&\quad \le C_1 A^2 T_0^{-\frac{d}{2}} \sum_{i=1}^{d} \sup_{u\in\mathbf{R}^d} \int_0^{T_1} \int_{|u-z|\ge 3R} s^{-\frac{d+1}{2}} e^{-\frac{|u-z|^2}{16s}} \\
&\qquad\quad \times (|U_n^i|(z)dz + |\mu^i|(dz))ds \\
&\quad \le C_1 A^2 T_0^{-\frac{d}{2}} \sum_{i=1}^{d} \Bigg(\sup_{u\in\mathbf{R}^d} \int_0^{\delta} \int_{\mathbf{R}^d} s^{-\frac{d+1}{2}} e^{-\frac{|u-z|^2}{16s}} (|U_n^i|(z)dz + |\mu^i|(dz))ds \\
&\qquad\quad + \sup_{u\in\mathbf{R}^d} \int_{\delta}^{T_1} \int_{|u-z|\ge 3R} s^{-\frac{d+1}{2}} e^{-\frac{|u-z|^2}{16s}} (|U_n^i|(z)dz + |\mu^i|(dz))ds \Bigg) < \frac{\varepsilon}{4}.
\end{aligned}$$

On the other hand, since $\delta < T_0/2$, by (3.15)–(3.16) we have

$$\begin{aligned}
&\mathrm{II}(n,t,x,y) \\
&\qquad \le A^2 (\frac{2}{T_0})^{\frac{d+1}{2}} \sum_{i=1}^{d} \sup_{u\in\mathbf{R}^d} \int_0^{\delta} \int_{\mathbf{R}^d} s^{-\frac{d}{2}} e^{-\frac{|u-z|^2}{8s}} (|U_n^i|(z)dz + |\mu^i|(dz))ds.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathrm{III}(n,t,x,y) \\
&\qquad \le A^2 (\frac{2}{T_0})^{\frac{d}{2}} \sum_{i=1}^{d} \sup_{u\in\mathbf{R}^d} \int_0^{\delta} \int_{\mathbf{R}^d} s^{-\frac{d+1}{2}} e^{-\frac{|u-z|^2}{8s}} (|U_n^i|(z)dz + |\mu^i|(dz))ds.
\end{aligned}$$

Therefore, by (3.23), we conclude

$$\sup_{\substack{|x|,|y|\le r, n\ge 1,\\ T_0\le t\le T_1}} (\mathrm{II}(n,t,x,y) + \mathrm{III}(n,t,x,y)) < \frac{\varepsilon}{4}.$$

Now we estimate IV. Let

$$f_i(t,x,y,z) := \int_\delta^{t-\delta} p(s,x,z)\partial_{z_i} p(t-s,z,y)ds.$$

By the continuity of $p(s,x,z)$ and $\partial_{z_i} p(t-s,z,y)$ and (3.16)–(3.15), $f_i(t,x,y,z)$ is continuous on $[T_0,T_1]\times\overline{B(0,r)}\times\overline{B(0,r)}\times\overline{B(0,7R)}$. Therefore, by Lemma 3.2,

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{\substack{|x|,|y|\le r,\\ T_0\le t\le T_1}} \mathrm{IV}(n,t,x,y)\\
&\quad = \lim_{n\to\infty} \sum_{i=1}^d \sup_{\substack{|x|,|y|\le r,\\ T_0\le t\le T_1}} \left| \int_{B(0,7R)} f_i(t,x,y,z)(\mu_n^i-\mu^i)(dz)\right| = 0. \qquad \Box
\end{aligned}$$

LEMMA 3.7. *For any compact subsets $K_1,K_2\subset\mathbf{R}^d$ and $T_0\in(0,T_1]$, we have*

$$\lim_{n\to\infty} \sup_{\substack{t\in[T_0,T_1],\\ (x,y)\in K_1\times K_2}} |I_k^n(t,x,y)-I_k(t,x,y)| \;=\; 0, \quad k\ge 1.$$

Proof. Without loss of generality we may assume that $T_0<2$. We will prove this lemma for the case $K_1=K_2=\overline{B(0,r)}$ only. The previous lemma implies that the present lemma is valid for $k=1$. We assume that the lemma is true for k, which implies that $I_k(s,x,z)$ is continuous. Given $\varepsilon>0$, we choose $r_1>0$ as in (3.22) and then, using Proposition 2.2 and Lemma 3.1, choose $\delta=\delta(r_1,d)<\min(\frac12 T_0,1)$ such that

$$\begin{aligned}
(3.25)\quad &\sum_{i=1}^d \sup_{x\in\mathbf{R}^d}\int_0^\delta\int_{\mathbf{R}^d} s^{-\frac{d+1}{2}}\exp\left(-\frac{|z-x|^2}{16s}\right)(|U_n^i|(z)dz+|\mu^i|(dz))ds\\
&\qquad < \frac{T_0^{d/2}\varepsilon}{8AM_0}(C_1^{-1}\wedge 2^{-\frac{d+1}{2}}T_0^{1/2}), \quad n\ge 1,
\end{aligned}$$

where C_1 is the constant in Lemma 3.4. Then for this δ, using Lemma 3.5 and (2.4) we choose $R>2r$ large enough so that

$$\begin{aligned}
(3.26)\quad &\int_\delta^{T_1}\int_{|z-x|\ge 3R} s^{-\frac{d+1}{2}}\exp\left(-\frac{|z-x|^2}{16s}\right)(|U_n^i|(z)dz+|\mu^i|(dz))ds\\
&\qquad < \frac{T_0^{d/2}\varepsilon}{8C_1AM_0}, \quad n\ge 1.
\end{aligned}$$

Let

$$\mathrm{I}(n,t,x,y) := \sum_{i=1}^{d} \left(\int_0^t \int_{|z|\geq 7R} |I_k^n(t,x,y)||\nabla_z p(t-s,z,y)||U_n^i|(z)dzds \right.$$
$$\left. + \int_0^t \int_{|z|\geq 7R} |I_k(t,x,y)||\nabla_z p(t-s,z,y)||\mu^i|(dz)dzds \right),$$

$$\mathrm{II}(n,t,x,y) := \sum_{i=1}^{d} \left(\int_0^\delta \int_{|z|<7R} |I_k^n(t,x,y)||\nabla_z p(t-s,z,y)||U_n^i|(z)dzds \right.$$
$$\left. + \int_0^\delta \int_{|z|<7R} I_k(t,x,y)|\nabla_z p(t-s,z,y)||\mu^i|(dz)dzds \right),$$

$$\mathrm{III}(n,t,x,y) := \sum_{i=1}^{d} \left(\int_{t-\delta}^t \int_{|z|<7R} |I_k^n(t,x,y)||\nabla_z p(t-s,z,y)||U_n^i|(z)dzds \right.$$
$$\left. + \int_{t-\delta}^t \int_{|z|<7R} I_k(s,x,z)|\nabla_z p(t-s,z,y)||\mu^i|(dz)dzds \right),$$

$$\mathrm{IV}(n,t,x,y) := \sum_{i=1}^{d} \left| \int_\delta^{t-\delta} \int_{|z|<7R} I_k(s,x,z)\partial_{z_i} p(t-s,z,y)U_n^i(z)dzds \right.$$
$$\left. - \int_\delta^{t-\delta} \int_{|z|<7R} I_k(s,x,z)\partial_{z_i} p(t-s,z,y)\mu^i(dz))ds \right|$$

and

$$\mathrm{V}(n,t,x,y) := \sum_{i=1}^{d} \int_\delta^{t-\delta} \int_{|z|<7R} |I_k^n(s,x,z) - I_k(s,x,z)| \times$$
$$\times |\partial_{z_i} p(t-s,z,y)||U_n^i(z)|dzds.$$

Then we have

$$|I_{k+1}^n(t,x,y) - I_{k+1}(t,x,y)|$$
$$\leq \mathrm{I}(n,t,x,y) + \mathrm{II}(n,t,x,y) + \mathrm{III}(n,t,x,y) +$$
$$+ \mathrm{IV}(n,t,x,y) + \mathrm{V}(n,t,x,y).$$

Since $|x| \le r < R$, we have $|x-z| > |z|-|x| > 6R$ for $|z| \ge 7R$. So by Lemma 3.4, (3.17), (3.20) and (3.15), we have

$$\begin{aligned}
&\sup_{|x|,|y|\le r, n\ge 1} \mathrm{I}(n,t,x,y) \\
&\qquad \le 2^{-k} C_1 A M_0 t^{-\frac{d}{2}} \sum_{i=1}^d \sup_{|u|<R/2} \int_0^t \int_{|u-z|\ge 3R} s^{-\frac{d+1}{2}} e^{-\frac{|u-z|^2}{16s}} \times \\
&\qquad\qquad \times (|U_n^i|(z)dz + |\mu^i|(dz))ds.
\end{aligned}$$

Therefore, by (3.25)–(3.26), we get

$$\begin{aligned}
&\sup_{\substack{|x|,|y|\le r, n\ge 1,\\ T_0\le t\le T_1}} \mathrm{I}(n,t,x,y) \\
&\le C_1 A M_0 T_0^{-\frac{d}{2}} \sum_{i=1}^d \sup_{u\in\mathbf{R}^d} \int_0^{T_1} \int_{|u-z|\ge 3R} s^{-\frac{d+1}{2}} e^{-\frac{|u-z|^2}{16s}} \times \\
&\quad \times (|U_n^i|(z)dz + |\mu^i|(dz))ds \\
&\le C_1 A M_0 T_0^{-\frac{d}{2}} \sum_{i=1}^d \left(\sup_{u\in\mathbf{R}^d} \int_0^{\delta} \int_{\mathbf{R}^d} s^{-\frac{d+1}{2}} e^{-\frac{|u-z|^2}{16s}} (|U_n^i|(z)dz + |\mu^i|(dz))ds \right. \\
&\quad \left. + \sup_{u\in\mathbf{R}^d} \int_{\delta}^{T_1} \int_{|u-z|\ge 3R} s^{-\frac{d+1}{2}} e^{-\frac{|u-z|^2}{16s}} (|U_n^i|(z)dz + |\mu^i|(dz))ds \right) < \frac{\varepsilon}{4}.
\end{aligned}$$

On the other hand, since $\delta < T_0/2$, by (3.17), (3.20) and (3.15) we have

$$\begin{aligned}
\mathrm{II}(n,t,x,y) \le 2^{-k} A M_0 (\frac{2}{T_0})^{\frac{d+1}{2}} \sum_{i=1}^d \sup_{u\in\mathbf{R}^d} \int_0^{\delta} \int_{\mathbf{R}^d} s^{-\frac{d}{2}} e^{-\frac{|u-z|^2}{8s}} \times \\
\times (|U_n^i|(z)dz + |\mu^i|(dz))ds.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{III}(n,t,x,y) \le 2^{-k} A M_0 (\frac{2}{T_0})^{\frac{d}{2}} \sum_{i=1}^d \sup_{u\in\mathbf{R}^d} \int_0^{\delta} \int_{\mathbf{R}^d} s^{-\frac{d+1}{2}} e^{-\frac{|u-z|^2}{8s}} \times \\
\times (|U_n^i|(z)dz + |\mu^i|(dz))ds.
\end{aligned}$$

Therefore, by (3.25), we conclude

$$\sup_{|x|,|y|\le r, n\ge 1, T_0\le t\le T_1} (\mathrm{II}(n,t,x,y) + \mathrm{III}(n,t,x,y)) < \frac{\varepsilon}{4}.$$

Now we estimate IV. Let

$$f_i(t,x,y,z) := \int_{\delta}^{t-\delta} I_k(s,x,z) \partial_{z_i} p(t-s,z,y)ds.$$

By the continuity of $I_k(s,x,z)$ and $\partial_{z_i} p(t-s,z,y)$ and (3.15) and (3.17), $f_i(t,x,y,z)$ is continuous on $[T_0,T_1]\times\overline{B(0,r)}\times\overline{B(0,r)}\times\overline{B(0,7R)}$. Therefore, by Lemma 3.2,

$$\lim_{n\to\infty}\sup_{|x|,|y|\le r, T_0\le t\le T_1} \mathrm{IV}(n,t,x,y)$$
$$=\lim_{n\to\infty}\sum_{i=1}^d \sup_{|x|,|y|\le r,T_0\le t\le T_1}\left|\int_{B(0,7R)} f_i(t,x,y,z)(\mu_n^i-\mu^i)(dz)\right| = 0.$$

Finally, we estimate V. From (3.15), we easily see that

$$\mathrm{V}(n,t,x,y)\le M_0T_1(\frac{2}{T_0})^{\frac{d+1}{2}}\sup_{|x|<r,|z|\le 7R,\delta\le t\le T_1}|I_k^n(s,x,z)-I_k(s,x,z)|\times$$
$$\times\sup_{n\ge1}\sum_{i=1}^d\int_0^t\int_{\mathbf{R}^d}(t-s)^{-\frac{d+1}{2}}e^{-\frac{|z-y|^2}{4(t-s)}}|U_n^i(z)|dzds.$$

By Proposition 2.2 and Lemma 3.1,

$$\sup_{n\ge1,y\in\mathbf{R}^d}\sum_{i=1}^d\int_0^t\int_{\mathbf{R}^d}(t-s)^{-\frac{d+1}{2}}e^{-\frac{|z-y|^2}{4(t-s)}}|U_n^i(z)|dzds$$

is bounded. Therefore

$$\lim_{n\to\infty}\sup_{\substack{|x|,|y|\le r,\\ T_0\le t\le T_1}}\mathrm{V}(n,t,x,y)=0$$

by the assumption on I_k^n. □

THEOREM 3.8. *The sequence $q_n(t,x,y)$ converges uniformly on any compact subset of $(0,\infty)\times\mathbf{R}^d\times\mathbf{R}^d$.*

Proof. By (3.17), for every $T_0\in(0,T_1)$ and compact subsets $K_1,K_2\subset\mathbf{R}^d$

$$\sup_{(t,x,y)\in[T_0,T_1]\times K_1\times K_2}\sum_{k=0}^\infty|I_k^n(t,x,y)|<\infty.$$

Therefore Lemma 3.7 and a standard ε-δ argument give the uniform convergence of $q_n(t,x,y)$ on any compact subset of $(0,T_1]\times\mathbf{R}^d\times\mathbf{R}^d$.

