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MIXING AND DESCRIPTIVE SET THEORY

ROBERT KAUFMAN

If you seek his monument, look around you.

ABSTRACT. A complete analytic set is defined through the concepts of invariant measure mixing.

0. Introduction

Let X be a compact metric space and $H(X)$ the group of all homeomorphisms of X. The subset $H_{\mathbf{m}}(X)$ is then defined as follows: $T \in H_{\mathbf{m}}(X)$ *provided there is a T-invariant probability measure μ such that T defines a mixing operator in $L^2(\mu)$.*

The main results are then:

EXAMPLE (Siboni). For a certain space X_1, $H_{\mathbf{m}}$ isn't closed in H.

THEOREM 1. *$H_{\mathbf{m}}$ is always an analytic subset of H.*

THEOREM 2. *For a certain space X_2, $H_{\mathbf{m}}(X_2)$ is a complete analytic subset of $H(X_2)$.*

Theorem 2 suffers from a stylistic defect: the space X_2 has infinite topological dimension. This is remedied by re-working the example with a space X_3 of dimension 0, and Theorem 2 with a space of dimension 1.

We now provide some detail on the group $H(X)$; we write $h(X)$ for the set of continuous maps of X into X (sometimes called *self-maps* of X). Then $h(X)$ is provided with the uniform metric

$$d^*(f, g) \equiv \sup\{d(f(x), g(x))\}.$$

$h(X)$ is then a Polish space [2, I, p. 244] whose open sets depend only on the open sets of X, not on the metric d. Composition is jointly continuous in

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$h(X)$ as we see from the inequalities

$$\begin{aligned} d^*(f_n \cdot g_n, f \cdot g) &\le d^*(f_n \cdot g_n, f \cdot g_n) + d^*(f \cdot g_n, f \cdot g) \\ &\le d^*(f_n, f) + d^*(f \cdot g_n, f \cdot g). \end{aligned}$$

Supposing then that $\lim f_n = f$, $\lim g_n = g$, we have only to observe that the last distance tends to 0 because f is uniformly continuous.

The metric in $H(X)$ is defined by

$$d_*(h_1, h_2) \equiv d^*(h_1, h_2) + d^*(h_1^{-1}, h_2^{-1}).$$

Then $H(X)$ is a G_δ set in $h(X)$ and is homeomorphic to a closed subset of $h(X) \times h(X)$. We observe a whimsical definition of the metric

$$d^*(I, h_2 h_1^{-1}) + d^*(I, h_2^{-1} h_1).$$

A classical source for analytic sets is [2], especially [2, I, pp. 446, 447, 453]. Theorem 2 means that for each analytic set A in a Polish space W, there is a continuous map φ of W into $H(X_2)$, such that $\varphi^{-1}(H_{\mathbf{m}}(X_2)) = A$.

1. Example

Let K be the Cantor set, represented as infinite sequences $y = (\epsilon_p)_{-\infty}^{\infty}$ ($\epsilon_p = -1, +1$), and S^1 a circle represented as $R/2\pi$. Let $0 < \alpha < 1$, and let (α_n) be a sequence of numbers converging to 0 such that each $\alpha_n^{-1}\pi$ is irrational, and let σ be the left shift on K. We define

$$T_n(y, z) \equiv (\sigma y, z + \alpha + \alpha_n \epsilon_0(y)).$$

Skew products of this type have been investigated by Siboni, for example in [5]. Here each T_n is mixing for the usual product measure on $K \times S^1$, while its limit, T_∞, has a nonconstant eigenfunction (and thus isn't in $H_{\mathbf{m}}$).

Thanks are due to Manfred Denker for references to the literature on skew products.

2. Theorem 1

Let $\mathbb{P}(X)$ be the set of probability measures in X, endowed with its usual w^* topology, and $(g_p)_1^\infty$ a dense sequence in the real B-space $C(X)$. Let Mix be the subset of $H(X) \times \mathbb{P}(X)$ containing pairs (T, μ) such that

(i) μ is T-invariant,
(ii) T is mixing in $L^2(\mu)$.