The uniform upper bounds of q_n and q imply that for large R,

$$\begin{aligned}\sup_{\substack{n\geq 1, t\in(T_1,\frac{3}{2}T_1],\\ (x,y)\in K_1\times K_2}} \int_{|z|\geq R} \Big(& q_n(\frac{T_1}{2},x,z)q_n(t-\frac{T_1}{2},z,y)+ \\ & +q(\frac{T_1}{2},x,z)q(t-\frac{T_1}{2},z,y)\Big)\,dz \\ \leq\ & c_1(T_1)^{-d}\int_{|z|\geq R} e^{-c_2\frac{|z|^2}{T_1}}dz,\end{aligned}$$

for some positive constants c_1 and c_2. For any given $\varepsilon > 0$, we can choose R large such that

$$c_1(T_1)^{-d}\int_{|z|\geq R} e^{-c_2\frac{|z|^2}{T_1}}dz < \frac{\varepsilon}{2}.$$

By the Chapman-Kolmogorov equation, we have for $(t,x,y) \in (T_1, \frac{3}{2}T_1] \times K_1 \times K_2$,

$$\begin{aligned}&\left|q_n(t,x,y) - \int_{\mathbf{R}^d} q(\frac{T_1}{2},x,z)q(t-\frac{T_1}{2},z,y)dz\right| \\ &= \left|\int_{\mathbf{R}^d} q_n(\frac{T_1}{2},x,z)q_n(t-\frac{T_1}{2},z,y) - q(\frac{T_1}{2},x,z)q(t-\frac{T_1}{2},z,y)dz\right| \\ &< c_3R^d \sup_{\substack{t\in[\frac{1}{2}T_1,T_1],\\ (x,y,z)\in K_1\times K_2\times\overline{B(0,R)}}} \Big| q_n(\frac{T_1}{2},x,z)q_n(t-\frac{T_1}{2},z,y) \\ &\qquad\qquad -q(\frac{T_1}{2},x,z)q(t-\frac{T_1}{2},z,y)\Big| + \frac{\varepsilon}{2}\end{aligned}$$

for some positive constant c_3. The first term in the last line above goes zero as $n \to \infty$ by the uniform convergence of $q_n(t,x,y)$ on compact subsets of $(0,T_1] \times \mathbf{R}^d \times \mathbf{R}^d$. The general case can be proved by induction. □

We define q on $(0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d$ by

$$q(t,x,y) := \lim_{n\to\infty} q_n(t,x,y).$$

Using (3.15), the continuity of $\nabla_x q_n(t,x,y)$ and (3.13), we can easily show that for any positive integer n and any $(t,x,y) \in (0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d$,

$$\begin{aligned}\nabla_x q_n(t,x,y) &= \nabla_x p(t,x,y) \\ &\quad + \int_0^t \int_{\mathbf{R}^d} \nabla_x q_n(s,x,z)U_n \cdot \nabla_z p(t-s,z,y)dzds.\end{aligned} \tag{3.27}$$

We define vector-valued functions $J_k^n(t,x,y) = (J_k^{n,(1)}, \cdots, J_k^{n,(d)})(t,x,y)$ recursively for $k \geq 0$ and $(t,x,y) \in (0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d$:

$$J_0^n(t,x,y) := \nabla_x p(t,x,y),$$
$$J_{k+1}^n(t,x,y) := \int_0^t \int_{\mathbf{R}^d} J_k^n(s,x,z) U_n(z) \cdot \nabla_z p(t-s,z,y) dz ds.$$

Then iterating (3.27) gives

$$\nabla_x q_n(t,x,y) = \sum_{k=0}^{\infty} J_k^n(t,x,y), \quad (t,x,y) \in (0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d. \tag{3.28}$$

Using Lemma 3.1(b) in [28] and (3.15), one can show that there exists positive constant M_{10} such that for $k = 0, 1, \cdots$ and $(t,x,y) \in (0,T_1] \times \mathbf{R}^d \times \mathbf{R}^d$

$$|J_k^n(t,x,y)| \leq 2^{-k-1} M_{10} t^{-\frac{d+1}{2}} e^{-\frac{|x-y|^2}{8t}}. \tag{3.29}$$

The proof is similar to the proof of (3.17), so we skip the details.

Now we define $J_k(t,x,y) := (J_k^{(1)}, \cdots, J_k^{(d)})(t,x,y)$ recursively for $k \geq 0$ and $(t,x,y) \in (0,T_1] \times \mathbf{R}^d \times \mathbf{R}^d$:

$$J_0(t,x,y) := \nabla_x p(t,x,y),$$
$$J_{k+1}(t,x,y) := \int_0^t \int_{\mathbf{R}^d} J_k(s,x,z) \nabla_z p(t-s,z,y) \cdot \mu(dz) ds.$$

Let

$$r(t,x,y) := \sum_{k=0}^{\infty} J_k(t,x,y), \quad (t,x,y) \in (0,T_1] \times \mathbf{R}^d \times \mathbf{R}^d. \tag{3.30}$$

Using an argument similar to the proof of (3.17), we also have that for $k = 0, 1, \cdots$ and $(t,x,y) \in (0,T_1] \times \mathbf{R}^d \times \mathbf{R}^d$

$$|J_k(t,x,y)| \leq 2^{-k-1} M_{10} t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{8t}}. \tag{3.31}$$

So $\sum_{k=0}^{\infty} J_k(t,x,y)$ converges absolutely on $(t,x,y) \in (0,T_1] \times \mathbf{R}^d \times \mathbf{R}^d$ and converges uniformly on $(t,x,y) \in [T_0,T_1] \times \mathbf{R}^d \times \mathbf{R}^d$ for every $0 < T_0 < T_1$, which implies that $r(t,x,y)$ is jointly continuous on $(0,T_1] \times \mathbf{R}^d \times \mathbf{R}^d$. Moreover

$$r(t,x,y) \leq M_{10} t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{8t}}, \qquad (t,x,y) \in (0,T_1] \times \mathbf{R}^d \times \mathbf{R}^d. \tag{3.32}$$

Similar to q_n, we will show that $\nabla_x q_n(t,x,y)$ converges uniformly on each compact subset of $(0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d$ through several lemmas.

LEMMA 3.9. *For any compact subsets $K_1, K_2 \subset \mathbf{R}^d$ and $T_0 \in (0,T_1]$, we have*

$$\lim_{n\to\infty} \sup_{(t,x,y)\in[T_0,T_1]\times K_1\times K_2} |J_1^n(t,x,y) - J_1(t,x,y)| = 0.$$

Proof. The proof of this lemma is similar to that of Lemma 3.6. We omit the details. □

LEMMA 3.10. *For any compact subsets $K_1, K_2 \subset \mathbf{R}^d$ and $T_0 \in (0, T_1]$, we have*

$$\lim_{n\to\infty} \sup_{(t,x,y)\in[T_0,T_1]\times K_1\times K_2} |J_k^n(t,x,y) - J_k(t,x,y)| \,=\, 0, \quad k \geq 1.$$

Proof. The proof of this lemma is similar to that of Lemma 3.7. We omit the details. □

THEOREM 3.11. *$\nabla_x q_n(t,x,y)$ converges uniformly on any compact subset of $(0,\infty)\times\mathbf{R}^d\times\mathbf{R}^d$.*

Proof. Recall that $q(t,x,y) = \lim_{n\to\infty} q_n(t,x,y)$. By Theorem 3.8, the above convergence is uniform on any compact subset of $(0,\infty)\times\mathbf{R}^d\times\mathbf{R}^d$. Using this fact, one can prove this theorem using an argument similar to that used in the proof of Theorem 3.8 (without using induction). We omit the details. □

Define

$$S_n^\lambda f(x) = \mathbf{E}_x \int_0^\infty e^{-\lambda t} f(X_t^n)dt, \quad \lambda > 0.$$

Using the estimates in (3.4) and (3.5), we can give simpler proofs of Theorem 4.2 and Theorem 4.3 in [4] without assuming that μ has compact support.

COROLLARY 3.12. *For every $\lambda > M_8$, the family of functions $\{S_n^\lambda g : n \geq 1, \|g\|_{L^\infty(\mathbf{R}^d)} = 1\}$ is equicontinuous.*

Proof. By (3.5) and the continuity of $\nabla_x q_n(t,x,y)$, we have

$$\begin{aligned}
&|\nabla_x S_n^\lambda g(x)| = \left| \int_{\mathbf{R}^d} \int_0^\infty e^{-\lambda t} \nabla_x q_n(t,x,y) g(y) dy dt \right| \\
&\leq M_7 \|g\|_{L^\infty(\mathbf{R}^d)} \int_{\mathbf{R}^d} \int_0^\infty e^{(M_8-\lambda)t} t^{-\frac{d+1}{2}} \exp\left(-\frac{M_9|x-y|^2}{2t}\right) dy dt \\
&= M_7 \int_{\mathbf{R}^d} \int_0^\infty e^{(M_8-\lambda)t} t^{-\frac{d+1}{2}} \exp\left(-\frac{M_9|y|^2}{2t}\right) dy dt \\
&:= C(M_7, M_8, M_9, \lambda).
\end{aligned}$$

□

COROLLARY 3.13. *For any positive constants β, ε and T, there exists $\delta > 0$ independent of x and n such that*

$$\mathbf{P}_x \left(\sup_{s,t\leq T, |t-s|<\delta} |X_t^n - X_s^n| > \beta \right) < \varepsilon.$$

Proof. As in the proof of Theorem 4.3 in [4], by the Markov property and Chebyshev's inequality, it is enough to show that

$$\sup_{n\geq 1}\left\|\mathbf{E}_x\int_0^\delta \sum_{i=1}^d |U_n^i|(X_t^n)dt\right\|_{L^\infty(\mathbf{R}^d)} < \frac{\varepsilon\beta}{4}.$$

In fact, by Lemma 3.1 and (3.4), the above expectation is bounded by

$$M_0e^{M_5\delta}\sum_{i=1}^d\int_{\mathbf{R}^d}\int_0^\delta t^{-\frac{d}{2}}e^{-\frac{M_6|x-y|^2}{2t}}|U_n^i|(y)dydt$$
$$\leq M_0M_6e^{M_5\delta}(2\pi)^{-\frac{d}{2}}\sum_{i=1}^d N_\mu^1(\frac{\delta}{M_6}),$$

which is arbitrarily small as δ goes to zero. □

Now, we are ready to prove the main result of this section.

THEOREM 3.14. *For any $\mu\in\mathbf{K}_{d,1}$, the process X has a transition density $q^\mu(t,x,y)$ which is jointly continuous on $(0,\infty)\times\mathbf{R}^d\times\mathbf{R}^d$. Moreover, there exist positive constants M_i, $i=1,\dots,9$, depending on μ only via the rate at which $\max_{1\leq i\leq d}M_{\mu^i}^1(t)$ goes to zero, such that*

$$M_1e^{-M_2t}t^{-\frac{d}{2}}e^{-\frac{M_3|x-y|^2}{2t}}\leq q^\mu(t,x,y)\leq M_0e^{M_5t}t^{-\frac{d}{2}}e^{-\frac{M_6|x-y|^2}{2t}} \tag{3.33}$$

and

$$|\nabla_x q^\mu(t,x,y)| \leq M_7e^{M_8t}t^{-\frac{d+1}{2}}e^{-\frac{M_9|x-y|^2}{2t}} \tag{3.34}$$

for all $(t,x,y)\in(0,\infty)\times\mathbf{R}^d\times\mathbf{R}^d$.

Proof. Let X^n be the Brownian motion with drift U_n. The transition density $q_n(t,x,y)$ of X^n is the heat kernel of $\frac{1}{2}\triangle+U_n\cdot\nabla$. Using the estimates (3.4) and (3.5), we have proved in Corollaries 3.12 and 3.13 above the conclusions of Theorems 4.2 and 4.3 in [4] without any extra assumption on μ. So the conclusions of Theorems 4.2 and 4.3 in [4] are valid without any extra assumption on μ, and the proof of Theorem 4.5 in [4] goes through under the assumption $\lambda>M_8$. Thus for every subsequence n_k there is a sub-subsequence n_{k_m} such that $X^{n_{k_m}}$ converges weakly under $\mathbf{P}_x$ for every $x\in\mathbf{R}^n$ in $C([0,\infty),\mathbf{R}^d)$. Theorem 3.8 tells us that all subsequence limits of the density X^n are the same, which implies that X_t^n converges weakly to the X_t in $C([0,\infty),\mathbf{R}^d)$. In particular, for every open set B in $\mathbf{R}^d$,

$$\begin{aligned}\mathbf{P}_x(X_t \in B) &\leq \liminf_{n\to\infty} \mathbf{E}_x\left[1_B(X_t^n)\right] \\ &= \liminf_{n\to\infty} \int_B q_n(t,x,y)dy \\ &\leq M_0 e^{M_5 t} t^{-\frac{d}{2}} \int_B dy.\end{aligned}$$

Therefore $\mathbf{P}_x(X_t \in dy)$ is absolutely continuous with respect to Lebesgue measure. So by Theorem 3.8,

$$q^\mu(t,x,y) := \lim_{n\to\infty} q_n(t,x,y)$$

is the density for X_t. Since $q^\mu(t,x,y)$ is the uniform limit of jointly continuous functions on any compact subset of $(0,\infty)\times\mathbf{R}^d\times\mathbf{R}^d$, it is jointly continuous on $(0,\infty)\times\mathbf{R}^d\times\mathbf{R}^d$. Now (3.33) follows immediately from Theorem 3.8 and (3.4). Using Theorems 3.8 and 3.11 we see that for any $(t,y)\in(0,\infty)\times\mathbf{R}^d$, the sequence of functions $\{q_n(t,\cdot,y) : n \geq 1\}$ is a Cauchy sequence in the Banach space $C_b^1(\overline{B(0,r)})$ for every $r>0$. Thus $\nabla_x q^\mu(t,x,y)$ exists for every $(t,x,y)\in(0,\infty)\times\mathbf{R}^d\times\mathbf{R}^d$ and it satisfies (3.34). □

Using Lemmas 3.1, 3.3 and 3.6, and Theorems 3.8, 3.11 and 3.14, we can easily show the following result:

THEOREM 3.15. *For any $\mu \in \mathbf{K}_{d,1}$, the density $q^\mu(t,x,y)$ of X satisfies the equations*

$$\begin{aligned}q^\mu(t,x,y) &= p(t,x,y) + \int_0^t \int_{\mathbf{R}^d} q^\mu(s,x,z)\cdot\nabla_z p(t-s,z,y)\mu(dz)ds, \\ \nabla_x q^\mu(t,x,y) &= \nabla_x p(t,x,y) + \int_0^t \int_{\mathbf{R}^d} \nabla_x q^\mu(s,x,z)\cdot\nabla_z p(t-s,z,y)\mu(dz)ds\end{aligned}$$

for all $(t,x,y)\in(0,\infty)\times\mathbf{R}^d\times\mathbf{R}^d$.

Proof. We omit the details. □

4. Two-sided estimates for the density of X^D

In this section we assume that D is a bounded $C^{1,1}$ domain. We will use $\rho(x)$ to denote the distance between x and ∂D. Let $p^D(t,x,y)$ be the density of a standard Brownian motion killed upon exiting D. It is obvious that

$$p^D(t,x,y) \leq p(t,x,y), \quad (t,x,y)\in(0,\infty)\times D\times D.$$

We let $a \wedge b := \min\{a, b\}$. It is known that for any $T > 0$, there exist positive constants $C_i, i = 1, \dots, 5$, depending on T and D such that

$$\begin{aligned}(4.1)\qquad C_1(1 \wedge \frac{\rho(x)}{\sqrt{t}})(1 \wedge \frac{\rho(y)}{\sqrt{t}})t^{-\frac{d}{2}}e^{-\frac{C_2|x-y|^2}{t}} &\le p^D(t, x, y) \\ &\le C_3(1 \wedge \frac{\rho(x)}{\sqrt{t}})(1 \wedge \frac{\rho(y)}{\sqrt{t}})t^{-\frac{d}{2}}e^{-\frac{C_4|x-y|^2}{t}}\end{aligned}$$

and

$$(4.2)\qquad |\nabla_x p^D(t, x, y)| \le C_5 t^{-\frac{d+1}{2}} e^{-\frac{C_4|x-y|^2}{t}}$$

for all $(t, x, y) \in (0, T] \times D \times D$. (4.1) was proved in [10] and [29], while a proof of (4.2) can be found in [14]. Differentiating with respect to x in the equation

$$p^D(t, x, y) = \int_D p^D(\frac{t}{2}, x, z)p^D(\frac{t}{2}, z, y)dz$$

and using the above estimates on $p^D(t, x, y)$ and $\nabla_x p^D(t, x, y)$ we get

$$\begin{aligned}|\nabla_x p^D(t, x, y)| &\le 2^{d+1}C_3C_5 \int_D t^{-\frac{d+1}{2}} e^{-\frac{2C_4|x-z|^2}{t}} \rho(y) t^{-\frac{d+1}{2}} e^{-\frac{2C_4|z-y|^2}{t}} dz \\ &\le 2^{d+1}C_3C_5\rho(y) \int_{\mathbf{R}^d} t^{-\frac{d+1}{2}} e^{-\frac{2C_4|x-z|^2}{t}} t^{-\frac{d+1}{2}} e^{-\frac{2C_4|z-y|^2}{t}} dz \\ &:= C_6\rho(y)t^{-\frac{d+2}{2}} e^{-\frac{C_4|x-y|^2}{t}}.\end{aligned}$$

Combining this with (4.2) we see that, for any $T > 0$, there exists a positive constant C_7 such that

$$(4.3)\qquad |\nabla_x p^D(t, x, y)| \le C_7(1 \wedge \frac{\rho(y)}{\sqrt{t}})t^{-\frac{d+1}{2}} e^{-\frac{C_4|x-y|^2}{t}}$$

for all $(t, x, y) \in (0, T] \times D \times D$. By the translation and scaling property of p^D, we see that, with properly scaled T, the constants in (4.1)–(4.3) are invariant under translation and Brownian scaling.

Let

$$\psi(t, x, y) := (1 \wedge \frac{\rho(x)}{\sqrt{t}})(1 \wedge \frac{\rho(y)}{\sqrt{t}}), \quad (t, x, y) \in (0, \infty) \times D \times D.$$

The following result is an analog of Lemma 3.1 of [28].

LEMMA 4.1. *For any $a > 0$, there exist positive constants C_1 and C_2 depending only on a and d such that for any measure ν on $\mathbf{R}^d$ and any*

$(t, x, y) \in (0, \infty) \times D \times D$,

$$\int_0^t \int_D \psi(s,x,z) s^{-\frac{d}{2}} e^{-\frac{a|x-z|^2}{2s}} (1 \wedge \frac{\rho(y)}{\sqrt{t-s}})(t-s)^{-\frac{d+1}{2}} e^{-\frac{a|z-y|^2}{t-s}} \nu(dz)ds \tag{4.4}$$
$$\leq C_1 \psi(t,x,y) t^{-\frac{d}{2}} e^{-\frac{a|x-y|^2}{2t}} \sup_{u \in \mathbf{R}^d} \int_0^t \int_{\mathbf{R}^d} s^{-\frac{d+1}{2}} e^{-\frac{a|u-z|^2}{4s}} \nu(dz)ds$$

and

(4.5)
$$\int_0^t \int_D (1 \wedge \frac{\rho(z)}{\sqrt{s}}) s^{-\frac{d+1}{2}} e^{-\frac{a|x-z|^2}{2s}} (1 \wedge \frac{\rho(y)}{\sqrt{t-s}})(t-s)^{-\frac{d+1}{2}} e^{-\frac{a|z-y|^2}{t-s}} \nu(dz)ds$$
$$\leq C_2 (1 \wedge \frac{\rho(y)}{\sqrt{t}}) t^{-\frac{d+1}{2}} e^{-\frac{a|x-y|^2}{2t}} \sup_{u \in \mathbf{R}^d} \int_0^t \int_{\mathbf{R}^d} s^{-\frac{d+1}{2}} e^{-\frac{a|u-z|^2}{4s}} \nu(dz)ds.$$

Proof. The proof of (4.4) (for $\nu(dz) = f(x)dz$) is contained in the proof of Theorem 2.1 (pages 389–391) in [22]. (4.5) can be proved similarly if one notes that, for any $a > 0$, there exists $c > 0$ depending only on a and d such that

$$\int_0^t \int_{\mathbf{R}^d} s^{-\frac{d+2}{2}} e^{-\frac{a|x-z|^2}{2s}} (t-s)^{-\frac{d+1}{2}} e^{-\frac{a|z-y|^2}{t-s}} \nu(dz)ds$$
$$\leq c\, t^{-\frac{d+2}{2}} e^{-\frac{a|x-y|^2}{2t}} \int_0^t \int_{\mathbf{R}^d} s^{-\frac{d+1}{2}} e^{-\frac{a|u-z|^2}{4s}} \nu(dz)ds,$$

which can proved using the same argument in the proof of Lemma 3.1 in [28]. □

The proof of next theorem is similar to the proof of Theorem 2.1 in [22]. However, since some observations on the proof will be made later in Sections 4 and 5, we give a sketch of the proof.

THEOREM 4.2. *Suppose that $U(x) = (U^1(x), \dots, U^d(x))$ is such that each component U^i is smooth and bounded. Then the Brownian motion with drift U killed upon exiting from D has a transition density $q^{U,D}(t,x,y)$. $q^{U,D}$ is the fundamental solution of the problem*

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = \frac{1}{2}\Delta_x u(t,x) + U(x) \cdot \nabla_x u(t,x), & (t,x) \in (0,\infty) \times D, \\ u(t,x) = 0, & (t,x) \in (0,\infty) \times \partial D \end{cases}$$

and is also called the Dirichlet heat kernel for $\frac{1}{2}\triangle + U \cdot \nabla$ in D. For each $T > 0$ there exist positive constants $M_j, 11 \leq j \leq 15$, depending on U only via the rate at which $\max_{1 \leq i \leq d} M^1_{U^i}(r)$ goes to zero, such that

$$M_{11} t^{-\frac{d}{2}} \psi(t,x,y) e^{-\frac{M_{12}|x-y|^2}{2t}} \;\leq\; q^{U,D}(t,x,y) \leq M_{13} t^{-\frac{d}{2}} \psi(t,x,y) e^{-\frac{M_{14}|x-y|^2}{2t}}$$

and

$$|\nabla_x q^{U,D}(t,x,y)| \le M_{15}(1 \wedge \frac{\rho(y)}{\sqrt{t}})t^{-\frac{d+1}{2}} e^{-\frac{M_{14}|x-y|^2}{2t}}$$

for all $(t,x,y) \in (0,T] \times D \times D$.

Proof. The existence of the fundamental solution $q^{U,D}$ is well known. Recall that $p^D(t,x,y)$ is the density of a killed Brownian motion in D. It is easy to check that the function defined by

$$\tilde{q}^{U,D}(t,x,y) := p^D(t,x,y) + \int_0^t \int_D q^{U,D}(s,x,z)U(z) \cdot \nabla_z p^D(t-s,z,y)dzds$$

is a fundamental solution of

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta_x u(t,x) + U(x) \cdot \nabla_x u(t,x), & (t,x) \in (0,\infty) \times D, \\ u(t,x) = 0, & (t,x) \in (0,\infty) \times \partial D. \end{cases}$$

Thus it follows from Theorem 6 of [2] that

$$\begin{aligned} q^{U,D}(t,x,y) = {} & p^D(t,x,y) \\ & + \int_0^t \int_D q^{U,D}(s,x,z)U(z) \cdot \nabla_z p^D(t-s,z,y)dzds. \end{aligned} \tag{4.6}$$

We define $\tilde{I}_k(t,x,y)$ recursively for $k \ge 0$ and $(t,x,y) \in (0,\infty) \times D \times D$:

$$\begin{aligned} \tilde{I}_0(t,x,y) &:= p^D(t,x,y), \\ \tilde{I}_{k+1}(t,x,y) &:= \int_0^t \int_D \tilde{I}_k(s,x,z)U(z) \cdot \nabla_z p^D(t-s,z,y)dzds. \end{aligned}$$

Then iterating (4.6) gives

$$q^{U,D}(t,x,y) = \sum_{k=0}^{\infty} \tilde{I}_k(t,x,y), \quad (t,x,y) \in (0,\infty) \times D \times D. \tag{4.7}$$

By induction, and using Lemma 4.1, (4.1) and (4.3), one can show that there exist positive constants C_0, C_1 and M_{14} depending only on the constants in (4.1), (4.3) and (4.4) such that for $k = 0, 1, \cdots$ and $(t,x,y) \in (0,1] \times D \times D$

$$|\tilde{I}_k(t,x,y)| \le C_0\, \psi(t,x,y)\, t^{-\frac{d}{2}}\, e^{-\frac{M_{14}|x-y|^2}{2t}} \left(C_1 \sum_{i=1}^{d} N^1_{U^i}(\frac{2t}{M_{14}}) \right)^k \tag{4.8}$$

(see [22] for details). Choose $t_0 < 1$ small so that

$$C_1 \sum_{i=1}^{d} N^1_{U^i}(\frac{2t_0}{M_{14}}) < \frac{1}{2}. \tag{4.9}$$

By (2.5), t_0 depends on U only via the rate at which $\max_{1\le i\le d} M^1_{U^i}(r)$ goes to zero. (4.7) and (4.8) imply that for $(t,x,y)\in(0,t_0]\times D\times D$

$$q^{U,D}(t,x,y)\;\le\;\sum_{k=0}^{\infty}|\tilde{I}_k(t,x,y)|\;\le\;2C_0\psi(t,x,y)t^{-\frac{d}{2}}e^{-\frac{M_{14}|x-y|^2}{2t}}. \tag{4.10}$$

Now we are going to prove the lower estimate of $q^{U,D}(t,x,y)$. Combining (4.7), (4.8) and (4.9) we have for every $(t,x,y)\in(0,t_0]\times D\times D$,

$$\begin{aligned}|q^{U,D}(t,x,y)-p^D(t,x,y)| &\le \sum_{k=1}^{\infty}|\tilde{I}_k(t,x,y)|\\ &\le\; C_0C_1\sum_{i=1}^{d}N^1_{U^i}(\frac{2t_0}{M_{14}})\psi(t,x,y)t^{-\frac{d}{2}}e^{-\frac{M_{14}|x-y|^2}{2t}}.\end{aligned}$$

Since there exist C_2 and $C_3\le 1$ depending on t_0 such that

$$p^D(t,x,y)\ge 2C_3\psi(t,x,y)t^{-\frac{d}{2}}e^{-\frac{C_2|x-y|^2}{2t}},$$

we have for $|x-y|\le\sqrt{t}$ and $(t,x,y)\in(0,t_0]\times D\times D$,

$$q^{U,D}(t,x,y)\;\ge\;\left(2C_3e^{-2C_2}-C_0C_1\sum_{i=1}^{d}N^1_{U^i}(\frac{2t_0}{M_{14}})\right)\psi(t,x,y)t^{-\frac{d}{2}}. \tag{4.11}$$

Now we choose $t_1\le t_0$ small so that

$$C_0C_1\sum_{i=1}^{d}N^1_{U^i}(\frac{2t_1}{M_{14}})\;<\;C_3e^{-2C_2}. \tag{4.12}$$

Note that t_1 depends on U only via the rate at which $\max_{1\le i\le d} M^1_{U^i}(r)$ goes to zero. So for $(t,x,y)\in(0,t_1]\times D\times D$ and $|x-y|\le\sqrt{t}$, we have

$$q^{U,D}(t,x,y)\ge C_3e^{-2C_2}\psi(t,x,y)t^{-\frac{d}{2}}. \tag{4.13}$$

It is easy to check (see pages 420–421 of [29]) that there exists a positive constant t_2 depending only on the characteristics of the bounded $C^{1,1}$ domain D such that for any $t\le t_2$ and $x,y\in D$ with $\rho(x)\ge\sqrt{t},\rho(y)\ge\sqrt{t}$, one can find an arclength-parameterized curve $l\subset D$ connecting x and y such that the length $|l|$ of l is equal to $\lambda_1|x-y|$ with $\lambda_1\le\lambda_0$, a constant depending only on the characteristics of the bounded $C^{1,1}$ domain D. Moreover, l can be chosen so that

$$\rho(l(s))\ge\lambda_2\sqrt{t},\quad s\in[0,|l|]$$

for some positive constant λ_2 depending only on the characteristics of the bounded $C^{1,1}$ domain D. Put $t_3=t_1\wedge t_2$. Using this fact and (4.13), and following the proof of Theorem 2.7 in [13], we can show that there exists a positive constant C_4 depending only on d and the characteristics of the bounded $C^{1,1}$ domain D such that

$$q^{U,D}(t,x,y) \ge \frac{1}{2} C_3 e^{-2C_2} \psi(t,x,y) t^{-\frac{d}{2}} e^{-\frac{C_4|x-y|^2}{t}} \tag{4.14}$$

for all $t \in (0, t_3]$ and $x, y \in D$ with $\rho(x) \ge \sqrt{t}, \rho(y) \ge \sqrt{t}$.