In a moment we'll see that Mix is a set of type $F_{\sigma\delta}$ in $H \times \mathbb{P}$. Since its projection in $H(X)$ is $H_{\mathbf{m}}(X)$, the latter set is analytic. Clearly (i) defines a closed set. The meaning of (ii) is

$$\lim \int (g_p \cdot T^k) g_q d\mu = \int g_p d\mu \int g_q d\mu$$

for each p and q and clearly this defines an $F_{\sigma\delta}$ (and in fact implies that μ is T-invariant). We observe in passing that the set Weak Mix—defined by analogy with Mix—is a G_δ in $H \times \mathbb{P}$.

3. Theorem 2

Let N^* be the set $\{1, 2, 3, \dots, +\infty\}$ regarded as the Alexandroff compactification of N, and $Y = X_1 \times N^*$. We define an element T of $H(Y)$ by the formula

$$T(x, m) = (T_m x, m) \quad (x \in X_1, m \in N^*).$$

In the product space $Z = Y^N$, we use S to denote the homeomorphism which operates by T in each factor Y; finally, we set $X_2 = Z \times S^1$. Let f be a continuous mapping on Z into S^1; we define a skew product of S and f by the formula

$$S'(z, \theta) \equiv (Sz, \theta + f(z)) \quad (z \in Z, \theta \in S^1)$$

and we use the symbol $S \ltimes \rho(f)$ for S'. We observe that the inverse of $S \ltimes \rho(f)$ is $S^{-1} \ltimes \rho(-f \cdot S^{-1}))$. Thus, keeping S fixed, we see that $S \ltimes \rho(f)$, as an element of $H(X_2)$, depends continuously on f.

When f has a very special form, we can decide precisely when $S \ltimes \rho(f)$ belongs to $H_{\mathbf{m}}(X_2)$. An element $(m_1, m_2, m_3, \dots)$ of $\bar{N}$ will be called *irrational* (by analogy with continued fractions) if all $m_k < +\infty$. The set $\mathcal{I}$ of *irrationals* is therefore homeomorphic to the Baire null-space $N \times N \times N \times \cdots$, customarily denoted by $\mathcal{N}$. We impose the requirement that $f(z_1, m_1, z_2, m_2, z_3, m_3, \dots)$ depends only on $(m_1, m_2, m_3, \dots)$, so that f is effectively a continuous map of $\bar{N} \equiv N^* \times N^* \times N^* \times \cdots$ into S^1.

Lemma. *$S \ltimes \rho(f)$ belongs to $H_{\mathbf{m}}(X_2) \Longleftrightarrow f$ has a zero in $\mathcal{I}$.*

Proof. If $f = 0$ at an irrational $(m_1^0, m_2^0, m_3^0, \dots)$, we observe that $S \ltimes \rho(f)$ leaves the coefficients m_1, m_2, $m_3, \dots$ unchanged. Restricting $S \ltimes \rho(f)$ to the subset $m_1 = m_1^0$, $m_2 = m_2^0$, $m_3 = m_3^0, \dots, \theta = 0$, we see that $S \ltimes \rho(f)$ belongs to $H_{\mathbf{m}}(X_2)$.

Conversely, suppose that μ is invariant for $S \ltimes \rho(f)$, and that $S \ltimes \rho(f)$ is mixing in $L^2(\mu)$. Since μ is ergodic, the variables m_1, m_2, $m_3, \dots$ must be constant on the closed support of μ, hence f is constant μ-a.e. Since μ is mixing—or just weakly mixing—we see that $f = 0$ μ-a.e. and $(m_1, m_2, m_3, \dots)$ is a fixed irrational value μ-a.e. Hence f has a zero in $\mathcal{I}$, proving the lemma. $\square$

The last step is classical, and seems to go back to Mazurkiewicz and Sierpiński. Let A be an analytic set in a Polish space W, and let δ be a metric in $\bar{N} \times W$ of diameter $< 2\pi$. Let ψ be a continuous mapping of $\mathcal{I}$ onto A. Let r be the function on $\bar{N} \times W$ whose value at $(\bar{n}, w)$ is the distance to the graph of ψ, i.e., the set of pairs $(\bar{n}, \psi(\bar{n}))$, where $\bar{n} \in \mathcal{I}$. We observe that $\sup r < 2\pi$, that r is uniformly continuous on $\bar{N} \times W$; for each w, the partial function $f(\cdot, w)$ mapping $\bar{N}$ into $[0, 2\pi)$ has an irrational zero iff $w \in A$. (Here we use the continuity of ψ on $\mathcal{I}$.)