It is easy to check that there exists a positive constant t_4 depending only on the characteristics of the bounded $C^{1,1}$ domain D such that for $t \in (0, t_4]$ and arbitrary $x, y \in D$, one can find $x_0, y_0 \in D$ be such that $\rho(x_0) \ge \sqrt{t}, \rho(y_0) \ge \sqrt{t}$ and $|x - x_0| \le \sqrt{t}, |y - y_0| \le \sqrt{t}$. Put $t_5 = t_3 \wedge t_4$. Then, using (4.11) and (4.14) one can repeat the last paragraph of the proof of Theorem 2.1 in [22] to show that there exists a positive constant C_5 depending only on d and the characteristics of the bounded $C^{1,1}$ domain D such that

$$q^{U,D}(t,x,y) \ge C_3 C_5 e^{-2C_2} \psi(t,x,y) t^{-\frac{d}{2}} e^{-\frac{2C_4|x-y|^2}{t}} \tag{4.15}$$

for all $(t, x, y) \in (0, t_5] \times D \times D$.

Now we are going to prove the upper estimate of $\nabla_x q^{U,D}(t, x, y)$. Using (4.3), the continuity of $\nabla_x q^{U,D}(t, x, y)$ and (4.6) we can easily show that for $(0, 1] \times D \times D$,

$$\begin{aligned} \nabla_x q^{U,D}(t,x,y) &= \nabla_x p^D(t,x,y) + \\ &+ \int_0^t \int_D \nabla_x q^{U,D}(s,x,z) U(z) \cdot \nabla_z p^D(t-s,z,y) dz ds. \end{aligned} \tag{4.16}$$

We define $\tilde{J}_k(t, x, y)$ for $(t, x, y) \in (0, 1] \times D \times D$ recursively by

$$\begin{aligned} \tilde{J}_0(t,x,y) &:= \nabla_x p^D(t,x,y), \\ \tilde{J}_{k+1}(t,x,y) &:= \int_0^t \int_D \tilde{J}_k(s,x,z) U(z) \cdot \nabla_z p^D(t-s,z,y) dz ds. \end{aligned}$$

Then iterating (4.16) gives

$$\nabla_x q^{U,D}(t,x,y) = \sum_{k=0}^{\infty} \tilde{J}_k(t,x,y), \quad (t,x,y) \in (0,1] \times D \times D. \tag{4.17}$$

Using induction and Lemma 4.1, one can show that there exist constants C_6 and C_7 depending only on the constants in (4.1), (4.3) and (4.5) such that for $k = 0, 1, \cdots$ and $(t, x, y) \in (0, 1] \times D \times D$

$$|\tilde{J}_k(t,x,y)| \le C_6 \, (1 \wedge \frac{\rho(y)}{\sqrt{t}}) \, t^{-\frac{d+1}{2}} e^{-\frac{M_{14}|x-y|^2}{2t}} \left(C_7 \sum_{i=1}^{d} N^1_{U^i}(\frac{2t}{M_{14}}) \right)^k. \tag{4.18}$$

Now we choose $t_6 \le t_5$ small so that

$$C_7 \sum_{i=1}^{d} N^1_{U^i}(\frac{2t_6}{M_{14}}) < \frac{1}{2}. \tag{4.19}$$

By (2.5), the choice of the above constant t_6 depends only on C_1, M_{14} and U, with the dependence on U being only via the rate at which $\max_{1\le i\le d} M^1_{U^i}(r)$ goes to zero. Let

$$C_8 := C_6 \sum_{k=0}^{\infty} \left(C_7 \sum_{i=1}^{d} N^1_{U^i}(\frac{2t_6}{M_{14}}) \right)^k .$$

Now (4.17) and (4.18) imply that for $(t,x,y) \in (0,t_6] \times D \times D$

$$\left|\nabla_x q^{U,D}(t,x,y)\right| \le \sum_{k=1}^{\infty} |\tilde{J}_k(t,x,y)| \le C_8\,(1\wedge \frac{\rho(y)}{\sqrt{t}})\,t^{-\frac{d+1}{2}}\,e^{-\frac{M_{14}|x-y|^2}{2t}}. \tag{4.20}$$

Now we have proved that the conclusion of this theorem is valid for $t \le t_6$. To prove this theorem for a general $T > 0$, we can apply the Chapman-Kolmogorov equation and use the argument in the proof of Theorem 3.9 in [25]. We omit the details. □

Recall that $\mu^i \in \mathbf{K}_{d,1}$ and $\mu^i_n(dx) = U^i_n(x)dx = \int \varphi_n(x-y)\mu^i(dy)dx$. Let $q^D_n(t,x,y)$ be the Dirichlet heat kernel for $\frac{1}{2}\triangle + U_n \cdot \nabla$ in D. Since U_n is smooth and bounded, Lemma 3.1 and the above theorem imply that for each $T>0$ there exist positive constants $M_j, 11 \le j \le 15$, depending on T and μ only via the rate at which $\max_{1\le i\le d} M^1_{\mu^i}(r)$ goes to zero, such that

$$\begin{aligned} M_{11} t^{-\frac{d}{2}} \psi(t,x,y) e^{-\frac{M_{12}|x-y|^2}{2t}} &\le q^D_n(t,x,y) \\ &\le M_{13} t^{-\frac{d}{2}} \psi(t,x,y) e^{-\frac{M_{14}|x-y|^2}{2t}} \end{aligned} \tag{4.21}$$

and

$$|\nabla_x q^D_n(t,x,y)| \le M_{15}(1\wedge \frac{\rho(y)}{\sqrt{t}}) t^{-\frac{d+1}{2}} e^{-\frac{M_{14}|x-y|^2}{2t}} \tag{4.22}$$

for all $(t,x,y) \in (0,T] \times D \times D$.

We define $\widehat{I}^n_k(t,x,y)$, $\widehat{I}_k(t,x,y)$, $\widehat{J}^n_k(t,x,y)$ and $\widehat{J}_k(t,x,y)$ recursively for $k \ge 0$ and $(t,x,y) \in (0,\infty)\times D \times D$:

$$\begin{aligned} \widehat{I}^n_0(t,x,y) &:= p^D(t,x,y), \\ \widehat{I}^n_{k+1}(t,x,y) &:= \int_0^t \int_D \widehat{I}^n_k(s,x,z) U_n(z) \cdot \nabla_z p^D(t-s,z,y) dz ds, \\ \widehat{I}_0(t,x,y) &:= p^D(t,x,y), \\ \widehat{I}_{k+1}(t,x,y) &:= \int_0^t \int_D \widehat{I}_k(s,x,z) \nabla_z p^D(t-s,z,y) \cdot \mu(dz) ds, \\ \widehat{J}^n_0(t,x,y) &:= \nabla_x p^D(t,x,y), \\ \widehat{J}^n_{k+1}(t,x,y) &:= \int_0^t \int_D \widehat{J}^n_k(s,x,z) U_n(z) \cdot \nabla_z p^D(t-s,z,y) dz ds, \end{aligned}$$

and

$$\widehat{J}_0(t,x,y) := \nabla_x p^D(t,x,y),$$
$$\widehat{J}_{k+1}(t,x,y) := \int_0^t \int_D \widehat{J}_k(s,x,z)\nabla_z p^D(t-s,z,y)\cdot \mu(dz)ds.$$

By (4.6) and (4.16),

$$q_n^D(t,x,y) = \sum_{k=0}^{\infty} \widehat{I}_k^n(t,x,y)$$

and

$$\nabla_x q_n^D(t,x,y) = \sum_{k=0}^{\infty} \widehat{J}_k^n(t,x,y), \quad (t,x,y) \in (0,\infty)\times D\times D.$$

Recall the constant t_6 in the proof of Theorem 4.2. Let

$$\widehat{q}(t,x,y) = \sum_{k=0}^{\infty} \widehat{I}_k(t,x,y)$$

and

$$\widehat{r}(t,x,y) = \sum_{k=0}^{\infty} \widehat{J}_k(t,x,y), \quad (t,x,y) \in (0,t_6]\times D\times D.$$

Then, from (4.8), (4.18), (4.10) and (4.20), there exists a positive constant B such that for $k = 0, 1, \cdots$ and $(t,x,y) \in (0,t_6]\times D\times D$

$$|\widehat{I}_k(t,x,y)|,\ |\widehat{I}_k^n(t,x,y)| \le B\, 2^{-k-1}\psi(t,x,y)\, t^{-\frac{d}{2}}\, e^{-\frac{M_{14}|x-y|^2}{2t}}, \tag{4.23}$$

$$|\widehat{J}_k(t,x,y)|,\ |\widehat{J}_k^n(t,x,y)| \le B\, 2^{-k-1}\psi(t,x,y)\, t^{-\frac{d+1}{2}}\, e^{-\frac{M_{14}|x-y|^2}{2t}}, \tag{4.24}$$

and

$$q_n^D(t,x,y),\ \widehat{q}(t,x,y) \le B\,\psi(t,x,y)\, t^{-\frac{d}{2}}\, e^{-\frac{M_{14}|x-y|^2}{2t}},$$
$$|\nabla_x q_n^D(t,x,y)|,\ |\widehat{r}(t,x,y)| \le B\,\psi(t,x,y)\, t^{-\frac{d}{2}}\, e^{-\frac{M_{14}|x-y|^2}{2t}}.$$

In the remainder of this section, we fix this constant B. We will show that $q_n^D(t,x,y)$ and $\nabla_x q_n^D(t,x,y)$ converge uniformly on every compact subset of $(0,\infty)\times D\times D$.

In the remainder of this section, t_6 stands for the constant t_6 defined in the proof of Theorem 4.2.

LEMMA 4.3. *For any compact subsets K_1 and K_2 of D, and $T_0 \in (0,t_6]$, we have*

$$\lim_{n\to\infty} \sup_{(t,x,y)\in[T_0,t_6]\times K_1\times K_2} |\widehat{I}_1^n(t,x,y) - \widehat{I}_1(t,x,y)| = 0,$$
$$\lim_{n\to\infty} \sup_{(t,x,y)\in[T_0,t_6]\times K_1\times K_2} |\widehat{J}_1^n(t,x,y) - \widehat{J}_1(t,x,y)| = 0.$$

Proof. We only prove the first identity; the proof of the second identity is similar. Without loss of generality we may assume that $T_0 < 2$ and we will prove this lemma for the case $K := K_1 = K_2$ only. Given $\varepsilon > 0$, we choose $r_1 > 0$ as in (3.22) and then, using Proposition 2.2 and Lemma 3.1, choose $\delta = \delta(r_1, d, M_{14}) < \min(\frac{1}{2}T_0, 1)$ such that for every $n \geq 1$

(4.25)
$$\sum_{i=1}^{d} \sup_{x\in\mathbf{R}^d} \int_0^\delta \int_{\mathbf{R}^d} s^{-\frac{d+1}{2}} \exp\left(-\frac{M_{14}|z-x|^2}{4s}\right)(|U_n^i|(z)dz + |\mu^i|(dz))ds$$
$$< \frac{T_0^{d/2}}{8B^2}(C_1^{-1} \wedge 2^{-\frac{d+1}{2}} T_0^{1/2})\varepsilon,$$

where C_1 is the constant from Lemma 3.4 with $a = \frac{1}{4}$. For this δ, we choose a smooth domain $D_1 \subset \overline{D_1} \subset D$ such that for every $z \in D \setminus D_1$

$$\rho(z) < d_1 := \sqrt{\delta}(1 \wedge \frac{\varepsilon \delta^{d/2}}{4B^2 L_2}), \tag{4.26}$$

where

$$L_2 := \sup_{n\geq 1, u\in\mathbf{R}^d} \sum_{i=1}^{d} \int_0^{t_6} \int_{\mathbf{R}^d} s^{-\frac{d+1}{2}} e^{-\frac{M_{14}|u-z|^2}{2s}}(|U_n^i|(z)dz + |\mu^i|(dz))ds,$$

which is finite by Proposition 2.2 and Lemma 3.1. We split $|\widehat{I}_1^n(t,x,y) - \widehat{I}_1(t,x,y)|$ into four parts:

$$\begin{aligned}
&|\widehat{I}_1^n(t,x,y) - \widehat{I}_1(t,x,y)| \\
&\quad \leq \sum_{i=1}^{d} \int_\delta^t \int_{D\setminus D_1} p^D(s,x,z)|\nabla_z p^D(t-s,z,y)|(|U_n^i|(z)dz + |\mu^i|(dz))ds \\
&\qquad + \sum_{i=1}^{d} \int_0^\delta \int_D p^D(s,x,z)|\nabla_z p^D(t-s,z,y)|(|U_n^i|(z)dz + |\mu^i|(dz))ds \\
&\qquad + \sum_{i=1}^{d} \int_{t-\delta}^t \int_D p^D(s,x,z)|\nabla_z p^D(t-s,z,y)|(|U_n^i|(z)dz + |\mu^i|(dz))ds \\
&\qquad + \sum_{i=1}^{d} \bigg| \int_\delta^{t-\delta} \int_{D_1} p^D(s,x,z)\partial_{z_i} p^D(t-s,z,y)U_n^i(z)dzds \\
&\qquad\qquad - \int_\delta^{t-\delta} \int_{D_1} p^D(s,x,z)\partial_{z_i} p^D(t-s,z,y)\mu^i(dz))ds \bigg| \\
&\quad =: \mathrm{I}(n,t,x,y) + \mathrm{II}(n,t,x,y) + \mathrm{III}(n,t,x,y) + \mathrm{IV}(n,t,x,y).
\end{aligned}$$

By (4.23) and (4.24), we have

$$\begin{aligned}
&\sup_{\substack{(x,y)\in K\times K,\\ n\geq 1}} \mathrm{I}(n,t,x,y)\\
&\leq B^2 \sup_{y\in K, n\geq 1} \sum_{i=1}^{d} \int_{\delta}^{t} \int_{D\setminus D_1} (1\wedge \frac{\rho(z)}{s}) s^{-\frac{d}{2}} (t-s)^{-\frac{d+1}{2}} e^{-\frac{M_{14}|y-z|^2}{2(t-s)}} \times\\
&\qquad \times (|U_n^i|(z)dz + |\mu^i|(dz))ds\\
&\leq \frac{B^2 d_1}{\delta^{\frac{d+1}{2}}} \sup_{y\in K, n\geq 1} \sum_{i=1}^{d} \int_{\delta}^{t} \int_{\mathbf{R}^d} (t-s)^{-\frac{d+1}{2}} e^{-\frac{M_{14}|y-z|^2}{2(t-s)}} (|U_n^i|(z)dz + |\mu^i|(dz))ds\\
&\leq \frac{B^2 d_1}{\delta^{\frac{d+1}{2}}} \sup_{u\in \mathbf{R}^d, n\geq 1} \sum_{i=1}^{d} \int_{0}^{t_6} \int_{\mathbf{R}^d} s^{-\frac{d+1}{2}} e^{-\frac{M_{14}|u-z|^2}{2s}} (|U_n^i|(z)dz + |\mu^i|(dz))ds,
\end{aligned}$$

which is less than $\frac{\varepsilon}{4}$ by (4.26).

On the other hand, since $\delta < T_0/2$, by (4.23)–(4.24) we have

$$\begin{aligned}
&\mathrm{II}(n,t,x,y)\\
&\quad \leq B^2 (\frac{2}{T_0})^{\frac{d+1}{2}} \sum_{i=1}^{d} \sup_{u\in \mathbf{R}^d} \int_{0}^{\delta} \int_{\mathbf{R}^d} s^{-\frac{d}{2}} e^{-\frac{M_{14}|u-z|^2}{2s}} (|U_n^i|(z)dz + |\mu^i|(dz))ds.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathrm{III}(n,t,x,y)\\
&\quad \leq B^2 (\frac{2}{T_0})^{\frac{d}{2}} \sum_{i=1}^{d} \sup_{u\in \mathbf{R}^d} \int_{0}^{\delta} \int_{\mathbf{R}^d} s^{-\frac{d+1}{2}} e^{-\frac{M_{14}|u-z|^2}{2s}} (|U_n^i|(z)dz + |\mu^i|(dz))ds.
\end{aligned}$$

Therefore, by (4.25), we conclude

$$\sup_{(t,x,y)\in[T_0,t_6]\times K\times K, n\geq 1} (\mathrm{II}(n,t,x,y) + \mathrm{III}(n,t,x,y)) < \frac{\varepsilon}{4}.$$

Now we estimate IV. Let

$$f_i(t,x,y,z) := \int_{\delta}^{t-\delta} p^D(s,x,z)\partial_{z_i} p^D(t-s,z,y)ds.$$

Note that by the Schauder type estimates for parabolic equations (see, for instance, Theorem 3.26 in [12] and Theorems 4.8 and 4.27 in [18]), $\partial_{w_i} p^D(s,w,v)$ is jointly continuous in $(0,t_6]\times D\times D$. By the continuity of $p^D(s,x,z)$ and $\partial_{z_i} p^D(t-s,z,y)$ and (4.23)–(4.24), $f_i(t,x,y,z)$ is uniformly continuous on

$[T_0, t_6] \times K \times K \times \overline{D_1}$. Therefore, by Lemma 3.2,

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{(t,x,y)\in[T_0,t_6]\times K\times K} \mathrm{IV}(n,t,x,y) \\
&\quad = \lim_{n\to\infty} \sum_{i=1}^{d} \sup_{(t,x,y)\in[T_0,t_6]\times K\times K} \left| \int_{D_1} f_i(t,x,y,z)(\mu_n^i - \mu^i)(dz) \right| = 0.
\end{aligned}$$

□

LEMMA 4.4. *For any compact subsets K_1 and K_2 of D, and $T_0 \in (0, t_6]$, we have for all $k \geq 1$,*

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{(t,x,y)\in[T_0,t_6]\times K_1\times K_2} |\widehat{I}_k^n(t,x,y) - \widehat{I}_k(t,x,y)| = 0, \\
&\lim_{n\to\infty} \sup_{(t,x,y)\in[T_0,t_6]\times K_1\times K_2} |\widehat{J}_k^n(t,x,y) - \widehat{J}_k(t,x,y)| = 0.
\end{aligned}$$

Proof. We only prove the first identity; the proof of the second is similar. Without loss of generality we may assume that $T_0 < 2$ and we will prove this lemma for the case $K := K_1 = K_2$ only. The previous lemma implies that the present lemma is valid for $k = 1$. We assume that the lemma is true for k, which implies that $\widehat{I}_k(s,x,z)$ is continuous. Given $\varepsilon > 0$, we choose $r_1 > 0$ as in (3.22) and then, using Proposition 2.2 and Lemma 3.1, choose $\delta = \delta(r_1, d, M_{14}) < \min(\frac{1}{2}T_0, 1)$ as in (4.25). For this δ, we choose a smooth domain $D_1 \subset \overline{D_1} \subset D$ such that for every $z \in D \setminus D_1$, $\rho(z) < d_1$ as in (4.26). Let

$$\begin{aligned}
\mathrm{I}(n,t,x,y) := \sum_{i=1}^{d} \Bigg(&\int_0^t \int_{D\setminus D_1} |\widehat{I}_k^n(t,x,y)||\nabla_z p^D(t-s,z,y)||U_n^i|(z)dzds \\
&+ \int_0^t \int_{D\setminus D_1} |\widehat{I}_k(t,x,y)||\nabla_z p^D(t-s,z,y)||\mu^i|(dz)dzds \Bigg),
\end{aligned}$$

$$\begin{aligned}
\mathrm{II}(n,t,x,y) := \sum_{i=1}^{d} \Bigg(&\int_0^\delta \int_D |\widehat{I}_k^n(t,x,y)||\nabla_z p^D(t-s,z,y)||U_n^i|(z)dzds \\
&+ \int_0^\delta \int_D \widehat{I}_k(t,x,y)|\nabla_z p^D(t-s,z,y)||\mu^i|(dz)dzds \Bigg),
\end{aligned}$$

$$\begin{aligned}
\mathrm{III}(n,t,x,y) := \sum_{i=1}^{d} \Bigg(&\int_{t-\delta}^t \int_D |\widehat{I}_k^n(t,x,y)||\nabla_z p^D(t-s,z,y)||U_n^i|(z)dzds \\
&+ \int_{t-\delta}^t \int_D \widehat{I}_k(s,x,z)|\nabla_z p^D(t-s,z,y)||\mu^i|(dz)dzds \Bigg),
\end{aligned}$$

$$\begin{aligned}\mathrm{IV}(n,t,x,y) := \sum_{i=1}^{d} \bigg| &\int_{\delta}^{t-\delta} \int_{D_1} \widehat{I}_k(s,x,z)\partial_{z_i} p^D(t-s,z,y) U_n^i(z) dz ds \\ &- \int_{\delta}^{t-\delta} \int_{D_1} \widehat{I}_k(s,x,z)\partial_{z_i} p^D(t-s,z,y)\mu^i(dz)) ds \bigg|\end{aligned}$$

and

$$\begin{aligned}\mathrm{V}(n,t,x,y) := \sum_{i=1}^{d} \int_{\delta}^{t-\delta} \int_{D_1} &|\widehat{I}_k^n(s,x,z) - \widehat{I}_k(s,x,z)| \times \\ &\times |\partial_{z_i} p^D(t-s,z,y)||U_n^i(z)| dz ds.\end{aligned}$$

Then we have

$$\begin{aligned}&|\widehat{I}_{k+1}^n(t,x,y) - \widehat{I}_{k+1}(t,x,y)| \\ &\qquad \le \mathrm{I}(n,t,x,y) + \mathrm{II}(n,t,x,y) + \mathrm{III}(n,t,x,y) + \\ &\qquad\qquad + \mathrm{IV}(n,t,x,y) + \mathrm{V}(n,t,x,y).\end{aligned}$$

By (4.23) and (4.24), we have

$$\begin{aligned}&\sup_{(x,y)\in K\times K, n\ge 1} \mathrm{I}(n,t,x,y) \\ &\le 2^{-k} B^2 \sup_{y\in K, n\ge 1} \sum_{i=1}^{d} \int_{\delta}^{t} \int_{D\setminus D_1} (1 \wedge \frac{\rho(z)}{s}) s^{-\frac{d}{2}} (t-s)^{-\frac{d+1}{2}} e^{-\frac{M_{14}|y-z|^2}{2(t-s)}} \times \\ &\qquad \times (|U_n^i|(z)dz + |\mu^i|(dz)) ds \\ &\le \frac{B^2 d_1}{\delta^{\frac{d+1}{2}}} \sup_{y\in K, n\ge 1} \sum_{i=1}^{d} \int_{\delta}^{t} \int_{\mathbf{R}^d} (t-s)^{-\frac{d+1}{2}} e^{-\frac{M_{14}|y-z|^2}{2(t-s)}} (|U_n^i|(z)dz + |\mu^i|(dz)) ds \\ &\le \frac{B^2 d_1}{\delta^{\frac{d+1}{2}}} \sup_{u\in \mathbf{R}^d, n\ge 1} \sum_{i=1}^{d} \int_{0}^{t_6} \int_{\mathbf{R}^d} s^{-\frac{d+1}{2}} e^{-\frac{M_{14}|u-z|^2}{2s}} (|U_n^i|(z)dz + |\mu^i|(dz)) ds,\end{aligned}$$

which is less than $\frac{\varepsilon}{4}$ by (4.26).

On the other hand, since $\delta < T_0/2$, by (4.23)–(4.24) we have

$$\begin{aligned}&\mathrm{II}(n,t,x,y) \\ &\le 2^{-k} B^2 (\frac{2}{T_0})^{\frac{d+1}{2}} \sum_{i=1}^{d} \sup_{u\in \mathbf{R}^d} \int_{0}^{\delta} \int_{\mathbf{R}^d} s^{-\frac{d}{2}} e^{-\frac{M_{14}|u-z|^2}{2s}} (|U_n^i|(z)dz + |\mu^i|(dz)) ds.\end{aligned}$$

Similarly,

$$\begin{aligned}&\mathrm{III}(n,t,x,y)\\&\le 2^{-k}B^2(\frac{2}{T_0})^{\frac{d}{2}}\sum_{i=1}^{d}\sup_{u\in\mathbf{R}^d}\int_0^{\delta}\int_{\mathbf{R}^d}s^{-\frac{d+1}{2}}e^{-\frac{M_{14}|u-z|^2}{2s}}(|U_n^i|(z)dz+|\mu^i|(dz))ds.\end{aligned}$$

Therefore, by (4.25), we conclude

$$\sup_{\substack{t\in[T_0,t_6],\\(x,y)\in K\times K,n\ge 1}}(\mathrm{II}(n,t,x,y)+\mathrm{III}(n,t,x,y))<\frac{\varepsilon}{4}.$$

Now we estimate IV. Let

$$f_i(t,x,y,z):=\int_{\delta}^{t-\delta}\widehat{I}_k(s,x,z)\partial_{z_i}p^D(t-s,z,y)ds.$$

By the continuity of $\widehat{I}_k(s,x,z)$ and $\partial_{z_i}p^D(t-s,z,y)$ and (4.23)–(4.24), we know that $f_i(t,x,y,z)$ is uniformly continuous on $[T_0,t_6]\times K\times K\times\overline{D_1}$. Therefore, by Lemma 3.2,

$$\begin{aligned}&\lim_{n\to\infty}\sup_{\substack{t\in[T_0,t_6],\\(x,y)\in K\times K}}\mathrm{IV}(n,t,x,y)\\&\qquad=\lim_{n\to\infty}\sum_{i=1}^{d}\sup_{\substack{t\in[T_0,t_6],\\(x,y)\in K\times K}}\left|\int_{D_1}f_i(t,x,y,z)(\mu_n^i-\mu^i)(dz)\right|=0.\end{aligned}$$

Finally, we estimate V. From (4.24), we easily see that

$$\begin{aligned}\mathrm{V}(n,t,x,y)\le Bt_6(\frac{2}{T_0})^{\frac{d+1}{2}}&\sup_{\substack{t\in[T_0,t_6],\\(x,z)\in K\times\overline{D_1}}}|\widehat{I}_k^n(s,x,z)-\widehat{I}_k(s,x,z)|\times\\&\times\sup_{n\ge 1}\sum_{i=1}^{d}\int_0^t\int_{\mathbf{R}^d}(t-s)^{-\frac{d+1}{2}}e^{-\frac{M_{14}|z-y|^2}{2(t-s)}}|U_n^i(z)|dzds.\end{aligned}$$

By Proposition 2.2 and Lemma 3.1,

$$\sup_{n\ge 1,y\in\mathbf{R}^d}\sum_{i=1}^{d}\int_0^t\int_{\mathbf{R}^d}(t-s)^{-\frac{d+1}{2}}e^{-\frac{M_{14}|z-y|^2}{2(t-s)}}|U_n^i(z)|dzds$$

is bounded. Therefore

$$\lim_{n\to\infty}\sup_{(t,x,y)\in[T_0,t_6]\times K\times K}\mathrm{V}(n,t,x,y)=0$$

by the assumption on $\widehat{I}_k^n$. □

The proof of the next theorem is similar to the proofs of Theorems 3.8 and 3.11.

THEOREM 4.5. *$q^D_n(t,x,y)$ and $\nabla_x q^D_n(t,x,y)$ converge uniformly on any compact subset of* $(0,\infty)\times D\times D$.

Proof. We will prove the uniform convergence of $q^D_n(t,x,y)$ only. By (4.23), for every $T_0\in(0,t_6)$ and any compact subsets $K_1,K_2\subset D$

$$\sup_{(t,x,y)\in[T_0,t_6]\times K_1\times K_2}\sum_{k=0}^{\infty}|\widehat{I}^n_k(t,x,y)|<\infty.$$

Therefore Lemma 4.4 and a standard ε-δ argument give the uniform convergence of $q^D_n(t,x,y)$ on any compact subsets of $(0,t_6]\times D\times D$.