To define a continuous mapping φ of W into $H(X_2)$ we map w to $S \ltimes \rho(f(\cdot, w))$, so that $\varphi(w) \in H_m(X_2) \Longleftrightarrow$ the partial function has an irrational zero $\Longleftrightarrow w \in A$. This is Theorem 2.

4. Improving the example

If the example is obtained with a space of dimension 0 then the set X_2 will have dimension 1, and will in fact be homeomorphic to $K \times S^1$. Let α be any continuous map of the Cantor set K onto X_1. For each $m = 1, 2, 3, \ldots, +\infty$ we define a subset F_m of K^Z, which is shift-invariant and can be described as the "subshift over T_m by α." By definition an element (x_k) of K^Z belongs to F_m if for each k $\alpha(x_{k+1}) = T_m \alpha(x_k)$. Denoting by π_0 the mapping $x \mapsto \alpha(x_0)$ of K^Z onto X_1, we see that the elements of F_m satisfy the relation $\pi_0 \sigma(x) = T_m \pi_0(x)$. From this it follows that $\sigma \mid F_\infty$ doesn't belong to $H_{\mathbf{m}}(F_\infty)$. We shall see in a moment that the reverse is true for $1 \leq m < +\infty$.

To establish the last assertion we use a Borel *selector* of α, that is a Borel map τ of X_1 into K such that $\alpha\tau = \mathrm{id}$ on X_1. The existence of the map τ is discussed in a final paragraph. We use τ to define a map $\tau^\#$ of X_1 into K^Z: the k-th co-ordinate of $\tau^\# y$ is $\tau(T_m^k y)$ $(y \in X_1, k \in Z)$. Then $\tau^\# X_1 \subseteq F_m$ and $\sigma \circ \tau^\# = \tau^\# \circ T_m$. Hence an invariant measure μ for which T_m is mixing yields an invariant measure ν for σ (in F_m) for which σ is mixing, namely that defined by $\nu(B) \equiv \mu(\tau^{\#-1}(B))$.

We observe that the metrical lim sup of the sequence $(F_m)_1^\infty$ is contained in F_∞; stated differently, every neighborhood of F_∞ contains all but a finite number of the sets F_m. We now define a closed set L of $K^Z \times N^*$ as the union of the sets $F'_m = F_m \times \{m\}$ for $1 \leq m \leq +\infty$; σ acts on L, leaving the co-ordinate m unchanged.

Finally, we make the Cantor set K into an abelian topological group (it does not matter how) and look at a skew product $\sigma \ltimes \rho(f)$, where f is a continuous map of L into K. We suppose that f is constant on each of the sets F'_m, $1 \leq m \leq +\infty$. We see that $\sigma \ltimes \rho(f)$ is in $H_{\mathbf{m}}(L \times K)$ if and only if $f = 0$ on one of the sets F'_m, $1 \leq m < +\infty$. Hence $H_{\mathbf{m}}(L \times K)$ isn't closed in $H(L \times K)$, as claimed.

5. On selectors

Here is a simple way to find the selector τ. We place K in the interval $[0, 1]$ and define

$$\tau(y) = \sup\{x \in K, \alpha(x) = y\},$$

for each y in X_1. We observe that $\tau(y) \geq t \Longleftrightarrow y \in \alpha(K \cap [t, 1])$, for $0 \leq t \leq 1$, so that $\{\tau < t\}$ is open, and τ is upper semicontinuous. One can also use a theorem on selectors by Kuratowski and Ryll-Nardzewski, stated in [2, II, p. 74] and proved in [4], [3, pp. 570–576], [1, pp. 8–13].

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Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801, USA

E-mail address: rpkaufma@math.uiuc.edu