Let $D_r:=\{z\in D:\rho(z)<r\}$. The uniform upper bounds of q^D_n and q^D imply that for $r<(t_6/2)^{1/2}$,

$$\begin{aligned}
&\sup_{\substack{n\ge 1,t\in(t_6,\frac{3}{2}t_6],\\ (x,y)\in K_1\times K_2}}\int_{D_r}\Big(q^D_n(\frac{t_6}{2},x,z)q^D_n(t-\frac{t_6}{2},z,y)+\\
&\qquad+q^D(\frac{t_6}{2},x,z)q^D(t-\frac{t_6}{2},z,y)\Big)\,dz\\
&\le c_1(t_6)^{-d}\int_{D_r}(1\wedge\frac{\sqrt{2}\rho(z)}{\sqrt{t_6}})dz\le\sqrt{2}c_1t_6^{-d-\frac{1}{2}}|D|r
\end{aligned}$$

for some positive constants c_1 and c_2. For any given $\varepsilon>0$, we can choose $0<r<(t_6/2)^{1/2}$ small such that $\sqrt{2}c_1t_6^{-d-\frac{1}{2}}|D|r<\frac{\varepsilon}{2}$. By the Chapman-Kolmogorov equation, we have for $(t,x,y)\in(t_6,\frac{3}{2}t_6]\times K_1\times K_2$,

$$\begin{aligned}
&\left|q^D_n(t,x,y)-\int_D q^D(\frac{t_6}{2},x,z)q^D(t-\frac{t_6}{2},z,y)dz\right|\\
&=\left|\int_D q^D_n(\frac{t_6}{2},x,z)q^D_n(t-\frac{t_6}{2},z,y)-q^D(\frac{t_6}{2},x,z)q^D(t-\frac{t_6}{2},z,y)dz\right|\\
&<c_3|D|\sup_{\substack{t\in[\frac{1}{2}t_6,t_6],\\ (x,y)\in K_1\times K_2,\rho(z)\ge r}}\Big|q^D_n(\frac{t_6}{2},x,z)q^D_n(t-\frac{t_6}{2},z,y)\\
&\qquad-q^D(\frac{t_6}{2},x,z)q^D(t-\frac{t_6}{2},z,y)\Big|+\frac{\varepsilon}{2}
\end{aligned}$$

for some positive constant c_3. The first term in the last line above goes zero as $n\to\infty$ by the uniform convergence of $q^D_n(t,x,y)$ on compact subsets of $(0,t_6]\times D\times D$. The general case can be proved by induction. □

Recall that X is the Brownian motion with drift μ. Define $\tau_D:=\inf\{t>0: X_t\notin D\}$. Let $X^D_t(\omega)=X_t(\omega)$ if $t<\tau_D(\omega)$ and set $X^D_t(\omega)=\partial$ if $t\ge\tau_D(\omega)$, where ∂ is a cemetery point added to D. The process X^D is called a killed Brownian motion with drift μ in D.

The following is the main result of this section.

Theorem 4.6. *For any $\mu \in \mathbf{K}_{d,1}$, X^D has transition density $q^D(t,x,y)$, which is jointly continuous on $(0,\infty)\times D\times D$. For each $T>0$ there exist positive constants $M_j, 11\le j\le 15$, depending on μ only via the rate at which $\max_{1\le i\le d} M^1_{\mu^i}(r)$ goes to zero, such that*

$$M_{11}t^{-\frac{d}{2}}\psi(t,x,y)e^{-\frac{M_{12}|x-y|^2}{2t}} \le q^D(t,x,y) \le M_{13}t^{-\frac{d}{2}}\psi(t,x,y)e^{-\frac{M_{14}|x-y|^2}{2t}} \tag{4.27}$$

and

$$|\nabla_x q^D(t,x,y)| \le M_{15}(1\wedge \frac{\rho(y)}{\sqrt{t}})t^{-\frac{d+1}{2}}e^{-\frac{M_{14}|x-y|^2}{2t}} \tag{4.28}$$

for all $(t,x,y)\in(0,T)\times D\times D$.

Proof. Recall that X^n is the Brownian motion with drift U_n. Similar to X^D, we use $X^{n,D}$ to denote the process obtained by killing X^n upon exiting from D. The density function for $X^{n,D}_t$ is the heat kernel $q^D_n(t,x,y)$ for $\frac{1}{2}\Delta + U_n\cdot\nabla$ in D. In the proof of Theorem 3.14, we have shown that X^n_t converges weakly to X_t in $C([0,\infty),\mathbf{R}^d)$. Since X^n_t has continuous sample paths and D is a bounded $C^{1,1}$ domain, for every $t>0$ and $x\in\mathbf{R}^d$,

$$\mathbf{P}_x[\tau_D = t] \le \int_{\partial D} q(t,x,y)dy = 0 \tag{4.29}$$

Since $\{\tau_D > t\}$ is open in $C([0,\infty),\mathbf{R}^d)$, using (4.29), we get that for any bounded continuous function f in D,

$$\lim_{n\to\infty}\mathbf{E}_x[f(X^n_t)1_{\{t<\tau_D\}}] = \mathbf{E}_x[f(X_t)1_{\{t<\tau_D\}}], \quad \forall x\in D, t>0.$$

So by Theorem 4.5,

$$q^D(t,x,y) := \lim_{n\to\infty} q^D_n(t,x,y)$$

is the density for X^D_t. Since $q^D(t,x,y)$ is the uniform limit of jointly continuous functions on any compact subset of $(0,\infty)\times D\times D$, it is jointly continuous on $(0,\infty)\times D\times D$. Now (4.27) follows immediately from Theorem 4.5 and (4.21). Using Theorem 4.5 we see that for any $(t,y)\in(0,\infty)\times D$ and any relatively compact open subset D_0 of D, the sequence of functions $\{q_n(t,\cdot,y) : n\ge 1\}$ is a Cauchy sequence in the Banach space $C^1(\overline{D_0})$, thus $\nabla_x q^D(t,x,y)$ exists for every $(t,x,y)\in(0,\infty)\times D\times D$ and it satisfies (4.28). $\square$

Similar to Theorem 3.15, we have the following result.

THEOREM 4.7. *For any $\mu \in \mathbf{K}_{d,1}$, the density $q^D(t,x,y)$ of X^D satisfies the equations*

$$q^D(t,x,y) = p^D(t,x,y) + \int_0^t \int_D q^D(s,x,z) \cdot \nabla_z p^D(t-s,z,y)\mu(dz)ds,$$

$$\nabla_x q^D(t,x,y) = \nabla_x p^D(t,x,y) + \\ + \int_0^t \int_D \nabla_x q^D(s,x,z) \cdot \nabla_z p^D(t-s,z,y)\mu(dz)ds$$

for all $(t,x,y) \in (0,\infty) \times D \times D$.

Proof. We omit the details. □

5. Parabolic Harnack principle for X

In this section we shall prove the small time parabolic Harnack principle for X. With the density estimates of the last two sections in hand, one can follow the ideas in [13] (see also [27]) to prove the parabolic Harnack principle. For this reason, the proofs in this section will be omitted.

Throughout this section we assume that D is a bounded $C^{1,1}$ domain in $\mathbf{R}^d$. We start with the following observation on the proof of Theorem 4.2: We have chosen $t_0 \ge t_1 \ge t_6$ small enough so that (4.9), (4.12) and (4.19) are true, respectively. The estimates of the density of $q^D(t,x,y)$ for $t \le t_6$ we get depend only on t_6, the estimates of $p^D(t,x,y)$ and the characteristics (r_0,Λ) of D. So the constants $M_j, 11 \le j \le 15$, are invariant under translation and Brownian scaling:

For any $z \in \mathbf{R}^d$, $0 < r < \infty$ and bounded $C^{1,1}$ domain D, let

$$D^z_r := z + rD,$$

$$\psi_{D^z_r}(t,x,y) := (1 \wedge \frac{\rho_{D^z_r}(x)}{\sqrt{t}})(1 \wedge \frac{\rho_{D^z_r}(y)}{\sqrt{t}}), \quad (t,x,y) \in (0,\infty) \times D^z_r \times D^z_r,$$

where $\rho_{D^z_r}(x)$ denotes the distance between x and ∂D^z_r. Then, for any $T > 0$, there exist positive constants t_0 and $M_j, 11 \le j \le 15$, independent of z and r and depending on μ only via the rate at which $\max_{1 \le i \le d} M^1_{\mu^i}(r)$ goes to zero, such that

$$M_{11} t^{-\frac{d}{2}} \psi_{D^z_r}(t,x,y) e^{-\frac{M_{12}|x-y|^2}{2t}} \le q^{D^z_r}(t,x,y) \\ \le M_{13} t^{-\frac{d}{2}} \psi_{D^z_r}(t,x,y) e^{-\frac{M_{14}|x-y|^2}{2t}} \tag{5.1}$$

and

$$|\nabla_x q^{D^z_r}(t,x,y)| \le M_{15} (1 \wedge \frac{\rho_{D^z_r}(y)}{\sqrt{t}}) t^{-\frac{d+1}{2}} e^{-\frac{M_{14}|x-y|^2}{2t}} \tag{5.2}$$

for all $(t,x,y) \in (0, t_0 \wedge (r^2 T)] \times D^z_r \times D^z_r$. We will sometimes suppress the indices from D^z_r when it is clear from the context.

The next lemma is an easy consequence of (5.1).

LEMMA 5.1. *There exists a constant* $t_0 = t_0(d, \mu)$ *depending on* μ *only via the rate at which* $\max_{1\le i\le d} M^1_{\mu^i}(r)$ *goes to zero such that for each* $0 < \delta, r < 1$, *there exists* $\varepsilon = \varepsilon(d, \delta, r) > 0$ *such that*

$$q^{B(x_0,R)}(t, x, y) \ge \frac{\varepsilon}{|B(x_0, \delta R)|} \tag{5.3}$$

for all $x, y \in B(x_0, \delta R)$, *and* $(1 - r)R^2 \le t \le R^2 \le t_0$.

Proof. We omit the details. □

We adopt the notation from [6] and define a space-time process $Z_s := (T_s, X_s)$, where $T_s = T_0 + s$. The law of the space-time process Z_s starting from (t, x) will be denoted by $\mathbf{P}_{t,x}$.

DEFINITION 5.2. For any $(t, x) \in [0, \infty) \times \mathbf{R}^d$, $r > 0$ and bounded open subset D of $\mathbf{R}^d$, we say that a nonnegative continuous function u defined on $[t, t + r] \times D$ is parabolic in $[t, t + r] \times D$ if for any $[s_1, s_2] \subset [t, t + r)$ and $B(y, \delta) \subset \overline{B(y, \delta)} \subset D$ we have

$$u(s, z) = \mathbf{E}_{(s,z)} \left[u(Z_{\tau_{[s_1,s_2)\times B(y,r)}}) \right], \quad (s, z) \in [s_1, s_2) \times B(y, \delta), \tag{5.4}$$

where $\tau_{[s_1,s_2)\times B(y,\delta)} = \inf\{s > 0 : Z_s \notin [s_1, s_2) \times B(y, \delta)\}$.

LEMMA 5.3. *For each* $T > 0$ *and* $y \in D$, $(t, x) \to q^D(T - t, x, y)$ *is parabolic in* $[0, T) \times D$.

Proof. See the proof of Lemma 4.5 in [6]. □

COROLLARY 5.4. *For any* $T \in (0, \infty)$ *and any nonnegative bounded function* f *on* D, *the function*

$$u(t, x) := \int_D q^D(T - t, x, y) f(y) dy$$

is parabolic on $[0, T) \times D$.

Proof. The continuity of u follows from the continuity of q^D. (5.4) follows from Lemma 5.3 and Fubini's theorem. □

For $s \ge 0$, $x \in \mathbf{R}^d$ and $R > 0$, let

$$\begin{aligned}\mathrm{Osc}(u; s, x, R) = \sup \{&|u(t_1, x_1) - u(t_2, x_2)| : \\ &s < t_1, t_2 < s + R^2, x_1, x_2 \in B(x, R)\}.\end{aligned}$$

We assume that t_0 is the constant from Lemma 5.1 in the remainder of this section.

LEMMA 5.5. *For any $0 < \delta < 1$, there exist $0 < \rho < 1$ such that for all $s \in [0, \infty)$, $x_0 \in \mathbf{R}^d$, $0 < R \le \sqrt{t_0}$ and every function u which is parabolic in $[s, s+R^2) \times B(x_0, R)$ and continuous in $[s, s+R^2] \times \overline{B(x_0, R)}$*

$$\mathrm{Osc}(u; s, x_0, \delta R) \le \rho\ \mathrm{Osc}(u; s, x_0, R).$$

Proof. The proof is similar to that of the corresponding result in [13] and we omit the details. □

The above lemma implies:

THEOREM 5.6. *All parabolic functions are Hölder continuous. More precisely, for any $0 < \delta < 1$, there exist $C > 0$ and $\beta \in (0, 1)$ such that for all $s \in [0, \infty)$, $x_0 \in \mathbf{R}^d$, $0 < R \le \sqrt{t_0}$ and every function u which is parabolic in $[s, s+R^2) \times B(x_0, R)$ and continuous in $[s, s+R^2] \times \overline{B(x_0, R)}$ we have*

$$|u(t_1, x_1) - u(t_2, x_2)| \le C\|u\|_{L^\infty([s,s+R^2]\times\overline{B(x_0,R)})} \left(\frac{|t_1 - t_2|^2 + |x_1 - x_2|}{R}\right)^\beta$$

for any $(t_1, x_1), (t_2, x_2) \in [s, s+\delta R^2] \times \overline{B(x_0, \delta R)}$.

Proof. See Theorem 5.3 in [13]. □

Using Lemma 5.1 and 5.5, the proof of the next theorem is almost identical to Theorem 5.4 in [13]. So we omit the proof.

THEOREM 5.7. *For any $0 < \alpha < \beta < 1$ and $0 < \delta < 1$, there exists $M > 0$ such that for all $x_0 \in \mathbf{R}^d$, $s \in [0, \infty)$, $0 < R \le \sqrt{t_0}$ and every function u which is parabolic in $[s, s+R^2) \times B(x_0, R)$ and continuous in $[s, s+R^2] \times \overline{B(x_0, R)}$, we have*

$$u(t, y) \le M u(s, x_0), \quad (t, y) \in [s + \alpha R^2, s + \beta R^2] \times B(x_0, \delta R).$$

Now the parabolic Harnack inequality is an easy corollary of the above theorem.

THEOREM 5.8 (Parabolic Harnack principle). *For any $0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < 1$ and $0 < \delta < 1$, there exists $M > 0$ such that for all $x_0 \in \mathbf{R}^d$, $0 < R \le \sqrt{t_0}$ and every function u which is parabolic in $[0, R^2) \times B(x_0, R)$ and continuous in $[0, R^2] \times \overline{B(x_0, R)}$, we have*

$$\sup_{B_2} u \le M \inf_{B_1} u,$$

where $B_i = \{(t, y) \in [\alpha_i R^2, \beta_i R^2] \times B(x_0, \delta R)\}$.

DEFINITION 5.9. Suppose D is an open subset of $\mathbf{R}^d$. A nonnegative continuous function f on D is said to be harmonic in D with respect to X if for every relatively compact open subset B of D,

$$f(x) = \mathbf{E}_x[f(X_{\tau_B})], \quad x \in D, \tag{5.5}$$

where $\tau_B = \inf\{s > 0 : X_s \notin B\}$.

The parabolic Harnack inequality implies the Harnack inequality.

COROLLARY 5.10 (Harnack principle). *There exist $r_1 = r_1(d, \mu) > 0$ and $N = N(d, \mu) > 0$ depending on μ only via the rate at which $\max_{1\le i\le d} M^1_{\mu^i}(r)$ goes to zero such that for every harmonic function f in $B(x_0, R)$ with $R \in (0, r_1)$, we have*

$$\sup_{y\in B(x_0,R/4)} f(y) \le N \inf_{y\in B(x_0,R/4)} f(y).$$

6. Green function estimates and boundary Harnack principle

The main objective of this section is to obtain two-sided estimates for the Green function G_D of X^D and prove the boundary Harnack principle for nonnegative harmonic functions of X. Throughout this section we also assume that D is a bounded $C^{1,1}$ domain in $\mathbf{R}^d$. We start with the following simple result.

LEMMA 6.1. *There exist positive constants C_1 and C_2 such that*

$$q^D(t, x, y) \le C_1 e^{-C_2 t}, \quad (t, x, y) \in (1, \infty) \times D \times D.$$

Proof. Let $l < \infty$ be the diameter of D. Recall that $\tau_D := \inf\{t > 0 : X_t \notin D\}$. By (3.33), for every $x \in D$ we have

$$\begin{aligned}\mathbf{P}_x(\tau_D \le 1) \ge \mathbf{P}_x(X_1 \in \mathbf{R}^d \setminus D) &= \int_{\mathbf{R}^d\setminus D} q^\mu(1, x, y)dy \\ &\ge M_1 e^{-M_2} \int_{\mathbf{R}^d\setminus D} e^{-\frac{M_3}{2}|x-y|^2} dy \\ &\ge M_1 e^{-M_2} \int_{\{\,|z|\ge l\,\}} e^{-\frac{M_3}{2}|z|^2} dz \;>\; 0.\end{aligned}$$

Thus

$$\sup_{x\in D} \int_D q^D(1, x, y)dy \;=\; \sup_{x\in D} \mathbf{P}_x(\tau_D > 1) \;<\; 1.$$

The Markov property implies that there exist positive constants c_1 and c_2 such that

$$\int_D q^D(t, x, y)dy \;\le\; c_1 e^{-c_2 t}, \quad (t, x) \in (0, \infty) \times D.$$

It follows from Theorem 4.6 that there exists $c_3 > 0$ such that

$$q^D(1, x, y) \le c_3, \quad (x, y) \in D \times D.$$

Thus for any $(t, x, y) \in (1, \infty) \times D \times D$, we have

$$\begin{aligned} q^D(t, x, y) &= \int_D q^D(t-1, x, z) q^D(1, z, y) dz \\ &\le c_3 \int_D q^D(t-1, x, z) dz \le c_1 c_3 e^{-c_2(t-1)}. \end{aligned}$$

□

Combining the above result with (4.27) we know that there exist positive constants M_{16} and M_{17} such that for any $(t, x, y) \in (0, \infty) \times D \times D$,

$$q^D(t, x, y) \le M_{17} t^{-\frac{d}{2}} \exp\left(-\frac{M_{16}|x-y|^2}{2t}\right). \tag{6.1}$$

Therefore the Green function

$$G_D(x, y) := \int_0^\infty q^D(t, x, y) dt$$

is finite for $x \ne y$.

The following theorem is one of the main results of this section. It should be compared with Theorem 5.1 of [16], where the same conclusion is obtained under the assumptions that each μ^i is given by $\mu^i(dx) = U^i(x)dx$ with $U^i \in \mathbf{K}_{d,1}$ and that

$$\sup_{x \in D} \int_D \left(1 \wedge \frac{\rho(y)}{|x-y|}\right) \frac{|U^i(y)|}{|x-y|^{d-1}} dy$$

is sufficiently small for each $i = 1, \dots, d$. Recall that $\rho(x)$ is the distance between x and ∂D.

Theorem 6.2. *Let D be a bounded $C^{1,1}$ domain in $\mathbf{R}^d$. There exist constants $M_{18} = M_{18}(D, \mu) > 0$ and $M_{19} = M_{19}(D, \mu) > 0$ depending on μ only via the rate at which* $\max_{1 \le i \le d}$ *goes to zero such that for $x, y \in D$,*

$$\begin{aligned} M_{18} (1 \wedge \frac{\rho(x)\rho(y)}{|x-y|^2}) \frac{1}{|x-y|^{d-2}} &\le G_D(x, y) \\ &\le M_{19} (1 \wedge \frac{\rho(x)\rho(y)}{|x-y|^2}) \frac{1}{|x-y|^{d-2}}. \end{aligned} \tag{6.2}$$

Proof. As $\rho(x) \le \rho(y) + |x-y|$ for every $x, y \in D$, it is easy to see that

$$(1\wedge \frac{\rho(x)}{|x-y|})\,(1\wedge \frac{\rho(y)}{|x-y|}) \;\le\; 1\wedge \frac{\rho(x)\rho(y)}{|x-y|^2} \;\le\; 2\,(1\wedge \frac{\rho(x)}{|x-y|})\,(1\wedge \frac{\rho(y)}{|x-y|}). \tag{6.3}$$

Put $T := \mathrm{diam}(D)^2$, where $\mathrm{diam}(D)$ is the diameter of D. First we show the lower bound of G_D. By the change of variable $u = \frac{|x-y|^2}{t}$, we have

$$\begin{aligned} &\int_0^T (1\wedge \frac{\rho(x)}{\sqrt{t}})(1\wedge \frac{\rho(y)}{\sqrt{t}})t^{-\frac{d}{2}} e^{-\frac{M_{12}|x-y|^2}{2t}}dt \\ &\quad = \frac{1}{|x-y|^{d-2}} \int_{\frac{|x-y|^2}{T}}^\infty u^{\frac{d-4}{2}} (1\wedge \frac{\sqrt{u}\rho(x)}{|x-y|})(1\wedge \frac{\sqrt{u}\rho(y)}{|x-y|}) e^{-\frac{1}{2}M_{12}u}du. \end{aligned} \tag{6.4}$$

Then by (4.27) and the above identity,

$$\begin{aligned} G(x,y) &\ge \int_0^T q^D(t,x,y)dt \\ &\ge M_{11}\int_0^T (1\wedge \frac{\rho(x)}{\sqrt{t}})(1\wedge \frac{\rho(y)}{\sqrt{t}})t^{-\frac{d}{2}} e^{-\frac{M_{12}|x-y|^2}{2t}}dt \\ &\ge \frac{M_{11}}{|x-y|^{d-2}}\int_1^\infty u^{\frac{d-4}{2}} (1\wedge \frac{\sqrt{u}\rho(x)}{|x-y|})(1\wedge \frac{\sqrt{u}\rho(y)}{|x-y|}) e^{-\frac{1}{2}M_{12}u}du \\ &\ge C(1\wedge \frac{\rho(x)}{|x-y|})(1\wedge \frac{\rho(y)}{|x-y|})\frac{1}{|x-y|^{d-2}}, \end{aligned}$$

where $C = M_{11}\int_1^\infty u^{\frac{d-4}{2}} e^{-\frac{1}{2}M_{12}u}du$.

Now we show the upper bound of G_D. Using (4.27) and (6.4), we have

$$\begin{aligned} &\int_0^T q^D(t,x,y)dt \\ &\le M_{13}\int_0^T (1\wedge \frac{\rho(x)}{\sqrt{t}})(1\wedge \frac{\rho(y)}{\sqrt{t}})t^{-\frac{d}{2}} e^{-\frac{M_{14}|x-y|^2}{2t}}dt \\ &= \frac{M_{13}}{|x-y|^{d-2}}\int_{\frac{|x-y|^2}{T}}^\infty u^{\frac{d-4}{2}} (1\wedge \frac{\sqrt{u}\rho(x)}{|x-y|})(1\wedge \frac{\sqrt{u}\rho(y)}{|x-y|}) e^{-\frac{1}{2}M_{14}u}du \\ &\le \frac{M_{13}}{|x-y|^{d-2}}(1\wedge \frac{\rho(x)}{|x-y|})(1\wedge \frac{\rho(y)}{|x-y|})\int_0^\infty u^{\frac{d-4}{2}}(u\vee 1) e^{-\frac{1}{2}M_{14}u}du. \end{aligned} \tag{6.5}$$

On the other hand, by the Chapman-Kolmogorov equation, (4.27) and (6.1), if $t > T$ we have

$$
\begin{aligned}
q^D(t,x,y) &= \int_D\int_D q^D(\frac{T}{2},x,z)q^D(t-T,z,w)q^D(\frac{T}{2},w,y)dzdw \\
&\le 2M_{13}^2M_{17}(1\wedge\frac{\rho(x)}{\sqrt{T}})(1\wedge\frac{\rho(y)}{\sqrt{T}})\int_D\int_D(\frac{T}{2})^{-d}(t-T)^{-\frac{d}{2}}\times \\
&\quad \times e^{-\frac{M_{14}|x-z|^2}{T}}e^{-\frac{M_{16}|z-w|^2}{2(t-T)}}e^{-\frac{M_{14}|w-y|^2}{T}}dzdw \\
&\le 2M_{13}^2M_{17}(1\wedge\frac{\rho(x)}{|x-y|})(1\wedge\frac{\rho(y)}{|x-y|})\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}(\frac{T}{2})^{-d}(t-T)^{-\frac{d}{2}}\times \\
&\quad \times e^{-\frac{M_{14}|x-z|^2}{T}}e^{-\frac{M_{16}|z-w|^2}{2(t-T)}}e^{-\frac{M_{14}|w-y|^2}{T}}dzdw \\
&\le C_0M_{13}^2M_{17}(1\wedge\frac{\rho(x)}{|x-y|})(1\wedge\frac{\rho(y)}{|x-y|})t^{-\frac{d}{2}}e^{-\frac{C_1|x-y|^2}{2t}},
\end{aligned}
$$

where $C_0 = C_0(d)$ and $C_1 = C_1(M_{14}, M_{16})$. Since

$$
\int_T^\infty t^{-\frac{d}{2}}e^{-\frac{C_1|x-y|^2}{2t}}dt = \frac{1}{|x-y|^{d-2}}\int_0^{\frac{|x-y|^2}{T}} u^{\frac{d-4}{2}}e^{-\frac{1}{2}C_1u}du,
$$

we have

$$
\begin{aligned}
\int_T^\infty q^D(t,x,y)dt &\le C_0M_{13}^2M_{17}\frac{1}{|x-y|^{d-2}}\times \\
&\quad\times\left(1\wedge\frac{\rho(x)}{|x-y|}\right)\left(1\wedge\frac{\rho(y)}{|x-y|}\right)\int_0^\infty u^{\frac{d-4}{2}}e^{-\frac{1}{2}C_1u}du.
\end{aligned}
\tag{6.6}
$$

Combining (6.3), (6.5) and (6.6), we have proved the upper estimate for Green function. □

The next corollary is an easy consequences of Theorem 6.2.

Corollary 6.3 (3G Theorem). *For any bounded $C^{1,1}$ domain D in $\mathbf{R}^d$, there exists a constant $M_{20} > 0$ depending on μ only via the rate at which* $\max_{1\le i\le d}$ *goes to zero such that for $x, y \in D$,*

$$
\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \;\le\; M_{20}\,\frac{|x-z|^{d-2}}{|x-y|^{d-2}|y-z|^{d-2}}
\tag{6.7}
$$

and

$$
\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \;\le\; M_{20}\,\left(\frac{\rho(y)}{\rho(x)}G_D(x,y)+\frac{\rho(y)}{\rho(z)}G_D(y,z)\right).
$$

Proof. The proof of this result is elementary. We omit the details. For the proof of the last inequality, one can see the proof of Proposition 4.2 in [7]. □

In the remainder of this section, we will prove a boundary Harnack principle for nonnegative harmonic functions in D with respect to X.

The following proposition is also an easy consequence of the Green function estimate.

Proposition 6.4. *Suppose D be a bounded $C^{1,1}$ domain in $\mathbf{R}^d$, V is an open set of $\mathbf{R}^n$ and K is a compact subset of V. Then there exists a constant $c = c(D, V, K, \mu) > 1$ depending on μ only via the rate at which $\max_{1\le i\le d} M^1_{\mu^i}(t)$ goes to zero such that for every $x_1, x_2 \in K \cap D$ and $y_1, y_2 \in D \setminus V$,*

$$c^{-1}\,\frac{G_D(x_1,y_1)}{G_D(x_2,y_1)} \;\le\; \frac{G_D(x_1,y_2)}{G_D(x_2,y_2)} \;\le\; c\,\frac{G_D(x_1,y_1)}{G_D(x_2,y_1)}. \tag{6.8}$$

Proof. We skip the details. □

If D is a bounded $C^{1,1}$ domain, then it is easy to check that there exists $R > 0$ such that for any $x \in \partial D$ and $r \in (0, R)$, $B(x, r) \cap D$ is connected.

Theorem 6.5 (Boundary Harnack principle). *Suppose D be a bounded $C^{1,1}$ domain in $\mathbf{R}^d$ and R is the above constant. Then for any $r \in (0, R)$ and $z \in \partial D$, there exists a constant $c > 1$ depending on μ only via the rate at which $\max_{1\le i\le d}$ goes to zero such that for any nonnegative functions u, v which are harmonic in $D \cap B(z, r)$ with respect to X and vanish continuously on $\partial D \cap B(z, r)$, we have*

$$\frac{u(x)}{v(x)} \;\le\; c\,\frac{u(y)}{v(y)} \qquad \text{for any } x, y \in D \cap B(z, r/2).$$

Proof. One can find bounded $C^{1,1}$ domains $D_1 \subset D_2$, which are relatively compact in $D \cap B(z, r)$ such that $D \cap B(z, r/2) \subset D_1 \subset D_2 \subset D \cap B(z, r)$. Define $T_{D_1} := \inf\{t \ge 0 :\; X_t \in D_1\}$. Let

$$f(x) := \mathbf{E}_x\left[u(X^{D_2}_{T_{D_1}})\right], \quad x \in D_2.$$

Obviously $f = u$ in D_1. f is an excessive function of X^{D_2} which is harmonic in D_1 and vanishes continuously on ∂D_2, so by the Riesz decomposition theorem in [9] we know that there exists a Radon measure ν supported on $D_2 \setminus D_1$ such that

$$f(x) = \int_{D_2\setminus D_1} G_{D_2}(x, y)\nu(dy).$$

Fix a $z_0 \in D_2 \setminus D_1$. By (6.8), for every $x, y \in D \cap B(z, r/2)$ and $w \in D_2 \setminus D_1$,

$$c^{-1}\, G_{D_2}(x, w) \;\le\; \frac{G_{D_2}(x, z_0)}{G_{D_2}(y, z_0)} G_{D_2}(y, w) \;\le\; c\, G_{D_2}(x, w).$$

Therefore, integrating the above expression over $w \in D_2 \setminus D_1$ with respect to ν gives

$$c^{-1}\, u(x) \;\le\; \frac{G_{D_2}(x, z_0)}{G_{D_2}(y, z_0)} u(y) \;\le\; c\, u(x). \qquad \square$$

7. Martin kernel and Martin boundary

Throughout this section we will assume that D is a bounded $C^{1,1}$ domain in $\mathbf{R}^d$. We will show in this section that the Martin boundary and the minimal Martin boundary of X^D can all be identified with the Euclidean boundary ∂D of D. There are at least two existing arguments in the literature to identify the Martin boundary of killed diffusion processes corresponding to differential operators: the argument in [1] uses a harmonic space approach and the argument in [21] uses some deep results from PDE. Brownian motions with singular drift do not fit into the framework of harmonic spaces in general, so the argument in [1] does not apply to the present case. Some of the deep results from PDE used in [21] are not available for Brownian motions with singular drift, so the argument in [21] does not apply to the present case either.

Fix $x_0 \in D$ and define

$$M_D(x,y) := \begin{cases} \frac{G_D(x,y)}{G_D(x_0,y)}, & \text{if } x \in D \text{ and } y \in D \setminus \{x_0\}\ , \\ 1_{\{x_0\}}(x), & \text{if } y = x_0. \end{cases}$$

Note that for each $y \in D\setminus\{x_0\}$ and $\varepsilon > 0$, $M_D(\,\cdot\,, y)$ is a harmonic function with respect to X in $D \setminus B(y,\varepsilon)$ and

$$M_D(x,y) = \mathbf{E}_x \left[M_D(X_{\tau_{D\setminus B(y,\varepsilon)}}, y) \right], \qquad x \in D \setminus B(y,\varepsilon).$$

Using Theorem 6.2, we can easily see that

$$M_D(x,y) \le c\, \frac{\rho(x)(\rho(y) \vee 1)}{|x-y|^d}, \qquad (x,y) \in D \times D. \tag{7.1}$$

Combining Theorem 6.2 with the results of [9] we know that the Riesz decomposition theorem holds for the excessive functions of X^D. Using the Riesz decomposition theorem, the Harnack inequality (5.10) and the Hölder continuity of harmonic functions (Theorem 5.6), one can follow the arguments in [19] (see also Section 2.7 of [3] or [26]) to show that there is a unique compactification D^M of D up to homeomorphism satisfying the following properties:

- (M1) D is open and dense in D^M and its relative topology coincides with its original topology;
- (M2) $M_D(x,\cdot)$ can be extended to D^M uniquely in such a way that the function $M_D(x,\xi)$ is jointly continuous on $D \times (D^M \setminus \{x_0\})$, and for each $\xi \in D^M \setminus D$, $M_D(x,y)$ converges to $M_D(x,\xi)$ as $y \to \xi$, and $M_D(\cdot,\xi_1) \ne M_D(\cdot,\xi_2)$ if $\xi_1 \ne \xi_2$.

The boundary $\partial_M D := D^M \setminus D$ is called the Martin boundary of X^D. The bound (7.1) and the harmonicity of $M_D(\,\cdot\,, y)$ in $D\setminus\{y\}$ imply the harmonicity of $M_D(\,\cdot\,,\xi)$ in D for every $\xi \in \partial_M D$.

PROPOSITION 7.1. *For every $z \in \partial_M D$, $x \mapsto M(x,z)$ is harmonic with respect to X in D.*

Proof. Fix $z \in \partial_M D$ and a relatively compact open set U in D. Choose a sequence $\{y_n\}_{n\geq 1}$ in $D \setminus \overline{U}$ converging to z in $D \cup \partial_M D$ so that

$$M_D(x,z) = \lim_{n\to\infty} M_D(x,y_n).$$

Since $M_D(\,\cdot\,,y_n)$ is harmonic in a neighborhood of U for large n, we have

$$\mathbf{E}_x\left[M_D(X^D_{\tau_U},y_n)\right] \;=\; M_D(x,y_n), \quad x \in U.$$

By (7.1) and the bounded convergence theorem,

$$\lim_{n\to\infty} \mathbf{E}_x\left[M_D(X^D_{\tau_U},y_n)\right] \;=\; \mathbf{E}_x\left[M_D(X^D_{\tau_U},z)\right] \;=\; M_D(x,z), \quad x \in U. \qquad \square$$

Recall that a positive harmonic function u with respect to X in D is said to be minimal if, whenever v is positive harmonic with respect to X in D and $v \leq u$, then v is a constant multiple of u. The minimal Martin boundary of X^D is defined as

$$\partial_m D = \{\xi \in \partial_M D : M(\cdot,\xi) \text{ is minimal harmonic with respect to } X \text{ in } D\}.$$

Using Theorem 6.2 and Theorem 5.6 we can show that, for any compact subset K of D, the family $\{M_D(\cdot,y) : y \in \partial_M D\}$ is uniformly bounded and equicontinuous on K. One can then apply the Ascoli-Arzelà theorem to prove that, for every excessive function f of X^D, there are a unique Radon measure ν_1 on D and a unique finite measure ν_2 on $\partial_m D$ such that

$$f(x) = \int_D G_D(x,y)\nu_1(dy) + \int_{\partial_m D} M_D(x,z)\nu_2(dz), \tag{7.2}$$

and f is harmonic in D with respect to X if and only if $\nu_1 = 0$ (see Section 2.7 of [3]). When f is harmonic in D, the measure ν_2 above is called the Martin measure of f.

Now we will use the Green function estimates of Section 6 to show that there exists a homeomorphism between $\partial_M D \cup D$ and $\overline{D}$ which is an identity map in D. The next lemma is well-known.

LEMMA 7.2 (Maximum principle). *Suppose that h is a nonnegative function. If h is a harmonic function in D with respect to X and continuous on $\overline{D}$, then $\sup_{x\in\overline{D}} h(x) = \sup_{x\in\partial D} h(x)$.*

Proof. Take an increasing sequence of smooth domains $\{D_m\}_{m\geq 1}$ such that $\overline{D_m} \subset D_{m+1}$ and $\bigcup_{m=1}^{\infty} D_m = D$. By the bounded convergence theorem, we have

$$h(x) = \lim_{m\to\infty} \mathbf{E}_x\left[h(X_{\tau_{D_m}})\right] = \mathbf{E}_x\left[h\big(\lim_{t\uparrow\tau_D} X_t\big)\right].$$

Therefore, $\sup_{x\in\overline{D}} h(x) = \sup_{x\in\partial D} h(x)$. $\square$

LEMMA 7.3. *There exists a continuous map ι from $\partial_M D \cup D$ onto $\overline{D}$ which is an identity map in D. Moreover,*

$$M_D(x,w) \le c\,\frac{\rho(x)}{|x-\iota(w)|^d}, \quad w \in \partial_M D. \tag{7.3}$$

Proof. Note that $D \cup \partial_M D$ is a compact metric space. We will show that if a sequence $\{y_n\}_{n\ge 1}$ in D that converges to a point w in $\partial_M D$, it converges in $\overline{D}$. Assume that a subsequence $\{y_{n_k}\}_{k\ge 1}$ of $\{y_n\}_{n\ge 1}$ converges to $y_0 \in \overline{D}$. Let U be a neighborhood of y_0 and $x \in D \setminus \overline{U}$. By (7.1), we have

$$M_D(x,w) = \lim_{k\to\infty} M_D(x,y_{n_k}) \le c \liminf_{k\to\infty} \frac{\rho(x)}{|x-y_{n_k}|^d}.$$

So for every $y \in \partial D \setminus \overline{U}$, $\lim_{x\to y} M_D(x,w) = 0$. If there exists a subsequence of $\{y_n\}_{n\ge 1}$ that converges to a point in $\overline{D}$ different from y_0, the above argument shows that the Martin kernel would vanish continuously near ∂D. But this implies that $M_D(\,\cdot\,,w) \equiv 0$ by the maximum principle (Lemma 7.2), which is impossible. Therefore $y_0 \in \overline{D}$ must be unique. Moreover, y_0 must be in ∂D, for otherwise we could choose U in D and also argue that the Martin kernel would vanish continuously near ∂D.

Since $\{y_n\}_{n\ge 1}$ is bounded in $\overline{D}$, the above argument shows that every subsequence of $\{y_n\}_{n\ge 1}$ has a further subsequence converging to a unique point in $\overline{D}$. So the map ι defined by $\iota(w) = y_0$ is continuous and (7.3) is true.

Now we show that ι is onto. Fix a point $z_0 \in \partial D$ and choose a sequence $\{y_n\}_{n\ge 1}$ in D converging to a z_0 in $\overline{D}$. Since $\{y_n\}_{n\ge 1}$ is a sequence in the compact metric space $D \cup \partial_M D$, there exists a subsequence $\{y_{n_k}\}_{k\ge 1}$ of $\{y_n\}_{n\ge 1}$ that converges to a $w_0 \in D \cup \partial_M D$. By the continuity of ι, $\iota(w_0) = z_0$. □

LEMMA 7.4. *For each $w_0 \in \partial_M D$, the support of the Martin measure ν for $M_D(\,\cdot\,,w_0)$ is contained in $\iota^{-1}(\iota(w_0))$.*

Proof. Let $z_0 := \iota(w_0)$ and ν be the Martin measure ν for $M_D(\,\cdot\,,w_0)$. For any closed subset U of $\partial_M D$ such that $U \cap \iota^{-1}(z_0) = \emptyset$, define

$$h(x) = \int_U M_D(x,w)\nu(dw) \quad \text{ for } x \in D.$$

We will show that $h \equiv 0$, which implies that the support of ν is contained in $\iota^{-1}(z_0)$.

If $z \in \partial D$ is different from $z_0 = \iota(w_0)$, then by (7.3)

$$h(x) \le \int_{\partial_M D} M_D(x,w)\nu(dw) = M_D(x,w_0) \le c\,\frac{\rho(x)}{|x-z_0|^d} \to 0 \quad \text{as } x \to z.$$

On the other hand, for any $w \in U$,

$$\lim_{x\to z_0} M_D(x,w) \le c \lim_{x\to z_0} \frac{\rho(x)}{|x-\iota(w)|^d} = 0.$$

Moreover, for any $w \in U$ and $x \in D$ near z_0,

$$M_D(x,w) \le c\,\frac{\rho(x)}{|x-\iota(w)|^d}$$

is bounded. Therefore by the bounded convergence theorem, we get

$$\lim_{x\to z_0} h(x) = \lim_{x\to z_0}\int_U M_D(x,w)\nu(dw) = 0.$$

Thus the harmonic function $h(x)$ vanishes continuously on ∂D. So by Lemma 7.2, $h \equiv 0$ □

LEMMA 7.5. *The map ι is one-to-one. Moreover $\partial_m D = \partial_M D$.*

Proof. Fix $z_0 \in \partial D$. (7.2) and Lemma 7.4 imply that for each $w \in \iota^{-1}\{z_0\}$, there is a unique Martin measure ν_w for $M_D(\,\cdot\,,w)$ such that

$$M_D(x,w) \;=\; \int_{\iota^{-1}(z_0)\cap\partial_m D} M_D(x,a)\nu_w(da).$$

Therefore $\iota^{-1}(z_0)\cap\partial_m D \neq \emptyset$. Fix $w_0 \in \iota^{-1}(z_0)\cap\partial_m D$ and let $w_1,\ w_2 \in \iota^{-1}(z_0)$. Suppose $\{y^0_n\}_{n\ge1} \subset D$ converges to w_0 in $D\cup\partial_M D$ and $\{y^1_n\}_{n\ge1}$ in D converges to w_1 in $D\cup\partial_M D$. Then both $\{y^0_n\}_{n\ge1}$ and $\{y^1_n\}_{n\ge1}$ converge to z_0 in $\overline{D}$. By Theorem 6.2,

$$M_D(x,w_1) = \lim_{n\to\infty} M_D(x,y^1_n) \le c\lim_{n\to\infty} M_D(x,y^0_n) = cM_D(x,w_0).$$

The minimal harmonicity of $M_D(\,\cdot\,,w_0)$ implies that $M_D(x,w_1) = cM_D(x,w_0)$. But the two functions agree on $x = x_0$. So $M_D(\,\cdot\,,w_1) = M_D(\,\cdot\,,w_0)$. The same argument shows that $M_D(\,\cdot\,,w_2) = M_D(\,\cdot\,,w_0)$, thus $M_D(\,\cdot\,,w_2) = M_D(\,\cdot\,,w_1)$, which implies that $w_1 = w_2$. Therefore ι is one-to-one. The above argument also shows that every $w \in \iota^{-1}(\partial D)$ is a minimal Martin boundary point. Since ι is onto and $\iota(\partial_M D) = \partial D$ (Lemma 7.3), every $w \in \partial_M D$ is a minimal Martin boundary point of X^D. Therefore $\partial_m D = \partial_M D$ □

Lemmas 7.3 and 7.5 imply there exists a homeomorphism between $\partial_m D\cup D$ and $\overline{D}$ which is an identity map in D. Therefore we arrive the following result.

THEOREM 7.6. *There is a one-to-one correspondence between the minimal Martin boundary $\partial_m D$ and the Euclidean boundary ∂D.*

As a consequence of Theorem 6.2 and the above results, we immediately get the following result.

THEOREM 7.7. *There exists $c = c(x_0, D)$ such that for all $x \in D$ and $z \in \partial D$,*

$$\frac{1}{c}\,\frac{\rho(x)}{|x-z|^d} \;\le\; M_D(x,z) \;\le\; c\,\frac{\rho(x)}{|x-z|^d}.$$

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